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by

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Abstract. — We exhibit cases of a level fixing phenomenon for galoisian automorphic representations of a CM field F, with dimension $d \geqslant 2$. The proof rests on the freeness of the localized cohomology groups of KHT Shimura varieties and the strictness of its filtration induced by the spectral sequence associated to the filtration of stratification of the nearby cycles perverse sheaf at some fixed place v of F. The main point is the observation that the action of the unipotent monodromy operator at v is then given by those on the nearby cycles where its order of nilpotency modulo l equals those in characteristic zero. Finally we infer some consequences concerning level raising and Ihara's lemma.

Contents

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1. Introduction

In [**27**], Ribet considers an absolutely irreducible representation

$$\overline{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL_2(\overline{\mathbb{F}}_l),$$

which is modular of level N, meaning that it arises from a cusp form of weight 2 and trivial character on $\Gamma_0(N)$.

a) Then for a prime $p \nmid lN$ such that

$$\mathrm{Tr}\ \overline{\rho}(\mathrm{Frob}_p) \equiv \pm(p+1) \mod l,$$

he proves that $\overline{\rho}$ also arises from a modular form of level pN which is p-new, i.e. the automorphic representation associated to this modular form has a local component at p which is isomorphic to the Steinberg representation of $GL_2(\mathbb{Q}_p)$.

b) Using Cebotarev density theorem, there exists infinitely many primes $p \nmid lN$ such that $\overline{\rho}(\mathrm{Frob}_p)$ is conjugate to $\overline{\rho}(c)$ where c is the complex conjugation $c \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Then we have both

$$p \equiv -1 \mod l \text{ and } \mathrm{Tr}\ \overline{\rho}(\mathrm{Frob}_p) \equiv 0 \mod l,$$

so that $\overline{\rho}$ is p-new for infinitely many p.

In [**30**] Sorensen generalizes this level raising congruences in higher dimension for a connected reductive group G over a totally real field F^+ such that G_∞ is compact. One might also look at [**1**] theorem 1.1 and theorem 4.1, for the case of automorphic representations of unitary type of GL_{2n}.

In this article, in higher dimension, we rather try to find mild conditions where level lowering is impossible, i.e. where $\bar{\rho}$ is modular of level pN but not of level N. To do so we consider a finite CM extension $F = EF^+$ of $\mathbb{Q}$ with $E/\mathbb{Q}$ imaginary quadratic and F^+ totally real. Let $\rho : \mathrm{Gal}(\overline{F}/F) \longrightarrow GL_{\overline{\mathbb{Q}}_l}(V)$ be an irreducible l-adic representation of dimension d. Let Γ be a stable $\overline{\mathbb{Z}}_l$-lattice of V and suppose that the modulo l reduction $\overline{\rho}$ of ρ is still irreducible so that, up to homothety, Γ is uniquely defined.

Consider now $p \neq l$ and a place v of F above p: we denote by F_v the completion of F at v with $\mathcal{O}_v$ its ring of integers and κ_v its residue field with order q_v. We then denote by

$$e_v(l) = \begin{cases} l, & \text{if } q_v \equiv 1 \mod l \\ \text{the order of } q_v \text{ modulo } l, & \text{otherwise} \end{cases}$$

Thanks to Grothendieck's theorem, we know that the action of the inertia subgroup I_v at v is quasi-unipotent and let N_v be the nilpotent monodromy operator. Such a nilpotent operator defines a standard partition

$$\underline{d_{\rho,v}} = (n_1 \geqslant n_2 \geqslant \cdots n_r \geqslant 1)$$

of $d = n_1 + \cdots + n_r$ corresponding to the decomposition of the restriction ρ_v of ρ to the decomposition group at v:

$$\rho_v \simeq \mathrm{Sp}_{n_1}(\rho_{v,1}) \oplus \cdots \oplus \mathrm{Sp}_{n_r}(\rho_{v,r}),$$

where the $\rho_{v,i}$ are irreducible and

$$\mathrm{Sp}_{n_i}(\rho_{v,i}) = \rho_{v,i}(\frac{1-n_i}{2}) \oplus \rho_{v,i}(\frac{3-n_i}{2}) \oplus \cdots \oplus \rho_{v,i}(\frac{n_i-1}{2}),$$

where N_v induces isomorphisms $\rho_{v,i}(\frac{1-n_i+2\delta}{2}) \longrightarrow \rho_{v,i}(\frac{1-n_i+2(\delta+1)}{2})$ for $0 \leqslant \delta < n_i - 1$ and is trivial on $\rho_{v,i}(\frac{n_i-1}{2})$.

If $l \geqslant n_1$, then we can also consider the nilpotent operator $\overline{N}_v$ of the action of I_v modulo l which also defines a partition $\underline{d_{\bar{\rho},v}}$ of d which is smaller than $\underline{d_{\rho,v}}$ for the Bruhat order. As in [**11**], one may ask for a condition so that $\underline{d_{\bar{\rho},v}} = \underline{d_{\rho,v}}$, especially when there is as much irreducible constituents in the Frobenius semi-simplification of $\bar{\rho}_v$ as in ρ_v. To be relevant such a condition should not be about the place v.

To state our result, consider a similitude group $G/\mathbb{Q}$ as in §2, and for any open compact subgroup I of $G(\mathbb{A}^\infty)$, let denote by $\mathrm{Sh}_{I,v} \to \mathrm{Spec}\,\mathcal{O}_v$ the Kottwitz-Harris-Taylor Shimura variety with level I, cf. definition 2.4. For any finite set S of places of F, let $\mathbb{T}^S_\xi$ be the Hecke algebra defined, cf. 2.5, as the image of the abstract unramified $\overline{\mathbb{Z}}_l$-Hecke algebra of $G(\mathbb{A}^S)$ outside S acting on the cohomology of $\mathrm{Sh}_{I,v}$ with coefficients in a local system $V_{\xi,\overline{\mathbb{Z}}_l}$, where I is unramified away from S.

Consider then a maximal cohomological ideal $\mathfrak{m}$ of $\mathbb{T}^S_\xi$, and for a prime ideal $\widetilde{\mathfrak{m}} \subset \mathfrak{m}$, we denote by

$$\rho_{\widetilde{\mathfrak{m}}} : \mathrm{Gal}_{F,S} \longrightarrow GL_d(\overline{\mathbb{Q}}_l)$$

the Galois $\overline{\mathbb{Q}}_l$-representation associated to $\widetilde{\mathfrak{m}}$, cf. [**22**], where $\mathrm{Gal}_{F,S}$ is the Galois group of the maximal extension of F which is unramified outside S. By Cebotarev's density theorem and the fact that a semi-simple representation is determined , up to isomorphism, by characteristic polynomials, then the semi-simple class $\overline{\rho}_{\mathfrak{m}}$ of the reduction modulo l of $\rho_{\widetilde{\mathfrak{m}}}$ depends only of the maximal ideal $\mathfrak{m}$ of $\mathbb{T}$ containing $\widetilde{\mathfrak{m}}$. For all prime x

of $\mathbb{Z}$ split in E and a place $w \notin S$ of F above x, we moreover denote by $P_{\mathfrak{m},w}(X)$ the characteristic polynomial of $\overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_w)$ and let $S_{\mathfrak{m}}(w)$ be its multi-set of roots. We also denote by $N_{v,\tilde{\mathfrak{m}}}$ the monodromy opertator at v of $\rho_{\tilde{\mathfrak{m}}}$ and $\overline{N}_{v,\mathfrak{m}}$ its modulo l reduction.

1.1. Definition. — *(cf. the introduction of* [**12**]*)*
We say that $\mathfrak{m}$ *is KHT-free if the cohomology groups of the Kottwitz-Harris-Taylor Shimura variety of notation 2.4, localized at* $\mathfrak{m}$*, are free.*

From [**8**], any of the following properties ensure KHT-freeness of $\mathfrak{m}$.

(1) There exists a place $w_1 \notin S$ of F above a prime p_1 splits in E, such that the multi-set $S_{\mathfrak{m}}(w_1)$ of roots of $P_{\mathfrak{m},w_1}(X)$ does not contain any sub-multi-set of the shape $\{\alpha, q_{w_1}\alpha\}$ where q_{w_1} is the order of the residue field of F at w_1. This hypothesis is called *generic* in [**13**].
(2) When $[F(\exp(2i\pi/l)) : F] > d$, if we suppose the following property to be true, cf. [**8**] 4.17. If $\theta : G_F \longrightarrow GL_d(\overline{\mathbb{Q}}_l)$ is an irreducible continuous representation such that for all place $w \notin S$ above a prime $x \in \mathbb{Z}$ split in E, then $P_{\mathfrak{m},w}(\theta(\mathrm{Frob}_w)) = 0$ (resp. $P_{\mathfrak{m}^\vee,w}(\theta(\mathrm{Frob}_w)) = 0$) implies that θ is equivalent to $\overline{\rho}_{\mathfrak{m}}$ (resp. $\overline{\rho}_{\mathfrak{m}^\vee}$), where $\mathfrak{m}^\vee$ is the maximal ideal of $\mathbb{T}_S$ associated to the dual mutliset of Satake parameters, cf. [**8**] notation 4.4. In [**18**], the authors proved that the previous property is verified in each of the following cases:
   - either $\overline{\rho}_{\mathfrak{m}}$ is induced from a character of G_K where K/F is a cyclic galoisian extension;
   - or $l \geqslant d$ and $SL_d(k) \subset \overline{\rho}_{\mathfrak{m}}(G_F) \subset \overline{\mathbb{F}}_l^{\times} GL_d(k)$ for some subfield $k \subset \overline{\mathbb{F}}_l$.

1.2. Theorem. — *Suppose that*

– $\overline{\rho_{\mathfrak{m}}}$ *is irreducible,*
– *the set* $S_v(\mathfrak{m})$ *of irreducible sub-quotients of the restriction of* $\overline{\rho_{\mathfrak{m}}}$ *to the decomposition group at* v *is made of characters. Moreover, whatever is the* $\overline{\mathbb{F}}_l$*-character* χ_v*, we assume that* $S_v(\mathfrak{m})$ *does not contain* $\{\chi_v, \chi_v(1), \cdots, \chi_v(e_v(l)-1)\}$.
– $\mathfrak{m}$ *is KHT-free in the sense of definition 1.1.*

Then the two partitions $\underline{d_{\mathfrak{m},v}}$ *and* $\underline{d_{\tilde{\mathfrak{m}},v}}$ *associated respectively to the unipotent operator at* v *for* $\rho_{\tilde{\mathfrak{m}},v}$ *and* $\overline{\rho_{\mathfrak{m},v}}$*, share the same maximal nilpotency as explained below in definition 1.3.*

Two characters $\chi_v, \chi_v' : F_v^\times \longrightarrow \overline{\mathbb{Q}}_l^\times$ (resp. $\overline{\mathbb{F}}_l^\times$), viewed by the Artin isomorphism as characters of the Weil group at v, are said inertially equivalent, and we write $\chi_v' \sim^i \chi_v$, if there exists $\xi : \mathbb{Z} \longrightarrow \overline{\mathbb{Q}}_l^\times$ (resp. $\overline{\mathbb{F}}_l^\times$) such that $\chi_v' \simeq \chi_v \otimes (\xi \circ \mathrm{val})$. We then denote by $\mathcal{C}_v$ (resp. $\overline{\mathcal{C}}_v$) the set of characters $\chi_v : F_v^\times \longrightarrow \overline{\mathbb{Q}}_l^\times$ (resp. $\overline{\mathbb{F}}_l^\times$) modulo inertial equivalence.

For $\tilde{\mathfrak{m}} \subset \mathfrak{m}$ as above, we write

$$\rho_{\tilde{\mathfrak{m}},v} \simeq \bigoplus_{\chi_v \in \mathcal{C}_v} \rho_{\tilde{\mathfrak{m}},\chi_v},$$

where $\rho_{\tilde{\mathfrak{m}},\chi_v} \simeq \mathrm{Sp}_{n_{\tilde{\mathfrak{m}},1}(\chi_v)}(\chi_{v,1}) \oplus \cdots \oplus \mathrm{Sp}_{n_{\tilde{\mathfrak{m}},r}(\chi_v)}(\chi_{v,r})$ with $\chi_{v,1}, \cdots \chi_{v,r}$ inertially equivalent to χ_v, and $n_{\tilde{\mathfrak{m}},1}(\chi_v) \geqslant \cdots \geqslant n_{\tilde{\mathfrak{m}},r}(\chi_v)$. In the same way, we write

$$\overline{\rho}_{\mathfrak{m},v} \simeq \bigoplus_{\overline{\chi}_v \in \overline{\mathcal{C}}_v} \overline{\rho}_{\mathfrak{m},\overline{\chi}_v},$$

as well as $n_{\mathfrak{m},1}(\overline{\chi}_v) \geqslant \cdots \geqslant n_{\mathfrak{m},r}(\overline{\chi}_v)$.

1.3. Definition. — *We say that $\rho_{\tilde{\mathfrak{m}},v}$ and $\overline{\rho}_{\mathfrak{m},v}$, share the same maximal nilpotency if for every $\overline{\chi}_v \in \overline{\mathcal{C}}_v$ there exists χ_v whose modulo l reduction is inertially equivalent to $\overline{\chi}_v$, with $n_{\tilde{\mathfrak{m}},1}(\chi_v) = n_{\mathfrak{m},1}(\overline{\chi}_v)$.*

Remarks:

- For GL_2 and the proper Shimura curves of [**14**] associated to a totally real field $F^+ \neq \mathbb{Q}$, our second hypothesis is $q_v \not\equiv -1 \mod l$ and the conclusion is $d_{\tilde{\mathfrak{m}},v} = d_{\mathfrak{m},v}$.
- In [**10**] corollary 2.9, we explained how to construct automorphic congruences between tempered representations, one being ramified at v while the other is not. To do so we need to start from a non trivial torsion class in the cohomology of some KHT Shimura variety. In section 3 of [**10**], we construct torsion cohomology classes associated to some maximal ideal $\mathfrak{m}$ but in these constructions $\overline{\rho}_{\mathfrak{m}}$ is always reducible, as it is also in theorem 1.1 of [**1**].

The main idea to prove the theorem, is to analyze the monodromy action at the place v, inside the cohomology groups of the compact KHT Shimura variety $\mathrm{Sh}_{I,v} \longrightarrow \mathrm{Spec}\,\mathcal{O}_v$. This cohomology localized at $\mathfrak{m}$ is computed through the spectral sequence, cf. (2.11)

$$E_{1,\mathfrak{m}}^{p,q} \Longrightarrow E_{\infty,\mathfrak{m}}^{p+q},$$

associated to the filtration of stratification of the free perverse sheaf of nearby cycles. As $\overline{\rho}_{\mathfrak{m}}$ is supposed to be irreducible, it degenerates at

E_1 after tensoring with $\overline{\mathbb{Q}}_l$. Moreover, by definition of KHT-freeness of $\mathfrak{m}$, we know that the $E^n_{\infty,\mathfrak{m}}$ are all free. The main step is then to prove that all the $E^{p,q}_{1,\mathfrak{m}}$ are also free so that the monodromy action $N^{coho}_{v,\mathfrak{m}}$ on the cohomology groups is purely of local nature and can be read on the sheaf of nearby cycles, cf. proposition 4.1 and the proof of theorem 1.2 in §5. The result follows then from the theorem 2.12, originally proved by Carayol in [**15**] for GL_2, which describes the $\overline{\mathbb{Z}}_l$-cohomology localized at $\mathfrak{m}$ of our KHT Shimura variety as a tensor product $\sigma_{\mathfrak{m}} \otimes_{\mathbb{T}^S_{\xi,\mathfrak{m}}} \rho_{\mathfrak{m}}$, where

- $\sigma_{\mathfrak{m}}$ is a $\mathbb{T}^S_{\xi,\mathfrak{m}}$-module on which $\mathrm{Gal}_{F,S}$ acts trivially,
- and $\rho_{\mathfrak{m}} : \mathrm{Gal}_{F,S} \longrightarrow GL_d(\mathbb{T}^S_{\xi,\mathfrak{m}})$ is a stable lattice of $\bigoplus_{\tilde{\mathfrak{m}}\subset\mathfrak{m}} \rho_{\tilde{\mathfrak{m}}}$, cf. theorem 5.1 of [**29**].

More precisely, the freeness of the $E^{p,q}_{1,\mathfrak{m}}$ implies that the nilpotency of the modulo l reduction of $N^{coho}_{v,\mathfrak{m}}$ is maximal so that it has to be the same as those of $\overline{N}_{v,\mathfrak{m}}$.

One might expect similar results in other situations where enters arithmetic geometry. Although this level fixing phenomenon seems specific to these situations, cf. for example theorem 4.1 of [**1**].

Remark. Using the main results of [**9**], you could replace everywhere the characters by irreducible supercuspidal representations such that their modulo l reduction remains supercuspidal. The main issue is the lemma 3.2 which is obvious for characters and which is one of the main results of [**9**] for supercuspidal representations which remain supercuspidal after modulo l reduction.

In the work of Ribet, the main tool for proving the level raising, is Ihara's lemma. In higher dimension, Clozel, Harris and Taylor proposed a conjectural version of Ihara's lemma for compact unitary groups. In [**12**] we proved some instances of this conjecture implied by a non compact version of Ihara's lemma for $H^{d-1}(\mathrm{Sh}_{I,\bar{\eta}_v}, \overline{\mathbb{F}}_l)_{\mathfrak{m}}$. In the last section, we first prove that Ihara's lemma for compact unitary group implies Ihara's lemma for KHT unitary groups. When the order q_v of the residual field at the place v is such that $q_v \equiv 1 \mod l$, we then prove that Ihara's lemma implies that the partitions $\underline{d_{\mathfrak{m},v}} = \underline{d_{\tilde{\mathfrak{m}},v}}$ are maximal, cf. the remark after proposition 7.2 and, thanks to theorem 1.2, this level raising property is in fact equivalent to Ihara's lemma, cf. proposition 7.3.

2. The monodromy operator acting on the cohomology

Let $F = F^+E$ be a CM field with $E/\mathbb{Q}$ quadratic imaginary and F^+ totally real. Let B/F be a central division algebra with dimension d^2 with an involution of second kind $*$. For $\beta \in B^{*=-1}$, consider the similitude group $G/\mathbb{Q}$ defined for any $\mathbb{Q}$-algebra R by

$$G(R) := \{(\lambda, g) \in R^\times \times (B^{op} \otimes_{\mathbb{Q}} R)^\times \text{ such that } gg^{\sharp_\beta} = \lambda\},$$

with $B^{op} = B \otimes_{F,c} F$ where $c = *_{|F}$ is the complex conjugation and $\sharp_\beta$ is the involution $x \mapsto x^{\sharp\beta} := \beta x^* \beta^{-1}$. Following [**22**], we assume from now on that $G(\mathbb{R})$ has signatures $(1, d-1), (0, d), \cdots, (0, d)$.

2.1. Definition. — *Let* Spl *be the set of places v of F such that $p_v := v_{|\mathbb{Q}} \neq l$ is split in E and $B_v^\times \simeq GL_d(F_v)$.*

We now suppose that $p = uu^c$ splits in E so that

$$G(\mathbb{Q}_p) \simeq \mathbb{Q}_p^\times \times \prod_{w|u} (B_w^{op})^\times$$

where w describes the places of F above u and we fix a place $v \in$ Spl dividing p.

2.2. Definition. — *For a finite set S of places of $\mathbb{Q}$ containing the places where G is ramified, denote by $\mathbb{T}^S_{abs} := \prod_{x \notin S} \mathbb{T}_{x,abs}$ the abstract unramified Hecke algebra where $\mathbb{T}_{x,abs} \simeq \overline{\mathbb{Z}}_l[X^{un}(T_x)]^{W_x}$ for T_x a split torus, W_x the spherical Weyl group and $X^{un}(T_x)$ is the set of $\overline{\mathbb{Z}}_l$-unramified characters of T_x.*

Example. For $x = uu^c$ split in E we have

$$\mathbb{T}_{x,abs} = \prod_{w|u} \overline{\mathbb{Z}}_l\big[T_{w,i} : \ i = 1, \cdots, d\big],$$

where $T_{w,i}$ is the characteristic function of

$$GL_d(\mathcal{O}_w) \operatorname{diag}(\overbrace{\varpi_w, \cdots, \varpi_w}^{i}, \overbrace{1, \cdots, 1}^{d-i}) GL_d(\mathcal{O}_w) \subset GL_d(F_w).$$

We then denote by $\mathcal{I}$ the set of open compact subgroups

$$U^p(m_1, \cdots, m_r) = U^p \times \mathbb{Z}_p^\times \times \prod_{i=1}^{r} \operatorname{Ker}(\mathcal{O}_{B_{v_i}}^\times \longrightarrow (\mathcal{O}_{B_{v_i}}/\mathcal{P}_{v_i}^{m_i})^\times)$$

where U^p is any small enough open compact subgroup of $G(\mathbb{A}^{p,\infty})$ and $\mathcal{O}_{B_{v_i}}$ is the maximal order of B_{v_i} with maximal ideal $\mathcal{P}_{v_i}$ and where $v = v_1, \cdots, v_r$ are the places of F above u with $p = uu^c$.

2.3. Notation. — *For $I = U^p(m_1, \cdots, m_r) \in \mathcal{I}$, we will denote by $I^v(n) := U^p(n, m_2, \cdots, m_r)$. We also denote by* $\mathrm{Spl}(I)$ *the subset of* Spl *of places which does not divide the level I.*

2.4. Notation. — *As defined in* [**22**]*, attached to each $I \in \mathcal{I}$ is a Shimura variety called of KHT-type and denoted by*

$$\mathrm{Sh}_{I,v} \longrightarrow \mathrm{Spec}\, \mathcal{O}_v$$

where $\mathcal{O}_v$ denote the ring of integers of the completion F_v of F at v.

Let $\sigma_0 : E \hookrightarrow \overline{\mathbb{Q}}_l$ be a fixed embedding and write Φ for the set of embeddings $\sigma : F \hookrightarrow \overline{\mathbb{Q}}_l$ whose restriction to E equals σ_0. There exists then, cf. [**22**] p.97, an explicit bijection between irreducible algebraic representations ξ of G over $\overline{\mathbb{Q}}_l$ and $(d+1)$-uple $\big(a_0, (\overrightarrow{a_\sigma})_{\sigma\in\Phi}\big)$ where $a_0 \in \mathbb{Z}$ and for all $\sigma \in \Phi$, we have $\overrightarrow{a_\sigma} = (a_{\sigma,1} \leqslant \cdots \leqslant a_{\sigma,d})$. We then denote by

$$V_{\xi,\overline{\mathbb{Z}}_l}$$

the associated $\overline{\mathbb{Z}}_l$-local system on $\mathrm{Sh}_{I,v}$.

2.5. Notation. — *Let $\mathbb{T}^S_\xi$ be the image of $\mathbb{T}^S_{abs}$ inside*

$$\bigoplus_{i=0}^{2d-2} \varinjlim_{I} H^i(\mathrm{Sh}_{I,\bar{\eta}_v}, V_{\xi,\overline{\mathbb{Q}}_l})$$

where the limit concerned the ideals I which are maximal at each places outside S, and $\mathrm{Sh}_{I,\bar{\eta}_v}$ *is the geometric generic fiber of* $\mathrm{Sh}_{I,v}$.

To each maximal ideal $\widetilde{\mathfrak{m}}$ of $\mathbb{T}^S_\xi[1/l]$, or equivalently a minimal prime of $\mathbb{T}^S_\xi$, which we now supposed to be non-Eisenstein, is associated an irreducible automorphic representation $\Pi_{\widetilde{\mathfrak{m}}}$ which is ξ-cohomological, i.e. there exists an integer i such that

$$H^i\big((\mathrm{Lie}\ G(\mathbb{R})) \otimes_{\mathbb{R}} \mathbb{C}, U, \Pi_\infty \otimes \xi^\vee\big) \neq (0),$$

where U is a maximal open compact subgroup modulo the center of $G(\mathbb{R})$.

2.6. Notation. — *Let denote by* $\mathrm{Scusp}_v(\widetilde{\mathfrak{m}})$*, the supercuspidal support of its local component at* v*, denoted* $\Pi_{\widetilde{\mathfrak{m}},v}$*. Note*[(1)] *that the modulo* l *reduction of* $\mathrm{Scusp}_v(\widetilde{\mathfrak{m}})$ *is independent of the choice of* $\widetilde{\mathfrak{m}} \subset \mathfrak{m}$*: we denote it* $\mathrm{Scusp}_v(\mathfrak{m})$*.*

Recall that the geometric special fiber $\mathrm{Sh}_{I,\bar{s}_v}$ of $\mathrm{Sh}_{I,v}$, is equipped with the Newton stratification

$$\mathrm{Sh}^{\geqslant d}_{I,\bar{s}_v} \subset \mathrm{Sh}^{\geqslant d-1}_{I,\bar{s}_v} \subset \cdots \subset \mathrm{Sh}^{\geqslant 1}_{I,\bar{s}_v} = \mathrm{Sh}_{I,\bar{s}_v},$$

where for $1 \leqslant h \leqslant d$, $\mathrm{Sh}^{\geqslant h}_{I,\bar{s}_v}$ (resp. $\mathrm{Sh}^{=h}_{I,\bar{s}_v}$) is the closed (resp. the open) Newton stratum of height h and of pure dimension $d-h$, defined as the sub-scheme where the connected component of the universal Barsotti-Tate group is of rank greater or equal to h (resp. equal to h).

Moreover for $1 \leqslant h < d$, the Newton stratum $\mathrm{Sh}^{=h}_{\mathcal{I},\bar{s}_v}$ is geometrically induced under the action of the parabolic subgroup $P_{h,d-h}(F_v)$, defined as the stabilizer of the first h vectors of the canonical basis of F_v^d. Concretely, cf. [**2**] §10.4, this means that there exists a closed sub-scheme $\mathrm{Sh}^{=h}_{I,\bar{s}_v,\overline{1_h}}$ stabilized by the Hecke action of $P_{h,d-h}(F_v)$ and such that

$$\varprojlim_n \mathrm{Sh}^{=h}_{I^v(n),\bar{s}_v} \simeq \left(\varprojlim_n \mathrm{Sh}^{=h}_{I^v(n),\bar{s}_v,\overline{1_h}}\right) \times_{P_{h,d-h}(F_v)} GL_d(F_v).$$

2.7. Notation. — *For a representation* π_v *of* $GL_d(F_v)$ *with coefficients either* $\overline{\mathbb{Q}}_l$ *or* $\overline{\mathbb{F}}_l$*, and* $n \in \frac{1}{2}\mathbb{Z}$*, we set* $\pi_v\{n\} := \pi_v \otimes \nu^n$ *where* $\nu(g) := q_v^{-\operatorname{val}\det(g)}$*. Recall that the normalized induction of two representations* $\pi_{v,1}$ *and* $\pi_{v,2}$ *of respectively* $GL_{n_1}(F_v)$ *and* $GL_{n_2}(F_v)$ *is*

$$\pi_1 \times \pi_2 := \mathrm{ind}_{P_{n_1,n_2}(F_v)}^{GL_{n_1+n_2}(F_v)} \pi_{v,1}\{\frac{n_2}{2}\} \otimes \pi_{v,2}\{-\frac{n_1}{2}\}.$$

Recall that a representation π_v of $GL_d(F_v)$ is called *cuspidal* (resp. *supercuspidal*) if it is not a subspace (resp. a subquotient) of a proper parabolic induced representation. When the field of coefficients is of characteristic zero then these two notions coincides, but this is no more true over $\overline{\mathbb{F}}_l$. For example the modulo l reduction of an irreducible $\overline{\mathbb{Q}}_l$-representation is still cuspidal but not necessary supercuspidal, its supercuspidal support being a Zelevinsky segment.

(1) It follows, through the Langlands correspondence, from Cebotarev's theorem and the fact that a semi-simple representation is determined, up to isomorphism, by its characteristic polynomials.

2.8. Definition. — *(see* [**32**] *§9 and* [**4**] *§1.4) Let g be a divisor of $d = sg$ and π_v an irreducible cuspidal $\overline{\mathbb{Q}}_l$-representation of $GL_g(F_v)$. The induced representation*

$$\pi_v\{\frac{1-s}{2}\} \times \pi_v\{\frac{3-s}{2}\} \times \cdots \times \pi_v\{\frac{s-1}{2}\} \tag{2.8}$$

holds a unique irreducible quotient (resp. subspace) denoted $\mathrm{St}_s(\pi_v)$ *(resp.* $\mathrm{Speh}_s(\pi_v)$*); it is a generalized Steinberg (resp. Speh) representation.*

Remark. For χ_v a character, $\mathrm{Speh}_s(\chi_v)$ is the character $\chi_v \circ \det$ of $GL_s(F_v)$.

Let π_v be an irreducible cuspidal $\overline{\mathbb{Q}}_l$-representation of $GL_g(F_v)$ and fix $t \geqslant 1$ such that $tg \leqslant d$. Thanks to Igusa varieties, Harris and Taylor constructed a local system on $\mathrm{Sh}^{=tg}_{\mathcal{I},\bar{s}_v,\overline{1_{tg}}}$

$$\mathcal{L}_{\overline{\mathbb{Q}}_l}(\pi_v[t]_D)_{\overline{1_{tg}}} = \bigoplus_{i=1}^{e_{\pi_v}} \mathcal{L}_{\overline{\mathbb{Q}}_l}(\rho_{v,i})_{\overline{1_{tg}}}$$

where

- $\pi_v[t]_D$ is the representation of $D^\times_{v,tg}$ which is the image of the contragredient of $\mathrm{St}_t(\pi_v)$ by the Jacquet-Langlands correspondence,
- $D_{v,tg}$ is the central division algebra over F_v with invariant tg,
- with maximal order denoted by $\mathcal{D}_{v,tg}$
- and with $(\pi_v[t]_D)_{|\mathcal{D}^\times_{v,tg}} = \bigoplus_{i=1}^{e_{\pi_v}} \rho_{v,i}$ with $\rho_{v,i}$ irreductible.

The Hecke action of $P_{tg,d-tg}(F_v)$ is then given through its quotient

$$P_{tg,d-tg}(F_v) \twoheadrightarrow GL_{tg}(F_v) \times GL_{d-tg}(F_v) \twoheadrightarrow GL_{d-tg}(F_v) \times \mathbb{Z},$$

where $GL_{tg}(F_v) \times GL_{d-tg}(F_v)$ is the Levi quotient of the parabolic $P_{tg,d-tg}(F_v)$ and the second map is given by the valuation of the determinant map $GL_{tg}(F_v) \twoheadrightarrow \mathbb{Z}$. These local systems have stable $\overline{\mathbb{Z}}_l$-lattices and we will write simply $\mathcal{L}(\pi_v[t]_D)_{\overline{1_{tg}}}$ for any $\overline{\mathbb{Z}}_l$-stable lattice that we do not want to specify.

2.9. Notations. — *For Π_t any $\overline{\mathbb{Q}}_l$-representation of $GL_{tg}(F_v)$, and $\Xi : \frac{1}{2}\mathbb{Z} \longrightarrow \overline{\mathbb{Z}}_l^\times$ defined by $\Xi(\frac{1}{2}) = q^{1/2}$, we introduce*

$$\widetilde{HT}_{\overline{\mathbb{Q}}_l,\overline{1_{tg}}}(\pi_v, \Pi_t) := \mathcal{L}_{\overline{\mathbb{Q}}_l}(\pi_v[t]_D)_{\overline{1_{tg}}} \otimes \Pi_t \otimes \Xi^{\frac{tg-d}{2}}$$

and its induced version

$$\widetilde{HT}_{\overline{\mathbb{Q}}_l}(\pi_v, \Pi_t) := \left(\mathcal{L}_{\overline{\mathbb{Q}}_l}(\pi_v[t]_D)_{\overline{1_{tg}}} \otimes \Pi_t \otimes \Xi^{\frac{tg-d}{2}}\right) \times_{P_{tg,d-tg}(F_v)} GL_d(F_v),$$

where the unipotent radical of $P_{tg,d-tg}(F_v)$ acts trivially and the action of

$$(g^{\infty,v}, \begin{pmatrix} g_v^c & * \\ 0 & g_v^{et} \end{pmatrix}, \sigma_v) \in G(\mathbb{A}^{\infty,v}) \times P_{tg,d-tg}(F_v) \times W_v$$

where W_v is the Weil group at v, is given

- *by the action of g_v^c on Π_t and $\deg(\sigma_v) \in \mathbb{Z}$ on $\Xi^{\frac{tg-d}{2}}$, where $\deg : W_v \longrightarrow \mathbb{Z}$ sends geometric Frobenius to 1,*
- *and the action of $(g^{\infty,v}, g_v^{et}, \mathrm{val}(\det g_v^c) - \deg \sigma_v) \in G(\mathbb{A}^{\infty,v}) \times GL_{d-tg}(F_v) \times \mathbb{Z}$ on $\mathcal{L}_{\overline{\mathbb{Q}}_l}(\pi_v[t]_D)_{\overline{1_{tg}}} \otimes \Xi^{\frac{tg-d}{2}}$.*

We also introduce

$$HT_{\overline{\mathbb{Q}}_l,\overline{1_{tg}}}(\pi_v, \Pi_t) := \widetilde{HT}_{\overline{\mathbb{Q}}_l,\overline{1_{tg}}}(\pi_v, \Pi_t)[d-tg],$$

and the perverse sheaf

$$P_{\overline{\mathbb{Q}}_l}(t, \pi_v)_{\overline{1_{tg}}} := j^{=tg}_{\overline{1_{tg}},!*} HT_{\overline{\mathbb{Q}}_l,\overline{1_{tg}}}(\pi_v, \mathrm{St}_t(\pi_v)) \otimes \mathbb{L}(\pi_v),$$

and their induced version, $HT_{\overline{\mathbb{Q}}_l}(\pi_v, \Pi_t)$ and $P_{\overline{\mathbb{Q}}_l}(t, \pi_v)$, where

$$j^{=tg} = i^{tg} \circ j^{\geqslant tg} : \mathrm{Sh}^{=tg}_{\mathcal{I},\bar{s}_v} \hookrightarrow \mathrm{Sh}^{\geqslant tg}_{\mathcal{I},\bar{s}_v} \hookrightarrow \mathrm{Sh}_{\mathcal{I},\bar{s}_v}$$

and $\mathbb{L}$ is the local Langlands correspondence composed by contragredient.

We will also denote by $HT_{\overline{\mathbb{Q}}_l,\xi}(\pi_v, \Pi_t) := HT_{\overline{\mathbb{Q}}_l}(\pi_v, \Pi_t) \otimes V_\xi$ and similarly for the other notations as for example $P_{\overline{\mathbb{Q}}_l,\xi}(t, \pi_v) := P_{\overline{\mathbb{Q}}_l}(t, \pi_v) \otimes V_\xi$.

Remarks:

- We will simply denote by $P(t, \pi_v)$ any $\overline{\mathbb{Z}}_l$-lattice of $P_{\overline{\mathbb{Q}}_l}(t, \pi_v)$ that we do not want to precise except that it is stable under the various actions. We will use a similar convention for the other sheaves introduced before. When considering $\overline{\mathbb{F}}_l$-coefficients, we will put $\overline{\mathbb{F}}_l$ in place of $\overline{\mathbb{Q}}_l$ in the notations.
- Recall that π'_v is said inertially equivalent to π_v, and we write $\pi_v \sim_i \pi'_v$, if there exists a character $\zeta : \mathbb{Z} \longrightarrow \overline{\mathbb{Q}}_l^\times$ such that $\pi'_v \simeq \pi_v \otimes (\zeta \circ \mathrm{val} \circ \det)$. We denote by e_{π_v} the order of the inertial class of π_v.
- Note, cf. [**3**] 2.1.4, that $P_{\overline{\mathbb{Q}}_l}(t, \pi_v)$ depends only on the inertial class of π_v and

  $$P_{\overline{\mathbb{Q}}_l}(t, \pi_v) = e_{\pi_v} \mathcal{P}_{\overline{\mathbb{Q}}_l}(t, \pi_v)$$

  where $\mathcal{P}_{\overline{\mathbb{Q}}_l}(t, \pi_v)$ is an irreducible perverse sheaf.

– For a character[(2)] χ_v and when $t < e_v(l)$, up to homothety there is only one stable $\overline{\mathbb{Z}}_l$-stable lattice of $\mathcal{L}(\pi_v[t]_D)$. From the description of the modulo l reduction of $\mathrm{St}_t(\chi_v)$ in [**5**], the same is then true for $\mathcal{P}(t, \chi_v)$.
– Over $\overline{\mathbb{Z}}_l$, we also have the $p+$-perverse structure which is dual to the usual p-structure.

2.10. Notation. — *Let denote by*

$$\Psi_v := R\Psi_{\eta_v}(\overline{\mathbb{Z}}_l[d-1])(\frac{d-1}{2})$$

the nearby cycles autodual free perverse sheaf on the geometric special fiber $\mathrm{Sh}_{I,\bar{s}_v}$ *of* $\mathrm{Sh}_{I,v}$.

Following the constructions of [**6**] §2.3, we can then define a $\overline{\mathbb{Z}}_l$-filtration $\mathrm{Fil}^\bullet(\Psi_v)$ whose graded parts $\mathrm{gr}^r(\Psi_v)$ are free $\overline{\mathbb{Z}}_l$-perverse sheaves, cf. [**6**] §1, of the following shape

$$\begin{aligned} {}^p j_{!*}^{=tg} HT(\pi_v, \mathrm{St}_t(\pi_v))(\frac{1-t+2\delta}{2}) &\hookrightarrow \mathrm{gr}^r(\Psi_v) \\ &\hookrightarrow\!\!\!\!\rightarrow {}^{p+} j_{!*}^{=tg} HT(\pi_v, \mathrm{St}_t(\pi_v))(\frac{1-t+2\delta}{2}) \end{aligned}$$

for some $0 \leqslant \delta \leqslant t-1$, where $\hookrightarrow\!\!\!\!\rightarrow$ means a bimorphism[(3)], that is both a epimorphism and a monomorphism, and where the lattice $HT(\pi_v, \mathrm{St}_t(\pi_v))$ depends of the construction in the general situation but we will see that, with our hypothesis, up to isomorphism, there is only one such stable lattice.
Remarks:

– In [**9**], we prove that if you always use the adjunction maps $j_!^{=h} j^{=h,*} \to \mathrm{Id}$ (resp. $\mathrm{Id} \to j_*^{=h} j^{=h,*}$) then all the previous graded parts are isomorphic to p-intermerdiate (resp. $p+$) extensions. In our situation this issue disappear thanks to lemma 3.2.
– We can easily arrange the filtration so that it is compatible with the nilpotent monodromy operator N_v, i.e. so that for any r the

(2)For a general supercuspidal representation π_v whose modulo l reduction ϱ is still supercuspidal, the same is true if $t < m(\varrho)$ where $m(\varrho)$ is either the order of the Zelevinsky line of ϱ if it is > 1, otherwise $m(\varrho) := l$.
(3)by [**6**] corollary 1.4.6, the p and $p+$ intermediate extensions are free

image of $\mathrm{Fil}^r(\Psi_v)\otimes_{\overline{\mathbb{Z}}_l}\overline{\mathbb{Q}}_l$ under N_v is some $\mathrm{Fil}^{\phi(r)}(\Psi_v)\otimes_{\overline{\mathbb{Z}}_l}\overline{\mathbb{Q}}_l$ for some decreasing function ϕ.
– When dealing with sheaves, there is no need to introduce the local system $V_{\xi,\overline{\mathbb{Z}}_l}$ because it suffices to add $\otimes_{\overline{\mathbb{Z}}_l}V_{\xi,\overline{\mathbb{Z}}_l}$ to the formulas.

We now consider a fixed local system $V_{\xi,\overline{\mathbb{Z}}_l}$ and, following previous notations, we write $\Psi_{\xi,v}:=\Psi_v\otimes_{\overline{\mathbb{Z}}_l}V_{\xi,\overline{\mathbb{Z}}_l}$. We then have a spectral sequence

$$E_1^{p,q}=H^{p+q}(\mathrm{Sh}_{I,\bar{s}_v},\mathrm{gr}^{-p}(\Psi_{\xi,v}))\Rightarrow H^{p+q}(\mathrm{Sh}_{I,\bar{\eta}_v},V_{\xi,\overline{\mathbb{Z}}_l}). \tag{2.11}$$

As pointed out in [**11**], if for some $\mathfrak{m}$ the spectral sequence is concentrated in middle degree, i.e. $E_{1,\mathfrak{m}}^{p,q}=0$ for $p+q\neq d-1$, and all the $E_{1,\mathfrak{m}}^{p,d-1-p}$ are free, then, for $l>d$, the action of the monodromy operator $N_{v,\mathfrak{m}}^{coho}$ on $H^{d-1}(\mathrm{Sh}_{I,\bar{\eta}_v},V_{\xi,\overline{\mathbb{Z}}_l})_{\mathfrak{m}}$ comes from the action of N_v on Ψ_v, cf. §5.

We now want to understand the relation between the modulo l reduction $\overline{N}_{v,\mathfrak{m}}^{coho}$ of $N_{v,\mathfrak{m}}^{coho}$ and $\overline{N}_{v,\mathfrak{m}}$. We first write

$$H^{d-1}(\mathrm{Sh}_{I,\bar{\eta}_v},V_{\xi,\overline{\mathbb{Z}}_l})_{\mathfrak{m}}\otimes_{\overline{\mathbb{Z}}_l}\overline{\mathbb{Q}}_l\simeq\bigoplus_{\tilde{\mathfrak{m}}\subset\mathfrak{m}}\Pi_{\tilde{\mathfrak{m}}}^I\otimes\rho_{\tilde{\mathfrak{m}}},$$

where we view $\Pi_{\tilde{\mathfrak{m}}}$ as a $\mathbb{T}_\xi^S\otimes_{\overline{\mathbb{Z}}_l}\overline{\mathbb{Q}}_l$-module and $\rho_{\tilde{\mathfrak{m}}}$ as a representation of $\mathrm{Gal}_{F,S}$, the Galois group of the maximal extension of F which is unramified outside S.

The cohomology with $\overline{\mathbb{Z}}_l$-coefficients is then a lattice of this decomposition which is stable by the action of both $\mathbb{T}_{\xi,\mathfrak{m}}^S$ and $\mathrm{Gal}_{F,S}$. As $\overline{\rho}_{\mathfrak{m}}$ is supposed to be irreducible, we then have the following analog in higher dimension of a theorem of Carayol in [**15**].

2.12. Theorem. — *(cf.* [**29**] *theorem 5.6) As a* $\mathbb{T}_{S,\mathfrak{m}}[\mathrm{Gal}_{F,S}]$*-module,*

$$H^{d-1}(X_{U,\bar{\eta}},V_{\xi,\overline{\mathbb{Z}}_l})_{\mathfrak{m}}\simeq\sigma_{\mathfrak{m}}\otimes_{\mathbb{T}_{\xi,\mathfrak{m}}^S}\rho_{\mathfrak{m}},$$

for some $\mathbb{T}_{\xi,\mathfrak{m}}^S$*-module* $\sigma_{\mathfrak{m}}$ *on which* $\mathrm{Gal}_{F,S}$ *acts trivially and*

$$\rho_{\mathfrak{m}}:\mathrm{Gal}_{F,S}\longrightarrow GL_d(\mathbb{T}_{\xi,\mathfrak{m}}^S)$$

is a stable lattice of $\bigoplus_{\tilde{\mathfrak{m}}\subset\mathfrak{m}}\rho_{\tilde{\mathfrak{m}}}$*, cf. theorem 5.1 of* [**29**].

Note that the monodromy operator $N_{v,\mathfrak{m}}$ of $\sigma_{\mathfrak{m}}$ is then equals to $N_{v,\mathfrak{m}}^{coho}$ so that in particular $(N_{v,\mathfrak{m}}^{coho})^k$ is zero modulo l if and only if $\overline{N}_{v,\mathfrak{m}}^k=0$.

3. A saturated filtration of the cohomology

The aim of this section is the following proposition.

3.1. Proposition. — *Under the hypothesis of theorem 1.2, the $E_{1,\mathfrak{m}}^{p,q}$ are torsion free and trivial for $p+q \neq d-1$.*

Note that, as (2.11) after localization at $\mathfrak{m}$, degenerates at E_1 over $\mathbb{Q}_l$, then the spectral sequence gives us a saturated filtration of $H^{d-1}(\mathrm{Sh}_{I,\bar{\eta}_v}, V_{\xi,\overline{\mathbb{Z}}_l})_{\mathfrak{m}}$. As the proof uses Grothendieck-Verdier duality and we need to understand the difference between the p and $p+$ intermediate extensions of Harris-Taylor local systems $HT(\pi_v, \mathrm{St}_t(\pi_v))$.

3.2. Lemma. — *With the previous notations, we have an isomorphism*

$$ {}^{p}j^{\geqslant h}_{\overline{1_h},!*} HT_{\overline{1_h}}(\chi_v, \Pi_h) \simeq {}^{p+}j^{\geqslant h}_{\overline{1_h},!*} HT_{\overline{1_h}}(\chi_v, \Pi_h). $$

Proof. — By definition we have

$$ HT_{\overline{1_h}}(\chi_v, \Pi_h)[h-d] = (\overline{\mathbb{Z}}_l)_{|\,\mathrm{Sh}^{=h}_{I,\bar{s}_v,\overline{1_h}}} \otimes \Pi_h, $$

where the action of the fondamental group goes through the character χ_v. As, cf. [**24**], $\mathrm{Sh}^{\geqslant h}_{I,\bar{s}_v,\overline{1_h}}$ is smooth over $\mathrm{Spec}\,\overline{\mathbb{F}}_p$, then $P := (\overline{\mathbb{Z}}_l)_{|\,\mathrm{Sh}^{\geqslant h}_{I,\bar{s}_v,\overline{1_h}}} \otimes \Pi_h$ is perverse for the two t-structures with

$$ i^{h\leqslant+1,*}_{\overline{1_h}} P \in {}^{p}\mathcal{D}^{<0} \text{ and } i^{h\leqslant+1,!}_{\overline{1_h}} P \in {}^{p+}\mathcal{D}^{\geqslant 1}. $$

□

Proof. — (of proposition 3.1)
As $\mathfrak{m}$ is supposed to be KHT-free, then all the $E^n_{\infty,\mathfrak{m}}$ are free. Moreover, as $\overline{\rho}_{\mathfrak{m}}$ is irreducible, then, cf. [**4**] §3.6, the $E_{1,\mathfrak{m}}^{p,q} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ are all zero if $p+q \neq d-1$. Recall, cf. [**9**] proposition 3.1.3, that we have the following splitting

$$ \Psi_v \simeq \bigoplus_{g=1}^{d} \bigoplus_{\varrho \in \mathrm{Scusp}_{\overline{\mathbb{F}}_l}(g)} \Psi_\varrho $$

where $\mathrm{Scusp}_{\overline{\mathbb{F}}_l}(g)$ is the set of inertial equivalence classes of irreducible $\overline{\mathbb{F}}_l$-supercuspidal representations of $GL_g(F_v)$, with the property that the irreducible constituents of $\Psi_\varrho \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ are exactly the perverse Harris-Taylor sheaf associated to an irreducible cuspidal $\overline{\mathbb{Q}}_l$-representation of

some $GL_g(F_v)$ such that the supercuspidal support of the modulo l reduction of π_v is a segment associated to the inertial class ϱ. In particular for every $\varrho \in \mathrm{Scusp}_{\overline{\mathbb{F}}_l}(g)$, the cohomology groups of Ψ_ϱ are torsion free.
Remark. When computing the $\mathfrak{m}$-localized cohomology groups, we are only concerned with $\varrho \in \mathrm{Scusp}_v(\mathfrak{m})$ which are characters. Moreover as $e_v(l) > d$, in $\Psi_\varrho \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ we have only to deal with characters χ_v so that, by the previous lemma, the p and $p+$ intermediate extensions coincide.

3.3. Proposition. — *We have the following equivariant resolution*

$$0 \to j_!^{=d} HT(\chi_v, \mathrm{St}_h(\chi_v\{\frac{h-d}{2}\}) \times \mathrm{Speh}_{d-h}(\chi_v\{h/2\})) \otimes \Xi^{\frac{d-h}{2}} \longrightarrow \cdots$$
$$\longrightarrow j_!^{=h+1} HT(\chi_v, \mathrm{St}_h(\chi_v(-1/2)) \times \chi_v\{h/2\}) \otimes \Xi^{\frac{1}{2}} \longrightarrow$$
$$j_!^{=h} HT(\chi_v, \mathrm{St}_h(\chi_v)) \longrightarrow {}^p j_{!*}^{=h} HT(\chi_v, \mathrm{St}_h(\chi_v)) \to 0. \quad (3.4)$$

Note that

- as this resolution is equivalent to the computation of the sheaves cohomology groups of ${}^p j_{!*}^{=h} HT(\chi_v, \mathrm{St}_h(\chi_v))$ as explained for example in [**9**] proposition B.1.5 of appendice B, then, over $\overline{\mathbb{Q}}_l$, it follows from the main results of [**3**].
- Over $\overline{\mathbb{Z}}_l$, as every terms are free perverse sheaves, then all the maps are necessary strict.
- This resolution, for a a general supercuspidal representation with supercuspidal modulo l reduction, is one of the main result of [**9**] §2.3.

Proof. — For the case of a character χ_v as above, the argument is almost obvious. Indeed as the strata $\mathrm{Sh}^{\geqslant h}_{I^v,\bar{s}_v,1}$ are smooth, then, cf. the proof of the lemma 3.2, the constant sheaf, up to shift, is perverse and so equals to the intermediate extension of the constant sheaf, shifted by $d-h$, on $\mathrm{Sh}^{=h}_{I^v,\bar{s}_v,1}$. In particular its sheaves cohomology groups are well known so that the resolution is completely obvious for ${}^p j^{=h}_{\overline{1_h},!*} HT_{\overline{1_h}}(\chi_v, \mathrm{St}_h(\chi_v))$ if one remember that $\mathrm{Speh}_i(\chi_v)$ is just the character $\chi_v \circ \det$ of $GL_i(F_v)$.

The stated resolution is then simply the induced version of the resolution of ${}^p j^{=h}_{\overline{1_h},!*} HT_{\overline{1_h}}(\chi_v, \mathrm{St}_h(\chi_v))$: recall that a direct sum of intermediate extensions is still an intermediate extension. □

By adjunction property, the map

$$j_!^{=h+\delta} HT(\chi_v, \mathrm{St}_h(\chi_v\{\frac{-\delta}{2}\}) \times \mathrm{Speh}_\delta(\chi_v\{h/2\})) \otimes \Xi^{\delta/2}$$
$$\longrightarrow j_!^{=h+\delta-1} HT(\chi_v, \mathrm{St}_h(\chi_v\{\frac{1-\delta}{2}\}) \times \mathrm{Speh}_{\delta-1}(\chi_v\{h/2\})) \otimes \Xi^{\frac{\delta-1}{2}} \quad (3.5)$$

is given by

$$HT(\chi_v, \mathrm{St}_h(\chi_v\{\frac{-\delta}{2}\}) \times \mathrm{Speh}_\delta(\chi_v\{h/2\})) \otimes \Xi^{\delta/2} \longrightarrow$$
$$j^{=h+\delta,*}({}^p i^{h+\delta,!}(j_!^{=h+\delta-1} HT(\chi_v, \mathrm{St}_h(\chi_v\{\frac{1-\delta}{2}\}) \times \mathrm{Speh}_{\delta-1}(\chi_v\{h/2\})) \otimes \Xi^{\frac{\delta-1}{2}})) \quad (3.6)$$

To compute this last term we use the resolution (3.4). Precisely denote by $\mathcal{H} := HT(\chi_v, \mathrm{St}_h(\chi\{\frac{1-\delta}{2}\}) \times \mathrm{Speh}_{\delta-1}(\chi_v\{h/2\})) \otimes \Xi^{\frac{\delta-1}{2}}$, and write the previous resolution as follows

$$0 \to K \longrightarrow j_!^{=h+\delta} \mathcal{H}' \longrightarrow Q \to 0,$$
$$0 \to Q \longrightarrow j_!^{=h+\delta-1} \mathcal{H} \longrightarrow {}^p j_{!*}^{=h+\delta-1} \mathcal{H} \to 0,$$

with

$$\mathcal{H}' := HT\Big(\chi_v, \mathrm{St}_h(\chi_v\{\frac{1-\delta}{2}\}) \times \big(\mathrm{Speh}_{\delta-1}(\chi_v\{-1/2\}) \times \chi_v\{\frac{\delta-1}{2}\}\big)\{h/2\}\Big) \otimes \Xi^{\delta/2}.$$

As the support of K is contained in $\mathrm{Sh}_{I,\bar{s}_v}^{\geqslant h+\delta+1}$ then ${}^p i^{h+\delta,!} K = K$ and $j^{=h+\delta,*}({}^p i^{h+\delta,!} K)$ is zero. Moreover ${}^p i^{h+\delta,!}({}^p j_{!*}^{=h+\delta-1} \mathcal{H})$ is zero by construction of the intermediate extension. We then deduce that

$$j^{=h+\delta,*}({}^p i^{h+\delta,!}(j_!^{=h+\delta-1} HT(\chi_v, \mathrm{St}_t(\chi_v\{\frac{1-\delta}{2}\}) \times \mathrm{Speh}_{\delta-1}(\chi_v\{h/2\})) \otimes \Xi^{\frac{\delta-1}{2}}))$$
$$\simeq HT\Big(\chi_v, \mathrm{St}_h(\chi_v\{\frac{1-\delta}{2}\})$$
$$\times \big(\mathrm{Speh}_{\delta-1}(\chi_v\{-1/2\}) \times \chi_v\{\frac{\delta-1}{2}\}\big)\{h/2\}\Big) \otimes \Xi^{\delta/2} \quad (3.7)$$

3.8 — Fact. In particular, up to homothety, the map (3.7), and so those of (3.6), is unique. Finally as the maps of (3.4) are strict, the given maps (3.5) are uniquely determined, that is, if we forget the infinitesimal parts, these maps are independent of the chosen t in (3.4).

For every $1 \leqslant h \leqslant d$, let denote by $i(h)$ the smallest index i such that $H^i(\mathrm{Sh}_{I,\bar{s}_v}, {}^p j_{!*}^{=h} HT(\chi_v, \mathrm{St}_h(\chi_v)))_{\mathfrak{m}}$ has non trivial torsion: if it does not exist then we set $i(h) = +\infty$. By duality, as ${}^p j_{!*} = {}^{p+} j_{!*}$ for Harris-Taylor

local systems associated to characters, note that when $i(h)$ is finite then $i(h) \leqslant 0$. Suppose by absurdity there exists h with $i(h)$ finite and denote h_0 the biggest such h.

3.9. Lemma. — *For* $1 \leqslant h \leqslant h_0$ *then* $i(h) = h - h_0$.

Remark. A similar result is proved in [**8**] when the level is maximal at v.

Proof. — a) We first prove that for every $h_0 < h \leqslant d$, the cohomology groups of $j_!^{=h} HT(\chi_v, \Pi_h)$ are torsion free. Consider the following strict filtration in the category of free perverse sheaves

$$(0) = \mathrm{Fil}^{-1-d}(\chi_v, h) \hookrightarrow\!\!\!\!\rightarrow \mathrm{Fil}^{-d}(\chi_v, h) \hookrightarrow\!\!\!\!\rightarrow \cdots \hookrightarrow\!\!\!\!\rightarrow \mathrm{Fil}^{-h}(\chi_v, h) = j_!^{=h} HT(\chi_v, \Pi_h)$$

where the symbol $\hookrightarrow\!\!\!\!\rightarrow$ means a strict monomorphism, with graded parts

$$\mathrm{gr}^{-k}(\chi_v, h) \simeq {}^p j_{!*}^{=k} HT(\chi_v, \Pi_h\{\frac{h-k}{2}\} \otimes \mathrm{St}_{k-h}(\chi_v\{h/2\}))(\frac{h-k}{2}).$$

Over $\overline{\mathbb{Q}}_l$, the result is proved in [**3**] §4.3. From [**6**] such a filtration can be constructed over $\overline{\mathbb{Z}}_l$ up to the fact that the graduate parts are only known to verify

$$\begin{aligned} {}^p j_{!*}^{=k} HT(\chi_v, \Pi_h\{\frac{h-k}{2}\} \otimes \mathrm{St}_{k-h}(\chi_v\{h/2\}))(\frac{h-k}{2}) &\hookrightarrow\!\!\!\!\rightarrow \mathrm{gr}^{-k}(\chi_v, h) \\ &\hookrightarrow\!\!\!\!\rightarrow {}^{p+} j_{!*}^{=k} HT(\chi_v, \Pi_h\{\frac{h-k}{2}\} \otimes \mathrm{St}_{k-h}(\chi_v\{h/2\}))(\frac{h-k}{2}), \end{aligned}$$

and we can conclude thanks to lemma 3.2. The associated spectral sequence localized at $\mathfrak{m}$, is then concentrated in middle degree and torsion free which gives the claim.

b) Before watching the cases $h \leqslant h_0$, note that the spectral sequence associated to (3.4) for $h = h_0 + 1$, has all its E_1 terms torsion free and degenerates at its E_2 terms. As by hypothesis the aims of this spectral sequence is free and equals to only one E_2 terms, we deduce that all the

maps

$$H^0\big(\mathrm{Sh}_{I,\bar{s}_v}, j_!^{=h+\delta} HT_\xi(\chi_v, \mathrm{St}_h(\chi_v\{\frac{-\delta}{2}\}) \times \mathrm{Speh}_\delta(\chi_v\{h/2\})) \otimes \Xi^{\delta/2}\big)_{\mathfrak{m}}$$
$$\longrightarrow$$
$$H^0\big(\mathrm{Sh}_{I,\bar{s}_v}, j_!^{=h+\delta-1} HT_\xi(\chi_v, \mathrm{St}_h(\chi_v\{\frac{1-\delta}{2}\}) \times \mathrm{Speh}_{\delta-1}(\chi_v\{h/2\})) \otimes \Xi^{\frac{\delta-1}{2}}\big)_{\mathfrak{m}} \quad (3.10)$$

are saturated, i.e. their cokernel are free $\overline{\mathbb{Z}}_l$-modules. Then from the previous fact stressed after (3.7), this property remains true when we consider the associated spectral sequence for $1 \leqslant h' \leqslant h_0$.

c) Consider now $h = h_0$ and the spectral sequence associated to (3.4) where

$$E_2^{p,q} = H^{p+2q}(\mathrm{Sh}_{I,\bar{s}_v}, j_!^{=h+q} HT_\xi(\chi_v, \mathrm{St}_h(\chi_v(-q/2)) \times \mathrm{Speh}_q(\chi_v\{h/2\})) \otimes \Xi^{\frac{q}{2}})_{\mathfrak{m}} \quad (3.10)$$

By definition of h_0, we know that some of the $E_\infty^{p,-p}$ should have a non trivial torsion subspace. We saw that

- the contributions from the deeper strata are torsion free and
- $H^i(\mathrm{Sh}_{I,\bar{s}_v}, j_!^{=h_0} HT_\xi(\chi_v, \Pi_{h_0}))_{\mathfrak{m}}$ are zero for $i < 0$ and is torsion free for $i = 0$, whatever is Π_{h_0}.
- Then there should exist a non strict map $d_1^{p,q}$. But, we have just seen that it can not be maps between deeper strata.
- Finally, using the previous points, the only possibility is that the cokernel of

$$H^0\big(\mathrm{Sh}_{I,\bar{s}_v}, j_!^{=h_0+1} HT_\xi(\chi_v, \mathrm{St}_{h_0}(\chi_v\{\frac{-1}{2}\}) \times \chi_v\{h_0/2\})) \otimes \Xi^{1/2}\big)_{\mathfrak{m}}$$
$$\longrightarrow$$
$$H^0\big(\mathrm{Sh}_{I,\bar{s}_v}, j_!^{=h_0} HT_\xi(\chi_v, \mathrm{St}_{h_0}(\chi_v))\big)_{\mathfrak{m}} \quad (3.11)$$

  has a non trivial torsion subspace.

In particular we have $i(h_0) = 0$.

d) Finally using the fact 2.18 and the previous points, for any $1 \leqslant h \leqslant h_0$, in the spectral sequence (3.10)

- by point a), $E_2^{p,q}$ is torsion free for $q \geqslant h_0 - h + 1$ and so it is zero if $p + 2q \neq 0$;

– by affiness of the open strata, cf. [**8**] theorem 1.8, $E_2^{p,q}$ is zero for $p+2q<0$ and torsion free for $p+2q=0$;
– by point b), the maps $d_2^{p,q}$ are saturated for $q \geqslant h_0-h+2$;
– by point c), $d_2^{-2(h_0-h+1),h_0-h+1}$ has a cokernel with a non trivial torsion subspace.
– Moreover, over $\overline{\mathbb{Q}}_l$, the spectral sequence degenerates at E_3 and $E_3^{p,q}=0$ if $(p,q)\neq(0,0)$.

We then deduce that $H^i(\mathrm{Sh}_{I,\bar{s}_v}, {}^pj_{!*}^{=h}HT_\xi(\chi_v,\Pi_h))_{\mathfrak{m}}$ is zero for $i<h-h_0$ and for $i=h-h_0$ it has a non trivial torsion subspace. □

Consider now the filtration of stratification of Ψ_ϱ constructed using the adjunction morphisms $j_!^{=h}j^{=h,*}$ as in [**6**]

$$\mathrm{Fil}_!^1(\Psi_\varrho) \hookleftarrow \mathrm{Fil}_!^2(\Psi_\varrho) \hookleftarrow \cdots \hookleftarrow \mathrm{Fil}_!^d(\Psi_\varrho) \tag{3.11}$$

where $\mathrm{Fil}_!^h(\Psi_\varrho)$ is the saturated image of $j_!^{=h}j^{=h,*}\Psi_\varrho \longrightarrow \Psi_\varrho$. For our fixed χ_v, let denote $\mathrm{Fil}_{!,\chi_v}^1(\Psi) \hookleftarrow \mathrm{Fil}_!^1(\Psi_\varrho)$ such that $\mathrm{Fil}_{!,\chi_v}^1(\Psi)\otimes_{\overline{\mathbb{Z}}_l}\overline{\mathbb{Q}}_l \simeq \mathrm{Fil}_!^1(\Psi_{\chi_v})$ where Ψ_{χ_v} is the direct factor of $\Psi\otimes_{\overline{\mathbb{Z}}_l}\overline{\mathbb{Q}}_l$ associated to χ_v, cf. [**6**].

3.12. Proposition. — *We have the following resolution of* $\mathrm{Fil}_{!,\chi_v}^1(\Psi)$

$$\begin{aligned} 0 \to j_!^{=d}HT(\chi_v,\mathrm{Speh}_d(\chi_v))\otimes\mathbb{L}(\chi_v(\frac{d-1}{2})) \longrightarrow \\ j_!^{=d-1}HT(\chi_v,\mathrm{Speh}_{d-1}(\chi_v))\otimes\mathbb{L}(\chi_v(\frac{d-2}{2})) \longrightarrow \\ \cdots \longrightarrow j_!^{=1}HT(\chi_v,\chi_v)\otimes\mathbb{L}(\chi_v) \longrightarrow \mathrm{Fil}_{!,\chi_v}^1(\Psi_v)\to 0, \end{aligned} \tag{3.13}$$

where we recall that $\mathbb{L}$ *is the local Langlands correspondence composed by contragredient.*

Remarks:

– As explained after proposition 3.3, it amounts to describe the germs of the $\overline{\mathbb{Z}}_l$-sheaf cohomology of $\mathrm{Fil}_{!,\chi_v}^1(\Psi_{v,\xi})$. Over $\overline{\mathbb{Q}}_l$, the resolution (3.13) is then proved in [**3**].
– Over $\overline{\mathbb{Z}}_l$, it is proved in full generality in [**9**] for every irreducible supercuspidal representation π_v in place of χ_v. It amounts to prove that the germs of the sheaf cohomology of $\mathrm{Fil}_{!,\chi_v}^1(\Psi_{v,\xi})$ are free.

Proof. — We then just need to verify that every map is strict. Consider then the torsion part of the cokernel of one of these maps. Note that, thanks to lemma 3.2, such a cokernel must have non trivial invariants

under the action the Iwahori sub-group at v. We then work at Iwahori level at v. As said above, it amounts to understand the germs of the $\overline{\mathbb{Z}}_l$-sheaf cohomology of $\mathrm{Fil}^1_{!,\chi_v}(\Psi)$ which are described, cf. [**19**], by the cohomology of the Lubin-Tate tower. By the comparison theorem of Faltings-Fargues, cf. [**20**], one is reduced to compute the cohomology of the Drinfeld tower in Iwahori level which is already done in [**28**]: we then note that there are all free $\overline{\mathbb{Z}}_l$-modules. □

We can then apply the previous arguments a)-d) above, so that $H^i(\mathrm{Sh}_{I,\bar{s}_v}, \mathrm{Fil}^1_{!,\chi_v}(\Psi_{v,\xi})_{\mathfrak{m}}$ has non trivial torsion for $i = 1 - h_0$ and its free quotient is zero for $i \neq 0$.

Consider now the other graded parts. We also have a similar resolution

$$0 \to j_!^{=d} HT(\chi_v, LT_{h,d}(\chi_v)) \otimes L_g(\chi_v(\frac{d-h}{2})) \longrightarrow$$
$$j_!^{=d-1} HT(\chi_v, LT_{h,d-1}(\chi_v)) \otimes L_g(\chi_v(\frac{d-h-1}{2})) \longrightarrow$$
$$\cdots \longrightarrow j_!^{=h} HT(\chi_v, \mathrm{St}_h(\chi_v)) \otimes L_g(\chi_v) \longrightarrow \mathrm{Fil}^h_{!,\chi_v}(\Psi) \to 0, \quad (3.14)$$

where

$$LT_{h,h+\delta}(\chi_v) \hookrightarrow \mathrm{St}_h(\chi_v\{-\delta/2\}) \times \mathrm{Speh}_\delta(\chi_v\{h/2\}),$$

is the only irreducible sub-space of this induced representation. By the same arguments, for $h \leqslant h_0$ (resp. $h > h_0$) the torsion of $H^i(\mathrm{Sh}_{I,\bar{s}_v}, \mathrm{Fil}^h_{!,\chi_v}(\Psi_{v,\xi}))_{\mathfrak{m}}$ is trivial for any $i \leqslant h - h_0$ (resp. for all i) and the free parts are concentrated for $i = 0$. Using then the spectral sequence associated to the previous filtration, we can then conclude that $H^{1-t_0}(\mathrm{Sh}_{I,\bar{s}_v}, \Psi_{v,\xi})_{\mathfrak{m}}$ would have non trivial torsion which is false as $\mathfrak{m}$ is supposed to be KHT-free.

□

4. Local behavior of monodromy over $\overline{\mathbb{F}}_l$

Let ϱ be a $\overline{\mathbb{F}}_l$-character. In [**6**] §2.3, we explained how to construct a filtration $\mathrm{Fill}^\bullet(\Psi_\varrho)$ of Ψ_ϱ, called exhaustive, whose graded parts $\mathrm{grr}^r(\Psi_\varrho)$ are free $\overline{\mathbb{Z}}_l$-perverse sheaves of the following shape

$$\mathrm{grr}^r(\Psi_\varrho) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \simeq {}^p j_{!*}^{=h} HT_{\overline{\mathbb{Q}}_l}(\chi_v, \mathrm{St}_h(\chi_v))(\frac{1-h+2\delta}{2})$$

for some $0 \leqslant \delta \leqslant t-1$ and χ_v a $\overline{\mathbb{Q}}_l$-character whose modulo l reduction is inertially equivalent to ϱ. In order to compute the cohomology groups

localized at $\mathfrak{m}$ of $\mathrm{Sh}_{I,v}$, by hypothesis on $\mathfrak{m}$, we are only concerned with the cases where $1 \leqslant h < e_v(l)$. For a character χ_v with modulo l reduction inertially equivalent to ϱ, and for $2 \leqslant h \leqslant d$, consider $r_{\chi_v,\pm}(h)$ such that $\mathrm{grr}^{r_{\chi_v,\pm}(h)}(\Psi_\varrho)$ is isomorphic to $\mathcal{P}(h,\chi_v)(\pm\frac{1-h}{2})$. Then N_v^{h-1} induces a map

$$\begin{array}{ccc} \mathrm{Fill}^{r_{\chi_v,+}(h)}(\Psi_\varrho) & \xrightarrow{N_v^{h-1}} & \mathrm{Fill}^{r_{\chi_v,-}(h)}(\Psi_\varrho) \\ \downarrow & & \downarrow \\ \mathrm{Fill}^{r_{\chi_v,+}(h)}(\Psi_\varrho) & \xrightarrow{N(\chi_v,h)} & \mathrm{grr}^{r_{\chi_v,-}(h)}(\Psi_\varrho), \end{array}$$

so that $N(\chi_v,h)\otimes_{\overline{\mathbb{Z}}_l}\overline{\mathbb{Q}}_l$ is an isomorphism. In this section we want to prove that this property remains true modulo l.

4.1. Proposition. — *Suppose $e_v(q) > t$, then the morphism on $\mathrm{Fill}^{r_+(h)}(\Psi_\varrho)/\mathrm{Fill}^{r_-(h)-1}(\Psi_\varrho)$ induced by the monodromy operator N_v, is strict.*

Proof. — Recall first that the filtration $\mathrm{Fill}^\bullet(\Psi_\varrho)$ is constructed so that it is compatible with the action of N_v in the sense that, over $\overline{\mathbb{Q}}_l$, the image of $\mathrm{Fill}^r(\Psi_\varrho)$ is some $\mathrm{Fill}^{\phi(r)}(\Psi_\varrho)$. We then have to prove that for every $r_-(h) \leqslant r \leqslant r_+(h)$, we have a p-epimorphism $N_v : \mathrm{Fill}^r(\Psi_\varrho) \twoheadrightarrow \mathrm{Fill}^{\phi(r)}(\Psi_\varrho)$ which is clearly equivalent to prove that for every $1 \leqslant h' \leqslant h$, then N_v induces a isomorphism

$$\begin{aligned} {}^p j_{!*}^{=h'} HT(\chi_v, \mathrm{St}_{h'}(\chi_v))(\frac{1-h'+2\delta}{2}) \longrightarrow \\ {}^p j_{!*}^{=h'} HT(\chi_v, \mathrm{St}_{h'}(\chi_v))(\frac{1-h'+2(\delta-1)}{2}), \quad (4.2) \end{aligned}$$

for every $1 \leqslant \delta < h'$, where each of these two perverse sheaf is given by the graded parts $\mathrm{grr}^r(\Psi_\varrho)$ and $\mathrm{grr}^{\phi(r)}(\Psi_\varrho)$.

Recall that under the hypothesis that $e_v(q) > h \geqslant h'$, then the reduction modulo l of $\chi_v[h']_D \otimes L_g(\chi_v)\otimes \mathrm{St}_{h'}(\chi_v)$ is irreducible so that there exists, up to homothety, an unique stable lattice of $HT(\chi_v, \mathrm{St}_h(\chi_v))$ which means that to prove (4.2) is a isomorphism, it suffices to prove its reduction modulo l is non zero. To do so, it suffices to work in the Iwahori level and use the arguments of [**11**] §3.1 where the monodromy action is, thanks to Rapoport-Zink cf. [**23**] 3.6.13, described explicitly and is of maximal nilpotency. □

5. Level fixing property

Consider as before a maximal ideal $\mathfrak{m}$ associated to the modulo l reduction of an irreducible automorphic representation $\Pi_{\tilde{\mathfrak{m}}}$ with $\tilde{\mathfrak{m}} \subset \mathfrak{m}$: recall that $\mathfrak{m}$ corresponds to the multiset of modulo l Satake parameters at almost all places where $\Pi_{\tilde{\mathfrak{m}}}$ is unramified.

A classical question is then to know if it is possible to find another prime ideal $\tilde{\mathfrak{m}}'$ such that at a fixed place v, the level of $\Pi_{\tilde{\mathfrak{m}}',v}$ is strictly lower (resp. higher) of whose of $\Pi_{\tilde{\mathfrak{m}},v}$: we then speak of level lowering (resp. raising) phenomenon.

In this section we are interested in the opposite situation where whatever is $\tilde{\mathfrak{m}} \subset \mathfrak{m}$, then the level at v is constrained by $\mathfrak{m}$.

Proof. — (of theorem 1.2)
By KHT-freeness of $\mathfrak{m}$, we are in the situation of proposition 3.1 so that the nilpotency of the monodromy operator $\overline{N}_{\mathfrak{m},v}$ on $\overline{\rho_{\mathfrak{m},v}}$ is given by those of $N_{v,\mathfrak{m}}^{coho} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ on the cohomology which is controlled by the action of the local monodromy operator N_v on the vanishing cycle sheaves. More precisely, consider a $\overline{\mathbb{F}}_l$-character ϱ in the support of $\overline{\rho}_{\mathfrak{m},v}$. For any prime ideal $\tilde{\mathfrak{m}} \subset \mathfrak{m}$ we have

$$\rho_{\tilde{\mathfrak{m}},v} \simeq \mathrm{Sp}_{t_1}(\chi_{v,1}) \oplus \cdots \oplus \mathrm{Sp}_{t_r}(\chi_{v,r}),$$

where $\chi_{v,1}, \cdots, \chi_{v,r}$ are characters some of them being isomorphic to ϱ modulo l. Let then $t_{\tilde{\mathfrak{m}}}(\varrho) := \max\{t_i \text{ s.t. } \chi_{v,i} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l \simeq \varrho\}$ and

$$t := \max\{t_{\tilde{\mathfrak{m}}}(\varrho) \text{ s.t. } \tilde{\mathfrak{m}} \subset \mathfrak{m}\}.$$

Choose any $\tilde{\mathfrak{m}}$ such that $t = t_{\tilde{\mathfrak{m}}}(\varrho)$, then as $\rho_{\tilde{\mathfrak{m}}}$ is irreducible, from [**25**]

$$\Pi_{\tilde{\mathfrak{m}},v} \simeq \mathrm{St}_{t_1}(\chi_{v,1}) \times \cdots \mathrm{St}_{t_r}(\chi_{v,r}),$$

with, say, $t_1 = t$ and $\chi_{v,1} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l \simeq \varrho$. Let $r_\pm$ be the indexes such that $\mathrm{gr}^{r_\pm}(\Psi_\varrho) \simeq \mathcal{P}(t_1, \chi_{v,1})(\pm\frac{1-t_1}{2})$. By proposition 4.1, $N_v^{t_1-1}$ induces an isomorphism

$$N_v^{t_1-1} : \mathrm{gr}^{r_+}(\Psi_\varrho) \simeq \mathrm{gr}^{r_-}(\Psi_\varrho).$$

By proposition 3.1, the spectral sequence associated to the filtration of Ψ_ϱ induces a saturated filtration of $H^0(\mathrm{Sh}_{I,\bar{s}_v}, \Psi_{\varrho,\xi})_{\mathfrak{m}}$, and thus a $\overline{\mathbb{F}}_l$-filtration of

$$H^0(\mathrm{Sh}_{I,\bar{s}_v}, \Psi_{\varrho,\xi})_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l \simeq H^0(\mathrm{Sh}_{I,\bar{s}_v}, \Psi_{\varrho,\xi} \otimes^{\mathbb{L}}_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l)_{\mathfrak{m}}.$$

Then we deduce that $(\overline{N}_v^{coho})^{t-1} \neq (0)$ and then $\overline{N}_{\mathfrak{m}}^{t-1}$ is non zero on the direct factor $\overline{\rho}_{\mathfrak{m},\varrho}$ as defined in the introduction. Note also as N_v^t is zero on $\rho_{\tilde{\mathfrak{m}},\chi_v,1}$ then $\overline{N}_{\mathfrak{m},v}^t$ is necessary zero on $\overline{\rho}_{\mathfrak{m},\varrho}$.

Consider now any $\tilde{\mathfrak{m}}' \subset \mathfrak{m}$ and let χ_v be a character such that $\chi_v \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l \simeq \varrho$. As the modulo l reduction of N_v^{t-1} acting on $\rho_{\tilde{\mathfrak{m}},\chi_v}$ is non zero then there must exist $\chi_{v,1} \sim^i \chi_v$ such that $\mathrm{Sp}_{t_1}(\chi_{v,1})$ is a direct factor of $\rho_{\tilde{\mathfrak{m}},v}$ with $t_1 \geqslant t$: by maximality of the definition of t we then have $t_1 = t$, and the theorem is proved. □

Remark. Using the main result of [**9**], one can state and prove in the same way, the above statement without the hypothesis that $\mathrm{Scusp}_v(\mathfrak{m})$ contains only characters but cuspidal representation with modulo l reduction still supercuspidal.

6. Automorphic congruences

Consider a KHT-free maximal ideal $\mathfrak{m}$ as in theorem 1.2. From the proof of proposition 3.1 we then deduce that the $H^i(\mathrm{Sh}_{I,\bar{s}_v}, {}^p j_{!*}^{=h} HT_\xi(\chi_v,h))_{\mathfrak{m}}$ are free and concentrated in degree $i=0$ so that

$$H^0(\mathrm{Sh}_{I,\bar{s}_v}, {}^p j_{!*}^{=h} HT_\xi(\chi_v,h))_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l \simeq H^0(\mathrm{Sh}_{I,\bar{s}_v}, {}^p j_{!*}^{=h} HT_{\xi,\overline{\mathbb{F}}_l}(r_l(\chi_v),h))_{\mathfrak{m}} \simeq H^0(\mathrm{Sh}_{I,\bar{s}_v}, {}^p j_{!*}^{=h} HT_\xi(\chi'_v,h))_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l, \quad (6.1)$$

whatever is χ'_v such that the modulo l reduction $r_l(\chi'_v)$ of χ'_v is isomorphic to those of χ_v. Recall then from [**4**], the description of the $\overline{\mathbb{Q}}_l$-cohomology groups of ${}^p j_{!*}^{=h} HT_\xi(\chi_v,h)$ localized at $\mathfrak{m}$.

6.2. Proposition. — *(cf.* [**4**] *§3.6 with*[(4)] *$s=1$)*
For χ_v an unitary character of $F_v^\times$, then, for $1 \leqslant h \leqslant d$, as a $\mathbb{T}_{\mathfrak{m}}[GL_d(F_v)]$-module, we have

$$\varinjlim_{I_v} H^0(\mathrm{Sh}_{I^v I_v,\bar{s}_v}, {}^p j_{!*}^{=h} HT_{\xi,\overline{\mathbb{Q}}_l}(\chi_v,h))_{\mathfrak{m}} \simeq \bigoplus_{\Pi \in \mathcal{A}(I,\xi,h,\chi_v,\mathfrak{m})} m(\Pi)(\Pi^{\infty,v})^{I^v} \otimes \Pi_v,$$

where

(4) As $\overline{\rho}_{\mathfrak{m}}$ is supposed to be irreducible, the integer s of [**4**] §3.6 is necessary equal to 1.

– $\mathcal{A}(I, h, \chi_v, \mathfrak{m})$ *is the set of irreducible* ξ*-cohomological automorphic representations* Π *of* $G(\mathbb{A})$ *with non zero invariants under* I^v *with modulo* l *Satake's parameters prescribed by* $\mathfrak{m}$*,*
– *such that* Π_v *is of the following shape*

$$\Pi_v \simeq \mathrm{St}_h(\chi_v) \times \Psi_v$$

where Ψ_v *is a representation of* $GL_{d-h}(F_v)$*,*
– *and* $m(\Pi)$ *is the multiplicity of* Π *in the space of automorphic forms.*

Remark. We write the local component Π_v of $\Pi \in \mathcal{A}(I, \xi, h, \chi_v, \mathfrak{m})$ as

$$\Pi_v \simeq \mathrm{St}_{t_1}(\chi_{v,1}) \times \cdots \times \mathrm{St}_{t_r}(\chi_{v,r}) \times \Psi'_v,$$

where

– the $\chi_{v,i}$ are inertially equivalent characters,
– Ψ'_v is an irreducible representation of $GL_{d-\sum_{i=1}^r t_i}(F_v)$ whose cuspidal support, made of character by hypothesis, does not contain a character inertially equivalent to $\chi_{v,1}$.

Then Π contributes k times in the isomorphism of the previous proposition, where $k = \sharp\{1 \leqslant i \leqslant r \text{ such that } t_i = h\}$.

We are now in the same situation as in [**7**] where we prove that the conjecture 5.4.3 implies the conjecture 5.2.1 and the translation in terms of automorphic congruences explained at the end of §5.2 The situation here is much more simple as $s = 1$.

6.3. Corollary. — *Let* Π *be an irreducible automorphic representation of* $G(\mathbb{A})$ *which is* ξ*-comological of level* K *and such that*

– *its modulo* l *Satake's parameters are given by* $\mathfrak{m}$*,*
– *and its local component* Π_v *at* v *is isomorphic to* $\Pi_v \simeq \mathrm{St}_h(\chi_v) \times \Psi_v$*, where* χ_v *is a characters and* Ψ_v *is an irreducible representation of* $GL_{d-h}(F_v)$*.*

Consider then any character χ'_v *of* $F_v^\times$ *which is congruent to* χ_v *modulo* l*. Then there exists an irreducible automorphic representation* Π' *of* $G(\mathbb{A})$ *which is* ξ*-cohomological of the same level* K *and such that*

– *its modulo* l *Satake's parameters are given by* $\mathfrak{m}$*,*
– *its local component at* v *is of the following shape*

$$\Pi'_v \simeq \mathrm{St}_h(\chi'_v) \times \Psi'_v.$$

7. Ihara's lemma and level raising

In the Taylor-Wiles method, Ihara's lemma is the key ingredient to extend a $R = T$ property from the minimal case to a non minimal one. In higher dimension, Clozel, Harris and Taylor in their first proof of Sato-Tate theorem [**16**], proposed a generalization which involves, for example, a similitude group $\overline{G}$ over $\mathbb{Q}$ verifying the following hypothesis.

- there exists a prime number $p' = u'(u')^c$ decomposed in E, such that $p' \neq p$ and $\overline{G}(\mathbb{A}^{\infty,p'})$ is isomorphic to our previous $G(\mathbb{A}^{\infty,p'})$,
- the associated unitary group of $\overline{G}(\mathbb{R})$ is compact and
- $\overline{G}(\mathbb{Q}_{p'}) \simeq \mathbb{Q}_{p'}^{\times} \times \prod_{w'|u'} (\overline{B}_{w'}^{op})^{\times}$ where w' describe the places of F above u',
- where there exists a place $v'|u'$ such that $\overline{B}_{w'} \simeq B_{w'}$ for all $w'|u'$ distinct from v' and $\overline{B}_{v'} \simeq GL_d(F_{v'})$ while $B_{v'}$ is a division algebra with invariant $1/d$.

With the previous notations, consider a finite set S of places of F and a maximal ideal $\mathfrak{m}$ of the anemic Hecke algebra $\mathbb{T}_{\xi}^{S}$ such that $\overline{\rho}_{\mathfrak{m}}$ is absolutely irreducible.

7.1. Conjecture. — *Let $\overline{U}$ be an open compact subgroup of $\overline{G}(\mathbb{A})$ unramified outside S and let $\bar{\pi}$ be an irreducible sub-representation of $\mathcal{C}^{\infty}(\overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A})/\overline{U}^{v}, \overline{\mathbb{F}}_l)_{\mathfrak{m}}$, where $\overline{U} = \overline{U}_v\overline{U}^{v}$. Then its local component $\bar{\pi}_v$ at v is generic.*

More precisely the conjecture is supposed to be true for all similitude groups $\overline{G}/\mathbb{Q}$ such that the associated unitary group of $\overline{G}(\mathbb{R})$ is compact. We can also formulate a similar conjecture for similitude groupe G as in §2 to define a KHT Shimura variety. We could then hope that any irreducible sub-space of

$$\varprojlim_{n} H^{d-1}(\mathrm{Sh}_{I^v(n),\bar{\eta}_v}, \overline{\mathbb{F}}_l)_{\mathfrak{m}},$$

as a representation of $GL_d(F_v)$, is necessary generic. In [**12**] §2.3, we prove that if this KHT version of Ihara's conjecture would be true then the Clozel-Harris-Taylor usual version would also be true. We now first want to prove a reciprocal statement for G and $\overline{G}$ as above.

7.2. Proposition. — *Take $l > d$ such that $p' \equiv 1 \mod l$ and consider a maximal ideal $\mathfrak{m}$ which is KHT-free. Then the Ihara's conjecture for $\overline{G}$ implies the Ihara's conjecture for G.*

Proof. — Thanks to [**26**] theorem 6.23, we hace a rigid-analytic uniformization of $\mathrm{Sh}_{I,\bar{\eta}_{v'}}$ as

$$\overline{G}(\mathbb{Q})\backslash\Big(\breve{\Omega}_d \times G(\mathbb{A}^{\infty,p'}) \times \prod_{w|u',\ w\neq v'} (B_w^{op})^\times\Big)/I^{v'},$$

where $\breve{\Omega}_d$ is the Drinfeld rigid-analytic space of GL_d. By [**21**] §4.5.3, this uniformization allows us to compute the cohomology of $\mathrm{Sh}_{I,\bar{\eta}_{v'}}$ through a spectral sequence

$$E_1^{p,q} = \mathrm{Ext}^p_{GL_d(F_{v'})}\Big(H_c^{2d-q-2}(\breve{\Omega}_d, \overline{\mathbb{F}}_l)(d-1), \overline{\mathcal{A}}^{I^{v'}}\Big) \Rightarrow H^{p+q}(\mathrm{Sh}_{I,\bar{\eta}_{v'}}, \overline{\mathbb{F}}_l),$$

where $I_{v'}$ is supposed to be the inversible group of the maximal order of $B_{v'}$.

As $l > d$ and $p' \equiv 1 \mod l$, we are then in the banal case where the theory of $\overline{\mathbb{F}}_l$-representations and those over $\overline{\mathbb{Q}}_l$ are the same, cf. [**31**]. Recall, cf. [**28**], that the $H^i(\breve{\Omega}_d, \overline{\mathbb{Z}}_l)$ are torsion free so that

$$H^i(\breve{\Omega}_d, \overline{\mathbb{F}}_l) \simeq H^i(\breve{\Omega}_d, \overline{\mathbb{Z}}_l) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l.$$

We then localize the previous spectral sequence at $\mathfrak{m}$ so that, $E_{1,\mathfrak{m}}^{p,q} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l = (0)$ if $p+q \neq d-1$: in particular it degenerates at E_1. Indeed as a $GL_d(F_{v'})$-representation, $H^{2d-2-q}(\breve{\Omega}_d, \overline{\mathbb{Q}}_l)$ is isomorphic to $LT_{\mathbb{1}_{v'}}(d,q)$ defined as the unique irreducible sub-space of $\mathrm{St}_{q+1}(\mathbb{1}_{v'}(\frac{q-d}{2})) \times \mathrm{Speh}_{d-q-1}(\mathbb{1}_{v'}(\frac{q}{2}))$ so that, as $\mathfrak{m}$ is not pseudo Eisenstein, then the only irreducible automorphic representations Π of $\overline{G}(\mathbb{A})$ giving a non zero term in the spectral sequence are those for which $\Pi_{v'} \simeq \mathrm{St}_d(\chi_v)$ with χ_v a character inertially equivalent to the trivial one. Then it is well-known, cf. [**17**] theorem 1.3 for example, that for $0 \leqslant q \leqslant d-1$, the only $E_{1,\mathfrak{m}}^{p,q} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ which is non zero, is for $p = d-1-q$ with $p, q \geqslant 0$.

By the previous remark, the same is then true over $\overline{\mathbb{F}}_l$ so that we have a filtration of $H^{d-1}(\mathrm{Sh}_{I,\bar{\eta}_{v'}}, \overline{\mathbb{F}}_l)_{\mathfrak{m}}$ with graded parts the $E_{1,\mathfrak{m}}^{d-1-q,q}$. By hypothesis each of these $E_{1,\mathfrak{m}}^{d-1-q,q}$ satisfies the Ihara property at the place v, so that the same is true for $H^{d-1}(\mathrm{Sh}_{I,\bar{\eta}_{v'}}, \overline{\mathbb{F}}_l)_{\mathfrak{m}}$.

□

We now want to give some consequences of Ihara's lemma in relation with theorem 1.2. Consider then a maximal ideal $\mathfrak{m}$ of the anemic Hecke algebra $\mathbb{T}_{S,\xi}$ which is KHT-free and such that $\overline{\rho}_{\mathfrak{m}}$ is irreducible. We

moreover suppose that $q_v \equiv 1 \mod l$ with $l > d$, and that $S_v(\mathfrak{m})$ is made of characters. Let then $\varrho \in S_v(\mathfrak{m})$ and we denote by s its multiplicity.

7.3. Proposition. — *If Ihara's lemma is true for G, then there exists $\widetilde{\mathfrak{m}} \subset \mathfrak{m}$ such that $\Pi_{\widetilde{\mathfrak{m}},v} \simeq \mathrm{St}_s(\chi_v) \times \Psi_v$ where $\chi_v \equiv \varrho \mod l$, and Ψ_v is a irreducible representation of $GL_{d-s}(F_v)$ whose modulo l reduction has a supercuspidal support disjoint from*[5] *ϱ.*

Proof. — The strategy is to suppose that the conclusion is false and then prove that Ihara's lemma is then not verified. Let then $t < s$ maximal such that there exists $\widetilde{\mathfrak{m}} \subset \mathfrak{m}$ with $\Pi_{\widetilde{\mathfrak{m}}} \simeq \mathrm{St}_t(\chi_v) \times \Gamma_v$ such that $\chi_v \equiv \varrho \mod l$. Then, by proposition 6.2, for all $h > t$, the $H^i(\mathrm{Sh}_{I,\bar{s}_v}, \mathcal{P}_\xi(\chi_v, h))_{\mathfrak{m}}$ are zero and the spectral sequence associated to $\mathrm{Fill}^\bullet(\Psi_\varrho)$ gives

$$H^0(\mathrm{Sh}_{I,\bar{s}_v}, \mathcal{P}_\xi(\chi_v, t))_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l \hookrightarrow H^0(\mathrm{Sh}_{I,\bar{s}_v}, \Psi_{\varrho,\xi})_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l.$$

The idea is then to construct an irreducible $\overline{\mathbb{F}}_l[GL_d(F_v)]$ sub-module of $H^0(\mathrm{Sh}_{I,\bar{s}_v}, \mathcal{P}_\xi(\chi_v, t))_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ which is not generic. By proposition 6.2, $\Pi_{\widetilde{\mathfrak{m}}}$ is a $\overline{\mathbb{Q}}_l[GL_d(F_v)]$ sub-module of $H^0(\mathrm{Sh}_{I,\bar{s}_v}, \mathcal{P}_\xi(\chi_v, t))_{\mathfrak{m}}$ so that, by taking a saturated lattice, $\Pi_{\widetilde{\mathfrak{m}}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ is a $\overline{\mathbb{F}}_l[GL_d(F_v)]$ sub-module of $H^0(\mathrm{Sh}_{I,\bar{s}_v}, \mathcal{P}_\xi(\chi_v, t))_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$. Using the fact that the strata are induced and $s \leqslant d < l$ so that $\mathrm{St}_t(\chi_v) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ is irreducible isomorphic to $\mathrm{St}_t(\varrho)$, we are then reduced to prove that $\mathrm{St}_t(\varrho) \times (\Gamma_v \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l)$ has an irreducible sub-space which is not generic, whatever is the stable lattice of Γ_v taken to compute $(\Gamma_v \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l)$.

As $\mathfrak{m}$ is not pseudo-Eisenstein and $S_v(\mathfrak{m})$ is made of characters, then Γ_v is of the following shape $\mathrm{St}_{t_1}(\chi_{v,1}) \times \cdots \times \mathrm{St}_{t_r}(\chi_{v,r})$ where, as we supposed $t < s$, at least one of the characters $\chi_{v,i}$ is congruent to ϱ modulo l. Take then an irreducible sub-space $\bar{\pi}$ of $(\Gamma_v \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l)$. If $\bar{\pi}$ is not generic we are done, otherwise $\bar{\pi} \simeq \mathrm{St}_{s-t}(\varrho) \times \bar{\psi}$ where the supercuspidal support of $\bar{\psi}$ does not contain ϱ. We then conclude by noting that $\mathrm{St}_t(\varrho) \times \mathrm{St}_{s-t}(\varrho)$ has a non generic subspace. Indeed $\mathrm{St}_t(\varrho) \times \mathrm{St}_{s-t}(\varrho)$ is the modulo l reduction of $\mathrm{St}_t(\chi_v(\frac{t-s}{2})) \times \mathrm{St}_{s-t}(\chi_v(\frac{t}{2}))$ which has an unique irreducible subspace which is non generic, and we conclude as an irreducible non generic representation does not have any irreducible generic subquotient. □

[5] Recall that as $q_v \equiv 1 \mod l$, then the $\overline{\mathbb{F}}_l$-Zelevinsky line of ϱ is $\{\varrho\}$.

Remark. In the previous proof, note that $\mathrm{St}_t(\varrho) \times \mathrm{St}_{s-t}(\varrho)$ is also the modulo l reduction of $\mathrm{St}_t(\chi_v(-\frac{t-s}{2})) \times \mathrm{St}_{s-t}(\chi_v(-\frac{t}{2}))$ which has an unique irreducible sub-space which is $\mathrm{St}_s(\chi_v)$, so that $\mathrm{St}_t(\varrho) \times \mathrm{St}_{s-t}(\varrho)$ is the direct sum of a generic representation with a non generic one.

Regarding theorem 1.2, Ihara's lemma tells us that $\underline{d_{\mathfrak{m},v}} = \underline{d_{\tilde{\mathfrak{m}},v}}$ and is maximal, i.e. the order of nilpotency of N_v on each of the block associated to an irreducible $\overline{\mathbb{F}}_l$-representation ϱ is equal to the multiplicity of ϱ inside $S_v(\mathfrak{m})$ viewed as a multi-set. The reciprocal is also true and could be stated as follows.

7.4. Proposition. — *Let $\mathfrak{m}$ be as in theorem 1.2 with $q_v \equiv 1 \mod l$. We suppose that for every $\varrho \in S_v(\mathfrak{m})$ with multiplicity s, there exists $\tilde{\mathfrak{m}} \subset \mathfrak{m}$, such that $\Pi_{\tilde{\mathfrak{m}},v} \simeq \mathrm{St}_s(\chi_v) \times \Gamma_v$ where $\chi_v \equiv \varrho \mod l$, and Γ_v is a irreducible representation of $GL_{d-s}(F_v)$. Then Ihara's lemma is true.*

Proof. — We consider the filtration of $H^{d-1}(\mathrm{Sh}_{I,\bar{s}_v}, \Psi_{\varrho,\xi})_{\mathfrak{m}}$ induced by the spectral sequence associated to the previous filtration $\mathrm{Fill}^\bullet(\Psi_\varrho)$ of Ψ_ϱ. Recall that it degenerates at E_1 and each of the graded parts are torsion free. We then have a $\overline{\mathbb{F}}_l$-filtration of the $\overline{\mathbb{F}}_l$-cohomology and we want to prove that each of these graded parts verifies the Ihara's property.

By theorem 1.2, the only non zero contributions are those of

$$H^0(\mathrm{Sh}_{I,\bar{s}_v}, \mathcal{P}_\xi(s, \chi_v)(\frac{1-s+2k}{2}))_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l,$$

with $0 \leqslant k \leqslant s-1$. By proposition 6.2, this is, up to multiplicty, isomorphic to $\mathrm{St}_s(\varrho) \times (\Delta_v \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l)$ where Δ_v is a $\overline{\mathbb{Z}}_l$-stable lattice of $\Gamma_v := \mathrm{St}_{t_1}(\chi_{v,1}) \times \cdots \times \mathrm{St}_{t_r}(\chi_{v,r})$ where the $\chi_{v,i}$ are characters non isomorphic to ϱ modulo l. It then suffices to prove that all the $\chi_{v,i}$ are pairwise non isomorphic modulo l so that $\Delta_v \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ is generic.

Take for example $\varrho_1 := \chi_{v,1} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$. By the level raising property, there must exists $\tilde{\mathfrak{m}}_1$ such that $\Pi_{\tilde{\mathfrak{m}}_1,v} \simeq \mathrm{St}_{s_1}(\chi'_{v,1}) \times \Psi_{v,1}$ with $\chi'_{v,1} \equiv \chi_{v,1} \mod l$ and where s_1 equals the multiplicity of ϱ_1 in $S_v(\mathfrak{m})$. Then theorem 1.2 implies that $t_1 = s_1$, so that all the others $\chi_{v,i}$ for $2 \leqslant i \leqslant r$ are not congruent to $\chi_{v,1}$ modulo l.

□

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