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Disturbance Decoupling in Nonlinear Impulsive Systems

Elena Zattoni, Anna Maria Perdon, Giuseppe Conte and Claude H. Moog

Abstract— **This work deals with the problem of structural disturbance decoupling by state feedback for nonlinear impulsive systems. The dynamical systems addressed exhibit a hybrid behavior characterized by a nonlinear continuous-time state evolution interrupted by abrupt discontinuities at isolated time instants. The problem considered consists in finding a state feedback such that the system output is rendered totally insensitive to the disturbance. Both the case of static state feedback and that of dynamic state feedback are considered. A necessary and sufficient condition for the existence of a static state feedback that solves the problem in the multivariable case is proven by defining suitable tools in the context of the differential geometric approach. The situation concerning solvability by a dynamic state feedback is examined in the framework of the differntial algeraic approach. A necessary and sufficient solvaility condition is conjectured and discussed.**

I. Introduction

Impulsive systems are hybrid dynamical systems that feature a continuous-time state evolution except at isolated points of the time axis, where they show abrupt state discontinuities. Due to their effectiveness in modeling the behavior of mechanical, electrical or biological systems in several peculiar situations, such as collisions, switches and shocks, respectively , they have attracted considerable interest in the recent years (see, e.g., [1]–[5]).

In particular, this work is focused on nonlinear impulsive systems whose continuous-time state evolution is interrupted by discontinuities owed to time-triggered jumps. In this context, the available studies are almost exclusively focused on stability analysis and stabilization methods. The current literature provides some results about stability analysis for nonlinear impulsive systems based on discontinuous Lyapunov functions [6], switched Lyapunov functions and LMIs [7], average impulse interval [8], 2D time domain representation and vector Lyapunov functions [9]. The design of stabilizing feedbacks is investigated in [7] and optimal control problems for nonlinear impulsive systems are discussed in [3].

Differently from the papers mentioned above, the aim of this work is to investigate and solve, in the context of nonlinear impulsive systems, the problem of disturbance decoupling: namely, the problem of making the output totally insensitive to a disturbance input. Indeed, the attention will be restricted to nonlinear impulsive systems whose continuous-time dynamics is affine in the control and in the disturbance. Nevertheless, nonlinear continuous-time dynamics of this kind are sufficiently general to model, at least locally, a large number of behaviors of interest in control theory and applications (see, e.g., [10], [11]).

As to the problem of disturbance decoupling, this is a classic problem in control theory and, in the framework of input-affine nonlinear systems, its solvability by static state feedback was first characterized in [12], through a differential geometric approach (see also [13]). A key ingredient of that approach is the notion of controlled invariant distribution, which can be extended to the present framework as it will be shown in Section III. This makes it possible to tackle the disturbance decoupling problem by static state feedback for nonlinear impulsive systems, as formally stated in Section II, by means of a differential geometric approach and to provide, as proven in Section IV (Theorem 1), a constructive solvability condition .

It is worthwhile noting that the differential geometric approach of [13] generalizes to the nonlinear context the geometric approach originally developed for linear systems (see [14] and [15]). As far as disturbance decoupling is concerned, the geometric approach of [14] and [15] has proven to be particularly efficient in providing the methodological background and the tools needed to handle the problem for different classes of hybrid linear systems: specifically, switching linear systems (like, e.g., in [16] and [17]) and impulsive linear systems (like in [18]). As to hybrid nonlinear systems, some pioneering results grounded on the differential geometric approach of [13] are found in the context of switching nonlinear systems [19]. Instead, the investigation of the more convenient tools to deal with disturbance decoupling for nonlinear impulsive systems is, to the best of the authors' knowledge, totally open.

In this regard, it is well known that, beside the differential geometric approach, also the differential algebraic approach developed in [20] can be successfully applied to achieve the solution of the disturbance decoupling problem, as well as of other noninteractive control problems, in the nonlinear framework. Indeed, the differential algebraic approach is especially suitable to search for dynamic state feedback solutions. In light of this consideration, in Section V, we conjecturaly state in Theorem 2 a necessary and sufficient condition for solvability of the disturbance decoupling problem by a dynamic state feedback, we discuss its motivations and we outline a possible way to prove it. An example that illustrates our reasoning is worked out.

The work described in this paper focuses only on the the

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structural aspects of the disturbance decoupling problem, without taking into account the aspects relatated to the stability of the closed-loop system. In this regard, it is worth mentioning that setting forth the conditions that characterize the problem solvability from the structural point of view is, in any case, a preliminary, necessary step towards the investigation of the case (obviously more interesting for its impact on applications) where also the stability of the compensated system is required. The analysis of this more general case will be the object of future work.

Notation The symbols $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{R}^+$ are used to denote the sets of natural numbers (including 0), real numbers, and nonnegative real numbers, respectively. Given a smooth real valued function $\lambda(x)$ defined on a subset $U \subseteq \mathbb{R}^n$, the symbol $d\lambda$ denotes the co-vector field, called also the gradienf of λ, defined for any $x \in U$ by

$$d\lambda(x) = \left[\begin{array}{ccc} \dfrac{\partial \lambda}{\partial x_1}(x) & \dots & \dfrac{\partial \lambda}{\partial x_n}(x) \end{array} \right].$$

Given a smooth real valued function λ and a smooth vector field f defined on a subset $U \subseteq \mathbb{R}^n$, the derivative of λ along f is the smooth real valued function $L_f\lambda$ defined for any $x \in U$ by

$$L_f\lambda(x) = \sum_{i=1}^{n} \frac{\partial \lambda}{\partial x_i} f_i(x),$$

where f_i is the i-th component of f. Given two smooth vector fields f_1 and f_2 defined on a subset $U \subseteq \mathbb{R}^n$, the symbol $[f_1, f_2]$ denotes their Lie product: i.e., the vector field defined for any $x \in U$ by

$$[f_1, f_2](x) = \frac{\partial f_2}{\partial x} f_1(x) - \frac{\partial f_1}{\partial x} f_2(x).$$

Given a smooth vector field f and a co-vector ω defined on a subset $U \subseteq \mathbb{R}^n$, the derivative of ω along f is the co-vector field $L_f\omega$ defined for any $x \in U$ by

$$L_f\omega(x) = \left[f^\top(x) \frac{\partial \omega^\top}{\partial x} \right] + \omega(x) \frac{\partial f}{\partial x}$$

where f_i is the i-th component of f.

II. Preliminaries and Problem Statement

A basic element in the mathematical description of an impulsive system is the structure given to the hybrid time domain. In order to describe it, assume that the time variable t belongs to the nonnegative real axis $\mathbb{R}^+$ and that $\mathscr{T}$ is a countably infinite, ordered set of isolated points of $\mathbb{R}^+$. In particular, let

$$\mathscr{T} = \{t_k, t_k \in \mathbb{R}^+\}_{k \in \mathbb{N}},$$

with $t_0 = 0$ and assume $\tau_{\mathscr{T}} = \inf \{(t_{k+1} - t_k), t_{k+1}, t_k \in \mathscr{T}\} > 0$.

With a slight abuse of notation, the union of the right-open intervals of $\mathbb{R}^+$ between consecutive elements of $\mathscr{T}$ will be denoted by $\mathbb{R}^+ \setminus \mathscr{T}$.

Hence, with the notation introduced above, a nonlinear (input-affine) impulsive system $\Sigma_{\mathscr{T}}$ is described by

$$\Sigma_{\mathscr{T}} \equiv \left\{ \begin{array}{rcl} \dot{x}(t) & = & f(x(t)) + \sum_{i=1,\dots,m} g_i(x(t))\, u(t) \\ & & \text{with } t \in \mathbb{R}^+ \setminus \mathscr{T}, \\ x(t_k) & = & \phi(x^-(t_k)) \quad \text{with } t_k \in \mathscr{T}, \\ y(t) & = & h(x(t)) \quad \text{with } t \in \mathbb{R}^+, \end{array} \right. \tag{1}$$

where $x \in \mathbf{X} = \mathbb{R}^{\mathbf{n}}$, $u \in \mathbf{U} = \mathbb{R}^{\mathbf{m}}$ and $y \in \mathbf{Y} = \mathbb{R}^{\mathbf{p}}$ respectively denote the state, the input and the output variables; f, ϕ and the columns of the $(n \times m)$-matrix $g = [g_1 \ \dots \ g_m]$ are smooth vector fields; the components $h_1, \dots, h_p$ of the map h are smooth functions; $x^-(t_k)$ denotes the limit of $x(t)$ when t goes to t_k from the left: i.e.,

$$x^-(t_k) = lim_{t \to t_k^-} x(t).$$

It is understood that the state evolution of $\Sigma_{\mathscr{T}}$ is given by the solutions of (1) which are continuous on the left and limited on the right.

The first block of equations in (1) defines the so-called *flow dynamics* of $\Sigma_{\mathscr{T}}$, while the second block defines the so-called *jump behavior*.

The parameter $\tau_{\mathscr{T}}$ introduced above is generally referred to as the *dwell-time*. The condition $\tau_{\mathscr{T}} > 0$ implies that $\mathscr{T}$ has no accumulation points in $\mathbb{R}^+$ and its relevance lies in that this avoids the occurrence of Zeno phenomena in the behavior of $\Sigma_{\mathscr{T}}$.

It is worthwhile to point out that the overall behavior of $\Sigma_{\mathscr{T}}$ depends on the vector fields and functions that appear in (1) as well as on the specific set $\mathscr{T}$. Nevertheless, in many circumstances (as it will be the case in this work), the design goals are required to be achieved *for all* $\mathscr{T}$.

In the following, the considered nonlinear impulsive system will be subject to an additional disturbance input. Hence, it will be denoted by $\Sigma_{\mathscr{T}\mathcal{D}}$ and described by

$$\Sigma_{\mathscr{T}\mathcal{D}} \equiv \left\{ \begin{array}{rcl} \dot{x}(t) & = & f(x(t)) + g(x(t))\, u(t) + p(x(t))\, w(t) \\ & & \text{with } t \in \mathbb{R}^+ \setminus \mathscr{T}, \\ x(t_k) & = & \phi(x^-(t_k)) \quad \text{with } t_k \in \mathscr{T}, \\ y(t) & = & h(x(t)) \quad \text{with } t \in \mathbb{R}^+, \end{array} \right. \tag{2}$$

where $w \in \mathcal{D} = \mathbb{R}^q$ denotes the disturbance input and the columns of the $(n \times q)$-matrix $p = [p_1 \ \dots \ p_q]$ are smooth vector fields.

With reference to $\Sigma_{\mathscr{T}\mathcal{D}}$, the problem of making the output $y(t)$ insensitive to the disturbance $w(t)$ (or, in other words, the problem of *decoupling* the output from the disturbance), will be given two different statements.

The former statement deals with a *static* state feedback compensation scheme.

Problem 1: Disturbance Decoupling by Static State Feedback. Let the system $\Sigma_{\mathscr{T}\mathcal{D}}$, described by (2), be given. The Disturbance Decoupling Problem by Static State Feedback (DDPSF) for $\Sigma_{\mathscr{T}\mathcal{D}}$ consists in finding a state feedback

$$u(t) = F_s\left(x(t), v(t)\right)$$

of the form

$$u(t) = \alpha(x(t)) + \beta(x(t))\, v(t), \tag{3}$$

where $v \in \mathcal{V} = \mathbb{R}^m$, while the components $\alpha_1, \ldots, \alpha_m$ of the map α and the entries β_{ij} of the $(m \times m)$-matrix β are smooth functions, in such a way that the disturbance $w(t)$ does not affect the output $y(t)$ of the compensated system

$$\Sigma^{F_s}_{\mathscr{T}\mathcal{D}} \equiv \begin{cases} \dot{x}(t) \;=\; f(x(t)) + g(x(t))\, \alpha(x(t)) + \\ \qquad\qquad g(x(t))\beta(x(t))\, v(t) + p(x(t))\, w(t) \\ \qquad\qquad \text{with } t \in \mathbb{R}^+ \setminus \mathscr{T}, \\ x(t_k) \;=\; \phi(x^-(t_k)) \quad \text{with } t_k \in \mathscr{T}, \\ y(t) \;=\; h(x(t)) \quad \text{with } t \in \mathbb{R}^+ \end{cases} \tag{4}$$

for all $\mathscr{T}$ with $\tau_{\mathscr{T}} > 0$.

The second statement deals with *dynamic* state feedback compensation.

Problem 2: Disturbance Decoupling by Dynamic State Feedback. Let the system $\Sigma_{\mathscr{T}\mathcal{D}}$, described by (2), be given. The Disturbance Decoupling Problem by Dynamic State Feedback (DDPDF) for $\Sigma_{\mathscr{T}\mathcal{D}}$ consists in finding a dynamic feedback rrr of the form

$$F_d \equiv \begin{cases} \dot{z}(t) \;=\; \gamma(x(t), z(t)) + \delta(x(t), z(t))\, v(t) \\ \qquad\qquad \text{with } t \in \mathbb{R}^+ \setminus \mathscr{T}, \\ z(t_k) \;=\; 0 \quad \text{with } t_k \in \mathscr{T}, \\ u(t) \;=\; \alpha(x(t), z(t)) + \beta(x(t), z(t))\, v(t) \\ \qquad\qquad \text{with } t \in \mathbb{R}^+, \end{cases} \tag{5}$$

where $z \in \mathcal{Z} = \mathbb{R}^s$, while γ and the columns of the $(s \times m)$-matrix $\delta = [\delta_1 \;\ldots\; \delta_m]$ are smooth vector fields and the components $\alpha_1, \ldots, \alpha_m$ of the map α and the entries β_{ij} of the $(m \times m)$-matrix β are smooth functions, in such a way that the disturbance $w(t)$ does not affect the output $y(t)$ of the compensated system

$$\Sigma^{F_d}_{\mathscr{T}\mathcal{D}} \equiv \begin{cases} \dot{x}(t) \;=\; f(x(t)) + g(x(t))\, \alpha(x(t), z(t)) + \\ \qquad\qquad g(x(t))\beta(x(t), z(t))\, v(t) + p(x(t))\, w(t) \\ \dot{z}(t) \;=\; \phi(x(t), z(t)) + \gamma(x(t), z(t))\, v(t) \\ \qquad\qquad \text{with } t \in \mathbb{R}^+ \setminus \mathscr{T}, \\ x(t_k) \;=\; \phi(x^-(t_k)) \\ z(t_k) \;=\; 0 \quad \text{with } t_k \in \mathscr{T}, \\ y(t) \;=\; h(x(t)) \quad \text{with } t \in \mathbb{R}^+, \end{cases} \tag{6}$$

for all $\mathscr{T}$ with $\tau_{\mathscr{T}} > 0$.

Remark 1: As is shown by (5), the controller F_d considered in the statement of Problem 2 has the same impulsive structure as the disturbed system $\Sigma_{\mathscr{T}\mathcal{D}}$. In particular, the jump behavior of the controller F_d consists of a reset action. The reason is that it is assumed that the controller acts on the system on each time interval $[t_k, t_{k+1})$ in the same way as it does on the first time interval $[0, t_1)$, where it is natural to take $z(0) = 0$.

Remark 2: In expressing solvability conditions to nonlinear control problem, it is generally necessary to assume a local point of view, i.e. to restrict the attention only to a neighborhood U of a given point of interest x_0. In the present context, due the presence of jumps in the state behavior, the adoption of a local point of view needs some care. In facts, all trajectories starting from x_0 at $t = 0$ could leave U at time t_1, making the local point of view inapplicable or deprived of practical interest. This can be avoided by assuming, in each specific case, that the jumps do not drive the state out of the considered neighborhood, i.e. by assuming that $\phi(U) \subseteq U$. This additional condition will therefore be taken into account in stating the results of Section IV.

III. Differential Geometric Approach to DDP for Nonlinear Impulsive Systems

The notion of *distribution*, as given in [13, Section 1.3], plays a fundamental role in analyzing the structure of systems characterized by nonlinear dynamics.

Given the smooth vector fields $f_1, \ldots, f_d$ defined on a subset $U \subseteq \mathbb{R}^n$, the symbol Δ will be used to denote the smooth distribution that assigns the vector space

$$\Delta(x) = \operatorname{span}\{f_1(x), ..., f_d(x)\}$$

to each point $x \in U$. Consequently, the short notation

$$\Delta = \operatorname{span}\{f_1, ..., f_d\}$$

will also be used.

In the following, some definitions and properties functional to the statement of the necessary and sufficient condition for the solution of DDPSF are reviewed.

A distribution Δ defined on a subset $U \subseteq \mathbb{R}^n$ is said to be *nonsingular* if

$$\dim \Delta(x) = d,$$

for some nonnegative integer d and for all $x \in U$. The nonnegative integer d is called the dimension of Δ.

A point $x_0 \in U$ is said to be *regular* for Δ if there exists a neighborhood $U_0 \subseteq U$ of x_0 on which Δ is nonsingular.

If x_0 is a regular point for a smooth distribution Δ, then, for all x belonging to a neighborhood U_0 of x_0, Δ is of the form

$$\Delta(x) = \operatorname{span}\{f_1(x), \ldots, f_d(x)\},$$

for some nonnegative integer d, where f_i is a smooth vector field defined on $U_0 \subseteq \mathbb{R}^n$ for $i = 1, \ldots, d$, and the vectors $f_1(x), \ldots, f_d(x)$ are linearly independent.

A vector field f is said to belong to a distribution Δ defined on $U \subseteq \mathbb{R}^n$, if $f(x) \in \Delta(x)$ for all $x \in U$. If x_0 is a regular point for Δ, it follows that

$$f(x) = \sum_{i=1}^{d} c_i(x)\, f_i(x),$$

for all $x \in U_0$, for a neighborhood U_0 of x_0 and smooth real functions $c_1, \ldots, c_d$.

A distribution Δ is said to be *involutive* if the vector field $[f_1, f_2]$ belongs to Δ for any pair of vector fields f_1, f_2 in Δ.

Let $[f, \Delta]$ denote the distribution consisting of all the vector fields of the form $[f, f_1]$, where f is a given vector field and $f_1 \in \Delta$. Then, the following definitions and results are reported from [13] for the reader's convenience.

Definition 1: [13, Section 1.6] Given a vector field f and a distibution Δ defined on $U \subseteq \mathbb{R}^n$, Δ is said to be *invariant* with respect to f if $[f, \Delta]$ is contained in Δ: i.e., if

$$[f, \Delta](x) \subseteq \Delta(x) \tag{7}$$

for all $x \in U$.

Definition 2: [13, Section 6.2] Given the vector fields f and g_i, with $i = 1, \ldots, m$, and a distribution Δ defined on $U \subseteq \mathbb{R}^n$, Δ is said to be

(i) *controlled invariant* with respect to f and $g = [g_1 \ \ldots \ g_m]$ if there exists maps $\alpha : U \to \mathbb{R}^m$ and $\beta_i : U \to \mathbb{R}^m$ for $i = 1, ...m$ whose components α_i and β_{ij}, for $i = 1, ..., m$ and $j = 1, ..., m$ are smooth functions, such that

$$[\tilde{f}, \Delta](x) \subseteq \Delta(x) \tag{8}$$

$$[\tilde{g}_i, \Delta](x) \subseteq \Delta(x) \quad \text{with } i = 1, \ldots, m, \tag{9}$$

where $\tilde{f} = f + g\,\alpha$ and $\tilde{g}_i = (g\,\beta)_i$, with $i = 1, \ldots, m$, hold for all $x \in U$;

(ii) *locally controlled invariant* with respect to f and $g = [g_1 \ \ldots \ g_m]$ if, for any $x_0 \in U$, there exists a neighborhood U_0 of x_0 such that Δ is controlled invariant on U_0.

The key result about controlled invariant distributions is given by the following proposition.

Proposition 1: [13, Lemma 6.2.1] Let the vector fields f and g_i, with $i = 1, \ldots, m$, and a distribution Δ be given. Let $G = \text{span}\{g_1, ..., g_m\}$. Let Δ, G and $\Delta + G$ be nonsingular on U. Then, Δ is locally controlled invariant with respect to f and $g = [g_1 \ \ldots \ g_m]$ if and only if

$$[f, \Delta](x) \subseteq \Delta(x) + G(x) \tag{10}$$

$$[g_i, \Delta](x) \subseteq \Delta(x) + G(x) \quad \text{with } i = 1, \ldots, m \tag{11}$$

hold for all $x \in U$.

If the vector fields f and g_i define a nonlnear control dynamics Σ by the equation $\dot{x}(t) = f(x(t)) + g(x(t))u(t)$ with $g = [g_1 \ldots g_m]$, a locally controlled invariant distribution with respect to f and g is said locally controlled invariant for Σ and the maps α and β_i of Definition 2 are understood to define locally a feedback $u(t) = \alpha(x(t)) + \beta(x(t))v(t)$, with $\beta = [\beta_i \ldots \beta_m]$. This makes the notion of controlled invariant distribution crucial to characterize the local solvability (i.e., the solvability in a neighborhood of a given point $x_0 \in \mathbf{X}$) of the disturbance decoupling problem in the framework of nonlinear systems [12] (see the discussion of the problem in [13]). In order to derive suitable solvability conditions in the framework of the nonlinear impulsive systems considered herein, the following definition is introduced.

Definition 3: Let the system $\Sigma_{\mathscr{FD}}$, described by (2) and a distribution Δ defined on $U \subseteq \mathbb{R}^n$ be given. The distribution Δ is said to be an hybrid controlled invariant distribution (respectively, an hybrid locally controlled invariant distribution) for $\Sigma_{\mathscr{FD}}$ if it is controlled invariant (respectively, locally controlled invariant) with respect to f and $g = [g_1 \ \ldots \ g_m]$ and invariant with respect to ϕ: i.e., if (10) and (11) hold together with

$$[\phi, \Delta](x) \subseteq \Delta(x) \tag{12}$$

In addition to the concept of distribution, the dual notion of co-distribution will also be considered and used.

A *codistribution* Ω defined on a subset $U \subseteq \mathbb{R}^n$ is the assignment of a subspace of the dual vector space $(\mathbb{R}^n)^*$ to each point $x \in U$.

The *annihilator* of a given distribution Δ is the co-distribution $\Delta^\perp$ defined by

$$\Delta^\perp(x) = \{\omega^* \in (\mathbb{R}^n)^* \text{ such that } \langle \omega^*, v \rangle = 0 \text{ for all } v \in \Delta(x)\}$$

and conversely the *annihilator* of a given co-distribution Ω is the distribution $\Omega^\perp$ defined by

$$\Omega^\perp(x) = \{v \in \mathbb{R}^n \text{ such that } \langle \omega^*, v \rangle = 0 \text{ for all } v^* \in \Omega(x)\}$$

With the obvious meaning of the terms, a co-distribution can be *smooth* and a co-distribution can be *nonsigular*.

The short notation

$$\Omega = \text{span}\{\omega_1^*, ..., \omega_d^*\}$$

is used to indicate that the codistribution Ω is spanned by the co-vector fields $\omega_1^*, \ldots, \omega_d^*$, that is

$$\Omega(x) = \text{span}\{\omega_1^*(x), ..., \omega_d^*(x)\}.$$

The short notation $\text{Ker}\,(dh)$ is used to denote the distribution defined by

$$\text{Ker}\,(dh) = (\text{span}\{dh_i, ..., dh_m\})^\perp.$$

Then, the main result of this section is established.

Proposition 2: Let $\Omega_0 = \text{span}\{dh_i, ..., dh_m\}$ and consider the sequence of co-distributions Ω_k, recursively defined by

$$\Omega_k = \Omega_{k-1} + L_f\,(\Omega_{k-1} \cap G^\perp) + \sum_{i=1}^{m} L_{g_i}\,(\Omega_{k-1}) + \\ + L_\phi\,(\Omega_{k-1}) \quad \text{with } k = 1, 2, \ldots \tag{13}$$

If there exists an integer $\bar{k}$ such that $\Omega_{\bar{k}+1} = \Omega_{\bar{k}}$, then the sequence is stationary, i.e. $\Omega_k = \Omega_{\bar{k}}$ for all $k \geq \bar{k}$. If, in addition,

(i) $\Omega_{\bar{k}} \cap G^\perp$ and $\Omega_{\bar{k}}$ are smooth,
(ii) $\Omega_{\bar{k}}^\perp$, G and $\Omega_{\bar{k}}^\perp + G$ are nonsingular,

then $\Omega_{\bar{k}}^\perp$ is involutive and it is the maximal hybrid locally controlled invariant distribution contained in $\text{Ker}\,(dh)$.

Proof: The result follows from [13, Lemma 6.3.4] and [13, Lemma 6.3.2]. In fact, with respect to the controlled invariant distribution algorithm considered in [13, Chapter 6, equation (3.2)], the term $L_\phi\,(\Omega_{k-1})$ is added in (13) to guarantee invariance of $\Omega_{\bar{k}}$ with respect to ϕ. ■

Notation Under the assumptions of Proposition 2, the notion of *maximal hybrid locally controlled invariant distribution*

contained in $\mathrm{Ker}\,(dh)$ makes sense and this specific distribution is denoted henceforth by the symbol Δ^*. That is:

$$\Delta^* = \Omega_{\bar{k}}^{\perp}.$$

IV. A Local Solvability Condition to the DDPSF in MIMO Nonlinear Impulsive Systems

The differential geometric approach to nonlinear impulsive systems illustrated in Section III and, specifically, the newly introduced notion of maximal hybrid locally controlled invariant distribution set the methodological context for establishing a local solvability conditions for the DDPSF in the following terms.

Theorem 1: Let the disturbed hybdrid system $\Sigma_{\mathscr{T}\mathcal{D}}$, described by (2), be given. Let the assumptions of Proposition 2 hold. Let U be a neighborhood of x_0 such that $\phi(U) \subseteq U$. Then, the DDPSF is locally solvable for $\Sigma_{\mathscr{T}\mathcal{D}}$ in U if and only if the relation

$$\mathrm{span}\,\{p_1(x), \ldots, p_q(x)\} \subseteq \Delta^*(x) \tag{14}$$

holds for all $x \in U$.

Proof: Sufficiency. First, note that, under the hypotheses of the statement, there exists, on the neighborhood U, a feedback

$$u(t) = F_s(x(t), v(t)) = \alpha(x(t)) + \beta(x(t))\, v(t)$$

such that Δ^* is invariant with respect to the vector fields $\tilde{f} = f + g\,\alpha$ and $\tilde{g}_i = (g\beta)_i$, for $i = 1, ..., m$. Let d denote the dimension of Δ^*. As in [13, Proposition 1.7.1]), since

$$\mathrm{span}\,\{p_1(x), \ldots, p_q(x)\} \subseteq \Delta^* \subseteq (\mathrm{Ker}\,(dh)),$$

there exists a local coordinates transformation $\zeta = \Psi(x)$ on U such that, in the new coordinates, the equations of the compensated system $\Sigma_{\mathscr{T}\mathcal{D}}^{F_s}$ have the following structure:

$$\Sigma_{\mathscr{T}\mathcal{D}}^{F_s} \equiv \begin{cases} \dot{\zeta}_1(t) = f_1(\zeta_1(t), \zeta_2(t)) + g_1(\zeta_1(t), \zeta_2(t))\, v(t) \\ \qquad\qquad + p_1(\zeta_1(t), \zeta_2(t))\, w(t) \\ \dot{\zeta}_2(t) = f_2(\zeta_2(t)) + g_2(\zeta_2(t))\, v(t) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{with } t \in \mathbb{R}^+ \setminus \mathscr{T}, \\ \zeta_1(t_k) = \phi_1(\zeta_1^-(t_k), \zeta_2^-(t_k)) \\ \zeta_2(t_k) = \phi_2(\zeta_2^-(t_k)) \qquad\qquad \text{with } t_k \in \mathscr{T}, \\ y(t) = h_1(\zeta_2(t)) \qquad\qquad\qquad \text{with } t \in \mathbb{R}^+, \end{cases} \tag{15}$$

where $\zeta = \left[\zeta_1^\top \;\; \zeta_2^\top\right]^\top$, $\dim \zeta_1 = d$, the components of f_1, f_2, ϕ_1, ϕ_2, h_1 and the entries of g_1, g_2, p_1 are smooth functions. Equations (15) show that the output $y(t)$ is not affected by the disturbance $w(t)$.

Necessity. It follows from the maximality of Δ^*. ■

V. Toward a Differential Algebraic Approach to DDP for Nonlinear Impulsive Systems

The purpose of this section is to develop a methodological backgroung for tackling the disturbance decoupling problem by dynamic state feedback for multivariable nonlinear impulsive systems. To this aim, we propose to adopt a differential algebraic approach (see [20]) in place of the differential geometric approach that has effectively led to a solvability condition of the problem by static state feedback in Section IV.

To begin with, some notions and results from [20] are reviewed with reference to the disturbed nonlinear impulsive system $\Sigma_{\mathscr{T}\mathcal{D}}$, described by (2). In particular, let $\mathcal{C}$ denote the set of abstract indeterminates

$$\mathcal{C} = \Big\{x_i,\, i = 1, \ldots, n;\, u_j^{(r)},\, j = 1, \ldots, m,\, r \geq 0; \\ w_l^{(r')},\, l = 1, \ldots, q,\, r' \geq 0\Big\}$$

and let $\mathcal{K}$ denote the field of meromorphic functions in a finite number of indeterminates of $\mathcal{C}$. An element of $\mathcal{K}$ will be indicated, by abuse of notation, by $\lambda(x_i, u_j^{(r)}, w_l^{(r')})$. For all $i = 1, ..., n$, $j = 1, ..., m$, $l = 1, ..., q$ and all $r \geq 0$ and $r' \geq 0$, the partial derivative operators $\frac{\partial}{\partial x_i}$, $\frac{\partial}{\partial u_j^{(r)}}$ and $\frac{\partial}{\partial w_l^{(r')}}$ act naturaly on the elements of $\mathcal{K}$.

Note that the time derivative operator $\frac{d}{dt}$ can be applied repeatedly to the components $y_k(t)$, for $k = 1, ..., p$, of the output of $\Sigma_{\mathscr{T}\mathcal{D}}$ on any interval (t_k, t_{k+1}) by computing the time derivative and replacing $\dot{x}_i(t)$ by $f_i(x(t)) + \sum_{j=1,\ldots,m} g_{ij}(x) u_j^{(0)}$, i.e by applying the so-called *chain rule*. Then, the time derivatives $\frac{d^s y_k(t)}{dt^s} = y_k^{(s)}(t) = y_k^{(s)}(x_i(t)), u_j^{(s-1)}(t), w_l^{(s-1)}(t))$ of any order s of $y_k(t)$ defines the element $y_k^{(s)} = y_k^{(s)}(x_i, u_j^{(s-1)}, w_l^{(s-1)})$ of $\mathcal{K}$ by substituting the time functions $x_i(t)$, $u_j^{(r)}(t)$, $w_l^{(r')}(t)$ with the corresponding element of $\mathcal{C}$.

In addition, we can introduce another operator Φ which acts on the time derivatives $y_k^{(s)}(t))$ of any order $s \geq 0$ of $y_k(t)$ by letting $\Phi((y_k^{(s)}(t)) = \Phi(y_k^{(s)}(x_i(t)), u_j^{(s-1)}(t), w_l^{(s-1)}(t))) = y_k^{(s)}(\phi_i(x_i(t)), u_j^{(s-1)}(t), w_l^{(s-1)}(t))$. In other words, the action of Φ consists in substituting $x_i(t)$ with $\phi_i(x)$.

Note that the operators $\frac{d}{dt}$ and Φ can be applied repeatedly in any order to any component $y_k(t)$ of the output and, then, substituting the time functions $x_i(t)$, $u_j^{(s-1)}(t)$, $w_l^{(s-1)}(t)$ in the argument of the result by the corresponding elements of $\mathcal{C}$, one gets an element of $\mathcal{K}$. In order to represent this procedure more formally, let us denote by $\mathcal{A}$ the set of words constructed by means of the alphabet $\{\frac{d}{dt}, \Phi\}$, that is the set of all finite, ordered sequences of any length of the elements $\frac{d}{dt}$, Φ, including the empty sequence. Then, for any word $\mathbf{w} \in \mathcal{A}$, the time function $\mathbf{w}(y_k(t)$ defines, substituting the time functions $x_i(t)$, $u_j^{(s-1)}(t)$, $w_l^{(s-1)}(t)$ in its argument by the corresponding elements of $\mathcal{C}$, an element, denoted by $\mathbf{w}(y_k)$, of $\mathcal{K}$.

Let the infinite set of symbols $d\mathcal{C}$ be defined by

$$d\mathcal{C} = \Big\{dx_i,\, i = 1, \ldots, n;\, du_j^{(r)},\, j = 1, \ldots, m,\, r \geq 0; \\ dw_l^{(r')},\, l = 1, \ldots, q,\, r' \geq 0\Big\}$$

Hence, the linear space of *one-forms* $\mathcal{E}$ over $\mathcal{K}$ is defined as

$$\mathcal{E} = \mathrm{span}_{\mathcal{K}}\{dx, du, dw, \cdots, du^{(k)}, dw^{(k)},\ \text{for } k \geq 0\}.$$

To any element $\lambda(x_i, u_j^{(r)}, w_l^{(r')})$ in $\mathcal{K}$ we associate a one-form that is denoted by $d\lambda$ and is defined by

$$d\lambda \;=\; d\lambda(x_i, u_j^{(r)}, w_l^{(r')}) =$$

$$\sum_{i=1}^{n} \frac{\partial \lambda}{\partial x_i} dx_i + \sum_{\substack{j=1\\ r\geq 0}}^{n} \frac{\partial \lambda}{\partial u_j^r} du_j^r + \sum_{\substack{l=1\\ r'\geq 0}}^{q} \frac{\partial \lambda}{\partial w_l^{r'}} w_l^{r'}.$$

Note that the above notation is the same we have introduced in the differential geometric approach to indicate the differential of $\lambda(x_i(t), u_j^{(r)}(t), w_l^{(r')}(t))$, hence this is an abuse of notation. However, the information contained in the one-form and in the differential is the same and, since the framework is different no confusion arises.

$\mathcal{X} = \text{span}\,\{dx_1, ..., dx_n\}$ and $\mathcal{U} = \text{span}\,\{du_1^{(r)}, ..., du_m^{(r)}\}$, for $r \geq 0$, are linear subspaces of $\mathcal{E}$ and $d\mathbf{w}(y_k)$, for any $k = 1, ...p$ and for any $\mathbf{w} \in \mathcal{A}$ (as, in particular $dy_k = dh_k$ and $dy_k^{(s)}$ for any s), can be seen to belong to $\mathcal{E}$. We can therefore consider the subspace of one-forms $\mathcal{Y}_{\bar{i}}^{+} \subseteq \mathcal{E}$ defined, for $\bar{i} \in \mathbb{N}$, by

$$\mathcal{Y}_{\bar{i}}^{+} = span\{d\mathbf{w}y_k \in \mathcal{E}; k = 1, ...p; length(\mathbf{w}) \leq \bar{i}\}$$

and the chain of subspaces

$$\{0\} \subseteq \mathcal{O}_0^{+} \subseteq \mathcal{O}_1^{+} \subseteq \mathcal{O}_2^{+} \subseteq ... \subseteq \mathcal{O}_{\bar{i}}^{+} \subseteq ...$$

where $\mathcal{O}_{\bar{i}}^{+} = \mathcal{X} \cap (\mathcal{Y}_{\bar{i}}^{+} + \mathcal{U})$. Letting $\mathcal{Y}^{+} = \cup_{\bar{i}\geq 0} \mathcal{Y}_{\bar{i}}^{+}$, the limit $\mathcal{O}_{\infty}^{+}$ of the above chain is given by $\mathcal{O}_{\infty}^{+} = \mathcal{X} \cap (\mathcal{Y}^{+} + \mathcal{U})$.

Now, on the basis of [20, Theorem 10.2], the following result can be conjectured.

Theorem 2: Let the disturbed hybdrid system $\Sigma_{\mathscr{TD}}$, described by (2), be given. Then, the DDPSF is locally solvable for $\Sigma_{\mathscr{TD}}$ if and only if the distribution $span\{p_1(x), \ldots, p_q(x)\}$ is orthogonal to $\mathcal{X} \cap \mathcal{Y}^{+}$.

Before discussing Theorem 2, it may be helpful to consider a specific numerical example.

Example 1: Let the nonlinear impulsive system $\Sigma_{\mathscr{TD}}$ be defined by

$$\Sigma_{\mathscr{TD}} \equiv \begin{cases} \dot{x}_1(t) \;=\; x_2(t)\, u_1(t) \\ \dot{x}_2(t) \;=\; x_5(t) \\ \dot{x}_3(t) \;=\; x_2(t) + x_4(t) + x_4(t)\, u_1(t) \\ \dot{x}_4(t) \;=\; u_2(t) \\ \dot{x}_5(t) \;=\; x_1(t)\, u_1(t) + w(t) \text{ with } t \in \mathbb{R}^{+} \setminus \mathscr{T}, \\ x_1(t_k) \;=\; x_3^{-}(t_k) \\ x_2(t_k) \;=\; x_3^{-}(t_k) \\ x_3(t_k) \;=\; x_1^{-}(t_k) \\ x_4(t_k) \;=\; x_4^{-}(t_k) \\ x_5(t_k) \;=\; x_1^{-}(t_k) \text{ with } t_k \in \mathscr{T}, \\ y_1(t) \;=\; x_1(t) \\ y_2(t) \;=\; x_3(t). \end{cases}$$

Note that the disturbance $w(t)$ affects the outputs, since $w(t)$ appears in the time derivatives of sufficiently high order of $y_1(t)$, $y_2(t)$, $\Phi(y_1(t))$, $\Phi(y_2(t))$. Simple computations yeld $y_1 = x_1$, $y_2 = x_3$, $\Phi(y_1) = x_3$, $\Phi(y_2) = x_1$, $\dot{y}_1 = x_2 u_1$ and hence $\dot{y}_2 = x_2 + x_4 + \dot{y}_1 x_4 / x_2$, $\Phi(\dot{y}_2) = x_3 + x_4 + \Phi(\dot{y}_1) x_4 / x_3$. From this, we can conclude that

$$\mathcal{X} \cap \mathcal{Y}^{+} = \text{span}\left\{dx_1, dx_3, \left(1 - \frac{\dot{y}_1 x_4}{x_2^2}\right) dx_2 + \left(1 + \frac{\dot{y}_1}{x_2}\right) dx_4, \left(1 - \frac{\Phi(\dot{y}_1) x_4}{x_3^2}\right) dx_3 + \left(1 + \frac{\Phi(\dot{y}_1)}{x_3}\right) dx_4\right\} = \text{span}\,\{dx_1, dx_2, dx_3, dx_4\}.$$

By applying the dynamic feedback compensator F_d, of the form (5), given by

$$F_d \equiv \begin{cases} \dot{z}(t) \;=\; \frac{v_1(t) - x_5(t)\, z(t)}{x_2(t)} \quad \text{with } t \in \mathbb{R}^{+} \setminus \mathscr{T}, \\ z(t_k) \;=\; 0 \quad \text{with } t_k \in \mathscr{T}, \\ u_1(t) \;=\; z(t) \\ u_2(t) \;=\; \frac{x_2(t)\,(v_2(t) - x_5(t)) - x_4(t)(v_1(t) - x_5(t)\, z(t))}{x_2(t)(1 + z(t))} \\ \qquad\qquad \text{with } t \in \mathbb{R}^{+}, \end{cases}$$

we get the compensated system $\Sigma_{\mathscr{TD}}^{F_d}$ defined by

$$\Sigma_{\mathscr{TD}} \equiv \begin{cases} \dot{x}_1(t) \;=\; x_2(t)\, z(t) \\ \dot{x}_2(t) \;=\; x_5(t) \\ \dot{x}_3(t) \;=\; x_2(t) + x_4(t) + x_4(t)\, z(t) \\ \dot{x}_4(t) \;=\; \frac{x_2(t)\,(v_2(t) - x_5(t)) - x_4(t)(v_1(t) - x_5(t)\, z(t))}{x_2(t)(1 + z(t))} \\ \dot{x}_5(t) \;=\; x_1(t)\, z(t) + w(t) \\ \dot{z}(t) \;=\; \frac{v_1(t) - x_5(t)\, z(t)}{x_2(t)} \text{ with } t \in \mathbb{R}^{+} \setminus \mathscr{T}, \\ x_1(t_k) \;=\; x_3^{-}(t_k) \\ x_2(t_k) \;=\; x_3^{-}(t_k) \\ x_3(t_k) \;=\; x_1^{-}(t_k) \\ x_4(t_k) \;=\; x_4^{-}(t_k) \\ x_5(t_k) \;=\; x_1^{-}(t_k) \\ z(t) \;=\; 0 \text{ with } t_k \in \mathscr{T}, \\ y_1(t) \;=\; x_1(t) \\ y_2(t) \;=\; x_3(t). \end{cases}$$

Applying the change of coordinates defined by

$$\begin{cases} \xi_1 \;=\; x_1 \\ \xi_2 \;=\; x_2 z \\ \xi_3 \;=\; x_3 \\ \xi_4 \;=\; x_2 + x_4(1 + z) \\ \xi_5 \;=\; x_5 \\ \xi_6 \;=\; x_z \end{cases}$$

the compensated system take the form

$$\Sigma_{\mathscr{TD}} \equiv \begin{cases} \dot{\xi}_1(t) \;=\; \xi_2(t) \\ \dot{\xi}_2(t) \;=\; v_1(t) \\ \dot{\xi}_3(t) \;=\; \xi_4(t) \\ \dot{\xi}_4(t) \;=\; v_2(t) \\ \dot{\xi}_5(t) \;=\; \xi_1(tk)\xi_6(k) + w(t) \\ \dot{\xi}_6(t) \;=\; \frac{\xi_6(t)(v_1(t) - \xi_5(t)\xi_6(t))}{\xi_2(t)} \text{ with } t \in \mathbb{R}^{+} \setminus \mathscr{T}, \\ \xi_1(t_k) \;=\; \xi_3^{-}(t_k) \\ \xi_2(t_k) \;=\; 0 \\ \xi_3(t_k) \;=\; \xi_1^{-}(t_k) \\ \xi_4(t_k) \;=\; sin(\xi_3^{-}(t_k)) + \xi_4^{-}(t_k) \\ \xi_5(t_k) \;=\; \xi_1^{-}(t_k) \\ \xi_6(t) \;=\; 0 \text{ with } t_k \in \mathscr{T}, \\ y_1(t) \;=\; \xi_1(t) \\ y_2(t) \;=\; \xi_3(t). \end{cases}$$

and inspection shows clearly that the output $y_1(t)$, $y_2(t)$ is decoupled from the disturbance $w(t)$.

Basically, the result stated in Theorem 2 is motivated by the analogy between the present situation and those concerning, on one side, impulse free nonlinear systems, i.e. nonlinear systems which present no jumps ([20, Chapter 10]), and, on the other side, linear impulsive systems ([18]). The first thing to remark regarding the analogy is that $\mathcal{O}_\infty^+$ can be though of as the observable space of $\Sigma_{\mathcal{TD}}$. This is the case if $\phi(x)$ is the identity map, i.e. the system is actually impulse free (see [20, Section 4.4]), as well as if the system is impulsive but linear, i.e. $f(x) = Ax$, $\phi(x) = Jx$, $g(x) = B$, $p(x) = P$, $h(x) = Cx$ for suitable real matrices A, J, B, P, C. In the last case, $\mathcal{O}_\infty^+$ is seen to be given by $span\{C\bar{\mathbf{w}}dx, \bar{\mathbf{w}} \in \bar{\mathcal{A}}\}$, where $\bar{\mathcal{A}}$ is the set of words, including the empty one, constructed by means of the alphabet $\{A, J\}$, that is the set of all $n \times n$ real matrices that can be obtained by multiplying between them the matrices A and J any finite number of times and in any possible order.

The action of a dynamic feedback compensator F_d of the form (5) that is intended to solve the DDPDF consists in modifying the observable space of $\Sigma_{\mathcal{TD}}$ by shrinking it, in such a way to hide, so to say, the effect of the disturbance in the unobservable subsystem. In other terms, this means to make the compensated system $\Sigma_{\mathcal{TD}}^{F_d}$ maximally unobservable. The maximal reduction that can be achieved makes the observable space equal to $\mathcal{X} \cap \mathcal{Y}^+$ and, therefore, the DDPDF is solved if the distribution $span\{p_1(x), \ldots, p_q(x)\}$ is orthogonal to $\mathcal{X} \cap \mathcal{Y}^+$.

The main difficulties in pursuing this line of proof are related to the fact the distribution $(\mathcal{X} \cap \mathcal{Y}^+)^\perp$ is not always integrable and it may be difficult to construct (by considering the components of the output and their derivatives, taking into account the jumps action of the jump behaviour and eliminating the input variables whenever it is possible, as illustrated in the above example), except in the single output case, and that the definition of a suitable dynamic feedback compensator which solves the problem necessitates of a generalization to nonlinear impulsive systems of the structure algorithm described in [20, Section 5.4]. This, together with a complete proof of Theorem 2, is left for a forthcoming paper.

VI. Conclusions

The disturbance decoupling problem for nonlinear impulsive system has been considered from a structural point of view, taking into account both static state feedback soutions and dynamic state feedback solutions.

The main contribution of the paper consists in providing a complete solution in the static state feedback case by means of a differential geometric approach. The differential algebraic approach is potentially able to provide a solution to the considered problem in the dynamic state feedback case. In this sense, a result quite similar to that obtained for impulse free nonlinear systems has been conjectured. A major problem that remains open is that of the construction of a dynamic state feedback solution in case the condition of Theorem 2 is satisfied. This will be the subject of future researches.

The requirement of stability of the compensated system will also be taken into account in future works.

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