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LOCAL $\overline{\mathbb{F}}_l$ -MONODROMY AND LEVEL FIXING

by

Boyer Pascal

Abstract. — We tackle three related problems. The first deals with freeness of localized cohomology groups of Harris-Taylor perverse sheaves, defined on the special fiber of some Kottwitz-Harris-Taylor Shimura variety. We then study the nilpotent monodromy operator acting both on the global cohomology of KHT Shimura variety and on the perverse sheaf of vanishing cycles. We then exhibit cases of level fixing phenomenon in the sense where the level at some fixed place of any rise in characteristic zero of an irreducible automorphic representation, is fixed equals to the one modulo l .

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1. Introduction

Let F be a finite extension of \mathbb{Q} and $\rho : \text{Gal}(\bar{F}/F) \longrightarrow GL_{\bar{\mathbb{Q}}_l}(V)$ an irreducible l -adic representation of dimension d . Let Γ be a stable $\bar{\mathbb{Z}}_l$ -lattice of V and suppose that the modulo l reduction $\bar{\rho}$ of ρ is still irreducible so that, up to homothety, Γ is uniquely defined.

Consider now $p \neq l$ and a place v of F above p : we denote F_v the completion of F at v and \mathcal{O}_v its ring of integers. Thanks to Grothendieck's theorem, we know the action of inertia subgroup I_v at v to be unipotent, defining the nilpotent monodromy operator N_v . Such a nilpotent operator defines a partition $\underline{d}_{\rho,v} = (n_1 \geq n_2 \geq \dots \geq n_r \geq 1)$ of $d = n_1 + \dots + n_r$. If $l \geq n_1$, then we can also consider the nilpotent operator \bar{N}_v of the action of I_v modulo l which also defines a partition $\underline{d}_{\bar{\rho},v}$ of d which is smaller than $\underline{d}_{\rho,v}$ for the Bruhat order. As in [8], one may ask for a condition so that $\underline{d}_{\bar{\rho},v} = \underline{d}_{\rho,v}$, especially when there is as much irreducible constituents in the semi-simplification of $\bar{\rho}_v$ as in ρ_v .

To state our result, let \mathbb{T}_S be the unramified $\bar{\mathbb{Z}}_l$ -Hecke algebra of $G(\mathbb{A}_F^S)$ outside a finite number of place S of F and consider a maximal cohomological ideal \mathfrak{m} of \mathbb{T}_S and a prime ideal $\tilde{\mathfrak{m}} \subset \mathfrak{m}$ and denote $\rho := \rho_{\tilde{\mathfrak{m}}}$. The semi-simple class $\bar{\rho}_{\mathfrak{m}}$ of the reduction modulo l of $\rho_{\tilde{\mathfrak{m}}}$ depends only of the maximal ideal \mathfrak{m} of \mathbb{T} containing $\tilde{\mathfrak{m}}$. For all prime x of \mathbb{Z} split in E and a place $w \notin S$ of F above x , we then denote $P_{\mathfrak{m},w}(X)$ the characteristic polynomial of $\bar{\rho}_{\mathfrak{m}}(\text{Frob}_w)$ and $S_{\mathfrak{m}}(w)$ its multiset of roots.

1.1. Definition. — We say that \mathfrak{m} is free if any of this following property is verified:

- $\alpha \in S_{\mathfrak{m}}(w) \Rightarrow q_w \alpha \notin S_{\mathfrak{m}}(w)$ where q_w is the cardinal of the residue field at w ;
- $\bar{\rho}_{\mathfrak{m}}$ is induced from a character of G_K where K/F is a cyclic galoisian extension;
- $l \geq d$ and $SL_d(k) \subset \bar{\rho}_{\mathfrak{m}}(G_F) \subset \bar{\mathbb{F}}_l^\times GL_d(k)$ for some subfield $k \subset \bar{\mathbb{F}}_l$.

Remark. These conditions come from [4] so that the cohomology of KHT-Shimura varieties localized at \mathfrak{m} are torsion free, see the next section. In particular you can replace them by any other property which insures freeness of the cohomology.

1.2. Theorem. — Suppose that

- $\bar{\rho}_{\mathfrak{m}}$ is irreducible,

- for every irreducible subquotient ϱ of the restriction to $\overline{\rho}_{\mathfrak{m}}$ to the decomposition group at v , then the cardinal $m(\varrho)$ of the Zelevinsky line of ϱ , cf. notation 3.1, is strictly greater than d ,
- and \mathfrak{m} is free in the sense of definition 1.1.

Then the two partitions $\underline{d}_{\mathfrak{m},v}$ and $\underline{d}_{\tilde{\mathfrak{m}},v}$ associated respectively to the unipotent operator at v for $\rho_{\tilde{\mathfrak{m}},v}$ and $\overline{\rho}_{\mathfrak{m},v}$, are the same.

Such a property is orthogonal to those of [8] to have some level lowering statement and we will prove that indeed, under the hypothesis of the previous theorem, we obtain some level fixing statement, cf. proposition 4.1, which can also be stated as follow.

1.3. Corollary. — Consider two automorphic representations Π_1 and Π_2 whose Satake's parameters at places outside S , are congruent modulo l and prescribed by \mathfrak{m} . Then let K_1 and K_2 be open compact subgroups of $GL_d(\mathbb{A}^\infty)$ such that, $\Pi_i^{K_i} \neq (0)$ for $i = 1, 2$. Suppose moreover that $K_{1,v} = P_i(F_v) \cap GL_d(\mathcal{O}_v)$ where for $i = 1, 2$, P_i is a standard parabolic subgroup associated to a partition \underline{d}_i of d which we suppose to be maximal for the previous property. Then up to conjugacy, we have $\underline{d}_1 = \underline{d}_2$.

The main idea to prove previous theorem, is to deal with the cohomology groups of some compact Kottwitz-Harris-Taylor Shimura variety $X_{I,v} \rightarrow \text{Spec } \mathcal{O}_v$ with level I , an open compact subgroup of $G(\mathbb{A}^\infty)$ where G/\mathbb{Q} is some similitude group with signatures $(1, n-1), (0, n), \dots, (0, n)$ at infinity, cf. [9]. To each maximal ideal $\tilde{\mathfrak{m}}$ as before, is associated an irreducible automorphic representation $\Pi_{\tilde{\mathfrak{m}}}$ with local component at v denoted $\Pi_{\tilde{\mathfrak{m}},v}$ with supercuspidal support $\text{Scusp}_v(\tilde{\mathfrak{m}})$ whose modulo l reduction is independant of the choice of $\tilde{\mathfrak{m}} \subset \mathfrak{m}$: we denote it $\text{Scusp}_v(\mathfrak{m})$.

Consider the spectral sequence of vanishing cycles at the place v localized at \mathfrak{m} ; as by hypothesis \mathfrak{m} is such that $\rho_{\mathfrak{m}}$ is irreducible, then $E_{1,\mathfrak{m}}^{p,q} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \Rightarrow E_{\infty,\mathfrak{m}}^{p+q}$ degenerates at E_1 and is concentrated in degree $p+q = d-1$. The hypothesis \mathfrak{m} free implies that $E_{\infty,\mathfrak{m}}^n$ are all free; in the next section we will prove that all the $E_{1,\mathfrak{m}}^{p,q}$ are also free. Recall that each of these $E_1^{p,q}$ can be described as a successive extension of the middle cohomology group of a Harris-Taylor perverse sheaves and we are lead to prove that the localization at \mathfrak{m} of the cohomology groups of Harris-Taylor perverse sheaves are free. As an application of this freeness, we then deduce that the monodromy $\overline{\mathbb{Z}}_l$ -action is purely of local nature and

can be read on the sheaf of nearby cycles. In §3, we then study this local monodromy cf. proposition 3.2.

2. Uniformity of nilpotency on the cohomology

From now on we fix a maximal ideal \mathfrak{m} of \mathbb{T}_S verifying the hypothesis of theorem 1.2. We want to prove that the partition $d_{\tilde{\mathfrak{m}}}$ associated to any minimal prime ideal $\tilde{\mathfrak{m}} \subset \mathfrak{m}$ depends only on \mathfrak{m} .

Let then $F = F^+E$ be a CM field with E/\mathbb{Q} quadratic imaginary and let B/F be a central division algebra with dimension d^2 with a involution of second specie $*$. For $\beta \in B^{*-1}$, consider the similitude groupe G/\mathbb{Q} defined for any \mathbb{Q} -algebra R by

$$G(R) := \{(\lambda, g) \in R^\times \times (B^{op} \otimes_{\mathbb{Q}} R)^\times \text{ such that } gg^{\sharp_\beta} = \lambda\},$$

where $B^{op} = B \otimes_{F,c} F$ where $c = *|_F$ is the complex conjugation and \sharp_β is the involution $x \mapsto x^{\sharp_\beta} := \beta x^* \beta^{-1}$. Following [9], we can manage so that $G(\mathbb{R})$ has signatures $(1, d-1), (0, d), \dots, (0, d)$. We moreover suppose that $p = uu^c$ splits in E so that

$$G(\mathbb{Q}_p) \simeq \mathbb{Q}_p^\times \times \prod_{w|u} (B_w^{op})^\times$$

where w describes the places of F above u . We ask then B_v to be split, isomorphic to $GL_d(F_v)$.

We then denote \mathcal{I} the set of open compact subgroup

$$U^p(m) = U^p \times \mathbb{Z}_p^\times \times \prod_{i=1}^r \text{Ker}(\mathcal{O}_{B_{v_i}}^\times \longrightarrow (\mathcal{O}_{B_{v_i}}/\mathcal{P}_{v_i}^{m_i})^\times)$$

where U^p is any small enough open compact subgroup of $G(\mathbb{A}^{p,\infty})$ and $\mathcal{O}_{B_{v_i}}$ is the maximal order of B_{v_i} with maximal ideal \mathcal{P}_{v_i} and where v_1, \dots, v_r are the places of F above u with $p = uu^c$.

Attached to each $I \in \mathcal{I}$ is a Shimura variety $X_I \longrightarrow \mathcal{O}_v$ where \mathcal{O}_v denote the ring of integers of the completion F_v of F at v . Let denote

$$\Psi_v := R\Psi_{\eta_v}(\overline{\mathbb{Z}}_l[d-1])\left(\frac{d-1}{2}\right)$$

the nearby cycles autodual free perverse sheaf on the geometric special fiber X_{I, \bar{s}_v} of X_I . Using the Newton stratification and following the constructions of [7], we can define a $\overline{\mathbb{Z}}_l$ -filtration $\text{Fil}^\bullet(\Psi_v)$ whose graduates are free isomorphic to some free perverse Harris-Taylor sheaf defined as follows.

2.1. Notation. — For a representation π_v of $GL_d(F_v)$ and $n \in \frac{1}{2}\mathbb{Z}$, set $\pi_v\{n\} := \pi_v \otimes q_v^{-n \text{val} \circ \det}$. Recall that the normalized induction of two representations $\pi_{v,1}$ and $\pi_{v,2}$ of respectively $GL_{n_1}(F_v)$ and $GL_{n_2}(F_v)$ is

$$\pi_1 \times \pi_2 := \text{ind}_{P_{n_1, n_1+n_2}(F_v)}^{GL_{n_1+n_2}(F_v)} \pi_{v,1}\left\{\frac{n_2}{2}\right\} \otimes \pi_{v,2}\left\{-\frac{n_1}{2}\right\}.$$

Recall that a representation π_v of $GL_d(F_v)$ is called *cuspidal* (resp. *supercuspidal*) if it's not a subspace (resp. subquotient) of a proper parabolic induced representation. When the field of coefficients is of characteristic zero then these two notions coincides, but this is no more true for $\overline{\mathbb{F}}_l$.

2.2. Definition. — More precisely let π_v be an irreducible $\overline{\mathbb{Q}}_l$ -representation then its modulo l reduction is still cuspidal but not necessary supercuspidal: in this case its supercuspidal support is a segment $[\varrho, \varrho\nu^{s-1}]$ where ϱ is irreducible supercuspidal and s is either equal to 1 or of the following shape $s = m(\varrho)l^k$ for $k \in \mathbb{N}$. In the former case we say that π_v is of ϱ -type -1 and otherwise of ϱ -type k .

Let π_v be an irreducible cuspidal $\overline{\mathbb{Q}}_l$ -representation of $GL_g(F_v)$ and fix $t \geq 1$ such that $tg \leq d$. Thanks to Igusa varieties, Harris and Taylor constructed a local system on $X_{\mathcal{I}, \overline{s}, \overline{1}_h}^{=tg}$

$$\mathcal{L}_{\overline{\mathbb{Q}}_l}(\pi_v[t]_D)_{\overline{1}_h} = \bigoplus_{i=1}^{e_{\pi_v}} \mathcal{L}_{\overline{\mathbb{Q}}_l}(\rho_{v,i})_{\overline{1}_h}$$

where $(\pi_v[t]_D)_{|\mathcal{D}_{v,h}^\times} = \bigoplus_{i=1}^{e_{\pi_v}} \rho_{v,i}$ with $\rho_{v,i}$ irreducible. The Hecke action of $P_{tg, d-tg}(F_v)$ is then given through its quotient $GL_{d-tg} \times \mathbb{Z}$. These local systems have stable $\overline{\mathbb{Z}}_l$ -lattices and we will write simply $\mathcal{L}(\pi_v[t]_D)_{\overline{1}_h}$ for any $\overline{\mathbb{Z}}_l$ -stable lattice that we don't want to specify.

2.3. Notations. — For Π_t any representation of GL_{tg} and $\Xi : \frac{1}{2}\mathbb{Z} \longrightarrow \overline{\mathbb{Z}}_l^\times$ defined by $\Xi(\frac{1}{2}) = q^{1/2}$, we introduce

$$\widetilde{HT}_1(\pi_v, \Pi_t) := \mathcal{L}(\pi_v[t]_D)_{\overline{1}_h} \otimes \Pi_t \otimes \Xi^{\frac{tg-d}{2}}$$

and its induced version

$$\widetilde{HT}(\pi_v, \Pi_t) := \left(\mathcal{L}(\pi_v[t]_D)_{\overline{1}_h} \otimes \Pi_t \otimes \Xi^{\frac{tg-d}{2}} \right) \times_{P_{tg, d-tg}(F_v)} GL_d(F_v),$$

where the unipotent radical of $P_{tg,d-tg}(F_v)$ acts trivially and the action of

$$(g^{\infty,v}, \begin{pmatrix} g_v^c & * \\ 0 & g_v^{et} \end{pmatrix}, \sigma_v) \in G(\mathbb{A}^{\infty,v}) \times P_{tg,d-tg}(F_v) \times W_v$$

is given

- by the action of g_v^c on Π_t and $\deg(\sigma_v) \in \mathbb{Z}$ on $\Xi^{\frac{tg-d}{2}}$, and
- the action of $(g^{\infty,v}, g_v^{et}, \text{val}(\det g_v^c) - \deg \sigma_v) \in G(\mathbb{A}^{\infty,v}) \times GL_{d-tg}(F_v) \times \mathbb{Z}$ on $\mathcal{L}_{\overline{\mathbb{Q}}_l}(\pi_v[t]_D)_{\overline{1}_h} \otimes \Xi^{\frac{tg-d}{2}}$.

We also introduce

$$HT(\pi_v, \Pi_t)_{\overline{1}_h} := \widetilde{HT}(\pi_v, \Pi_t)_{\overline{1}_h}[d - tg],$$

and the perverse sheaf

$$P(t, \pi_v)_{\overline{1}_h} := j_{1_h, !}^{=tg} HT(\pi_v, \text{St}_t(\pi_v))_{\overline{1}_h} \otimes \mathbb{L}(\pi_v),$$

and their induced version, $HT(\pi_v, \Pi_t)$ and $P(t, \pi_v)$, where

$$j^{=h} = i^h \circ j^{\geq h} : X_{\mathcal{I}, \overline{s}}^{=h} \hookrightarrow X_{\mathcal{I}, \overline{s}}^{\geq h} \hookrightarrow X_{\mathcal{I}, \overline{s}}$$

and \mathbb{L}^\vee is the local Langlands correspondence.

Remark. Recall that π'_v is said inertially equivalent to π_v if there exists a character $\zeta : \mathbb{Z} \rightarrow \overline{\mathbb{Q}}_l^\times$ such that $\pi'_v \simeq \pi_v \otimes (\zeta \circ \text{val} \circ \det)$. Note, cf. [1] 2.1.4, that $P_{\overline{\mathbb{Q}}_l}(t, \pi_v)$ depends only on the inertial class of π_v and

$$P_{\overline{\mathbb{Q}}_l}(t, \pi_v) = e_{\pi_v} \mathcal{P}_{\overline{\mathbb{Q}}_l}(t, \pi_v)$$

where $\mathcal{P}_{\overline{\mathbb{Q}}_l}(t, \pi_v)$ is an irreducible perverse sheaf.

Over $\overline{\mathbb{Z}}_l$, we also have the p -perverse structure which is dual to the usual p -structure and the graduate $\text{gr}^r(\Psi_v)$ of $\text{Fill}^\bullet(\Psi_v)$ are free $\overline{\mathbb{Z}}_l$ -perverse sheaves of the following shape

$$\begin{aligned} {}^p j_{!}^{=h} HT_{\overline{\mathbb{Z}}_l}(\pi_v, \text{St}_t(\pi_v)) \left(\frac{1-t+2\delta}{2} \right) &\hookrightarrow \text{gr}^r(\Psi_v) \\ &\hookrightarrow {}^{p+} j_{!}^{=h} HT_{\overline{\mathbb{Z}}_l}(\pi_v, \text{St}_t(\pi_v)) \left(\frac{1-t+2\delta}{2} \right) \end{aligned}$$

for some $0 \leq k \leq t-1$, where \hookrightarrow means a bimorphism, that is both an epimorphism and a monomorphism, and where $HT_{\overline{\mathbb{Z}}_l}(\pi_v, \text{St}_t(\pi_v))$ means a certain lattice of the Harris-Taylor local system.

Remarks:

- In [6], we proved that if you always use the adjunction maps $j_!^h j^{=h,*} \rightarrow \text{Id}$ then all the previous graduates are isomorphic to p -intermediate extensions.
- Moreover if we are only interested in maximal ideal \mathfrak{m} such that $\text{Scusp}_v(\mathfrak{m})$ is multiplicity free, then for all $\pi_v \in \text{Scusp}_v(\mathfrak{m})$ the two intermediate extensions coincide.
- Finally we can easily arrange the filtration so that it is compatible with the nilpotent monodromy operator N_v , i.e. so that for any r the image of $\text{Fil}^r(\Psi_v) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ is some $\text{Fil}^{\phi(r)}(\Psi_v) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ for some decreasing function ϕ .

We then have a spectral sequence

$$E_1^{p,q} = H^{p+q}(X_{I,\overline{s}_v}, \text{gr}^{-p}(\Psi_v)) \Rightarrow H^{p+q}(X_{I,\overline{\eta}_v}, \overline{\mathbb{Z}}_l).$$

Note moreover it's straightforward to replace the constant sheaf $\overline{\mathbb{Z}}_l$ by the sheaf $V_{\overline{\mathbb{Z}}_l, \xi}$ attached to any irreducible algebraic representation ξ of $G(\mathbb{Q})$.

As pointed out in [8], if for some \mathfrak{m} the spectral sequence is concentrated in middle degree, i.e. $E_{1,\mathfrak{m}}^{p,q} = 0$ for $p+q \neq d-1$, and all the $E_{1,\mathfrak{m}}^{p,d-1-p}$ are free, then the action of the monodromy operator on $H^{d-1}(X_{I,\overline{\eta}_v}, \overline{\mathbb{Z}}_l)_{\mathfrak{m}}$ comes from its action on Ψ_v : we can then say that the nilpotency of the monodromy Galois action is \mathfrak{m} -uniform i.e. for every $\tilde{\mathfrak{m}} \subset \mathfrak{m}$ the partition $d_{\tilde{\mathfrak{m}}}$ is equal to $d_{\mathfrak{m}}$. The local question of the nilpotency of N_v will be studied in the next section.

2.4. Proposition. — *Under the hypothesis of theorem 1.2, the $E_{1,\mathfrak{m}}^{p,q}$ are torsion free and trivial for $p+q \neq d-1$.*

Proof. — In [4], we proved that under any of the conditions of definition 1.1, $H^i(X_{I,\overline{\eta}_v}, \overline{\mathbb{Z}}_l)_{\mathfrak{m}}$ is torsion free concentrated in degree $i = d-1$. Moreover $E_{1,\mathfrak{m}}^{p,q} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ is zero if $p+q \neq d-1$ so we are lead to prove that $E_{1,\mathfrak{m}}^{p,q}$ is torsion free. In [5] we prove it if $\text{Scusp}_v(\mathfrak{m})$ is multiplicity free. Here using the main result of [6], we will prove that if all the $E_{\infty,\mathfrak{m}}^n$ are torsion free then the same is true for all the $E_{1,\mathfrak{m}}^{p,q}$.

Recall, cf. [6] proposition 3.1.3, that we have the following splitting

$$\Psi_v \simeq \bigoplus_{g=1}^d \bigoplus_{\varrho \in \text{Scusp}_{\overline{\mathbb{F}}_l}(g)} \Psi_{\varrho}$$

where $\text{Scusp}_{\overline{\mathbb{F}}_l}(g)$ is the set of inertial equivalence classes of irreducible $\overline{\mathbb{F}}_l$ -supercuspidal representation of $GL_g(F_v)$, with the property the irreducible constituent of $\Psi_\varrho \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ are exactly the perverse Harris-Taylor sheaf associated to a irreducible cuspidale $\overline{\mathbb{Q}}_l$ -representations of some $GL_g(F_v)$ such that the supercuspidal support of the modulo l reduction of π_v is a segment associated to the inertial class ϱ , cf. definition 2.2. In particular for every $\varrho \in \text{Scusp}_{\overline{\mathbb{F}}_l}(g)$, the cohomology groups of Ψ_ϱ are torsion free.

Consider now an irreducible $\overline{\mathbb{Q}}_l$ -cuspidal representation $\pi_{v,u}$ of ϱ -type u . In [6] proposition 2.4.2, we proved the following equality in the Grothendieck group of equivariant $\overline{\mathbb{F}}_l$ -perverse sheaves

$$\mathbb{F}\left({}^p j_{l*}^{=tg_u(\varrho)} HT(\pi_{v,u}, \Pi_t)\right) = m(\varrho) l^u \sum_{r=0}^{s-tm(\varrho)l^u} {}^p j_{l*}^{=tg_u(\varrho)+rg_{-1}(\varrho)} HT(\varrho, r_l(\Pi_t) \overrightarrow{\times} V_\varrho(r+tm(\varrho)l^u, < \underline{\delta}_u)) \otimes \Xi^{r \frac{g-1}{2}}. \quad (2.5)$$

where with the notation of [2] propositions 3.2.2 et 3.2.7, $V_\varrho(r+tm(\varrho)l^u, < \underline{\delta}_u)$ is the sum of some of the irreducible subquotients of the modulo l reduction of the Steinberg representation $\text{St}_{r_{g-1}(\varrho)}(\pi_{v,-1})$ where $r_l(\pi_{v,-1}) \simeq \varrho$.

Remark. In particular, in order to prove that the cohomology of Harris-Taylor perverse is concentrated in middle degree, we only have to deal with those associated to a fixed cuspidal $\pi_{v,-1}$ of ϱ -type -1 where, by one of the main result of [6], the p and $p+$ intermediate extensions coincide.

Recall the following resolution of ${}^p j_{l*}^{=tg} HT(\pi_{v,-1}, \text{St}_t(\pi_{v,-1}))$

$$\begin{aligned} 0 \rightarrow j_l^{=sg} HT(\pi_{v,-1}, \text{St}_s(\pi_{v,-1} \left\{ \frac{t-s}{2} \right\}) \times \text{Speh}_{s-t}(\pi_{v,-1} \{t/2\})) \otimes \Xi^{\frac{s-t}{2}} \rightarrow \dots \\ \rightarrow j_l^{=(t+1)g} HT(\pi_{v,-1}, \text{St}_t(\pi_{v,-1}(-1/2)) \times \pi_{v,-1} \{t/2\}) \otimes \Xi^{\frac{1}{2}} \rightarrow \\ j_l^{=tg} HT(\pi_{v,-1}, \text{St}_t(\pi_{v,-1})) \rightarrow {}^p j_{l*}^{=tg} HT(\pi_{v,-1}, \text{St}_t(\pi_{v,-1})) \rightarrow 0. \end{aligned} \quad (2.6)$$

which is proved in [1] for $\overline{\mathbb{Q}}_l$ -coefficients and in [6] over $\overline{\mathbb{Z}}_l$. By adjunction property, the map

$$\begin{aligned} & j_!^{=(t+\delta)g} HT(\pi_{v,-1}, \text{St}_t(\pi_{v,-1}\{\frac{-\delta}{2}\}) \times \text{Speh}_\delta(\pi_{v,-1}\{t/2\})) \otimes \Xi^{\delta/2} \\ \longrightarrow & j_!^{=(t+\delta-1)g} HT(\pi_{v,-1}, \text{St}_t(\pi_{v,-1}\{\frac{1-\delta}{2}\}) \times \text{Speh}_{\delta-1}(\pi_{v,-1}\{t/2\})) \otimes \Xi^{\frac{\delta-1}{2}} \end{aligned} \quad (2.7)$$

is given by

$$\begin{aligned} & HT(\pi_{v,-1}, \text{St}_t(\pi_{v,-1}\{\frac{-\delta}{2}\}) \times \text{Speh}_\delta(\pi_{v,-1}\{t/2\})) \otimes \Xi^{\delta/2} \longrightarrow \\ & p_i^{(t+\delta)g, !} j_!^{=(t+\delta-1)g} HT(\pi_{v,-1}, \text{St}_t(\pi_{v,-1}\{\frac{1-\delta}{2}\}) \times \text{Speh}_{\delta-1}(\pi_{v,-1}\{t/2\})) \otimes \Xi^{\frac{\delta-1}{2}} \end{aligned} \quad (2.8)$$

From [6], we have

$$\begin{aligned} & p_i^{(t+\delta)g, !} j_!^{=(t+\delta-1)g} HT(\pi_{v,-1}, \text{St}_t(\pi_{v,-1}\{\frac{1-\delta}{2}\}) \times \text{Speh}_{\delta-1}(\pi_{v,-1}\{t/2\})) \otimes \Xi^{\frac{\delta-1}{2}} \\ & \simeq HT\left(\pi_{v,-1}, \text{St}_t(\pi_{v,-1}\{\frac{1-\delta}{2}\}) \right. \\ & \left. \times (\text{Speh}_{\delta-1}(\pi_{v,-1}\{-1/2\}) \times \pi_{v,-1}\{\frac{\delta-1}{2}\})\{t/2\}\right) \otimes \Xi^{\delta/2} \end{aligned} \quad (2.9)$$

Fact. In particular, up to homothety, the map (2.9), and so those of (2.8), is unique. Finally as the map of (2.6) are strict, the given maps (2.7) are uniquely determined, that is if we forget the infinitesimal parts, these maps are independent of chosen t in (2.6).

For every $1 \leq t \leq d$, let denote $i(t)$ the smaller index i such that $H^i(X_{I, \overline{s}_v}, {}^p j_{!*}^{-tg} HT_{\overline{\mathbb{Z}}_l}(\pi_{v,-1}, \text{St}_t(\pi_{v,-1})))_{\mathfrak{m}}$ has non trivial torsion: if it doesn't exists then set $i(t) = +\infty$. By duality, as ${}^p j_{!*} = {}^{p+} j_{!*}$ for cuspidal representation $\pi_{v,-1}$ of ϱ -type -1 , note that when $i(t)$ is finite then $i(t) \leq 0$. Suppose by absurdity there exists t with $i(t)$ finite and denote t_0 the bigger such t .

2.10. Lemma. — *For $1 \leq t \leq t_0$ then $i(t) = t - t_0$.*

Remark. A similar result is proved in [4] when $\pi_{v,-1}$ is a character and when the level is maximal at v .

Proof. — Note first that for every $t_0 \leq t \leq s$, then the cohomology groups of $j_!^{=tg} HT(\pi_{v,-1}, \Pi_t)$ are torsion free. Indeed there exists a filtration

$$(0) = \text{Fil}^0(\pi_{v,-1}, t) \hookrightarrow \text{Fil}^{-d}(\pi_{v,-1}, t) \hookrightarrow \cdots \\ \hookrightarrow \text{Fil}^{-t}(\pi_{v,-1}, t) = j_!^{=tg} HT(\pi_{v,-1}, \Pi_t)$$

with graduates

$$\text{gr}^{-k}(\pi_{v,-1}, t) \simeq {}^p j_{!*}^{=kg} HT(\pi_{v,-1}, \Pi_t \left\{ \frac{t-k}{2} \right\}) \otimes \text{St}_{k-t}(\pi_{v,-1} \{t/2\}) \left(\frac{t-k}{2} \right).$$

The associated spectral sequence localized at \mathfrak{m} is then concentrated in middle degree and torsion free. Then the spectral sequence associated to (2.6) has all its E_1 terms torsion free and degenerates at its E_2 terms. As by hypothesis the aims of this spectral sequence is free and equals to only one E_2 terms, we deduce that all the maps

$$H^0(X_{I,\bar{s},v}, j_!^{=(t+\delta)g} HT(\pi_{v,-1}, \text{St}_t(\pi_{v,-1} \left\{ \frac{-\delta}{2} \right\}) \times \text{Speh}_\delta(\pi_{v,-1} \{t/2\})) \otimes \Xi^{\delta/2})_{\mathfrak{m}} \\ \longrightarrow \\ H^0(X_{I,\bar{s},v}, j_!^{=(t+\delta-1)g} HT(\pi_{v,-1}, \text{St}_t(\pi_{v,-1} \left\{ \frac{1-\delta}{2} \right\}) \\ \times \text{Speh}_{\delta-1}(\pi_{v,-1} \{t/2\})) \otimes \Xi^{\frac{\delta-1}{2}})_{\mathfrak{m}} \quad (2.11)$$

are strict. Then from the previous fact stressed after (2.9), this property remains true when we consider the associated spectral sequence for $1 \leq t' \leq t_0$.

Consider now $t = t_0$ where we know the torsion to be non trivial. From what was observed above we then deduce that the map

$$H^0(X_{I,\bar{s},v}, j_!^{=(t_0+1)g} HT(\pi_{v,-1}, \text{St}_{t_0}(\pi_{v,-1} \left\{ \frac{-1}{2} \right\}) \times \pi_{v,-1} \{t_0/2\})) \otimes \Xi^{1/2})_{\mathfrak{m}} \\ \longrightarrow \\ H^0(X_{I,\bar{s},v}, j_!^{=t_0g} HT(\pi_{v,-1}, \text{St}_{t_0}(\pi_{v,-1}))_{\mathfrak{m}} \quad (2.12)$$

has a non trivial torsion cokernel so that $i(t_0) = 0$.

Finally for any $1 \leq t \leq t_0$, the map like (2.12) for $t + \delta - 1 < t_0$ are strict so that the $H^i(X_{I,\bar{s},v}, {}^p j_{!*}^{=tg} HT(\pi_{v,-1}, \Pi_t))_{\mathfrak{m}}$ are zero for $i < t - t_0$ while when $t + \delta - 1 = t_0$ its cokernel has non trivial torsion which gives then a non trivial torsion class in $H^{t-t_0}(X_{I,\bar{s},v}, {}^p j_{!*}^{=tg} HT(\pi_{v,-1}, \Pi_t))_{\mathfrak{m}}$. \square

Remark. Note from (2.5) that for any π_v of type ϱ and for every t , then the torsion of $H^i(X_{I, \overline{s}_v}, {}^p j_{!*}^{=tg} HT(\pi_v, \Pi_t))_{\mathfrak{m}}$ is trivial for any $i \leq 1 - t_0$ which gives us the corresponding informations about the associated map (2.12).

Let consider now the filtration of stratification of Ψ_ϱ constructed using the adjunction morphisms $j_!^{=tg} j^{=tg,*}$ as in [3]

$$\mathrm{Fil}_!^0(\Psi_\varrho) \hookrightarrow \mathrm{Fill}_!^g(\Psi_\varrho) \hookrightarrow \mathrm{Fill}_!^{2g}(\Psi_\varrho) \hookrightarrow \dots \hookrightarrow \mathrm{Fill}_!^{sg}(\Psi_\varrho)$$

where $\mathrm{Fill}_!^{tg}(\Psi_\varrho)$ is the saturated image of $j_!^{=tg} j^{=tg,*} \Psi_\varrho \longrightarrow \Psi_\varrho$. For our fixed $\pi_{v,-1}$, let denote $\mathrm{Fil}_{!, \pi_{v,-1}}^g(\Psi) \hookrightarrow \mathrm{Fil}_!^g(\Psi_\varrho)$ such that $\mathrm{Fil}_{!, \pi_{v,-1}}^g(\Psi) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \simeq \mathrm{Fil}_!^g(\Psi_{\pi_{v,-1}})$ where $\Psi_{\pi_{v,-1}}$ is the direct factor of $\Psi \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ associated to $\pi_{v,-1}$, cf. [3]. From the main result of [6], we have then the following resolution of $\mathrm{Fil}_{!, \pi_{v,-1}}^g(\Psi)$

$$\begin{aligned} 0 \rightarrow j_!^{=sg} HT(\pi_{v,-1}, \mathrm{Speh}_s(\pi_{v,-1})) \otimes L_g(\pi_{v,-1}(\frac{s-1}{2})) \longrightarrow \\ j_!^{=(s-1)g} HT(\pi_{v,-1}, \mathrm{Speh}_{s-1}(\pi_{v,-1})) \otimes L_g(\pi_{v,-1}(\frac{s-2}{2})) \longrightarrow \\ \dots \longrightarrow j_!^{=g} HT(\pi_{v,-1}, \pi_{v,-1}) \otimes L_g(\pi_{v,-1}) \longrightarrow \mathrm{Fil}_{!, \pi_{v,-1}}^g(\Psi) \rightarrow 0 \end{aligned} \quad (2.13)$$

We can then apply the previous arguments so that $H^i(X_{I, \overline{s}_v}, \mathrm{Fil}_{!, \pi_{v,-1}}^g)_{\mathfrak{m}}$ has non trivial torsion for $i = 1 - t_0$ and with free quotient zero for $i \neq 0$.

Consider now the other graduates which are either associated to some π_v of type ϱ or to $\mathrm{gr}_!^{tg}(\Psi_\varrho)$ for $t \geq 2$. In the last case, we also have similar resolution like before

$$\begin{aligned} 0 \rightarrow j_!^{=sg} HT(\pi_{v,-1}, LT_{t,s}(\pi_{v,-1})) \otimes L_g(\pi_{v,-1}(\frac{s-t}{2})) \longrightarrow \\ j_!^{=(s-1)g} HT(\pi_{v,-1}, LT_{t,s-1}(\pi_{v,-1})) \otimes L_g(\pi_{v,-1}(\frac{s-t-1}{2})) \longrightarrow \\ \dots \longrightarrow j_!^{=g} HT(\pi_{v,-1}, \mathrm{St}_t(\pi_{v,-1})) \otimes L_g(\pi_{v,-1}) \longrightarrow \mathrm{Fil}_{!, \pi_{v,-1}}^{tg}(\Psi) \rightarrow 0 \end{aligned} \quad (2.14)$$

where $LT_{t,t+\delta}(\pi_{v,-1})$ is the irreducible subspace of $\mathrm{St}_t(\pi_{v,-1}\{-\delta/2\}) \times \mathrm{Speh}_\delta(\pi_{v,-1}\{t/2\})$. By the same arguments the torsion of $H^i(X_{I, \overline{s}_v}, \mathrm{Fil}_{!, \pi_{v,-1}}^{tg}(\Psi))_{\mathfrak{m}}$ is trivial for any $i \leq 1 - t_0$ and the free parts are concentrated for $i = 0$. The case of the other π_v can then be deduce from the previous remark. Using then the spectral sequence associated to the previous filtration, we can then conclude that $H^{1-t_0}(X_{I, \overline{s}_v}, \Psi_\varrho)_{\mathfrak{m}}$ would have non trivial torsion which is false from our choice of \mathfrak{m} , so that the proposition is proved.

□

3. Local behavior of monodromy over $\overline{\mathbb{F}}_l$

Recall for each $1 \leq g \leq d$ with $s := \lfloor \frac{d}{g} \rfloor$, over $\overline{\mathbb{Q}}_l$, that N_v^{s-1} induces an isomorphism from the graduate of $\Psi_v(\overline{\mathbb{Q}}_l)$ isomorphic to $\mathcal{P}(s-1, \pi_v)(\frac{s-1}{2})$ to the one isomorphic to $\mathcal{P}(s-1, \pi_v)(\frac{1-s}{2})$. In this section we want to prove that this property remains true modulo l . In order to define N_v over $\overline{\mathbb{Z}}_l$, we suppose that $l \geq d$. Recall from [6] that we have a filtration $\text{Fil}^\bullet(\Psi_\varrho)$ whose graduate $\text{gr}^r(\Psi_\varrho)$ are free $\overline{\mathbb{Z}}_l$ -perverse sheaves of the following shape

$$\text{gr}^r(\Psi_v) \simeq {}^p j_{!*} HT_{\overline{\mathbb{Z}}_l}(\pi_v, \text{St}_t(\pi_v)) \left(\frac{1-t+2\delta}{2} \right)$$

for some $0 \leq k \leq t-1$ and cuspidal irreducible π_v of type ϱ .

3.1. Notation. — We denote $m(\varrho)$ to be equal to the cardinal of the Zelevinsky line of ϱ if it is not equal to 1, otherwise $m(\varrho) = l$.

3.2. Proposition. — Suppose $m(\varrho) > d$, then for all r , the morphism induced by the monodromy operator N_v from $\text{Fil}^r(\Psi_\varrho)$ to Ψ_ϱ , is strict.

Proof. — Recall first that the filtration $\text{Fil}^\bullet(\Psi_\varrho)$ is constructed so that it is compatible with the action of N_v in the sense that, over $\overline{\mathbb{Q}}_l$, the image of $\text{Fil}^r(\Psi_\varrho)$ is some $\text{Fil}^{\phi(r)}(\Psi_\varrho)$. We then have to prove that for every r , we have a p -epimorphism $N_v : \text{Fil}^r(\Psi_\varrho) \rightarrow \text{Fil}^{\phi(r)}(\Psi_\varrho)$ which is clearly equivalent to prove that for every irreducible cuspidal representation π_v of $GL_g(F_v)$ and for every $1 \leq t \leq d/g$, then N_v induces a isomorphism

$$\begin{aligned} {}^p j_{!*} HT_{\overline{\mathbb{Z}}_l}(\pi_v, \text{St}_t(\pi_v)) \left(\frac{1-t+2\delta}{2} \right) &\longrightarrow \\ {}^p j_{!*} HT_{\overline{\mathbb{Z}}_l}(\pi_v, \text{St}_t(\pi_v)) \left(\frac{1-t+2(\delta-1)}{2} \right), &\quad (3.3) \end{aligned}$$

where each of these two perverse sheaf is given by some graduates $\text{gr}^r(\Psi_v)$ and $\text{gr}^{\phi(r)}(\Psi_v)$. Under the hypothesis that $m(\varrho) > d$, then the reduction modulo l of $\pi_v[t]_D \otimes L_g(\pi_v) \otimes \text{St}_t(\pi_v)$ is irreducible so that there exists, up to homothety, an unique stable lattice of $HT(\pi_v, \text{St}_t(\pi_v))$ which means that to prove (3.3) is a isomorphism, it suffices to prove its reduction modulo l is non zero.

Remark. This is the only place in the proof where we use the hypothesis $m(\varrho) > d$, cf. also the remark after the proof.

To do so, consider then a maximal ideal \mathfrak{m} as in the previous section, i.e. such that for any $\tilde{\mathfrak{m}} \subset \mathfrak{m}$ such that $\Pi_{\tilde{\mathfrak{m}}}$ is cohomological and its local component at v is isomorphic to $\mathrm{St}_t(\pi_v) \times \pi'_v$ for some representation π'_v , we will have to choose such that it doesn't interact with π_v . Take for example $\pi'_v = \chi_{1,v} \times \cdots \times \chi_{r,v}$ such that the supercuspidal support of the modulo l reduction of π'_v is disjoint from the Zelevinsky line of ϱ . Then from the previous section, its monodromy is given by the local monodromy on Ψ_v so that in particular, if it happens that the monodromy on the modulo l reduction of $\rho_{\tilde{\mathfrak{m}},v}$ is maximal on the part associated to $\mathrm{St}_t(\pi_v)$, then (3.3) is necessary non trivial for such (π_v, t) . We are then reduced to study the nilpotency of $\rho_{\tilde{\mathfrak{m}},v}$.

Consider now a finite extension L_v/F_v such that the restriction of ρ_v to the Galois group of L_v is unramified and let L/F be a CM field globalizing L_v/F_v . By base change the local component of the automorphic representation $\Pi' := BC_{L/F}(\Pi)$ associated to $\rho' := \rho|_{G_L}$ is of the following form

$$\Pi'_v \simeq \mathrm{St}_s(\xi_1) \times \cdots \times \mathrm{St}_s(\xi_g) \times \chi_1 \times \cdots \times \chi_r$$

where ξ_1, \dots, ξ_g and χ_1, \dots, χ_r are characters. We are then reduce to the case of (ρ', Π') ; the idea is then to realize ρ' in the cohomology of a K-H-T Shimura variety over L through a spectral sequence like in the previous section, i.e. which degenerates in E_1 with all $E_1^{p,q}$ free. As remarked before, the monodromy is then obtained from those on the vanishing cycle sheaves. As we are in Iwahori level, we can use the arguments of [8] §3.1 where the monodromy action is, thanks to Rapoport-Zink cf. [10], described explicitly and is of maximal nilpotency. □

4. Level fixing property

Consider as before a maximal ideal \mathfrak{m} associated to the modulo l reduction of an irreducible automorphic representation $\Pi_{\tilde{\mathfrak{m}}}$ with $\tilde{\mathfrak{m}} \subset \mathfrak{m}$: recall that \mathfrak{m} corresponds to the multiset of modulo l Satake parameters at almost all places where $\Pi_{\tilde{\mathfrak{m}}}$ is unramified.

A classical question is then to know if it is possible to find another prime ideal $\tilde{\mathfrak{m}}'$ such that at a fixed place v , the level of $\Pi_{\tilde{\mathfrak{m}}',v}$ is strictly

lower (resp. higher) of those of $\Pi_{\tilde{\mathfrak{m}},v}$: we then speak of level lowering (resp. rising) phenomenon.

In this section we are interested in the opposite situation where whatever is $\tilde{\mathfrak{m}} \subset \mathfrak{m}$, then the level at v is given by \mathfrak{m} : we call it a level fixing phenomenon.

4.1. Proposition. — *Let \mathfrak{m} a maximal ideal of \mathbb{T}_S cohomological verifying the following properties:*

- *the order of q_v modulo l is strictly greater than d ;*
- *there exists a prime ideal $\tilde{\mathfrak{m}} \subset \mathfrak{m}$ such that*

$$\Pi_{\tilde{\mathfrak{m}},v} \simeq \text{St}_{t_1}(\chi_{v,1}) \times \cdots \times \text{St}_{t_r}(\chi_{v,r}),$$

- with $t_1 \geq \cdots \geq t_r$ and where $\chi_{v,1}, \dots, \chi_{v,r}$ are characters;*
- *the supercuspidal support of the modulo l reduction of $\Pi_{\tilde{\mathfrak{m}},v}$ is the union of r unlinked Zelevinsky segments*

$$\bigcup_{i=1}^r [r_l(\chi_{v,i}), r_l(\chi_{v,i} \nu^{t_i-1})];$$

- *$\overline{\rho}_{\mathfrak{m}}$ is irreducible and \mathfrak{m} is free in the sense of definition 1.1.*

For any prime ideals $\tilde{\mathfrak{m}}_1, \tilde{\mathfrak{m}}_2$ contained in \mathfrak{m} and for any open compact subgroup K_v of $GL_d(F_v)$, then $\Pi_{\tilde{\mathfrak{m}}_1,v}$ has non trivial K_v -invariants vectors if and only if the same is true for $\Pi_{\tilde{\mathfrak{m}}_2,v}$.

Proof. — From the last property we are in the situation of proposition 2.4 so that the nilpotency of the monodromy operator N_v on $\overline{\rho}_{\mathfrak{m},v}$ is given by its action on the vanishing cycle sheaves. From the second hypothesis, proposition 3.2 applies so that the partition $\underline{d}_{\mathfrak{m},v}$ is $t_1 \geq \cdots \geq t_r$.

Finally the third hypothesis tells us that $\underline{d}_{\mathfrak{m},v}$ is formally maximal in particular for any $\tilde{\mathfrak{m}}' \subset \mathfrak{m}$ with

$$\Pi_{\tilde{\mathfrak{m}}',v} \simeq \text{St}_{t'_1}(\chi'_{v,1}) \times \cdots \times \text{St}_{t'_r}(\chi'_{v,r}),$$

then $\underline{d}_{\tilde{\mathfrak{m}}',v} = (t'_1 \geq \cdots \geq t'_r)$ is necessary lower than $\underline{d}_{\mathfrak{m},v}$ and as it is trivially greater, then it's the same that is $(t'_1, \dots, t'_r) = (t_1, \dots, t_r)$. \square

Remark. Of course it should be possible to formulate a statement replacing characters by irreducible cuspidal representations by stressing the partition $(t_1 \geq \cdots \geq t_r)$, cf. corollary 1.3 of the introduction.

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