



# DIRICHLET PRODUCT OF DERIVATIVE ARITHMETIC WITH AN ARITHMETIC FUNCTION MULTIPLICATIVE

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# DIRICHLET PRODUCT OF DERIVATIVE ARITHMETIC WITH AN ARITHMETIC FUNCTION MULTIPLICATIVE

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A PREPRINT

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## ABSTRACT

We define the derivative of an integer to be the map sending every prime to 1 and satisfying the Leibniz rule. The aim of this article is to calculate the Dirichlet product of this map with a function arithmetic multiplicative.

## 1 Introduction

Barbeau [1] defined the arithmetic derivative as the function  $\delta : \mathbb{N} \rightarrow \mathbb{N}$ , defined by the rules :

1.  $\delta(p) = 1$  for any prime  $p \in \mathbb{P} := \{2, 3, 5, 7, \dots, p_i, \dots\}$ .
2.  $\delta(ab) = \delta(a)b + a\delta(b)$  for any  $a, b \in \mathbb{N}$  (the Leibnitz rule).

Let  $n$  a positive integer, if  $n = \prod_{i=1}^s p_i^{\alpha_i}$  is the prime factorization of  $n$ , then the formula for computing the arithmetic derivative of  $n$  is (see, e.g., [1, 3]) giving by :

$$\delta(n) = n \sum_{i=1}^s \frac{\alpha_i}{p_i} = n \sum_{p^\alpha || n} \frac{\alpha}{p} \quad (1)$$

A brief summary on the history of arithmetic derivative and its generalizations to other number sets can be found, e.g., in [4].

First of all, to cultivate analytic number theory one must acquire a considerable skill for operating with arithmetic functions. We begin with a few elementary considerations.

**Definition 1** (arithmetic function). An **arithmetic function** is a function  $f : \mathbb{N} \rightarrow \mathbb{C}$  with domain of definition the set of natural numbers  $\mathbb{N}$  and range a subset of the set of complex numbers  $\mathbb{C}$ .

**Definition 2** (multiplicative function). A function  $f$  is called an **multiplicative function** if and only if :

$$f(nm) = f(n)f(m) \quad (2)$$

for every pair of coprime integers  $n, m$ . In case (2) is satisfied for every pair of integers  $n$  and  $m$ , which are not necessarily coprime, then the function  $f$  is called **completely multiplicative**.

Clearly, if  $f$  are a multiplicative function, then  $f(n) = f(p_1^{\alpha_1}) \dots f(p_s^{\alpha_s})$ , for any positive integer  $n$  such that  $n = p_1^{\alpha_1} \dots p_s^{\alpha_s}$ , and if  $f$  is completely multiplicative, so we have :  $f(n) = f(p_1)^{\alpha_1} \dots f(p_s)^{\alpha_s}$ .

**Example 3.** Let  $n \in \mathbb{N}^*$ , This is the same classical arithmetic functions used in this article :

1. **Identity function** : The function defined by  $Id(n) = n$  for all positive integer  $n$ .

2. **The Euler phi function** :  $\varphi(n) = \sum_{\substack{k=1 \\ \gcd(k,n)=1}}^n 1.$

3. **The number of distinct prime divisors of  $n$**  :  $\omega(n) = \sum_{p|n} 1.$

4. **The Mobius function** :  $\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } p^2 | n \text{ for some prime } p \\ (-1)^{\omega(n)} & \text{otherwise} \end{cases}$

5. **number of positive divisors of  $n$  defined by** :  $\tau(n) = \sum_{d|n} 1.$

6. **sum of divisors function of  $n$  defined by** :  $\sigma(n) = \sum_{d|n} d.$

Now ,if  $f, g : \mathbb{N} \longrightarrow \mathbb{C}$  are two arithmetic functions from the positive integers to the complex numbers, the Dirichlet convolution  $f * g$  is a new arithmetic function defined by:

$$(f * g)(n) = \sum_{d|n} f(d)g\left(\frac{n}{d}\right) = \sum_{ab=n} f(a)g(b) \quad (3)$$

where the sum extends over all positive divisors  $d$  of  $n$  , or equivalently over all distinct pairs  $(a, b)$  of positive integers whose product is  $n$ .

In particular, we have  $(f * g)(1) = f(1)g(1)$  ,  $(f * g)(p) = f(1)g(p) + f(p)g(1)$  for any prime  $p$  and for any power prime  $p^m$  we have :

$$(f * g)(p^m) = \sum_{j=0}^m f(p^j)g(p^{m-j}) \quad (4)$$

This product occurs naturally in the study of Dirichlet series such as the Riemann zeta function. It describes the multiplication of two Dirichlet series in terms of their coefficients:

$$\left( \sum_{n \geq 1} \frac{(f * g)(n)}{n^s} \right) = \left( \sum_{n \geq 1} \frac{f(n)}{n^s} \right) \left( \sum_{n \geq 1} \frac{g(n)}{n^s} \right) \quad (5)$$

with Riemann zeta function or is defined by :

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}$$

These functions are widely studied in the literature (see, e.g., [5, 6, 7]).

Now before to past to main result we need this propriety , if  $f$  and  $g$  are multiplicative function , then  $f * g$  is multiplicative.

## 2 Main results

In this section we give the new result of Dirichlet product between derivative arithmetic and an arithmetic function multiplicative  $f$  , and we will give the relation between  $\tau$  and the derivative arithmetic .

**Theorem 4.** *Given a multiplicative function  $f$ , and lets  $n$  and  $m$  two positive integers such that  $\gcd(n, m) = 1$  , Then we have :*

$$(f * \delta)(nm) = (Id * f)(n) \cdot (f * \delta)(m) + (Id * f)(m) \cdot (f * \delta)(n) \quad (6)$$

*Proof.* Lets  $n$  and  $m$  two positive integers such that  $\gcd(n, m) = 1$ , and let  $f$  an arithmetic function multiplicative , then we have :

$$\begin{aligned}
 (f * \delta)(nm) &= \sum_{d|nm} f\left(\frac{nm}{d}\right) \delta(d) = \sum_{\substack{d_1|n \\ d_2|m}} f\left(\frac{nm}{d_1 d_2}\right) \delta(d_1 d_2) = \sum_{\substack{d_1|n \\ d_2|m}} f\left(\frac{n}{d_1}\right) f\left(\frac{m}{d_2}\right) \left(d_1 \delta(d_2) + d_2 \delta(d_1)\right) \\
 &= \sum_{\substack{d_1|n \\ d_2|m}} \left(d_1 f\left(\frac{n}{d_1}\right) f\left(\frac{m}{d_2}\right) \delta(d_2) + d_2 f\left(\frac{m}{d_2}\right) f\left(\frac{n}{d_1}\right) \delta(d_1)\right) \\
 &= \left(\sum_{d_1|n} d_1 f\left(\frac{n}{d_1}\right)\right) \left(\sum_{d_2|m} f\left(\frac{m}{d_2}\right) \delta(d_2)\right) + \left(\sum_{d_2|m} d_2 f\left(\frac{m}{d_2}\right)\right) \left(\sum_{d_1|n} f\left(\frac{n}{d_1}\right) \delta(d_1)\right) \\
 &= (Id * f)(n) \cdot (f * \delta)(m) + (Id * f)(m) \cdot (f * \delta)(n)
 \end{aligned}$$

□

**Lemma 5.** For any natural number  $n$  , if  $n = \prod_{i=1}^s p_i^{\alpha_i}$  is the prime factorization of  $n$ , then :

$$(f * \delta)(n) = (Id * f)(n) \sum_{i=1}^s \frac{(f * \delta)(p_i^{\alpha_i})}{(Id * f)(p_i^{\alpha_i})} \quad (7)$$

*Proof.* Let  $n$  a positive integer such that  $n = p_1^{\alpha_1} \dots p_s^{\alpha_s}$  and let  $f$  an arithmetic function , Then :

$$\begin{aligned}
 (f * \delta)(n) &= (f * \delta)(p_1^{\alpha_1} \dots p_s^{\alpha_s}) \\
 &= (Id * f)(p_2^{\alpha_2} \dots p_s^{\alpha_s}) \cdot (f * \delta)(p_1^{\alpha_1}) + (Id * f)(p_1^{\alpha_1}) \cdot (f * \delta)(p_2^{\alpha_2} \dots p_s^{\alpha_s}) \\
 &= (Id * f)(n) \cdot \frac{(f * \delta)(p_1^{\alpha_1})}{(Id * f)(p_1^{\alpha_1})} + (Id * f)(p_1^{\alpha_1}) \cdot \left[ (Id * f)(p_3^{\alpha_3} \dots p_s^{\alpha_s}) \cdot (f * \delta)(p_2^{\alpha_2}) + \right. \\
 &\quad \left. + (Id * f)(p_2^{\alpha_2}) \cdot (f * \delta)(p_3^{\alpha_3} \dots p_s^{\alpha_s}) \right] \\
 &= (Id * f)(n) \frac{(f * \delta)(p_1^{\alpha_1})}{(Id * f)(p_1^{\alpha_1})} + (Id * f)(n) \frac{(f * \delta)(p_2^{\alpha_2})}{(Id * f)(p_2^{\alpha_2})} + \\
 &\quad + (Id * f)(p_1^{\alpha_1}) (Id * f)(p_2^{\alpha_2}) (f * \delta)(p_3^{\alpha_3} \dots p_s^{\alpha_s}) \\
 &\vdots \\
 &= (Id * f)(n) \frac{(f * \delta)(p_1^{\alpha_1})}{(Id * f)(p_1^{\alpha_1})} + (Id * f)(n) \frac{(f * \delta)(p_2^{\alpha_2})}{(Id * f)(p_2^{\alpha_2})} + \dots + (Id * f)(n) \frac{(f * \delta)(p_s^{\alpha_s})}{(Id * f)(p_s^{\alpha_s})} \\
 &= (Id * f)(n) \left[ \frac{(f * \delta)(p_1^{\alpha_1})}{(Id * f)(p_1^{\alpha_1})} + \dots + \frac{(f * \delta)(p_s^{\alpha_s})}{(Id * f)(p_s^{\alpha_s})} \right] \\
 &= (Id * f)(n) \sum_{i=1}^s \frac{(f * \delta)(p_i^{\alpha_i})}{(Id * f)(p_i^{\alpha_i})}
 \end{aligned}$$

□

an other prof by induction on  $s$  that if  $n = \prod_{i=1}^s p_i^{\alpha_i}$  then  $(f * \delta)(n) = (Id * f)(n) \sum_{i=1}^s \frac{(f * \delta)(p_i^{\alpha_i})}{(Id * f)(p_i^{\alpha_i})}$ .

*Proof.* Consider  $n \in \mathbb{N}$  and express  $n = \prod_{i=1}^s p_i^{\alpha_i}$  where all  $p_i$  are distinct .

where  $s = 1$  , it is clear that  $(f * \delta)(n) = (Id * f)(n) \sum_{i=1}^1 \frac{(f * \delta)(p_i^{\alpha_i})}{(Id * f)(p_i^{\alpha_i})} = (Id * f)(p_1^{\alpha_1}) \frac{(f * \delta)(p_1^{\alpha_1})}{(Id * f)(p_1^{\alpha_1})} = (f * \delta)(p_1^{\alpha_1})$ .

Assume that  $n = \prod_{i=1}^s p_i^{\alpha_i}$ , then we have :

$$\begin{aligned}
 (id * \delta)(n.p_{s+1}^{\alpha_{s+1}}) &= (Id * f)(p_{s+1}^{\alpha_{s+1}}) \cdot (f * \delta)(n) + (Id * f)(n) \cdot (f * \delta)(p_{s+1}^{\alpha_{s+1}}) \\
 &= (Id * f)(p_{s+1}^{\alpha_{s+1}}) \cdot (Id * f)(n) \sum_{i=1}^s \frac{(f * \delta)(p_i^{\alpha_i})}{(Id * f)(p_i^{\alpha_i})} + (Id * f)(p_{s+1}^{\alpha_{s+1}}) \cdot (Id * f)(n) \frac{(f * \delta)(p_{s+1}^{\alpha_{s+1}})}{(Id * f)(p_{s+1}^{\alpha_{s+1}})} \\
 &= (Id * f)(n.p_{s+1}^{\alpha_{s+1}}) \sum_{i=1}^s \frac{(f * \delta)(p_i^{\alpha_i})}{(Id * f)(p_i^{\alpha_i})} + (Id * f)(n.p_{s+1}^{\alpha_{s+1}}) \frac{(f * \delta)(p_{s+1}^{\alpha_{s+1}})}{(Id * f)(p_{s+1}^{\alpha_{s+1}})} \\
 &= (Id * f)(n.p_{s+1}^{\alpha_{s+1}}) \left[ \sum_{i=1}^s \frac{(f * \delta)(p_i^{\alpha_i})}{(Id * f)(p_i^{\alpha_i})} + \frac{(f * \delta)(p_{s+1}^{\alpha_{s+1}})}{(Id * f)(p_{s+1}^{\alpha_{s+1}})} \right] \\
 &= (Id * f)(n.p_{s+1}^{\alpha_{s+1}}) \sum_{i=1}^{s+1} \frac{(f * \delta)(p_i^{\alpha_i})}{(Id * f)(p_i^{\alpha_i})}
 \end{aligned}$$

□

**Proposition 6.** Let  $f$  a function arithmetic multiplicative, and  $\delta$  the derivative arithmetic, then we have :

$$(Id * \delta)(n) = \frac{1}{2} \tau(n) \delta(n) \quad (8)$$

*Proof.* Since  $(Id * Id)(n) = \sum_{d|n} \frac{n}{d} = n \sum_{d|n} 1 = n \tau(n)$ .

and :  $(Id * \delta)(p^\alpha) = \sum_{j=1}^{\alpha} \delta(p^j) Id(p^{\alpha-j}) = \sum_{j=1}^{\alpha} j p^{j-1} p^{\alpha-j} = \frac{1}{2} \alpha(\alpha+1) p^{\alpha-1}$ .

Then for every a positive integer  $n$  such that  $n = p_1^{\alpha_1} \dots p_s^{\alpha_s}$ , we have :

$$\begin{aligned}
 (Id * \delta)(n) &= (Id * Id)(n) \sum_{i=1}^s \frac{(Id * \delta)(p_i^{\alpha_i})}{(Id * Id)(p_i^{\alpha_i})} \\
 &= n \tau(n) \sum_{i=1}^s \frac{\frac{1}{2} \alpha_i(\alpha_i+1) p_i^{\alpha_i-1}}{p_i^{\alpha_i} \tau(p_i^{\alpha_i})} \\
 &= n \tau(n) \sum_{i=1}^s \frac{\frac{1}{2} \alpha_i(\alpha_i+1) p_i^{\alpha_i-1}}{p_i^{\alpha_i} (\alpha_i+1)} \\
 &= \frac{1}{2} n \tau(n) \sum_{i=1}^s \frac{\alpha_i}{p_i} = \frac{1}{2} \tau(n) \delta(n)
 \end{aligned}$$

□

So by the proposition 6, and the equality 5 we have this relation between arithmetic derivative and the function  $\tau$  :

$$2\zeta(s-1) \sum_{n \geq 1} \frac{\delta(n)}{n^s} = \sum_{n \geq 1} \frac{\delta(n) \tau(n)}{n^s} \quad (9)$$

Let defined the new function arithmetic called **En-naoui** function, by :

$$\Phi_\varphi(n) = n \sum_{p|n} \left( 1 - \frac{1}{p} \right) \quad (10)$$

Then we have this equality related between 8 arithmetic function :

$$(\mu * \delta)(n) = \varphi(n) \left( \delta(n) - 2\omega(n) + B(n) + \frac{\Phi_\varphi(n)}{n} + \frac{(B * Id)(n)}{\sigma(n)} \right). \quad (11)$$

with  $B$  is the arithmetic function defined by :  $B(n) = \sum_{p^\alpha || n} \alpha p$ . In next article i will prove this equality, just I need to submit an article about this new function.

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### References

- [1] E. J. Barbeau. Remark on an arithmetic derivative, *Canad. Math. Bull.* 4 (1961), 117–122.
- [2] A. Buium. Arithmetic analogues of derivations, *J. Algebra*, 198 (1997), 290–299.
- [3] V. Ufnarovski and B. Åhlander. How to differentiate a number, *J. Integer Seq.*, 6 , (2003).
- [4] M. Stay. Generalized number derivatives, *J. Integer Seq.*, 8 , (2005).
- [5] T. M. Apostol. Introduction to Analytic Number Theory, *Springer-Verlag, New York* , (1976).
- [6] M.Lewinter and J.Meyer. Elementary number theory with programming , *Wiley* , (2016).
- [7] Heng Huat Chan, Analytic number theory for undergraduates , *World Scientific Publishing Co. Pte. Ltd.* , (2009).