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Local Reasoning about Parametric and Reconfigurable Component-based Systems

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We introduce a logical framework for the specification and verification of component-based systems, in which finitely many component instances are active, but the bound on their number is not known. Besides specifying and verifying parametric systems, we consider the aspect of dynamic reconfiguration, in which components can migrate at runtime on a physical map, whose shape and size may change. We describe such parametric and reconfigurable architectures using resource logics, close in spirit to Separation Logic, used to reason about dynamic pointer structures. These logics support the principle of local reasoning, which is the key for writing modular specifications and building scalable verification algorithms, that deal with large industrial-size systems.

1 Introduction

We consider distributed computing systems consisting of white-box components, whose interfaces are sets of communication ports. A port controls an internal transition of the component and interacts with zero or more ports belonging to other components. The behavior of a component is a finite-state machine, whose transitions are labeled with ports, that abstracts the behavior of a real-life hardware or software component. An architecture describes all possible interactions in a system, however it gives no information regarding the partial order in which they may execute. The global behavior of the system is determined by the composition of the local behaviors of each component, in the natural sense: an interaction represents a set of actions that are executed simultaneously, whereas different interactions occur interleaved.

We aim at providing a framework for the modular specification and verification of such component-based systems. The building blocks of this framework are:

1. partial *architectures*, defined by a *domain* (set of ports) and a set of *interactions* between ports from the domain and external ports,
2. a *composition* operation on architectures,
3. a *modular* composition of *behaviors*, that mirrors the composition of architectures and agrees with the global behavior described by interactions,
4. a *separation logic* of architectures, that supports the principle of *local reasoning* and allows to describe the local updates induced by reconfiguration actions.

We describe architectures using a resource logic that views the active components of the system as resources, which can be dynamically created and disposed of, and whose interaction scheme can be changed at runtime. Typically, reasoning about resources requires a notion of locality, which is captured by the separating connectives (conjunction and implication) of Separation Logic. In a nutshell, the advantages of modeling systems using Separation Logic are:

- elegant and concise definitions of parametric architectures with recursive patterns. In particular, a recursive definition of an architecture provides support for verification, in terms of hints for automatic generation of network invariants, used to prove safety properties of the system (deadlock freedom, mutual exclusion).
- correctness proofs of reconfiguration sequences, based on the principle of local reasoning: only a small region of the system where the update takes place, needs to be considered by the proof, instead of the entire system.

In order to have practical applications, a system modeling and proof framework requires a certain degree of automation. Although complete automation is, in general, impossible due to the inherent undecidability limits, defining decidable fragments of the logic and studying their computational complexity constitute important ingredients for building provers that can handle dynamically reconfigurable concurrent/distributed systems.

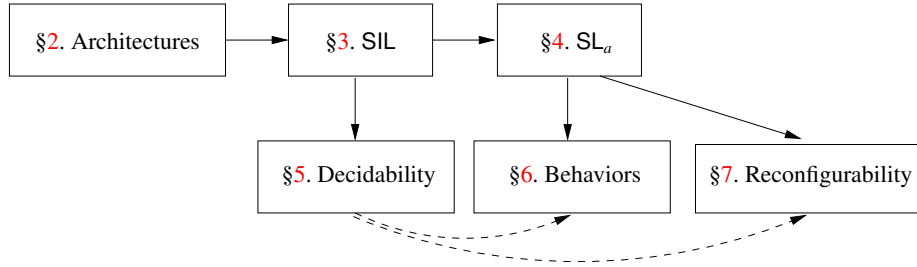


Fig. 1. Roadmap

1.1 Roadmap

The organization and reading flow of this paper are depicted in Figure 1. A solid edge between two sections A and B indicates that one needs to read A entirely before reading B. A dashed edge between A and B indicates that some results of A are used by B but reading of A is not necessary to understand B. Section 2 introduces the concept of architecture and defines the composition of architectures. Section 3 gives the formal syntax and semantics of the Separation Logic of Interactions (SIL), used to describe architectures and Section 5 deals with the decidability of two fragments of quantifier-free SIL. In Section 4 we extend SIL with component identifiers and recursive predicates, in order to describe parametric component-based systems, consisting of an arbitrary number of replicated components (SL_a). From here on, the reading flow splits in two separate directions, namely Section 6 introduces component behaviors and tackles the verification of safety properties (such as deadlock freedom, mutual exclusion, etc.) using the method of network invariants, and Section 7 introduces a framework for specifying and verifying dynamically reconfigurable systems, using a combination of two Separation Logics: classical SL interpreted over graphs, for describing the physical map and SL_a for describing the virtual architecture.

2 Architectures

Let Ports be a countably infinite alphabet of *ports*. An *interaction* is a finite set $I \in 2^{\text{Ports}}$ of ports. An *architecture* is a pair $\mathcal{A} = \langle D, \{I_1, \dots, I_k\} \rangle$, where $\text{dom}(\mathcal{A}) \stackrel{\text{def}}{=} D \in 2^{\text{Ports}}$ is a finite set of ports, called the *domain* of \mathcal{A} and $\text{inter}(\mathcal{A}) = \{I_1, \dots, I_k\}$ is a set of interactions, such that $I_i \cap D \neq \emptyset$, for all $i = 1, \dots, k$. An interaction $I \in \mathcal{I}$ is said to be *closed* if $I \subseteq D$ and *open*, otherwise. Intuitively, only closed interactions are executable in a given architecture, because the domain provides all the required ports. An architecture is *closed* if it contains only closed interactions, and *open*, otherwise. We write Arch for the set of architectures.

Example 1. Consider the architectures $\mathcal{A}_1 = \langle \{p\}, \{\{p, q\}\} \rangle$ and $\mathcal{A}_2 = \langle \{q\}, \{\{p, q\}\} \rangle$. Intuitively, \mathcal{A}_1 offers the port p , which is the only port in its domain, and requires the port q in order to perform the interaction $\{p, q\}$. On the other hand, \mathcal{A}_2 offers the port q and requires p to perform the same interaction $\{p, q\}$. In this case \mathcal{A}_1 and \mathcal{A}_2 have a match and their composition has domain $\{p, q\}$ and the only interaction $\{p, q\}$, which is closed and thus executable. ■

We move on to the formal definition of the composition of architectures. Because ports are viewed as resources distributed among architectures, we define composition only for architectures with disjoint domains. Allowing non-disjoint architectures to compose would require using multisets as architecture domains¹ and unnecessarily complicate the upcoming definitions. Two architectures $\mathcal{A}_1 = \langle D_1, \mathcal{I}_1 \rangle$ and $\mathcal{A}_2 = \langle D_2, \mathcal{I}_2 \rangle$ are *disjoint* if and only if $D_1 \cap D_2 = \emptyset$. For disjoint architectures, we define the following composition:

$$\mathcal{A}_1 \uplus \mathcal{A}_2 \stackrel{\text{def}}{=} \langle D_1 \cup D_2, (\mathcal{I}_1 \cap \mathcal{I}_2) \cup (\mathcal{I}_1 \cap 2^{\overline{D}_2}) \cup (\mathcal{I}_2 \cap 2^{\overline{D}_1}) \rangle$$

where $\overline{D}_i \stackrel{\text{def}}{=} \text{Ports} \setminus D_i$ is the complement of D_i , for $i = 1, 2$. The composition preserves the interactions of \mathcal{A}_i that are disjoint from the domain of \mathcal{A}_{3-i} , for $i = 1, 2$. However, an interaction I of \mathcal{A}_i that has a nonempty intersection with the domain of \mathcal{A}_{3-i} is kept in the composition if it matches an interaction of \mathcal{A}_i , i.e. formally $I \in \mathcal{I}_1 \cap \mathcal{I}_2$.

Recall that we require an interaction to be closed in order to be executable. Since the domain of an architecture is enlarged by composition, certain interactions may become closed, even if they do not match interactions from the other arty. To understand this point, consider the following example.

Example 2. Let $\mathcal{A}_1 = \langle \{p\}, \{\{p, q\}\} \rangle$ and $\mathcal{A}_2 = \langle \{q\}, \{q, r\} \rangle$ be architectures. Since the domain of the composition is $\text{dom}(\mathcal{A}_1 \uplus \mathcal{A}_2) = \{p, q\}$, the interaction $\{p, q\}$ of \mathcal{A}_1 is closed in $\mathcal{A}_1 \uplus \mathcal{A}_2$. However, this interaction is not executable, because it is not matched by any interaction from \mathcal{A}_2 . This is because \mathcal{A}_2 provides the required port q , but in a different interaction context $\{q, r\}$, that does not match $\{p, q\}$. The natural choice is thus to remove the interaction $\{p, q\}$ from $\text{inter}(\mathcal{A}_1 \uplus \mathcal{A}_2)$. The remaining interaction $\{q, r\}$ is kept because it might become executable in a future composition with an architecture \mathcal{A}_3 , provided that $r \in \text{dom}(\mathcal{A}_3)$ and $\{q, r\} \in \text{inter}(\mathcal{A}_3)$. ■

¹ Ports would be lost in composition, if domains are not disjoint and the domain of the composition is the union of domains.

We show that composition is well-defined and has natural algebraic properties:

Proposition 1. *Given disjoint architectures \mathcal{A}_1 and \mathcal{A}_2 , their composition $\mathcal{A}_1 \uplus \mathcal{A}_2$ is again an architecture. Moreover, the composition is commutative, associative and has neutral element $\langle \emptyset, \emptyset \rangle$.*

Proof: Let $\mathcal{A}_i = \langle D_i, \mathcal{I}_i \rangle$, for all $i = 1, 2, 3$. We have $\mathcal{A}_1 \uplus \mathcal{A}_2 = \langle D_1 \cup D_2, (\mathcal{I}_1 \cap \mathcal{I}_2) \cup (\mathcal{I}_1 \cap 2^{\bar{D}_2}) \cup (\mathcal{I}_2 \cap 2^{\bar{D}_1}) \rangle$. Let $I \in (\mathcal{I}_1 \cap \mathcal{I}_2) \cup (\mathcal{I}_1 \cap 2^{\bar{D}_2}) \cup (\mathcal{I}_2 \cap 2^{\bar{D}_1})$ be an interaction of $\mathcal{A}_1 \uplus \mathcal{A}_2$. To prove that $\mathcal{A}_1 \uplus \mathcal{A}_2$ is an architecture, we distinguish the cases:

- if $I \in \mathcal{I}_1 \cap \mathcal{I}_2$ then $I \cap D_1 \neq \emptyset$ and $I \cap D_2 \neq \emptyset$, hence $I \cap (D_1 \cup D_2) \neq \emptyset$.
- if $I \in \mathcal{I}_1 \cap 2^{\bar{D}_2}$ then $I \cap D_1 \neq \emptyset$, hence $I \cap (D_1 \cup D_2) \neq \emptyset$.
- if $I \in \mathcal{I}_2 \cap 2^{\bar{D}_1}$ then $I \cap D_2 \neq \emptyset$, hence $I \cap (D_1 \cup D_2) \neq \emptyset$.

Commutativity of \uplus follows from the symmetry of its definition. Associativity is proved by computing:

$$\begin{aligned}
(\mathcal{A}_1 \uplus \mathcal{A}_2) \uplus \mathcal{A}_3 &= \langle D_1 \cup D_2, (\mathcal{I}_1 \cap \mathcal{I}_2) \cup (\mathcal{I}_1 \cap 2^{\bar{D}_2}) \cup (\mathcal{I}_2 \cap 2^{\bar{D}_1}) \rangle \uplus \langle D_3, \mathcal{I}_3 \rangle \\
&= \langle D_1 \cup D_2 \cup D_3, (\mathcal{I}_1 \cap \mathcal{I}_2 \cap \mathcal{I}_3) \cup (\mathcal{I}_1 \cap \mathcal{I}_3 \cap 2^{\bar{D}_2}) \cup (\mathcal{I}_2 \cap \mathcal{I}_3 \cap 2^{\bar{D}_1}) \cup \\
&\quad (\mathcal{I}_1 \cap \mathcal{I}_2 \cap 2^{\bar{D}_3}) \cup (\mathcal{I}_1 \cap 2^{\bar{D}_2} \cap 2^{\bar{D}_3}) \cup (\mathcal{I}_2 \cap 2^{\bar{D}_1} \cap 2^{\bar{D}_3}) \cup \\
&\quad (\mathcal{I}_3 \cap 2^{\bar{D}_1} \cap 2^{\bar{D}_2}) \rangle \\
&= \langle D_1, \mathcal{I}_1 \rangle \uplus \langle D_2 \cup D_3, (\mathcal{I}_2 \cap \mathcal{I}_3) \cup (\mathcal{I}_2 \cap 2^{\bar{D}_3}) \cup (\mathcal{I}_3 \cap 2^{\bar{D}_2}) \rangle \\
&= \mathcal{A}_1 \uplus (\mathcal{A}_2 \uplus \mathcal{A}_3)
\end{aligned}$$

Finally, $\mathcal{A}_1 \uplus \langle \emptyset, \emptyset \rangle = \langle D_1, (\mathcal{I}_1 \cap \emptyset) \cup (\mathcal{I}_1 \cap 2^{\text{Ports}}) \cup (\emptyset \cap 2^{\bar{D}_1}) \rangle = \langle D_1, \mathcal{I}_1 \rangle$. \square

Sometimes it is convenient to define the *closure* of an architecture as the architecture obtained by removing all open interactions. Formally, we define closure by means of a transitive relation:

Definition 1. *Given architectures $\mathcal{A}_i = \langle D_i, \mathcal{I}_i \rangle$, for $i = 1, 2$, we have $\mathcal{A}_1 \triangleleft \mathcal{A}_2$ if and only if the following hold:*

1. $D_1 = D_2$ and
2. $\mathcal{I}_1 = \mathcal{I}_2^{\subseteq D_1}$.

Note that \triangleleft becomes the identity relation on *closed* architectures, i.e. architectures $\mathcal{A} = \langle D, \mathcal{I} \rangle$ with the property that $\mathcal{I} = \mathcal{I}^{\subseteq D}$.

3 Separation Logic of Interactions

We introduce a first logic to describe architectures, as defined in the previous section. Let $\text{PVars} = \{x, y, \dots\}$ be a countably infinite set of variables, ranging over ports. For each port $p \in \text{Ports}$ we consider a logical constant symbol with the same name and let PSym be the set of such constants². The *Separation Logic of Interactions* (SIL) is the

² We use the same symbol for a port and its corresponding constant symbol, with the convention that constant symbols are only used within logical formulae.

set of formulae ϕ generated by the following syntax:

$t := p \in \text{PSym} \mid x \in \text{PVars}$	port terms
$b := t \mid \bar{b}_1 \mid b_1 \cdot b_2$	boolean terms
$\phi := t_1 = t_2 \mid \text{emp} \mid t \multimap b \mid t \multimap\!\!\!\!\!\sqcap b \mid t \multimap\!\!\!\!\!\exists b \mid$	atomic propositions
$\langle \phi_1 \rangle \mid \phi_1 \wedge \phi_2 \mid \neg \phi_1 \mid \phi_1 * \phi_2 \mid \phi_1 \multimap * \phi_2 \mid \exists x \in \text{PVars} . \phi_1$	formulae

The derived connectives $\phi_1 \vee \phi_2$ and $\phi_1 \rightarrow \phi_2$ are defined as usual and we write \top (\perp) for $x = x$ ($\neg x = x$), where the choice of $x \in \text{PVars}$ is not important. The set of ports that occur in a formula ϕ is denoted as $\text{P}(\phi)$ and is defined recursively on the structure of ϕ , as usual.

To describe interactions, we use boolean terms built from port terms, connected with conjunction ($b_1 \cdot b_2$) and negation (\bar{b}). Boolean disjunction is defined as usual $b_1 + b_2 \stackrel{\text{def}}{=} \overline{\bar{b}_1 \cdot \bar{b}_2}$. Intuitively, $p \cdot q$ (written simply pq) denotes interactions in which both p and q occur, $p + q$ interactions in which p or q occurs, whereas $p\bar{q}$ denotes interactions in which p occurs, but not q , such as $\{p, r\}$. These boolean descriptors of interactions are used within atomic propositions that describe architectures with singleton domain, as illustrated by the following example.

Example 3. The atomic proposition $p \multimap qr$ describes those architectures \mathcal{A} with domain $\text{dom}(\mathcal{A}) = \{p\}$, whose interactions $I \in \text{inter}(\mathcal{A})$ contain both q and r . Moreover, by definition of architectures, p belongs to every interaction, if any. For instance $\langle \{p\}, \emptyset \rangle$, $\langle \{p\}, \{\{p, q, r\}\} \rangle$ and $\langle \{p\}, \{\{p, q, r\}, \{p, q, r, s\}\} \rangle$ are all models of $p \multimap qr$. Note that some interactions might contain ports other than p, q and r .

On the other hand, the atomic proposition $p \multimap\!\!\!\!\!\sqcap r(q + s)$ specifies those architectures whose domain is $\{p\}$ and each interaction is either $\{p, q, r\}$ or $\{p, r, s\}$, but not both: $\langle \{p\}, \emptyset \rangle$, $\langle \{p\}, \{p, q, r\} \rangle$ and $\langle \{p\}, \{p, r, s\} \rangle$. Since p belongs to every interaction, these must be *minimal* boolean models of the propositional formula $pr(q + s)$.

Finally, the atomic proposition $p \multimap\!\!\!\!\!\exists qr$ specifies those architectures whose domain is $\{p\}$ and whose interaction set contains at least one minimal model of pqr , for instance $\langle \{p\}, \{\{p, q\}, \{p, q, r\}\} \rangle$ but not $\langle \{p\}, \{\{p, q\}\} \rangle$. ■

Formally, a boolean term is interpreted over a valuation $\nu : \text{PVars} \rightarrow \text{Ports}$ and set of ports $I \subseteq \text{Ports}$, by the relation $I \vdash_\nu b$, defined recursively on the structure of b :

$$\begin{aligned}
I \vdash_\nu p &\iff p \in I \\
I \vdash_\nu x &\iff \nu(x) \in I \\
I \vdash_\nu \bar{b}_1 &\iff I \not\vdash_\nu b_1 \\
I \vdash_\nu b_1 b_2 &\iff I \vdash_\nu b_1 \text{ and } I \vdash_\nu b_2
\end{aligned}$$

We write $I \stackrel{\mu}{\vdash}_\nu b$ if and only if $I \vdash_\nu b$ and $I' \not\vdash_\nu b$ for all $I' \subsetneq I$, i.e. I is a minimal model of b , in the propositional sense. For a port term $t \in \text{PVars} \cup \text{PSym}$ and a valuation $\nu : \text{PVars} \rightarrow \text{Ports}$, we write $\nu(t)$ for $\nu(t)$ if $t \in \text{PVars}$ and t , if $t \in \text{PSym}$.

The semantics of SIL formulae is defined in terms of valuations $\nu : \text{PVars} \rightarrow \text{Ports}$ and architectures $\mathcal{A} = \langle D, \mathcal{I} \rangle$, by a satisfaction relation $\langle D, \mathcal{I} \rangle \models_\nu \phi$ defined recursively on the structure of ϕ as follows:

$$\begin{aligned}
\langle D, \mathcal{I} \rangle \models_v \text{emp} &\iff D = \emptyset \text{ and } \mathcal{I} = \emptyset \\
\langle D, \mathcal{I} \rangle \models_v t \multimap b &\iff D = \{\nu(t)\} \text{ and for all interactions } I \in \mathcal{I}, \text{ we have } I \vdash_v t \cdot b \\
\langle D, \mathcal{I} \rangle \models_v t \multimap b &\iff D = \{\nu(t)\} \text{ and for all interactions } I \in \mathcal{I}, \text{ we have } I \vdash_v^\mu t \cdot b \\
\langle D, \mathcal{I} \rangle \models_v t \overset{\exists}{\multimap} b &\iff D = \{\nu(t)\} \text{ and for some interaction } I \in \mathcal{I}, \text{ we have } I \vdash_v^\mu t \cdot b \\
\langle D, \mathcal{I} \rangle \models_v \langle \phi_1 \rangle &\iff \text{there exists } \mathcal{A}_1 \text{ such that } \mathcal{A}_1 \triangleleft \langle D, \mathcal{I} \rangle \text{ and } \mathcal{A}_1 \models_v \phi_1 \\
\langle D, \mathcal{I} \rangle \models_v \phi_1 \wedge \phi_2 &\iff \langle D, \mathcal{I} \rangle \models_v \phi_1 \text{ and } \langle D, \mathcal{I} \rangle \models_v \phi_2 \\
\langle D, \mathcal{I} \rangle \models_v \neg \phi_1 &\iff \langle D, \mathcal{I} \rangle \not\models_v \phi_1 \\
\langle D, \mathcal{I} \rangle \models_v \phi_1 * \phi_2 &\iff \text{there exist disjoint architectures } \mathcal{A}_i, \text{ such that} \\
&\quad \langle D, \mathcal{I} \rangle = \mathcal{A}_1 \uplus \mathcal{A}_2 \text{ and } \mathcal{A}_i \models_v \phi_i, \text{ for all } i = 1, 2. \\
\langle D, \mathcal{I} \rangle \models_v \phi_1 \multimap * \phi_2 &\iff \text{for each architecture } \mathcal{A}_1 \text{ disjoint from } \langle D, \mathcal{I} \rangle \text{ such that} \\
&\quad \mathcal{A}_1 \models_v \phi_1, \text{ we have } \mathcal{A}_1 \uplus \langle D, \mathcal{I} \rangle \models_v \phi_2. \\
\langle D, \mathcal{I} \rangle \models \exists x. \phi_1 &\iff \langle D, \mathcal{I} \rangle \models_{v[x \leftarrow p]} \phi_1, \text{ for some port } p \in \text{Ports}
\end{aligned}$$

Note that it is possible to define the existential counterpart of $p \multimap b$, as the derived formula $p \overset{\exists}{\multimap} b \stackrel{\text{def}}{=} p \multimap p \wedge \neg(p \multimap \bar{b})$. Since $p \multimap p$ defines those architectures with domain $\{p\}$, the meaning of $p \overset{\exists}{\multimap} b$ is the set of architectures with domain $\{p\}$ and interaction set containing at least one (not necessarily minimal) boolean model of b .

As a remark, the $\langle \cdot \rangle$ connective is the existential modality with respect to the closure relation \triangleleft between architectures. Sometimes, this connective can be used instead of $p \multimap b$ to describe closed interactions. Consider for instance the formula $\langle p \multimap q * q \overset{\exists}{\multimap} p \rangle$ whose only model is the architecture $\langle \{p, q\}, \{\{p, q\}\} \rangle$, equivalently defined by the formulae $p \multimap q * q \overset{\exists}{\multimap} p$, $p \multimap q * q \overset{\exists}{\multimap} p$ or $p \multimap q * q \overset{\exists}{\multimap} p$. However, the existential modality becomes more interesting in combination with recursive predicates (introduced next in §4), as one can use it to define closed interactions of unbounded size (Example 5).

Remark Using negation, one can also define the universal modality as $[\phi] \stackrel{\text{def}}{=} \neg \langle \neg \phi \rangle$, with the meaning "every open extension of the current architecture must be a model of ϕ ". However, we are currently not aware of any interesting property that may use the universal modality. ■

4 Component-based Architectures

The main purpose of using Separation Logic is the modeling of component-based systems consisting of finitely unbounded numbers of replicated components interacting according to a recursive pattern. We capture this aspects by the following extension of the SIL logic introduced previously in §3:

- the components are identified by the elements of an infinite countable set Id , ranged over by index variables $\text{IVars} = \{i, j, k, \dots\}$.

- the ports are associated to components via functions of type $\text{Id} \rightarrow \text{Ports}$, ranged over by the function symbols $\text{PFun} = \{\mathbf{p}, \mathbf{q}, \dots\}$. Intuitively, the term $\mathbf{p}(i)$ represents the \mathbf{p} ports of the i -th component. We formally require that $\mathbf{p}(i) = \mathbf{q}(j)$ if and only if $i = j$ and \mathbf{p} and \mathbf{q} are exactly the same function symbol.
- recursive interaction patterns are defined by means of predicate symbols $P(i_1, \dots, i_n)$, ranging over relations of type Id^n , where n is the arity of P , denoted $\#(P)$. We interpret these predicate symbols as the least solution of a system of inductive definitions, whose rules are written using a subset of the logic, defined next.

The syntax of this extended logic, called SL_a in the rest of the paper, is given below:

$$\begin{array}{ll}
t := p \in \text{PSym} \mid x \in \text{PVars} \mid \mathbf{p}(i), \mathbf{p} \in \text{PFun}, i \in \text{IVars} & \text{port terms} \\
\phi := t_1 = t_2 \mid \text{emp} \mid t \multimap b \mid t \multimap b \mid t \overset{\exists}{\multimap} b \mid P(i_1, \dots, i_{\#(P)}) & \text{atomic propositions} \\
\langle \phi_1 \rangle \mid \phi_1 \wedge \phi_2 \mid \neg \phi_1 \mid \phi_1 * \phi_2 \mid \phi_1 \multimap \phi_2 \mid \exists x \in \text{PVars} . \phi_1 \mid \exists i \in \text{IVars} . \phi_1 & \text{formulae}
\end{array}$$

The definition of boolean terms b is the same as for SIL , thus omitted.

A *rule* is a pair written as $P(i_1, \dots, i_{\#(P)}) \leftarrow \rho$, where $P(i_1, \dots, i_{\#(P)})$ is a predicate atoms and ρ , called the body of the rule, is a SL_a formula generated by the syntax:

$$\rho := i = j \mid i \neq j \mid \text{emp} \mid P(i_1, \dots, i_{\#(P)}) \mid \mathbf{p}(i) \multimap b \mid \mathbf{p}(i) \overset{\exists}{\multimap} b \mid \mathbf{p}(i) \multimap b \mid \mathbf{p}(i) \overset{\exists}{\multimap} b \mid \rho_1 * \rho_2 \mid \langle \rho_1 \rangle$$

Since this fragment of SL_a has no explicit negation, we consider $\mathbf{p}(i) \overset{\exists}{\multimap} b$ to be an atomic proposition, rather than a derived formula.

Using SL_a , a component-based system is described by the following methodology:

1. write a single predicate for each component type, which describes the local interactions of that component with its neighbourhood,
2. compose one or more component predicates in a recursive pattern, that is usually described by a single predicate.

This way of specifying architectures resembles the way in which programmers design recursive data structures (lists, trees and variations thereof), by specifying first the local links between a memory cell and its neighbours, before encapsulating this local specification into a recursively defined pattern. The following example provides some intuition, before moving on with the presentation of the formal details.

Example 4. Consider the parametric system from Figure 2, consisting of a Semaphore and a number of replicated Tasks. Each task interacts with the semaphore either by synchronizing its $t(\text{ake})$ port with the $\mathbf{p}(\text{roberen})$ port of the semaphore, or by synchronizing its $l(\text{eave})$ port with the $\mathbf{v}(\text{erhogen})$ port of the semaphore. To describe this system in SL_a , we define predicates for each of the two component types, describing their local interactions:

$$\begin{array}{l}
\text{Semaphore}(i) \leftarrow \exists j . \mathbf{p}(i) \multimap t(j) * \mathbf{v}(i) \multimap l(j) \\
\text{Task}(i, j) \leftarrow t(i) \overset{\exists}{\multimap} \mathbf{p}(j) * l(i) \overset{\exists}{\multimap} \mathbf{v}(j) \\
\text{Sys}(i, j, k) \leftarrow i = k * \text{Semaphore}(j) \\
\quad \leftarrow \exists \ell . \text{Task}(i, j) * \text{Sys}(\ell, j, k)
\end{array}$$

Intuitively, each component type (Semaphore, Task) is given the self-reference i as argument. Semaphore choses nondeterministically a Task to interact with, whereas Task

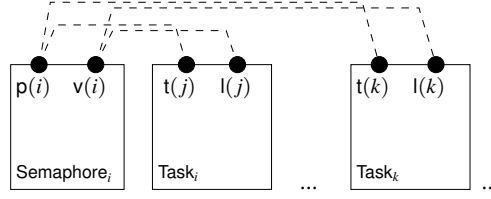


Fig. 2. Semaphore and Tasks

is given a reference j to the Semaphore it interacts with. Note that the composition between the atomic formulae $p(i) \multimap t(j)$ and $t(j) \multimap p(i)$ results in a closed interaction involving only $p(i)$ and $t(j)$ (similar for $v(i)$ and $l(j)$).

Finally, $\text{Sys}(i, j, k)$ is a recursive pattern whose arguments are understood as follows: i and k are the indices of the first and last Task in the system, whereas j is the reference to the unique Semaphore, specified by the base rule $\text{Sys}(i, j, k) \leftarrow i = k * \text{Semaphore}(j)$. The unfolding of the recursive rule $\text{Sys}(i, j, k) \leftarrow \exists \ell . \text{Task}(i, j) * \text{Sys}(\ell, j, k)$ creates arbitrarily many replicas of the component type Task. ■

The definition of the semantics for SL_a requires an *interpretation* of the predicate symbols, which is a function $\mathcal{X} : \text{Pred} \rightarrow \bigcup_{\alpha=1}^{\infty} 2^{\text{Id}^{\alpha} \times \text{Arch}}$ associating each predicate symbol $P \in \text{Pred}$ a set of pairs $\langle (k_1, \dots, k_{\#(P)}), \mathcal{A} \rangle$, where $k_1, \dots, k_{\#(P)} \in \text{Id}$ are component indices and \mathcal{A} is an architecture. Moreover, because there are two types of quantified variables in SL_a , we consider valuations $\nu : \text{PVars} \cup \text{IVars} \rightarrow \text{Ports} \cup \text{Id}$, such that $\nu(x) \in \text{Ports}$ if $x \in \text{PVars}$ and $\nu(x) \in \text{Id}$ if $x \in \text{IVars}$. The semantics of SL_a is given by a satisfaction relation $\models_{\nu}^{\mathcal{X}}$, whose definition is analogous to the one of \models_{ν} for SIL , except for the interpretation of predicate symbols, which is the following:

$$\mathcal{A} \models_{\nu}^{\mathcal{X}} P(i_1, \dots, i_{\#(P)}) \iff \langle (\nu(i_1), \dots, \nu(i_{\#(P)})), \mathcal{A} \rangle \in \mathcal{X}(P)$$

A set of rules of this form is called a *system of definitions*. From now on, we shall assume a given system of definitions Φ , that contains one or more rules for each predicate symbol used in a SL_a formula. Then a system of definitions Φ defines the following function \mathbb{X}_{Φ} on interpretations:

$$\mathbb{X}_{\Phi}(\mathcal{X}) \stackrel{\text{def}}{=} \lambda P . \{ \langle (k_1, \dots, k_{\#(P)}), \mathcal{A} \rangle \mid \mathcal{A} \models_{\nu_{[i_1 \leftarrow k_1] \dots [i_{\#(P)} \leftarrow k_{\#(P)}]}}^{\mathcal{X}} \rho, P(i_1, \dots, i_{\#(P)}) \leftarrow \rho \in \Phi \}$$

The set of interpretations, partially ordered by pointwise set inclusion, forms a complete lattice. Moreover, \mathbb{X}_{Φ} is monotone and continuous for each system Φ , thus it has a least fixed point, denoted as $\mu \mathbb{X}_{\Phi}$. In the following, we assume that the interpretation of each predicate symbol P , that occurs in a SL_a formula is the set $\mu \mathbb{X}_{\Phi}(P)$ and write \models_{ν}^{Φ} for $\models_{\nu}^{\mathcal{X}}$, whenever $\mathcal{X} = \mu \mathbb{X}_{\Phi}$.

We conclude this section with an example of a centralized controller-slaves architecture in which the interactions occur between an unbounded number of participants.

Example 5. The controller-slaves architecture in Figure 3 consists of a single interaction between the controller component and each of the slave components.

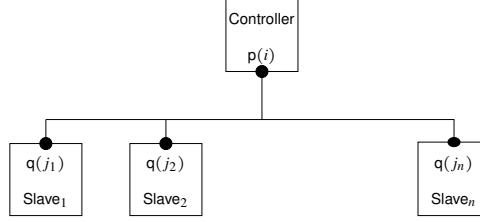


Fig. 3. Controller and Slaves

We describe such architectures using the following system of definitions:

$$\begin{aligned}
 \text{Controller}(i, j) &\leftarrow p(i) \multimap q(j) \\
 \text{Slave}(i, j) &\leftarrow q(i) \multimap p(j) \\
 \text{SysRec}(i, j, k) &\leftarrow i = k * \text{Slave}(k, j) * \text{Controller}(j, k) \\
 &\leftarrow \exists \ell . \text{Slave}(i, j) * \text{SysRec}(\ell, j, k) \\
 \text{Sys}() &\leftarrow \exists i \exists j \exists k . \langle \text{SysRec}(i, j, k) \rangle
 \end{aligned}$$

A Controller component takes the self-reference identifier i as argument and specifies only interactions involving its p port the q of a designated Slave component j . On the other hand, each Slave component i has only interactions involving the Controller, whose identifier is j . The SysRec rules create one Controller and an arbitrary number $n \geq 1$ of Slave components, whereas Sys uses the existential closure modality to ensure that the (unique, if any) interaction between the controller and the slaves is closed.

To understand why the models of Sys() are architectures with a single interaction, let us consider the following formula, describing the interactions between a controller and two slaves, obtained by applying the recursive rule for SysRec and the base rule once each:

$$\begin{aligned}
 \text{SysRec}(i, j, k) &\Rightarrow \exists \ell . \text{Slave}(i, j) * \text{SysRec}(\ell, j, k) \\
 &\Rightarrow \exists \ell . \text{Slave}(i, j) * \ell = k * \text{Slave}(k, j) * \text{Controller}(j, k) \\
 &\Rightarrow \text{Slave}(i, j) * \text{Slave}(k, j) * \text{Controller}(j, k) \\
 &\Rightarrow q(i) \multimap p(j) * q(k) \multimap p(j) * p(j) \multimap q(k)
 \end{aligned}$$

Let \mathcal{A} be a model of the above formula. Denoting $p(j) = p$, $q(i) = q$ and $q(k) = r$, we have $\text{dom}(\mathcal{A}) = \{p, q, r\}$ and $\text{inter}(\mathcal{A})$ contains at most one interaction I such that $p, q, r \in I$. Thus any model of the formula Sys() obtained by the above unfolding of the rules contains at most the closed interaction $\{p, q, r\}$. ■

5 Decidable Fragments of SIL

In order to automate checking the verification conditions expressed in SIL, or its extension SL_a , we study the decidability and computational complexity of the following decision problems:

- *satisfiability*: given a formula ϕ , is there an architecture \mathcal{A} and a valuation ν such that $\mathcal{A} \models_{\nu} \phi$?

- *entailment*: given formulae ϕ and ψ , for any architecture \mathcal{A} and valuation ν , does $\mathcal{A} \models_{\nu} \phi$ imply $\mathcal{A} \models \psi$?

Even though, in general, these problems are undecidable for SIL, in the presence of quantifiers, we identify two nontrivial quantifier-free fragments for which the problem is decidable. These fragments of SIL, denoted as SIL^+ and SIL^* , are defined by the syntax below, starting with the ϕ and ψ nonterminals, respectively:

$$\begin{aligned} \phi &:= \text{emp} \mid p \multimap b \mid p \multimap b \mid p \overset{\exists}{\multimap} b \mid p \overset{\exists}{\multimap} b \mid \langle \phi_1 \rangle \mid \phi_1 \wedge \phi_2 \mid \phi_1 \vee \phi_2 \mid \phi_1 * \phi_2 \mid \phi_1 \multimap^* \phi_2 \quad (\text{SIL}^+) \\ \psi &:= \text{emp} \mid p \multimap b \mid p \multimap b \mid p \overset{\exists}{\multimap} b \mid \psi_1 \wedge \psi_2 \mid \neg \psi_1 \mid \psi_1 * \psi_2 \quad (\text{SIL}^*) \end{aligned}$$

Note that, because SIL^+ does not have negation, we must consider the satisfiability and entailment problems separately. On the other hand, studying the satisfiability problem is sufficient for SIL^* , because of the negation connective allowing the encode entailment between ψ_1 and ψ_2 and the unsatisfiability of $\psi_1 \wedge \neg \psi_2$. The lack of negation is also the reason why we adopt the formula $p \overset{\exists}{\multimap} b$ as an atomic proposition of SIL^+ . Moreover, since there are no port variables in SIL^+ or SIL^* , we omit the valuation subscript and write $\mathcal{A} \models \phi$ instead of $\mathcal{A} \models_{\nu} \phi$, whenever ϕ is a formula of SIL^+ or SIL^* .

The proofs of decidability for SIL^+ and SIL^* follow essentially the same steps. First, we define an equivalence relation between architectures that is compatible (at least) with the (de)composition operation. Second, we define the equivalence classes of the relation using simple SIL formulae belonging to a small number of patterns, called *test formulae* and show that the equivalence relation is the same as the equivalence on a finite set of test formulae. Consequently, each formula in the given fragment of SIL is equivalent to a boolean combination of test formulae. Moreover, by considering each test formula as a propositional variable, one can transform the input formula into an equivalent QBF formula (modulo the interpretation of the propositional variables). The latter transformation yields the decidability result and a characterization of the complexity classes of the decision problems considered.

5.1 Decidability of SIL^+

We start by defining an equivalence relation on architectures. For any set of ports $P \subseteq \text{Ports}$ and a set of interactions $\mathcal{I} \subseteq 2^{\text{Ports}}$, we define the following sets of interactions:

$$\begin{aligned} \mathcal{I}^{\cap P} &\stackrel{\text{def}}{=} \{I \mid I \in \mathcal{I}, I \cap P \neq \emptyset\} & \mathcal{I}^{\subseteq P} &\stackrel{\text{def}}{=} \{I \mid I \in \mathcal{I}, I \subseteq P\} \\ \mathcal{I}^{\not\subseteq P} &\stackrel{\text{def}}{=} \mathcal{I} \setminus \mathcal{I}^{\subseteq P} & \mathcal{I} \cap P &\stackrel{\text{def}}{=} \{I \cap P \mid I \in \mathcal{I}\} \end{aligned}$$

Definition 2. Given architectures $\mathcal{A}_i = \langle D_i, \mathcal{I}_i \rangle$, for $i = 1, 2$ and a finite set of ports $P \in 2^{\text{Ports}}$, such that $D_1 \cup D_2 \subseteq P$, we have $\mathcal{A}_1 \sim_P \mathcal{A}_2$ if and only if the following hold:

1. $D_1 = D_2$,
2. $\mathcal{I}_1^{\subseteq P} = \mathcal{I}_2^{\subseteq P}$,
3. $\mathcal{I}_1^{\not\subseteq P} \cap P = \mathcal{I}_2^{\not\subseteq P} \cap P$.

Note that the relation \sim_P is defined only between architectures with domain included in P . It is easy to check that \sim_P is an equivalence relation, in this case. From now on, we shall silently assume that $\text{dom}(\mathcal{A}_1) \cup \text{dom}(\mathcal{A}_2) \subseteq P$, whenever $\mathcal{A}_1 \sim_P \mathcal{A}_2$ holds.

The next lemma shows that \sim_P is compatible with the decomposition of architectures:

Lemma 1. *Let $\mathcal{A} = \langle D, \mathcal{I} \rangle$ and $\mathcal{A}' = \langle D', \mathcal{I}' \rangle$ be architectures and $P \in 2^{\text{Ports}}$ be a set of ports such that $\mathcal{A} \sim_P \mathcal{A}'$. Then, for any two architectures $\mathcal{A}_i = \langle D_i, \mathcal{I}_i \rangle$, such that $\mathcal{A} = \mathcal{A}_1 \uplus \mathcal{A}_2$, there exist architectures $\mathcal{A}'_i = \langle D'_i, \mathcal{I}'_i \rangle$, for $i = 1, 2$ such that:*

1. $\mathcal{A}_i \sim_P \mathcal{A}'_i$, for each $i = 1, 2$ and
2. $\mathcal{A}' = \mathcal{A}'_1 \uplus \mathcal{A}'_2$.

Proof: From $\mathcal{A} = \mathcal{A}_1 \uplus \mathcal{A}_2$ we infer that:

$$D_1 \cup D_2 = D \text{ and } D_1 \cap D_2 = \emptyset \quad (\dagger)$$

$$\mathcal{I} = (\mathcal{I}_1 \cap \mathcal{I}_2) \cup (\mathcal{I}_1 \cap 2^{\overline{D_2}}) \cup (\mathcal{I}_2 \cap 2^{\overline{D_1}}) \quad (\ddagger)$$

Let $D'_i \stackrel{\text{def}}{=} D_i$ and $\mathcal{I}'_i \stackrel{\text{def}}{=} \mathcal{I}' \cap^{D_i} \cup X_i$, where $X_i \stackrel{\text{def}}{=} \mathcal{I}_i \setminus \mathcal{I}' \cap^{D_i}$, for $i = 1, 2$. We prove first that $\mathcal{A}_1 \sim_P \mathcal{A}'_1$, the proof for $\mathcal{A}_2 \sim_P \mathcal{A}'_2$ being identical. Note that $D'_1 = D_1 \subseteq P$, by the definition of D'_1 and (\dagger) . The two remaining points of Definition 2 are proved below:

(2) We compute:

$$\begin{aligned} \mathcal{I}'_1 \subseteq^P &= (\mathcal{I}' \cap^{D_1} \cup X_1) \subseteq^P = (\mathcal{I}' \cap^{D_1}) \subseteq^P \cup X_1 \subseteq^P \\ &= (\mathcal{I}' \subseteq^P) \cap^{D_1} \cup X_1 \subseteq^P, \text{ since } D_1 \subseteq P \\ &= (\mathcal{I} \subseteq^P) \cap^{D_1} \cup X_1 \subseteq^P, \text{ since } \mathcal{A} \sim_P \mathcal{A}' \\ &= (\mathcal{I} \cap^{D_1}) \subseteq^P \cup X_1 \subseteq^P, \text{ since } D_1 \subseteq P \\ &= (\mathcal{I} \cap^{D_1} \cup (\mathcal{I}_1 \setminus \mathcal{I}' \cap^{D_1})) \subseteq^P = (\mathcal{I} \cap^{D_1} \cup \mathcal{I}_1) \subseteq^P = \mathcal{I}_1 \subseteq^P, \text{ since } \mathcal{I} \cap^{D_1} \subseteq \mathcal{I}_1. \end{aligned}$$

(3) We prove that $\mathcal{I}'_1 \subseteq^P \cap P = (\mathcal{I}' \cap^{D_1}) \subseteq^P \cap P \cup X_1 \subseteq^P \cap P = \mathcal{I}_1 \subseteq^P \cap P$. “ \subseteq ” We distinguish the following cases:

- if $I = J \cup U' \in \mathcal{I}'$ such that $J \subseteq P$, $J \cap D_1 \neq \emptyset$, $U' \neq \emptyset$ and $U' \cap P = \emptyset$. Clearly every interaction $I \in (\mathcal{I}' \cap^{D_1}) \subseteq^P$ is of this form. Since $\mathcal{A} \sim_P \mathcal{A}'$, we have $(\mathcal{I} \cap^{D_1}) \subseteq^P \cap P = (\mathcal{I}' \cap^{D_1}) \subseteq^P \cap P$ and thus there exists $U \neq \emptyset$ such that $U \cap P = \emptyset$ and $J \cup U \in \mathcal{I}$. Moreover, since $J \cap D_1 \neq \emptyset$, we have that $J \cup U \in \mathcal{I}' \cap^{D_1}$ and $J \cup U \in \mathcal{I}_1$ follows, by (\ddagger) . Since $U \neq \emptyset$ and $U \cap P = \emptyset$, we obtain $J \cup U \in \mathcal{I}_1 \subseteq^P$ and thus $J \in \mathcal{I}_1 \subseteq^P \cap P$.

- else $I = J \cup U' \in X_1 = \mathcal{I}_1 \setminus \mathcal{I}' \cap^{D_1}$ then $J \in \mathcal{I}_1 \subseteq^P \cap P$ is immediate.

“ \supseteq ” Let $I = J \cup U \in \mathcal{I}_1$, such that $U \neq \emptyset$ and $U \cap P = \emptyset$. Then $I \cap D_1 \neq \emptyset$ and since $D_1 \subseteq P$, we have $J \cap D_1 \neq \emptyset$. We distinguish the following cases:

- if $J \cup U \in \mathcal{I}$, because $\mathcal{A} \sim_P \mathcal{A}'$, we have $\mathcal{I} \subseteq^P \cap P = \mathcal{I}' \subseteq^P \cap P$, thus there exists $U' \neq \emptyset$ such that $U' \cap P = \emptyset$ and $J \cup U' \in \mathcal{I}'$. Moreover, since $J \cap D_1 \neq \emptyset$, we have $J \cup U' \in \mathcal{I}' \cap^{D_1} \subseteq \mathcal{I}'_1$ and $J \in \mathcal{I}'_1 \subseteq^P \cap P$.
- else $J \cup U \notin \mathcal{I}$ then $J \cup U \in \mathcal{I}_1 \setminus \mathcal{I}' \cap^{D_1} = X_1 \subseteq \mathcal{I}'_1$ and $J \in \mathcal{I}'_1 \subseteq^P \cap P$.

Finally, we prove that $\mathcal{A}' = \mathcal{A}'_1 \uplus \mathcal{A}'_2$. We start by proving the following facts:

Fact 1 $X_1 \cap X_2 = \emptyset$

Proof:

$$\begin{aligned}
X_1 \cap X_2 &= (\mathcal{I}_1 \setminus \mathcal{I}^{\cap D_1}) \cap (\mathcal{I}_2 \setminus \mathcal{I}^{\cap D_2}) \\
&= (\mathcal{I}_1 \cap \mathcal{I}_2) \setminus (\mathcal{I}^{\cap D_1} \cup \mathcal{I}^{\cap D_2}) \\
&= (\mathcal{I}_1 \cap \mathcal{I}_2) \setminus \mathcal{I}^{\cap (D_1 \cup D_2)} \\
&= (\mathcal{I}_1 \cap \mathcal{I}_2) \setminus \mathcal{I} \\
&= \emptyset, \text{ the last step follows from } (\ddagger) \quad \square
\end{aligned}$$

Fact 2 For all $I \in X_i$, we have $I \cap D_1 \neq \emptyset$ and $I \cap D_2 \neq \emptyset$, for $i = 1, 2$.

Proof: We prove the case $i = 1$, the proof of the other case being identical. Let $I \in X_1$. Then $I \in \mathcal{I}_1$, thus $I \cap D_1 \neq \emptyset$, by the assumption that \mathcal{A}_1 is an architecture. Suppose $I \cap D_2 = \emptyset$. By (\ddagger) , we have $I \in \mathcal{I}$, thus $I \cap (D_1 \cup D_2) \neq \emptyset$. Since $I \cap D_2 = \emptyset$, we have $I \cap D_1 \neq \emptyset$, thus $I \in \mathcal{I}^{\cap D_1}$, which contradicts with $I \in X_1 = \mathcal{I}_1 \setminus \mathcal{I}^{\cap D_1}$. \square

We have:

$$\mathcal{I}' = (\mathcal{I}'^{\cap D_1} \cap \mathcal{I}'^{\cap D_2}) \cup (\mathcal{I}' \cap 2^{\overline{D_2}}) \cup (\mathcal{I}' \cap 2^{\overline{D_1}})$$

and compute, successively:

$$\begin{aligned}
\mathcal{I}'_1 \cap 2^{\overline{D_2}} &= (\mathcal{I}'^{\cap D_1} \cup X_1) \cap 2^{\overline{D_2}} \\
&= \mathcal{I}'^{\cap D_1} \cap 2^{\overline{D_2}}, \text{ by Fact 2} \\
&= \mathcal{I}' \cap 2^{\overline{D_2}}, \text{ since } \forall I \in \mathcal{I}' . I \cap (D_1 \cup D_2) \neq \emptyset
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}'_2 \cap 2^{\overline{D_1}} &= \mathcal{I}' \cap 2^{\overline{D_1}}, \text{ by a symmetric argument} \\
\mathcal{I}'_1 \cap \mathcal{I}'_2 &= (\mathcal{I}'^{\cap D_1} \cup X_1) \cap (\mathcal{I}'^{\cap D_2} \cup X_2) \\
&= (\mathcal{I}'^{\cap D_1} \cap \mathcal{I}'^{\cap D_2}) \cup (\mathcal{I}'^{\cap D_1} \cap X_2) \cup (\mathcal{I}'^{\cap D_2} \cap X_1), \text{ by Fact 1} \\
&= \mathcal{I}'^{\cap D_1} \cap \mathcal{I}'^{\cap D_2}
\end{aligned}$$

The last step follows from $\mathcal{I}'^{\cap D_1} \cap X_2 \subseteq \mathcal{I}'^{\cap D_2}$ and $\mathcal{I}'^{\cap D_2} \cap X_1 \subseteq \mathcal{I}'^{\cap D_1}$, which is proved below. Let $I \in \mathcal{I}'^{\cap D_1} \cap X_2$ (the other case is symmetric). If $I \in X_2$, we have $I \cap D_2 \neq \emptyset$, by Fact 2. Then $I \in \mathcal{I}'^{\cap D_2}$. This concludes the proof the Lemma. \square

Conversely, the next lemma shows that \sim_P is compatible with the composition of architectures:

Lemma 2. Let $\mathcal{A} = \langle D, \mathcal{I} \rangle$ and $\mathcal{A}' = \langle D', \mathcal{I}' \rangle$ be architectures and $P \in 2^{\text{Ports}}$ be a set of ports such that $\mathcal{A} \sim_P \mathcal{A}'$. Then, for any architecture $\mathcal{A}_1 = \langle D_1, \mathcal{I}_1 \rangle$ such that $D_1 \cap D = \emptyset$ and $D_1 \subseteq P$ there exists an architecture $\mathcal{A}'_1 = \langle D'_1, \mathcal{I}'_1 \rangle$ such that:

1. $\mathcal{A}_1 \sim_P \mathcal{A}'_1$ and
2. $\mathcal{A}_1 \uplus \mathcal{A} \sim_P \mathcal{A}'_1 \uplus \mathcal{A}'$.

Proof: Let $D'_1 \stackrel{\text{def}}{=} D_1$ and $\mathcal{I}'_1 \stackrel{\text{def}}{=} \mathcal{I}_1 \subseteq^P \cup X_1 \cup Y_1$, where:

$$\begin{aligned}
X_1 &\stackrel{\text{def}}{=} \{J \cup U' \mid U' \neq \emptyset, U' \cap P = \emptyset, J \in (\mathcal{I}_1 \cap \mathcal{I})^{\not\subseteq P}, J \cup U' \in \mathcal{I}'\} \\
Y_1 &\stackrel{\text{def}}{=} \{J \cup \{\alpha\} \mid J \in (\mathcal{I}_1 \setminus \mathcal{I})^{\not\subseteq P}, \forall U' . U' \neq \emptyset \wedge U' \cap P = \emptyset \Rightarrow J \cup U' \notin X_1\}
\end{aligned}$$

and $\alpha \in \text{Ports}$ is a fresh port, not occurring in either $\mathcal{A}, \mathcal{A}_1$ or \mathcal{A}' .

We prove that $\mathcal{A}_1 \sim_P \mathcal{A}'_1$. Note that $D'_1 = D_1$ by definition. The two remaining points of Definition 2 are proved below:

(2) We have $I_1^{\leq P} = I_1^{\leq P} \cup X_1^{\leq P} \cup Y_1^{\leq P} = I_1^{\leq P}$, because $X_1^{\leq P} = Y_1^{\leq P} = \emptyset$, by definition.

(3) By definition of I'_1 , we have $I_1^{\geq P} = X_1 \cup Y_1$. We have to prove that $X_1 \sqcap P \cup Y_1 \sqcap P = I_1^{\geq P} \sqcap P$. “ \subseteq ” Let $I = J \cup U'$, where $J \subseteq P$ and $U' \neq \emptyset, U' \cap P = \emptyset$, be an interaction.

We distinguish the following cases:

- if $J \cup U' \in X_1^{\geq P}$ then $J \in (I_1 \cap I)^{\geq P} \sqcap P$ and $J \in I_1^{\geq P} \sqcap P$ follows.
- else $J \cup U' \in Y_1^{\geq P}$ then $J \in (I_1 \setminus I)^{\geq P} \sqcap P$ and $J \in I_1^{\geq P} \sqcap P$ follows.

“ \supseteq ” Let $I = J \cup U \in I_1^{\geq P}$ be an interaction, such that $J \subseteq P, U \neq \emptyset$ and $U \cap P = \emptyset$.

We distinguish the following cases:

- if $J \cup U \in I$ then $J \cup U \in I_1 \cap I$. Moreover, because $\mathcal{A} \sim_P \mathcal{A}'$, we have $I^{\leq P} = I'^{\leq P}$, thus there exists $U' \neq \emptyset$ such that $U' \cap P = \emptyset$ and $J \cup U' \in I'$ and thus $J \cup U' \in X_1$, by the definition of X_1 . Consequently, we have $J \in X_1 \sqcap P$ in this case.
- else $J \cup U \notin I$, then $J \cup U \in (I_1 \setminus I)^{\geq P}$ and $J \in (I_1 \setminus I)^{\geq P} \sqcap P$. We distinguish two cases:
 - if there exists $U' \neq \emptyset$ such that $U' \cap P = \emptyset$ and $J \cup U' \in X_1$, then $J \in X_1 \sqcap P$.
 - else $J \cup U' \notin X_1$, for all $U' \neq \emptyset$ such that $U' \cap P = \emptyset$, then $J \cup \{\alpha\} \in Y_1$ and $J \in Y_1 \sqcap P$.

Finally, we prove that $\mathcal{A}_1 \uplus \mathcal{A} \sim_P \mathcal{A}'_1 \uplus \mathcal{A}'$. Note that $D_1 \cup D \subseteq P$ and $D_1 \cup D = D'_1 \cup D'$, by the definition of D'_1 and the fact that $\mathcal{A} \sim_P \mathcal{A}'$. We are left with proving the following two points of Definition 2:

(2) We have $I_1^{\leq P} = I_1^{\leq P}$, since $\mathcal{A}_1 \sim_P \mathcal{A}'_1$ and $I^{\leq P} = I'^{\leq P}$, since $\mathcal{A} \sim_P \mathcal{A}'$. Thus we obtain the following equalities:

$$\begin{aligned} (I_1 \cap I)^{\leq P} &= (I'_1 \cap I')^{\leq P} \\ (I_1 \cap 2^{\overline{D}})^{\leq P} &= (I'_1 \cap 2^{\overline{D}})^{\leq P} \\ (I \cap 2^{\overline{D}_1})^{\leq P} &= (I' \cap 2^{\overline{D}_1})^{\leq P} \end{aligned}$$

(3) We prove the following points:

- $(I_1 \cap I)^{\geq P} \sqcap P = (I'_1 \cap I')^{\geq P} \sqcap P$: “ \subseteq ” Let $I = J \cup U \in I_1 \cap I$, where $U \neq \emptyset$ and $U \cap P = \emptyset$ be an interaction. Since $\mathcal{A} \sim_P \mathcal{A}'$, we have $I^{\geq P} \sqcap P = I'^{\geq P} \sqcap P$ and thus there exists $U' \neq \emptyset$ such that $U' \cap P = \emptyset$ and $J \cup U' \in I'$. Consequently, $J \cup U' \in X_1 \subseteq I'_1$ and we obtain $J \cup U' \in I'_1 \cap I'$, thus $J \in (I'_1 \cap I')^{\geq P} \sqcap P$ follows. “ \supseteq ” Let $I = J \cup U' \in I'_1 \cap I'$, where $U' \neq \emptyset$ and $U' \cap P = \emptyset$ be an interaction. Since $J \cup U' \in I'_1$ then $J \cup U' \in X_1 \cup Y_1$ and since $J \cup U' \in I'$ it must be that $J \cup U' \in X_1$. By the definition of X_1 , we obtain $J \in (I_1 \cap I)^{\geq P} \sqcap P$.
- $(I_1 \cap 2^{\overline{D}})^{\geq P} \sqcap P = (I'_1 \cap 2^{\overline{D}})^{\geq P} \sqcap P$: Because $\mathcal{A}_1 \sim_P \mathcal{A}'_1$, we have $I_1^{\geq P} \sqcap P = I_1^{\geq P} \sqcap P$ and the result follows.
- $(I \cap 2^{\overline{D}_1})^{\geq P} \sqcap P = (I' \cap 2^{\overline{D}_1})^{\geq P} \sqcap P$: Because $\mathcal{A} \sim_P \mathcal{A}'$, we have $I^{\geq P} \sqcap P = I'^{\geq P} \sqcap P$ and the result follows. \square

Finally, we show that \sim_P is also compatible with the closure relation on architectures:

Lemma 3. *Let $\mathcal{A} = \langle D, \mathcal{I} \rangle$ and $\mathcal{A}' = \langle D', \mathcal{I}' \rangle$ be architectures and $P \in 2^{\text{Ports}}$ be a set of ports such that $\mathcal{A} \sim_P \mathcal{A}'$ and let $\mathcal{A}_1 = \langle D_1, \mathcal{I}_1 \rangle$ be an architecture such that $\mathcal{A}_1 \triangleleft \mathcal{A}$. Then $\mathcal{A}_1 \triangleleft \mathcal{A}'$ as well.*

Proof: Because $\mathcal{A}_1 \triangleleft \mathcal{A}$ we have $D_1 = D$ and $\mathcal{I}_1 = \mathcal{I}^{\subseteq D}$. Moreover, since $\mathcal{A} \sim_P \mathcal{A}'$, we have $D = D'$ and $\mathcal{I}^{\subseteq P} = \mathcal{I}'^{\subseteq P}$. Since $D \subseteq P$, we obtain $\mathcal{I}^{\subseteq D} = \mathcal{I}'^{\subseteq D} = \mathcal{I}_1$, hence $\mathcal{A}_1 \triangleleft \mathcal{A}'$. \square

The following theorem shows that \sim_P coincides with the equivalence of architectures with respect to SIL^+ formulae. The proof of the theorem requires that every model of a SIL^+ formula has only visible ports in its domain, which is proved below:

Lemma 4. *For each formula ϕ of SIL^+ and each architecture $\mathcal{A} = \langle D, \mathcal{I} \rangle$ such that $\mathcal{A} \models \phi$, we have $D \subseteq \text{P}(\phi)$.*

Proof: By induction on the structure of ϕ :

- emp: in this case $D = \emptyset$.
- $p \rightarrow b$, $p \rightarrow b$, $p \overset{\exists}{\rightarrow} b$ and $p \overset{\exists}{\rightarrow} b$: in this case $D = \{p\}$.
- $\langle \phi_1 \rangle$: in this case there exists an architecture $\mathcal{A}_1 = \langle D_1, \mathcal{I}_1 \rangle$ such that $\mathcal{A} \triangleleft \mathcal{A}_1$ and $\mathcal{A}_1 \models \phi_1$. Then $D = D_1$ and $D_1 \subseteq \text{P}(\phi_1)$, by the induction hypothesis. We conclude noticing that $\text{P}(\langle \phi_1 \rangle) = \text{P}(\phi_1)$.
- $\phi_1 \wedge \phi_2$: since $\mathcal{A} \models \phi_1$, by the induction hypothesis we have $D \subseteq \text{P}(\phi_1) \subseteq \text{P}(\phi_1 \wedge \phi_2)$.
- $\phi_1 \vee \phi_2$: if $\mathcal{A} \models \phi_1$, by the induction hypothesis we have $D \subseteq \text{P}(\phi_1) \subseteq \text{P}(\phi_1 \vee \phi_2)$. The case $\mathcal{A} \models \phi_2$ is symmetric.
- $\phi_1 * \phi_2$: in this case there exists $\mathcal{A}_i = \langle D_i, \mathcal{I}_i \rangle$ such that $\mathcal{A} = \mathcal{A}_1 \uplus \mathcal{A}_2$ and $\mathcal{A}_i \models \phi_i$, for both $i = 1, 2$. By the induction hypothesis, $D_i \subseteq \text{P}(\phi_i) \subseteq \text{P}(\phi_1 * \phi_2)$, for both $i = 1, 2$, thus $D = D_1 \cup D_2 \subseteq \text{P}(\phi_1 * \phi_2)$.
- $\phi_1 \multimap \phi_2$: let $\mathcal{A}_1 = \langle D_1, \mathcal{I}_1 \rangle$ be any architecture such that $D_1 \cap D = \emptyset$ and $\mathcal{A}_1 \models \phi_1$. Since $\mathcal{A} \models \phi_1 \multimap \phi_2$, we obtain $\mathcal{A}_1 \uplus \mathcal{A} \models \phi_2$. Again, by the induction hypothesis, $D_1 \cup D \subseteq \text{P}(\phi_2)$, hence $D \subseteq \text{P}(\phi_1 \multimap \phi_2)$ follows immediately. \square

Theorem 1. *Let $\mathcal{A} = \langle D, \mathcal{I} \rangle$ and $\mathcal{A}' = \langle D', \mathcal{I}' \rangle$ be architectures and $P \in 2^{\text{Ports}}$ be a set of ports such that $\mathcal{A} \sim_P \mathcal{A}'$. Then, for any formula ϕ of SIL^+ , such that $\text{P}(\phi) \subseteq P$, we have $\mathcal{A} \models \phi$ if and only if $\mathcal{A}' \models \phi$.*

Proof: By induction on the structure of ϕ :

- emp: if $\mathcal{A} \models \text{emp}$ then $D = \emptyset$ and $\mathcal{I} = \emptyset$. Since $\mathcal{A} \sim_P \mathcal{A}'$, we obtain $D' = D = \emptyset$, thus $\mathcal{I}' = \emptyset$ must be the case, otherwise every interaction $I \in \mathcal{I}'$ would have a non-empty intersection with D' . Consequently, $\mathcal{A}' \models \text{emp}$.

- $p \multimap b$: if $\mathcal{A} \models p \multimap b$, then $D = \{p\}$ and, since $\mathcal{A} \sim_P \mathcal{A}'$, we obtain $D' = D = \{p\}$. Let $I \in \mathcal{I}'$ be an interaction. If $I \subseteq P$ then $I \in \mathcal{I}'^{\subseteq P} = \mathcal{I}^{\subseteq P}$, because $\mathcal{A} \sim_P \mathcal{A}'$. Then $I \vdash p \cdot b$, because $\mathcal{A} \models p \multimap b$. Else, $I \not\subseteq P$ and $I \in \mathcal{I}'^{\not\subseteq P}$. Because $\mathcal{A} \sim_P \mathcal{A}'$, we obtain $I \cap P = J \cap P$, for some interaction $J \in \mathcal{I}$. Moreover, $J \vdash p \cdot b$ because $\mathcal{A} \models p \multimap b$ and, since $\mathsf{P}(p \multimap b) \subseteq P$, it must be the case that $I \vdash p \cdot b$ as well.
- $p \multimap b$: if $\mathcal{A} \models p \multimap b$, then $D = \{p\}$ and, since $\mathcal{A} \sim_P \mathcal{A}'$, we obtain $D' = D = \{p\}$. Let $I \in \mathcal{I}'$ be an interaction. The proof in the case $I \subseteq P$ is given at the point above, so we consider that $I \not\subseteq P$. Because $\mathcal{A} \sim_P \mathcal{A}'$, there exists an interaction $J \in \mathcal{I}^{\not\subseteq P}$ such that $I \cap P = J \cap P$. Moreover, since $\mathcal{A} \models p \multimap b$, we have that $J \vdash_\mu p \cdot b$. Then $J \subseteq \mathsf{P}(p \multimap b) \subseteq P$, which contradicts the fact that $J \in \mathcal{I}^{\not\subseteq P}$. Consequently, the only case possible is $I \subseteq P$, in which case $I \vdash_\mu p \cdot b$, thus $\mathcal{A}' \models p \multimap b$.
- $p \overset{\exists}{\multimap} b$: if $\mathcal{A} \models p \overset{\exists}{\multimap} b$, then $D = \{p\}$ and, since $\mathcal{A} \sim_P \mathcal{A}'$, we obtain $D' = D = \{p\}$. Moreover, there exists an interaction $I \in \mathcal{I}$ such that $I \vdash p \cdot b$. If $I \subseteq P$ then $I \in \mathcal{I}^{\subseteq P} = \mathcal{I}'^{\subseteq P}$, thus $\mathcal{A}' \models p \overset{\exists}{\multimap} b$. Else, $I \not\subseteq P$, hence $I \in \mathcal{I}^{\not\subseteq P}$. Since $\mathcal{A} \sim_P \mathcal{A}'$, there exists an interaction $J \in \mathcal{I}'^{\not\subseteq P}$ such that $J \cap P = I \cap P$. But then $J \vdash p \cdot b$, thus $\mathcal{A}' \models p \overset{\exists}{\multimap} b$.
- $p \overset{\exists}{\multimap} b$: if $\mathcal{A} \models p \overset{\exists}{\multimap} b$, then $D = \{p\}$ and, since $\mathcal{A} \sim_P \mathcal{A}'$, we obtain $D' = D = \{p\}$. Moreover, there exists an interaction $I \in \mathcal{I}$ such that $I \vdash_\mu p \cdot b$. Then $I \subseteq P$ and $I \in \mathcal{I}^{\subseteq P}$ follows. Since $\mathcal{A} \sim_P \mathcal{A}'$, we have $I \in \mathcal{I}'^{\subseteq P}$ and thus $\mathcal{A}' \models p \overset{\exists}{\multimap} b$.
- $\langle \phi_1 \rangle$: because $\mathcal{A} \models \langle \phi_1 \rangle$, there exists \mathcal{A}_1 such that $\mathcal{A} \triangleleft \mathcal{A}_1$ and $\mathcal{A}_1 \models \phi_1$. By Lemma 3, we have $\mathcal{A}' \triangleleft \mathcal{A}_1$, thus $\mathcal{A}' \models \langle \phi_1 \rangle$.
- $\phi_1 \wedge \phi_2$: since $\mathcal{A} \sim_P \mathcal{A}'$ and $\mathsf{P}(\phi_i) \subseteq \mathsf{P}(\phi_1 \wedge \phi_2) \subseteq P$, by the induction hypothesis, we obtain $\mathcal{A}' \models \phi_i$, for both $i = 1, 2$, thus $\mathcal{A}' \models \phi_1 \wedge \phi_2$.
- $\phi_1 \vee \phi_2$: assume that $\mathcal{A} \models \phi_1$, the case $\mathcal{A} \models \phi_2$ being symmetric. By the induction hypothesis, since $\mathsf{P}(\phi_1) \subseteq \mathsf{P}(\phi_1 \vee \phi_2) \subseteq P$, we obtain $\mathcal{A}' \models \phi_1$, thus $\mathcal{A}' \models \phi_1 \vee \phi_2$.
- $\phi_1 * \phi_2$: because $\mathcal{A} \models \phi_1 * \phi_2$, there exists disjoint architectures \mathcal{A}_1 and \mathcal{A}_2 such that $\mathcal{A} = \mathcal{A}_1 \uplus \mathcal{A}_2$ and $\mathcal{A}_i \models \phi_i$, for both $i = 1, 2$. Since $\mathcal{A} \sim_P \mathcal{A}'$, by Lemma 1, there exist architectures \mathcal{A}'_1 and \mathcal{A}'_2 , such that $\mathcal{A}' = \mathcal{A}'_1 \uplus \mathcal{A}'_2$ and $\mathcal{A}_i \sim_P \mathcal{A}'_i$, for both $i = 1, 2$. Since $\mathsf{P}(\phi_i) \subseteq \mathsf{P}(\phi_1 * \phi_2) \subseteq P$, by the induction hypothesis, we obtain $\mathcal{A}'_i \models \phi_i$, for both $i = 1, 2$, and consequently $\mathcal{A}' \models \phi_1 * \phi_2$.
- $\phi_1 \multimap \phi_2$: Let $\mathcal{A}'_1 = \langle D'_1, \mathcal{I}'_1 \rangle$ be any architecture such that $\mathcal{A}' \models \phi_1$ and $D'_1 \cap D' = \emptyset$. Because ϕ is in SIL^+ , by Lemma 4, we obtain $D'_1 \subseteq \mathsf{P}(\phi_1) \subseteq \mathsf{P}(\phi_1 \multimap \phi_2) \subseteq P$. By Lemma 2, there exists an architecture \mathcal{A}_1 such that $\mathcal{A}'_1 \sim_P \mathcal{A}_1$ and $\mathcal{A}_1 \uplus \mathcal{A} \sim_P \mathcal{A}'_1 \uplus \mathcal{A}'$. By the induction hypothesis, we have $\mathcal{A}_1 \models \phi_1$ and, since $\mathcal{A} \models \phi_1 \multimap \phi_2$

ϕ_2 , we obtain $\mathcal{A}_1 \uplus \mathcal{A} \models \phi_2$. Again, by the induction hypothesis, we obtain that $\mathcal{A}'_1 \uplus \mathcal{A}' \models \phi_2$, thus $\mathcal{A}' \models \phi_1 \multimap \phi_2$. \square

The rest of this section is concerned with the translation of any SIL^+ formula into an equivalent boolean combination of SIL formulae that are instances of a restricted set of patterns, called *test formulae*. As a remark, the test formulae are not SIL^+ formulae, as they contain quantification, negation and equality atoms. However, these constructs occur in a strictly controlled context and will not be used outside test formulae.

Definition 3. Given a set $P \subseteq \text{PSym}$ of port symbols, $p \in P$ and b a boolean term over the vocabulary P , the following are called test formulae:

$$\begin{aligned} \text{has}(p) &\stackrel{\text{def}}{=} p \multimap p \multimap \perp \\ p \xrightarrow{\exists} \circ b &\stackrel{\text{def}}{=} \exists x. \bigwedge_{q \in P} x \neq q \wedge p \multimap x \cdot b \multimap \top \\ p \xrightarrow{\exists} \square b &\stackrel{\text{def}}{=} p \multimap \square b \multimap \top \end{aligned}$$

Let $\text{TestForm}(P)$ be the set of test formulae ϕ such that $\text{P}(\phi) \subseteq P$. Given architectures \mathcal{A}_1 and \mathcal{A}_2 , we write $\mathcal{A}_1 \simeq_P \mathcal{A}_2$ for $\mathcal{A}_1 \models \phi \iff \mathcal{A}_2 \models \phi$, for any $\phi \in \text{TestForm}(P)$.

Intuitively, the test formulae $\text{has}(p)$ are true in an architecture \mathcal{A} whenever $p \in \text{dom}(\mathcal{A})$. The test formulae $p \xrightarrow{\exists} \circ b$ (resp. $p \xrightarrow{\exists} \square b$) are true in \mathcal{A} whenever $\text{inter}(\mathcal{A})$ contains an interaction I such that $I \vdash b$ and I is a non-minimal (resp. minimal) model of b . The following lemma states these properties formally:

Lemma 5. Given an architecture $\mathcal{A} = \langle D, I \rangle$, a set of port symbols $P \subseteq \text{PSym}$ and port symbol $p \in \text{PSym}$ and a boolean term b over the vocabulary PSym , the following hold:

1. $\mathcal{A} \models \text{has}(p) \iff p \in D$,
2. $\mathcal{A} \models p \xrightarrow{\exists} \circ b \iff$ there exists $I \in \mathcal{I}$ such that $I \vdash b$ and $I \not\vdash_\mu b$,
3. $\mathcal{A} \models p \xrightarrow{\exists} \square b \iff$ there exists $I \in \mathcal{I}$ such that $I \vdash_\mu b$.

Proof: (1) “ \implies ” If $\mathcal{A} \models \text{has}(p)$ then for no disjoint architecture \mathcal{A}_1 such that $\mathcal{A}_1 \models p \multimap p$, the composition $\mathcal{A}_1 \uplus \mathcal{A}$ is defined. Since $\text{dom}(\mathcal{A}_1) = \{p\}$, the only reason for \mathcal{A}_1 and \mathcal{A} not being disjoint is $p \in \text{dom}(\mathcal{A})$. “ \impliedby ” $p \in \text{dom}(\mathcal{A})$ means that any architecture \mathcal{A}_1 such that $\text{dom}(\mathcal{A}_1) = \{p\}$ cannot compose with \mathcal{A} , thus $\mathcal{A} \models \text{has}(p)$.

(2) For all valuations $\nu : \text{Var} \rightarrow \text{Ports}$, we have $\mathcal{A} \models_\nu p \xrightarrow{\exists} \circ b \iff \mathcal{A} \models p \multimap x \cdot b \multimap \top$, for some port $\nu(x)$ that is distinct from all $q \in P$. Since $I \vdash_\nu x \cdot b$, by induction on the structure of b , one shows that $I \vdash b$. Moreover, $I \not\vdash_\mu b$, because $I \setminus \{\nu(x)\} \subsetneq I$ and $I \setminus \{\nu(x)\} \vdash b$.

(3) Immediate, by the semantics of $p \xrightarrow{\exists} \square b$. \square

Clearly, \simeq_P is an equivalence relation between architectures. Below we show that \simeq_P is at least as fine as \sim_P :

Lemma 6. Given a set of ports $P \in 2^{\text{Ports}}$ and two architectures $\mathcal{A}_i = \langle D_i, I_i \rangle$, such that $D_i \subseteq P$, for each $i = 1, 2$, we have $\mathcal{A}_1 \sim_P \mathcal{A}_2$ if $\mathcal{A}_1 \simeq_P \mathcal{A}_2$.

Proof: Assume that $\mathcal{A}_1 \models \phi \iff \mathcal{A}_2 \models \phi$ for each $\phi \in \text{TestForm}(P)$ and prove the three points of Definition 2:

(1) Suppose, for a contradiction, that $D_1 \not\subseteq D_2$ and let $p \in D_1 \setminus D_2$ be a port. Then, by Lemma 5 (1), we have $\mathcal{A}_1 \models \text{has}(p)$ and $\mathcal{A}_2 \not\models \text{has}(p)$, which contradicts that $\mathcal{A}_1 \simeq_P \mathcal{A}_2$, since $p \in D_1 \subseteq P$ and, consequently, $\text{has}(p) \in \text{TestForm}(P)$. Then $D_1 \subseteq D_2$ and the proof for the other direction is symmetric.

(2) Suppose, for a contradiction, that $\mathcal{I}_1^{\subseteq P} \not\subseteq \mathcal{I}_2^{\subseteq P}$ and let $I \in \mathcal{I}_1^{\subseteq P} \setminus \mathcal{I}_2^{\subseteq P}$ be an interaction. Because $I \in \mathcal{I}_1^{\subseteq P}$, we have $I \cap D_1 \neq \emptyset$ and $I \subseteq P$. Let $p \in I \cap D_1$ be a port and let $\{q_1, \dots, q_k\} \stackrel{\text{def}}{=} I \setminus \{p\}$. Then $p, q_1, \dots, q_k \in P$, consequently $\mathbf{P}(p \xrightarrow{\exists} q_1 \dots q_k) \subseteq P$ and thus $p \xrightarrow{\exists} q_1 \dots q_k \in \text{TestForm}(P)$. Then, by Lemma 5 (3), we have $\mathcal{A}_1 \models p \xrightarrow{\exists} q_1 \dots q_k$ and $\mathcal{A}_2 \not\models p \xrightarrow{\exists} q_1 \dots q_k$, which contradicts with $\mathcal{A}_1 \simeq_P \mathcal{A}_2$. Then $\mathcal{I}_1^{\subseteq P} \subseteq \mathcal{I}_2^{\subseteq P}$. The proof for the other direction is symmetric.

(3) Suppose, for a contradiction, that $\mathcal{I}_1^{\subseteq P} \cap P \not\subseteq \mathcal{I}_2^{\subseteq P} \cap P$ and let $I \in (\mathcal{I}_1^{\subseteq P} \cap P) \setminus (\mathcal{I}_2^{\subseteq P} \cap P)$ be an interaction. Then there exists an interaction $J \in \mathcal{I}_1^{\subseteq P}$ such that $J \setminus P \neq \emptyset$, $I = J \cap P$. Since \mathcal{A}_1 is an architecture, $J \cap D_1 \neq \emptyset$ and let $p \in D_1 \cap J$ be a port. Because $D_1 \subseteq P$, we have $p \in P$ and thus $p \in D_1 \cap I$. Let $\{q_1, \dots, q_k\} \stackrel{\text{def}}{=} I \setminus \{p\}$ and $\{r_1, \dots, r_m\} \stackrel{\text{def}}{=} P \setminus \{p, q_1, \dots, q_k\}$. By Lemma 5 (2), we have $\mathcal{A}_1 \models p \xrightarrow{\exists \circ} q_1 \dots q_k \bar{r}_1 \dots \bar{r}_m$. Suppose, for a contradiction, that $\mathcal{A}_2 \not\models p \xrightarrow{\exists \circ} q_1 \dots q_k \bar{r}_1 \dots \bar{r}_m$. Then, by Lemma 5 (2), there exists an interaction $J \in \mathcal{I}_2^{\subseteq P}$ such that $J \cap P = I \cap P$, which contradicts with $I \notin \mathcal{I}_2^{\subseteq P} \cap P$. \square

The expressive completeness result of this section is stated below:

Corollary 1. *Each formula ϕ of SIL^+ is equivalent to a finite boolean combination of test formulae from $\text{TestForm}(\mathbf{P}(\phi))$.*

Proof: Let \mathcal{A} be a models of ϕ and define the formula:

$$\Phi(\mathcal{A}) \stackrel{\text{def}}{=} \bigwedge_{\substack{\phi \in \text{TestForm}(\mathbf{P}(\phi)) \\ \mathcal{A} \models \phi}} \phi \wedge \bigwedge_{\substack{\phi \in \text{TestForm}(\mathbf{P}(\phi)) \\ \mathcal{A} \not\models \phi}} \neg \phi$$

Since $\text{TestForm}(\mathbf{P}(\phi))$ is finite, there are finitely many such formulae. In the following, we prove the equivalence $\varphi \equiv \bigvee_{\mathcal{A} \models \varphi} \Phi(\mathcal{A})$. “ \Rightarrow ” Let $\mathcal{A} \models \varphi$ be an architecture. Then clearly $\mathcal{A} \models \Phi(\mathcal{A})$ by the definition of $\Phi(\mathcal{A})$, as a conjunction of formulae ψ , such that $\mathcal{A} \models \psi$. “ \Leftarrow ” Let $\mathcal{A} \models \Phi(\mathcal{A}')$, for some $\mathcal{A}' \models \varphi$. Then $\mathcal{A} \simeq_{\mathbf{P}(\varphi)} \mathcal{A}'$, by the definition of $\Phi(\mathcal{A})$. By Lemma 6, we obtain $\mathcal{A} \sim_{\mathbf{P}(\varphi)} \mathcal{A}'$ and, since $\mathcal{A}' \models \varphi$, by Theorem 1, we have $\mathcal{A} \models \varphi$. \square

Translation of SIL^+ into QBF Our decision procedure for SIL^+ is based on an equivalence-preserving translation in QBF, which enables the use of off-the-shelf QSAT solvers to

decide the satisfiability and entailment problem for SIL^+ . Moreover, since any QBF formula is a succinct encoding of a propositional formula, we obtain a finite representation of the set of models of a SIL^+ formula, that will become useful in designing a verification method for the safety properties of a system described by recursive predicates (§6).

From now until the end of this section, let $P = \{p_1, \dots, p_k\}$ be a set of visible ports and denote by \mathbf{B} the following set of boolean variables, parameterized by P :

- $\mathbf{h}(i)$ stands for the test formulae $\text{has}(p_i)$, for all $1 \leq i \leq k$,
- $\mathbf{o}(i_1, \dots, i_\ell)$, for all $1 \leq i_1 < \dots < i_\ell \leq k$, stands for the following boolean combination of test formulae: $\bigvee_{1 \leq h \leq \ell} \text{has}(p_{i_h}) \wedge p_{i_h} \xrightarrow{\exists} \circ p_{i_1} \dots p_{i_\ell}$.
- $\mathbf{c}(i_1, \dots, i_\ell)$, for all $1 \leq i_1 < \dots < i_\ell \leq k$, stands for the following boolean combination of test formulae: $\bigvee_{1 \leq h \leq \ell} \text{has}(p_{i_h}) \wedge p_{i_h} \xrightarrow{\exists} \square p_{i_1} \dots p_{i_\ell}$.

Clearly there are $2^{O(k)}$ boolean variables in \mathbf{B} . In the following, we use the shorthands $\mathbf{B}' \stackrel{\text{def}}{=} \{b' \mid b \in \mathbf{B}\}$, $\mathbf{B}'' \stackrel{\text{def}}{=} \{b'' \mid b \in \mathbf{B}\}$ and $\exists \mathbf{B} . \phi$ (resp. $\exists \mathbf{B}'$ and $\exists \mathbf{B}''$) for the formula obtained from ϕ by existentially quantifying every boolean variable from \mathbf{B} (resp. \mathbf{B}' and \mathbf{B}''). We write \bar{i} (resp. \bar{j}) for the strictly increasing sequence $i_1 < \dots < i_\ell$ (resp. $j_1 < \dots < j_\ell$). Since there are at most 2^k such sequences, we obtain that $\|\mathbf{B}\| = \|\mathbf{B}'\| = \|\mathbf{B}''\| = 2^{O(k)}$.

Before giving the translation of an arbitrary formula of SIL^+ into QBF, we need to introduce a number of shorthands. First, the boolean formula below characterizes those boolean valuations of \mathbf{B} that define valid architectures:

$$\mathcal{A}(\mathbf{B}) \stackrel{\text{def}}{=} \bigwedge_{1 \leq \bar{i} \leq k} (\mathbf{o}(\bar{i}) \vee \mathbf{c}(\bar{i})) \rightarrow \bigvee_{1 \leq h \leq \ell} \mathbf{h}(i_h)$$

More precisely, if $\beta : \mathbf{B} \rightarrow \{\perp, \top\}$ is a boolean valuation, such that $\beta \models \mathcal{A}(\mathbf{B})$, the architectures corresponding to β are the members of the set $\mathbb{A}(\beta)$, defined below:

Definition 4. For any architecture $\mathcal{A} = \langle D, \mathcal{I} \rangle$ and any boolean valuation $\beta : \mathbf{B} \rightarrow \{\perp, \top\}$, we have $\mathcal{A} \in \mathbb{A}(\beta)$ if and only if the following hold:

- $D = \{p_i \mid \beta(\mathbf{h}(i)) = \top\}$,
- $\mathcal{I}^{\circ P} = \{\{p_{i_1}, \dots, p_{i_\ell}\} \mid \beta(\mathbf{c}(i_1, \dots, i_\ell)) = \top\}$,
- $\mathcal{I}^{\square P} \cap P = \{\{p_{i_1}, \dots, p_{i_\ell}\} \mid \beta(\mathbf{o}(i_1, \dots, i_\ell)) = \top\}$.

It is not hard to prove that the set of boolean valuations $\{\beta : \mathbf{B} \rightarrow \{\perp, \top\} \mid \mathcal{A} \in \mathbb{A}(\beta)\}$ is closed under intersection and has a minimal element, denoted by $\beta_{\mathcal{A}}$ in the following.

The formula $\#(\mathbf{B}, \mathbf{B}')$ below states that the architectures defined by the boolean sets \mathbf{B} and \mathbf{B}' have disjoint domains:

$$\#(\mathbf{B}, \mathbf{B}') \stackrel{\text{def}}{=} \bigwedge_{1 \leq i \leq k} \neg(\mathbf{h}'(i) \wedge \mathbf{h}''(i))$$

Lemma 7. For any valuations $\beta : \mathbf{B} \rightarrow \{\perp, \top\}$ and $\beta' : \mathbf{B}' \rightarrow \{\perp, \top\}$, such that $\beta \cup \beta' \models \#(\mathbf{B}, \mathbf{B}')$ and any architectures $\langle D, \mathcal{I} \rangle \in \mathbb{A}(\beta)$ and $\langle D', \mathcal{I}' \rangle \in \mathbb{A}(\beta')$, we have $D \cap D' = \emptyset$.

Proof: Since $\mathcal{A} \in \mathbb{A}(\beta)$, we have $D = \{p \mid \beta(\mathbf{h}(j)) = \top\}$ and $D' = \{p \mid \beta(\mathbf{h}'(j)) = \top\}$, by a similar argument for $\mathbb{A}(\beta')$. Suppose, for a contradiction, that there exists a port $p_j \in D \cap D'$. Then $\beta(\mathbf{h}(j)) = \beta'(\mathbf{h}'(j)) = \top$, contradicting $\beta \cup \beta' \models \#(\mathbf{B}, \mathbf{B}')$. \square

The formula $\uplus(\mathbf{B}, \mathbf{B}', \mathbf{B}'')$ below states that, whenever \mathbf{B}' and \mathbf{B}'' define disjoint architectures \mathcal{A}' and \mathcal{A}'' , \mathbf{B} defines their composition $\mathcal{A}' \uplus \mathcal{A}''$:

$$\begin{aligned} \uplus(\mathbf{B}, \mathbf{B}', \mathbf{B}'') &\stackrel{\text{def}}{=} \bigwedge_{1 \leq i \leq k} (\mathbf{h}(i) \leftrightarrow (\mathbf{h}'(i) \vee \mathbf{h}''(i))) \\ &\quad \wedge \bigwedge_{1 \leq i \leq k} (\mathbf{c}'(\bar{i}) \wedge \mathbf{c}''(\bar{i}) \rightarrow \mathbf{c}(\bar{i})) \\ &\quad \wedge \bigwedge_{1 \leq i \leq k} (\mathbf{c}'(\bar{i}) \wedge \bigwedge_{1 \leq h \leq \ell} \neg \mathbf{h}''(i_h) \rightarrow \mathbf{c}(\bar{i})) \\ &\quad \wedge \bigwedge_{1 \leq i \leq k} (\mathbf{c}''(\bar{i}) \wedge \bigwedge_{1 \leq h \leq \ell} \neg \mathbf{h}'(i_h) \rightarrow \mathbf{c}(\bar{i})) \\ &\quad \wedge \bigwedge_{1 \leq i \leq k} (\mathbf{o}'(\bar{i}) \wedge \bigwedge_{1 \leq h \leq \ell} \neg \mathbf{h}''(i_h) \rightarrow \mathbf{o}(\bar{i})) \\ &\quad \wedge \bigwedge_{1 \leq i \leq k} (\mathbf{o}''(\bar{i}) \wedge \bigwedge_{1 \leq h \leq \ell} \neg \mathbf{h}'(i_h) \rightarrow \mathbf{o}(\bar{i})) \\ &\quad \wedge \bigwedge_{1 \leq i \leq k} (\neg \mathbf{c}'(\bar{i}) \wedge \neg \mathbf{c}''(\bar{i}) \rightarrow \neg \mathbf{c}(\bar{i})) \\ &\quad \wedge \bigwedge_{1 \leq i \leq k} (\mathbf{c}'(\bar{i}) \wedge \neg \mathbf{c}''(\bar{i}) \wedge \bigvee_{1 \leq h \leq \ell} \mathbf{h}''(i_h) \rightarrow \neg \mathbf{c}(\bar{i})) \\ &\quad \wedge \bigwedge_{1 \leq i \leq k} (\mathbf{c}''(\bar{i}) \wedge \neg \mathbf{c}'(\bar{i}) \wedge \bigvee_{1 \leq h \leq \ell} \mathbf{h}'(i_h) \rightarrow \neg \mathbf{c}(\bar{i})) \\ &\quad \wedge \bigwedge_{1 \leq i \leq k} (\neg \mathbf{o}'(\bar{i}) \wedge \neg \mathbf{o}''(\bar{i}) \rightarrow \neg \mathbf{o}(\bar{i})) \\ &\quad \wedge \bigwedge_{1 \leq i \leq k} (\mathbf{o}'(\bar{i}) \wedge \neg \mathbf{o}''(\bar{i}) \wedge \bigvee_{1 \leq h \leq \ell} \mathbf{h}''(i_h) \rightarrow \neg \mathbf{o}(\bar{i})) \\ &\quad \wedge \bigwedge_{1 \leq i \leq k} (\mathbf{o}''(\bar{i}) \wedge \neg \mathbf{o}'(\bar{i}) \wedge \bigvee_{1 \leq h \leq \ell} \mathbf{h}'(i_h) \rightarrow \neg \mathbf{o}(\bar{i})) \end{aligned}$$

Note that nothing can be stated about $\mathbf{o}(\bar{i})$ when $\mathbf{o}'(\bar{i})$ and $\mathbf{o}''(\bar{i})$ both hold, because these boolean variables denote interactions that coincide on their visible part, whereas $\mathbf{o}(\bar{i})$ holds only when those interactions coincide also on their invisible parts.

Lemma 8. *For any valuations $\beta' : \mathbf{B}' \rightarrow \{\perp, \top\}$ and $\beta'' : \mathbf{B}'' \rightarrow \{\perp, \top\}$, such that $\beta' \cup \beta'' \models \#(\mathbf{B}', \mathbf{B}'')$, the following hold:*

1. *for any architectures $\mathcal{A}' \in \mathbb{A}(\beta')$ and $\mathcal{A}'' \in \mathbb{A}(\beta'')$, there exists a valuation $\beta : \mathbf{B} \rightarrow \{\perp, \top\}$ such that $\beta \cup \beta' \cup \beta'' \models \uplus(\mathbf{B}, \mathbf{B}', \mathbf{B}'')$ and $\mathcal{A}' \uplus \mathcal{A}'' \in \mathbb{A}(\beta)$,*
2. *for any valuation $\beta : \mathbf{B} \rightarrow \{\perp, \top\}$ such that $\beta \cup \beta' \cup \beta'' \models \uplus(\mathbf{B}, \mathbf{B}', \mathbf{B}'')$ and any $\mathcal{A} \in \mathbb{A}(\beta)$ there exist $\mathcal{A}' \in \mathbb{A}(\beta')$ and $\mathcal{A}'' \in \mathbb{A}(\beta'')$, such that $\mathcal{A} = \mathcal{A}' \uplus \mathcal{A}''$.*

Proof: (1) Let $\mathcal{A}' = \langle D', I' \rangle$, $\mathcal{A}'' = \langle D'', I'' \rangle$ and $\mathcal{A}' \uplus \mathcal{A}'' \stackrel{\text{def}}{=} \langle D, I \rangle$. Let $\beta = \beta_{\mathcal{A}' \uplus \mathcal{A}''}$. Clearly, we have $\mathcal{A}' \uplus \mathcal{A}'' \in \mathbb{A}(\beta)$. It remains to show that $\beta \cup \beta' \cup \beta'' \models \uplus(\mathbf{B}, \mathbf{B}', \mathbf{B}'')$, by proving that each of the implications from the definition of $\uplus(\mathbf{B}, \mathbf{B}', \mathbf{B}'')$ is valid. We prove the most interesting cases below and leave the rest to the reader:

- $\bigwedge_{1 \leq i \leq k} (\mathbf{h}(i) \leftrightarrow (\mathbf{h}'(i) \vee \mathbf{h}''(i)))$: Because $\mathcal{A}' \in \mathbb{A}(\beta')$, we have $D' = \{p_i \mid \beta'(\mathbf{h}'(i)) = \top\}$ and, because $\mathcal{A}'' \in \mathbb{A}(\beta'')$, we have $D'' = \{p_i \mid \beta''(\mathbf{h}''(i)) = \top\}$. Because $D' \cup D'' = \{p_i \mid \beta(\mathbf{h}(i)) = \top\}$, by the definition of β , we have $\beta \cup \beta' \cup \beta'' \models \mathbf{h}(i) \leftrightarrow (\mathbf{h}'(i) \vee \mathbf{h}''(i))$, for each $1 \leq i \leq k$.
- $\bigwedge_{1 \leq i \leq k} (\mathbf{c}'(\bar{i}) \wedge \mathbf{c}''(\bar{i}) \rightarrow \mathbf{c}(\bar{i}))$: Because $\mathcal{A}' \in \mathbb{A}(\beta')$, we have $I'^{\subseteq P} = \{p(i_1, \dots, i_\ell) \mid \beta'(c'(i_1, \dots, i_\ell)) = \top\}$ and, because $\mathcal{A}'' \in \mathbb{A}(\beta'')$, we have $I''^{\subseteq P} = \{p(i_1, \dots, i_\ell) \mid \beta''(c''(i_1, \dots, i_\ell)) = \top\}$.

$\beta''(\mathbf{c}''(i_1, \dots, i_\ell)) = \top\}$. Assume that, for some $1 \leq i_1, \dots, i_\ell \leq k$, we have $\beta' \models \mathbf{c}'(i_1, \dots, i_\ell)$ and $\beta'' \models \mathbf{c}''(i_1, \dots, i_\ell)$. Then $\{p_{i_1}, \dots, p_{i_\ell}\} \in (\mathcal{I}' \cap \mathcal{I}'')^{\subseteq P}$ and $\beta \models \mathbf{c}(i_1, \dots, i_\ell)$ follows, by the definition of β .

- $\bigwedge_{1 \leq i \leq k} (\mathbf{c}'(\bar{i}) \wedge \bigwedge_{1 \leq h \leq \ell} \neg \mathbf{h}''(i_h) \rightarrow \mathbf{c}(\bar{i}))$: Because $\mathcal{A}' \in \mathbb{A}(\beta')$, we have $\mathcal{I}'^{\subseteq P} = \{p(i_1, \dots, i_\ell) \mid \beta'(\mathbf{c}'(i_1, \dots, i_\ell)) = \top\}$ and, because $\mathcal{A}'' \in \mathbb{A}(\beta'')$, we have $\mathcal{D}'' = \{p_i \mid \beta''(\mathbf{h}''(i)) = \top\}$. Assume that, for some $1 \leq i_1, \dots, i_\ell \leq k$, we have $\beta' \models \mathbf{c}'(i_1, \dots, i_\ell)$ and $\beta'' \not\models \mathbf{h}''(i_1, \dots, i_\ell)$. Then, $\{p_{i_1}, \dots, p_{i_\ell}\} \in \mathcal{I}'^{\subseteq P}$ and $\{p_{i_1}, \dots, p_{i_\ell}\} \cap \mathcal{D}'' = \emptyset$, i.e. $\{p_{i_1}, \dots, p_{i_\ell}\} \in (\mathcal{I}' \cap 2^{\overline{\mathcal{D}''}})^{\subseteq P}$. By the definition of β , we have $\beta \models \mathbf{c}(i_1, \dots, i_\ell)$.
- $\bigwedge_{1 \leq i \leq k} (\mathbf{o}'(\bar{i}) \wedge \neg \mathbf{o}''(\bar{i}) \wedge \bigvee_{1 \leq h \leq \ell} \mathbf{h}''(i_h) \rightarrow \neg \mathbf{o}(\bar{i}))$: Because $\mathcal{A}' \in \mathbb{A}(\beta')$, we have $\mathcal{I}'^{\subseteq P} \cap P = \{p(i_1, \dots, i_\ell) \mid \beta'(\mathbf{o}'(i_1, \dots, i_\ell)) = \top\}$ and, because $\mathcal{A}'' \in \mathbb{A}(\beta'')$, we have $\mathcal{D}'' = \{p_i \mid \beta''(\mathbf{h}''(i)) = \top\}$ and $\mathcal{I}''^{\subseteq P} \cap P = \{p(i_1, \dots, i_\ell) \mid \beta''(\mathbf{o}''(i_1, \dots, i_\ell)) = \top\}$. Assume that, for some $1 \leq i_1, \dots, i_\ell \leq k$, we have $\beta' \cup \beta'' \models \mathbf{o}'(i_1, \dots, i_\ell) \wedge \neg \mathbf{o}''(i_1, \dots, i_\ell) \wedge \bigvee_{1 \leq h \leq \ell} \mathbf{h}''(i_h)$. Then, by the definition of β , we have $\beta \not\models \mathbf{o}(i_1, \dots, i_\ell)$, because $\{p_{i_1}, \dots, p_{i_\ell}\} \notin (\mathcal{I}' \cap \mathcal{I}'')^{\subseteq P} \cap P \cup (\mathcal{I}' \cap 2^{\overline{\mathcal{D}''}})^{\subseteq P} \cap P \cup (\mathcal{I}'' \cap 2^{\overline{\mathcal{D}'}})^{\subseteq P} \cap P$, which is an easy check.

(2) Let $\mathcal{A} = \langle D, \mathcal{I} \rangle$ be a given architecture and $\mathcal{A}' = \langle D', \mathcal{I}' \rangle$, $\mathcal{A}'' = \langle D'', \mathcal{I}'' \rangle$, be architectures defined as follows:

$$\begin{aligned} D' &\stackrel{\text{def}}{=} \{p_i \mid \beta'(\mathbf{h}'(i)) = \top\} \\ D'' &\stackrel{\text{def}}{=} \{p_i \mid \beta''(\mathbf{h}''(i)) = \top\} \\ \mathcal{I}' &\stackrel{\text{def}}{=} \{\{i_1, \dots, i_\ell\} \mid \beta'(\mathbf{c}'(i_1, \dots, i_\ell)) = \top\} \cup X' \\ \mathcal{I}'' &\stackrel{\text{def}}{=} \{\{i_1, \dots, i_\ell\} \mid \beta''(\mathbf{c}''(i_1, \dots, i_\ell)) = \top\} \cup X'' \end{aligned}$$

where X' and X'' are defined below, for two distinct ports $\alpha', \alpha'' \in \text{Ports} \setminus \bigcup \mathcal{I}$:

- for all $1 \leq i_1, \dots, i_\ell \leq k$, such that $\beta'(\mathbf{o}'(i_1, \dots, i_\ell)) = \top$, if $\{i_1, \dots, i_\ell\} \cup Y \in \mathcal{I}$, for some $Y \subseteq \text{Ports} \setminus P$, we have $\{i_1, \dots, i_\ell\} \cup Y \in \mathcal{I}'$, else $\{i_1, \dots, i_\ell\} \cup \{\alpha'\} \in \mathcal{I}'$ and, moreover, nothing else is in \mathcal{I}' .
- for all $1 \leq i_1, \dots, i_\ell \leq k$, such that $\beta''(\mathbf{o}''(i_1, \dots, i_\ell)) = \top$, if $\{i_1, \dots, i_\ell\} \cup Y \in \mathcal{I}$, for some $Y \subseteq \text{Ports} \setminus P$, we have $\{i_1, \dots, i_\ell\} \cup Y \in \mathcal{I}''$, else $\{i_1, \dots, i_\ell\} \cup \{\alpha''\} \in \mathcal{I}''$ and, moreover, nothing else is in \mathcal{I}'' .

It is easy to check that $\mathcal{I}'^{\subseteq P} \cap P = \{\{i_1, \dots, i_\ell\} \mid \beta'(\mathbf{o}'(i_1, \dots, i_\ell)) = \top\}$ and $\mathcal{I}''^{\subseteq P} \cap P = \{\{i_1, \dots, i_\ell\} \mid \beta''(\mathbf{o}''(i_1, \dots, i_\ell)) = \top\}$. Consequently, $\mathcal{A}' \in \mathbb{A}(\beta')$ and $\mathcal{A}'' \in \mathbb{A}(\beta'')$ is again an easy check. We are left with proving that $\mathcal{A} = \mathcal{A}' \uplus \mathcal{A}''$. First, we compute:

$$\begin{aligned} D &= \{p_i \mid \beta(\mathbf{h}(i)) = \top\} && , \text{ because } \mathcal{A} \in \mathbb{A}(\beta) \\ &= \{p_i \mid \beta'(\mathbf{h}'(i)) = \top \vee \beta''(\mathbf{h}''(i)) = \top\} && , \text{ because } \beta \cup \beta' \cup \beta'' \models \mathbf{h}(i) \leftrightarrow (\mathbf{h}'(i) \vee \mathbf{h}''(i)) \\ &= \{p_i \mid \beta'(\mathbf{h}'(i)) = \top\} \cup \{p_i \mid \beta''(\mathbf{h}''(i)) = \top\} \\ &= D' \cup D'' && , \text{ by the definitions of } D' \text{ and } D'' \end{aligned}$$

To show that $\mathcal{I} = (\mathcal{I}' \cap \mathcal{I}'') \cup (\mathcal{I}' \cap 2^{\overline{\mathcal{D}''}}) \cup (\mathcal{I}'' \cap 2^{\overline{\mathcal{D}'}})$, we prove the following points:

- $\mathcal{I}^{\subseteq P} = (\mathcal{I}' \cap \mathcal{I}'')^{\subseteq P} \cup (\mathcal{I}' \cap 2^{\overline{D''}})^{\subseteq P} \cup (\mathcal{I}'' \cap 2^{\overline{D'}})^{\subseteq P}$: Because $\beta \cup \beta' \cup \beta'' \models \text{⋈}(\mathbf{B}, \mathbf{B}', \mathbf{B}'')$, we obtain, by a simple rewriting of the $\text{⋈}(\mathbf{B}, \mathbf{B}', \mathbf{B}'')$ formula:

$$\beta \cup \beta' \cup \beta'' \models \bigwedge_{1 \leq i \leq k} \left(\left((c'(i) \wedge c''(i)) \vee \left(c'(i) \wedge \bigwedge_{1 \leq h \leq \ell} \neg h''(i_h) \right) \vee \left(c''(i) \wedge \bigwedge_{1 \leq h \leq \ell} \neg h'(i_h) \right) \right) \leftrightarrow c(i) \right)$$

Because $\mathcal{A} \in \mathbb{A}(\beta)$, we have $\mathcal{I}^{\subseteq P} = \{\{i_1, \dots, i_\ell\} \mid \beta(c(i_1, \dots, i_\ell)) = \top\}$. Moreover, because $\mathcal{A}' \in \mathbb{A}(\beta')$, we have $D' = \{p_i \mid \beta'(h'(i)) = \top\}$ and $\mathcal{I}'^{\subseteq P} = \{\{i_1, \dots, i_\ell\} \mid \beta'(c'(i_1, \dots, i_\ell)) = \top\}$ and, because $\mathcal{A}'' \in \mathbb{A}(\beta'')$, we have $D'' = \{p_i \mid \beta''(h''(i)) = \top\}$ and $\mathcal{I}''^{\subseteq P} = \{\{i_1, \dots, i_\ell\} \mid \beta''(c''(i_1, \dots, i_\ell)) = \top\}$, which implies the required equality.

- $\mathcal{I}^{\subseteq P} = (\mathcal{I}' \cap \mathcal{I}'')^{\subseteq P} \cup (\mathcal{I}' \cap 2^{\overline{D''}})^{\subseteq P} \cup (\mathcal{I}'' \cap 2^{\overline{D'}})^{\subseteq P}$: “ \subseteq ” Let $I \in \mathcal{I}^{\subseteq P}$ be an interaction. Because $\mathcal{A} \in \mathbb{A}(\beta)$, there exists $Y \subseteq \text{Ports} \setminus P$ such that $I = \{i_1, \dots, i_\ell\} \cup Y$, where $\beta(o(i_1, \dots, i_\ell)) = \top$. Since \mathcal{A} is an architecture, it must be that $\{i_1, \dots, i_\ell\} \cap D \neq \emptyset$. Since $D = D' \cup D''$, by the previous point, we distinguish the cases below:
 - if $\{i_1, \dots, i_\ell\} \cap D' \neq \emptyset$ and $\{i_1, \dots, i_\ell\} \cap D'' \neq \emptyset$ then, because $\beta \cup \beta' \cup \beta'' \models \text{⋈}(\mathbf{B}, \mathbf{B}', \mathbf{B}'')$ we obtain:

$$\beta' \cup \beta'' \models (o'(i_1, \dots, i_\ell) \vee o''(i_1, \dots, i_\ell)) \wedge (\neg o'(i_1, \dots, i_\ell) \vee o''(i_1, \dots, i_\ell)) \wedge (\neg o''(i_1, \dots, i_\ell) \vee o'(i_1, \dots, i_\ell))$$

thus $\beta'(o'(i_1, \dots, i_\ell)) = \beta''(o''(i_1, \dots, i_\ell)) = \top$. By the definition of \mathcal{I}' and \mathcal{I}'' , we obtain $I \in \mathcal{I}' \cap \mathcal{I}''$.

- else, if $\{i_1, \dots, i_\ell\} \cap D' \neq \emptyset$ and $\{i_1, \dots, i_\ell\} \cap D'' = \emptyset$ then, because $\beta \cup \beta' \cup \beta'' \models \text{⋈}(\mathbf{B}, \mathbf{B}', \mathbf{B}'')$ we obtain:

$$\beta' \cup \beta'' \models (o'(i_1, \dots, i_\ell) \vee o''(i_1, \dots, i_\ell)) \wedge (\neg o''(i_1, \dots, i_\ell) \vee o'(i_1, \dots, i_\ell))$$

thus $\beta'(o'(i_1, \dots, i_\ell)) = \top$ and, by the definition of \mathcal{I}' , we have $I \in \mathcal{I}' \cap 2^{\overline{D''}}$.

- otherwise, if $\{i_1, \dots, i_\ell\} \cap D' = \emptyset$ and $\{i_1, \dots, i_\ell\} \cap D'' \neq \emptyset$ then, by a symmetric argument, we obtain $I \in \mathcal{I}'' \cap 2^{\overline{D'}}$.

“ \supseteq ” Let I be an interaction such that $I \cap P = \{i_1, \dots, i_\ell\}$ and $I \setminus P = Y$. We consider the following cases:

- if $I \in (\mathcal{I}' \cap \mathcal{I}'')^{\subseteq P}$ then $Y \neq \{\alpha'\}$ and $Y \neq \{\alpha''\}$, thus $I = \{i_1, \dots, i_\ell\} \cup Y \in \mathcal{I}$, by the definition of \mathcal{I}' and \mathcal{I}'' .
- else, if $I \in (\mathcal{I}' \cap 2^{\overline{D''}})^{\subseteq P}$ then, because $\beta \cup \beta' \cup \beta'' \models \text{⋈}(\mathbf{B}, \mathbf{B}', \mathbf{B}'')$, we obtain $\beta(o(i_1, \dots, i_\ell)) = \top$, thus $\{i_1, \dots, i_\ell\} \in \mathcal{I}^{\subseteq P} \cap P$. Moreover, by the definition of \mathcal{I}' , we have that $Y \neq \{\alpha'\}$, thus $I = \{i_1, \dots, i_\ell\} \cup Y \in \mathcal{I}$.
- the last case $I \in (\mathcal{I}'' \cap 2^{\overline{D'}})^{\subseteq P}$ is symmetric to the previous. \square

Before giving the effective translation of SIL^+ formulae to QBF, we define the formula $\triangleleft(\mathbf{B}, \mathbf{B}')$, stating that \mathbf{B} defines an architecture which is the closure of an architecture defined by \mathbf{B}' :

$$\triangleleft(\mathbf{B}, \mathbf{B}') \stackrel{\text{def}}{=} \bigwedge_{1 \leq j \leq k} (\mathbf{h}(j) \leftrightarrow \mathbf{h}'(j)) \wedge \bigwedge_{1 \leq i \leq k} (\neg \mathbf{o}(\bar{i}) \wedge (\mathbf{c}(\bar{i}) \leftrightarrow \mathbf{c}'(\bar{i})))$$

Lemma 9. For any valuations $\beta : \mathbf{B} \rightarrow \{\perp, \top\}$, such that $\beta \cup \beta' \models \triangleleft(\mathbf{B}, \mathbf{B}')$ and $\beta' : \mathbf{B}' \rightarrow \{\perp, \top\}$ and any architectures such that $\mathcal{A} \in \mathbb{A}(\beta)$ and $\mathcal{A}' \in \mathbb{A}(\beta')$, we have $\mathcal{A} \triangleleft \mathcal{A}'$.

Proof: Let $\mathcal{A} = \langle D, \mathcal{I} \rangle$ and $\mathcal{A}' = \langle D', \mathcal{I}' \rangle$. Because $\beta \cup \beta' \models \triangleleft(\mathbf{B}, \mathbf{B}')$, $\mathcal{A} \in \mathbb{A}(\beta)$ and $\mathcal{A}' \in \mathbb{A}(\beta')$, we have $D = \{p_i \mid \beta(\mathbf{h}(i)) = \top\} = \{p_i \mid \beta'(\mathbf{h}'(j)) = \top\} = D'$. Moreover, $\mathcal{I}^{\subseteq P} = \{\{i_1, \dots, i_\ell\} \mid \beta(\mathbf{c}(i_1, \dots, i_\ell)) = \top\} = \{\{i_1, \dots, i_\ell\} \mid \beta'(\mathbf{c}'(i_1, \dots, i_\ell)) = \top\} = \mathcal{I}'^{\subseteq P}$ and $\mathcal{I}^{\not\subseteq P} = \emptyset$, thus $\mathcal{I} = \mathcal{I}'^{\subseteq P}$. \square

Let us fix the set of visible port symbols $P = \{p_1, \dots, p_k\} \subseteq \text{PSym}$ for the rest of this section. We view the port symbols in P as propositional variables and write $\exists P [\forall P]$ for $\exists p_1 \dots \exists p_k [\forall p_1 \dots \forall p_k]$. Given a nonempty strictly increasing sequence $\bar{i} = i_1 < \dots < i_\ell \in \{1, \dots, k\}$ and a boolean term b over P , we define the propositional formulae below:

$$\begin{aligned} \pi(\bar{i}) &\stackrel{\text{def}}{=} \bigwedge_{j \in \{i_1, \dots, i_\ell\}} p_j \wedge \bigwedge_{j \notin \{i_1, \dots, i_\ell\}} \neg p_j \\ \theta(p_i) &\stackrel{\text{def}}{=} p_i \quad \theta(b_1 \cdot b_2) \stackrel{\text{def}}{=} \theta(b_1) \wedge \theta(b_2) \quad \theta(\bar{b}_1) \stackrel{\text{def}}{=} \neg \theta(b_1) \end{aligned}$$

The translation of a SIL^+ formula in QBF is defined recursively on its structure:

$$\begin{aligned} \text{tr}(\text{emp}, \mathbf{B}) &\stackrel{\text{def}}{=} \bigwedge_{1 \leq i \leq k} \neg \mathbf{h}(i) \\ \text{tr}(p_i \rightarrow b, \mathbf{B}) &\stackrel{\text{def}}{=} \mathbf{h}(i) \wedge \bigwedge_{1 \leq j \neq i \leq k} \neg \mathbf{h}(j) \wedge \bigwedge_{1 \leq \bar{i} \leq k} \forall P \cdot [\pi(\bar{i}) \wedge (\mathbf{o}(\bar{i}) \vee \mathbf{c}(\bar{i}))] \rightarrow \theta(b) \\ \text{tr}(p_i \rightarrow \square b, \mathbf{B}) &\stackrel{\text{def}}{=} \mathbf{h}(i) \wedge \bigwedge_{1 \leq j \neq i \leq k} \neg \mathbf{h}(j) \wedge \bigwedge_{1 \leq \bar{i} \leq k} \neg \mathbf{o}(\bar{i}) \wedge \\ &\quad \forall P \cdot [(\pi(\bar{i}) \wedge \mathbf{c}(\bar{i})) \rightarrow \theta(b)] \wedge \bigwedge_{\bar{j} \subseteq \bar{i}} \pi(\bar{j}) \rightarrow \neg \theta(b) \\ \text{tr}(p_i \overset{\exists}{\rightarrow} b, \mathbf{B}) &\stackrel{\text{def}}{=} \mathbf{h}(i) \wedge \bigwedge_{1 \leq j \neq i \leq k} \neg \mathbf{h}(j) \wedge \bigvee_{1 \leq \bar{i} \leq k} \exists P \cdot \pi(\bar{i}) \wedge \theta(b) \wedge (\mathbf{o}(\bar{i}) \vee \mathbf{c}(\bar{i})) \\ \text{tr}(p_i \overset{\exists}{\rightarrow} \square b, \mathbf{B}) &\stackrel{\text{def}}{=} \mathbf{h}(i) \wedge \bigwedge_{1 \leq j \neq i \leq k} \neg \mathbf{h}(j) \wedge \\ &\quad \bigvee_{1 \leq \bar{i} \leq k} \exists P \cdot \pi(\bar{i}) \wedge \theta(b) \wedge \mathbf{c}(\bar{i}) \wedge \forall P \cdot \bigwedge_{\bar{j} \subseteq \bar{i}} \pi(\bar{j}) \rightarrow \neg \theta(b) \\ \text{tr}(\langle \phi_1 \rangle, \mathbf{B}) &\stackrel{\text{def}}{=} \exists \mathbf{B}' \cdot \mathcal{A}(\mathbf{B}') \wedge \triangleleft(\mathbf{B}, \mathbf{B}') \wedge \text{tr}(\phi_1, \mathbf{B}') \\ \text{tr}(\phi_1 \wedge \phi_2, \mathbf{B}) &\stackrel{\text{def}}{=} \text{tr}(\phi_1, \mathbf{B}) \wedge \text{tr}(\phi_2, \mathbf{B}) \\ \text{tr}(\phi_1 \vee \phi_2, \mathbf{B}) &\stackrel{\text{def}}{=} \text{tr}(\phi_1, \mathbf{B}) \vee \text{tr}(\phi_2, \mathbf{B}) \\ \text{tr}(\phi_1 * \phi_2, \mathbf{B}) &\stackrel{\text{def}}{=} \exists \mathbf{B}' \exists \mathbf{B}'' \cdot \mathcal{A}(\mathbf{B}') \wedge \mathcal{A}(\mathbf{B}'') \wedge \#(\mathbf{B}', \mathbf{B}'') \wedge \uplus(\mathbf{B}, \mathbf{B}', \mathbf{B}'') \wedge \text{tr}(\phi_1, \mathbf{B}') \wedge \text{tr}(\phi_2, \mathbf{B}'') \\ \text{tr}(\phi_1 \dot{*} \phi_2, \mathbf{B}) &\stackrel{\text{def}}{=} \forall \mathbf{B}' \cdot \mathcal{A}(\mathbf{B}') \wedge \#(\mathbf{B}, \mathbf{B}') \wedge \text{tr}(\phi_1, \mathbf{B}') \rightarrow \exists \mathbf{B}'' \cdot \mathcal{A}(\mathbf{B}'') \wedge \uplus(\mathbf{B}'', \mathbf{B}, \mathbf{B}') \wedge \text{tr}(\phi_2, \mathbf{B}'') \end{aligned}$$

Note that, for any SIL^+ formula ϕ , $\text{tr}(\phi, \mathbf{B})$ is a QBF formula with free variables in \mathbf{B} . The following result proves the equivalence between SIL^+ formulae and their QBF translations.

Theorem 2. Given a SIL^+ formula ϕ , such that $\mathbf{P}(\phi) \subseteq P$, for any architecture $\mathcal{A} = \langle D, \mathcal{I} \rangle$, we have $\mathcal{A} \models \phi$ if and only if $\beta_{\mathcal{A}} \models \text{tr}(\phi, \mathbf{B})$.

Proof: We prove first the following fact:

Fact 3 Given an architecture \mathcal{A} and boolean valuations $\beta, \beta' : \mathbf{B} \rightarrow \{\perp, \top\}$, if $\mathcal{A} \in \mathbb{A}(\beta) \cap \mathbb{A}(\beta')$ then $\beta = \beta'$.

Proof: Necessarily β and β' agree on any propositional variable from \mathbf{B} . \square

The proof goes by induction on the structure of ϕ . We consider the cases below:

- emp : $\mathcal{A} \models \text{emp} \iff D = \emptyset \iff \beta_{\mathcal{A}} \models \bigwedge_{1 \leq i \leq k} \neg \mathbf{h}(i)$.
- $p_i \multimap b$: “ \Rightarrow ” If $\mathcal{A} \models p_i \multimap b$ then $D = \{p_i\}$ and $I \vdash b$, for all $I \in \mathcal{I}$ and $\beta_{\mathcal{A}} \models \text{tr}(p_i \multimap b, \mathbf{B})$ is an easy check. “ \Leftarrow ” Since $\beta_{\mathcal{A}} \models \mathbf{h}(i) \wedge \bigwedge_{1 \leq j \neq i \leq k} \neg \mathbf{h}(j)$, we have $D = \{p_i\}$. Let $I \in \mathcal{I}$ be an arbitrary interaction of \mathcal{A} . We distinguish two cases:
 - if $I \in \mathcal{I}^{\subseteq P}$ then let $\{p_{i_1}, \dots, p_{i_\ell}\} \stackrel{\text{def}}{=} I$. We obtain that $\beta_{\mathcal{A}}(\mathbf{c}(i_1, \dots, i_\ell)) = \top$, thus $\beta_{\mathcal{A}} \models \forall P. \pi(i_1, \dots, i_\ell) \rightarrow \theta(b)$, leading to $I \vdash b$.
 - else $I \in \mathcal{I}^{\not\subseteq P}$ and let $\{p_{i_1}, \dots, p_{i_\ell}\} \stackrel{\text{def}}{=} I \cap P$. We obtain $\beta(\mathbf{o}(i_1, \dots, i_\ell)) = \top$, thus $\beta \models \forall P. \pi(i_1, \dots, i_\ell) \rightarrow \theta(b)$, leading to $I \vdash b$.
 Consequently, we have $\mathcal{A} \models p_i \multimap b$.
- $p_i \multimap \square b$: “ \Rightarrow ” This direction is similar to the above point. “ \Leftarrow ” Similar to the above point, we obtain $D = \{p_i\}$. Let $I \in \mathcal{I}$ be an arbitrary interaction of \mathcal{A} and $\{p_{i_1}, \dots, p_{i_\ell}\} \stackrel{\text{def}}{=} I \cap P$. Since $\beta_{\mathcal{A}} \models \bigwedge_{1 \leq i \leq k} \neg \mathbf{o}(i_1, \dots, i_\ell)$ then $I \notin \mathcal{I}^{\not\subseteq P}$, so the only possibility is $I \in \mathcal{I}^{\subseteq P}$ and thus $I = \{p_{i_1}, \dots, p_{i_\ell}\}$. Then $\beta_{\mathcal{A}}(\mathbf{c}(i_1, \dots, i_\ell)) = \top$, thus $\beta \models \forall P. \pi(i_1, \dots, i_\ell) \rightarrow \theta(b)$, leading to $I \vdash b$. Moreover, for any interaction $J = \{p_{j_1}, \dots, p_{j_m}\} \subsetneq I$, we have $\beta \models \forall P. \pi(j_1, \dots, j_m) \rightarrow \neg \theta(b)$, from which we conclude that $J \not\vdash b$ and, consequently $I \vdash^\mu b$. Then, we have $\mathcal{A} \models p_i \multimap \square b$.
- $p_i \overset{\exists}{\multimap} b$: “ \Rightarrow ” $\beta_{\mathcal{A}} \models \text{tr}(p_i \overset{\exists}{\multimap} b, \mathbf{B})$ is an easy check. “ \Leftarrow ” Similar to the above point, we obtain $D = \{p_i\}$. Let $I = \{p_{i_1}, \dots, p_{i_\ell}\}$ be the interaction for which $\beta_{\mathcal{A}} \models \mathbf{o}(i_1, \dots, i_\ell) \vee \mathbf{c}(i_1, \dots, i_\ell)$ holds and conclude, since $I \vdash b$ follows from $\beta_{\mathcal{A}} \models \exists P. \pi(i_1, \dots, i_\ell) \wedge \theta(b)$.
- $p_i \overset{\exists}{\multimap} \square b$: “ \Rightarrow ” Similar to the above point. “ \Leftarrow ” Similar to the above point, we obtain $D = \{p_i\}$ and an interaction $I = \{p_{i_1}, \dots, p_{i_\ell}\}$ such that $I \vdash b$. Moreover, for any interaction $J = \{p_{j_1}, \dots, p_{j_m}\} \subsetneq I$ we have $\beta_{\mathcal{A}} \models \pi(j_1, \dots, j_m) \rightarrow \neg \theta(b)$, leading to $J \not\vdash b$, thus we obtain $I \vdash^\mu b$.
- $\langle \phi_1 \rangle$: “ \Rightarrow ” $\mathcal{A} \models \langle \phi_1 \rangle$ only if $\mathcal{A}_1 \models \phi_1$, for some architecture \mathcal{A}_1 such that $\mathcal{A} \triangleleft \mathcal{A}_1$. By Lemma 9, we obtain $\beta_{\mathcal{A}} \cup \beta_{\mathcal{A}_1} \models \triangleleft(\mathbf{B}, \mathbf{B}_1)$, thus $\beta_{\mathcal{A}} \models \text{tr}(\langle \phi_1 \rangle, \mathbf{B})$. “ \Leftarrow ” If $\beta_{\mathcal{A}} \models \text{tr}(\langle \phi_1 \rangle, \mathbf{B})$ then there exists a valuation $\beta' : \mathbf{B}' \rightarrow \{\perp, \top\}$ such that $\beta' \models \mathcal{A}(\mathbf{B}')$, $\beta_{\mathcal{A}} \cup \beta' \models \triangleleft(\mathbf{B}, \mathbf{B}')$ and $\beta' \models \text{tr}(\phi_1, \mathbf{B}')$. Since $\beta' \models \mathcal{A}(\mathbf{B}')$ there

exists an architecture $\mathcal{A}' \in \mathbb{A}(\beta')$ and, by Lemma 9, we obtain $\mathcal{A} \triangleleft \mathcal{A}'$. Moreover, by the induction hypothesis, we have $\mathcal{A}' \models \phi_1$, thus $\mathcal{A} \models \langle \phi_1 \rangle$.

- $\phi_1 \wedge \phi_2$: “ \Rightarrow ” If $\mathcal{A} \models \phi_1 \wedge \phi_2$, by the induction hypothesis, we have $\beta_{\mathcal{A}} \models \text{tr}(\phi_i, \mathbf{B})$, for $i = 1, 2$, thus $\beta_{\mathcal{A}} \models \text{tr}(\phi_1 \wedge \phi_2, \mathbf{B})$. “ \Leftarrow ” If $\beta_{\mathcal{A}} \models \text{tr}(\phi_1 \wedge \phi_2, \mathbf{B})$, by the induction hypothesis, we obtain $\mathcal{A} \models \phi_i$, for $i = 1, 2$, hence $\mathcal{A} \models \phi_1 \wedge \phi_2$.
- $\phi_1 \vee \phi_2$: similar to the above point, by direct application of the induction hypothesis.
- $\phi_1 * \phi_2$: “ \Rightarrow ” If $\mathcal{A} \models \phi_1 * \phi_2$ then there exist $\mathcal{A}_i = \langle D_i, \mathcal{I}_i \rangle$ such that $\mathcal{A}_1 \uplus \mathcal{A}_2 = \mathcal{A}$ and $\mathcal{A}_i \models \phi_i$, for $i = 1, 2$. By the induction hypothesis, $\beta_{\mathcal{A}_i} \models \text{tr}(\phi_i, \mathbf{B}_i)$, for $i = 1, 2$. Since $D_1 \cap D_2 = \emptyset$, we have $\beta_{\mathcal{A}_1} \cup \beta_{\mathcal{A}_2} \models \#(\mathbf{B}_1, \mathbf{B}_2)$ and, by Lemma 8 (1) there exists a boolean valuation $\beta : \mathbf{B} \rightarrow \{\perp, \top\}$ such that $\beta \cup \beta_1 \cup \beta_2 \models \uplus(\mathbf{B}, \mathbf{B}_1, \mathbf{B}_2)$ and $\mathcal{A} = \mathcal{A}_1 \uplus \mathcal{A}_2 \in \mathbb{A}(\beta)$. Moreover, $\beta \models \text{tr}(\phi_1 * \phi_2, \mathbf{B})$. Since $\mathcal{A} \in \mathbb{A}(\beta_{\mathcal{A}})$, by Fact 3, we conclude. “ \Leftarrow ” If $\beta_{\mathcal{A}} \models \text{tr}(\phi_1 * \phi_2, \mathbf{B})$, there exists valuations $\beta_i : \mathbf{B}_i \rightarrow \{\perp, \top\}$ such that $\beta_1 \cup \beta_2 \models \mathcal{A}(\mathbf{B}_1) \wedge \mathcal{A}(\mathbf{B}_2) \wedge \#(\mathbf{B}_1, \mathbf{B}_2) \wedge \uplus(\mathbf{B}, \mathbf{B}_1, \mathbf{B}_2)$ and $\beta_i \models \text{tr}(\phi_i, \mathbf{B}_i)$, for $i = 1, 2$. By Lemma 8 (2) there exist architectures $\mathcal{A}_i \in \mathbb{A}(\beta_i)$, such that $\mathcal{A} = \mathcal{A}_1 \uplus \mathcal{A}_2$. Since $\mathcal{A}_i \in \mathbb{A}(\beta_{\mathcal{A}_i})$, by Fact 3, we obtain $\beta_i = \beta_{\mathcal{A}_i}$, for $i = 1, 2$. By the inductive hypothesis, we obtain $\mathcal{A}_i \models \phi_i$, for $i = 1, 2$. Since, by Lemma 7, \mathcal{A}_1 and \mathcal{A}_2 are disjoint, we obtain $\mathcal{A} \models \phi_1 * \phi_2$.
- $\phi_1 \multimap \phi_2$: “ \Rightarrow ” Let $\beta_1 : \mathbf{B}_1 \rightarrow \{\perp, \top\}$ be any valuation such that $\beta_{\mathcal{A}} \cup \beta_1 \models \mathcal{A}(\mathbf{B}_1) \wedge \#(\mathbf{B}, \mathbf{B}_1) \wedge \text{tr}(\phi_1, \mathbf{B}_1)$. By the induction hypothesis, there exists an architecture $\mathcal{A}_1 \in \mathbb{A}(\beta_1)$ and, moreover, since $\beta_{\mathcal{A}} \cup \beta_1 \models \#(\mathbf{B}, \mathbf{B}_1)$, by Lemma 7, \mathcal{A} and \mathcal{A}_1 are disjoint. Since $\mathcal{A} \models \phi_1 \multimap \phi_2$, we have $\mathcal{A} \uplus \mathcal{A}_1 \models \phi_2$. By the inductive hypothesis, we have $\beta_{\mathcal{A} \uplus \mathcal{A}_1} \models \text{tr}(\phi_2, \mathbf{B}'')$. Moreover, by Lemma 8 (1), there exists a valuation $\beta'' : \mathbf{B}'' \rightarrow \{\perp, \top\}$ such that $\beta'' \cup \beta_{\mathcal{A}} \cup \beta_1 \models \uplus(\mathbf{B}'', \mathbf{B}, \mathbf{B}_1)$ and, by Fact 3, β'' and $\beta_{\mathcal{A} \uplus \mathcal{A}_1}$ are the same. Since the choice of β_1 was arbitrary, we obtain $\beta_{\mathcal{A}} \models \text{tr}(\phi_1 \multimap \phi_2, \mathbf{B})$. “ \Leftarrow ” Let \mathcal{A}_1 be any architecture disjoint from \mathcal{A} , such that $\mathcal{A}_1 \models \phi_1$. By the induction hypothesis, $\beta_{\mathcal{A}_1} \models \text{tr}(\phi_1, \mathbf{B}_1)$. Moreover, by Lemma 7, we have $\beta_{\mathcal{A}} \cup \beta_{\mathcal{A}_1} \models \#(\mathbf{B}, \mathbf{B}_1)$, thus, since $\beta_{\mathcal{A}} \models \text{tr}(\phi_1 \multimap \phi_2, \mathbf{B})$, there exists a valuation $\beta'' : \mathbf{B}'' \rightarrow \{\perp, \top\}$ such that $\beta_{\mathcal{A}} \cup \beta_{\mathcal{A}_1} \cup \beta'' \models \mathcal{A}(\mathbf{B}'') \wedge \uplus(\mathbf{B}'', \mathbf{B}, \mathbf{B}_1) \wedge \text{tr}(\phi_2, \mathbf{B}'')$. Then, by Lemma 8 (2), there exists an architecture $\mathcal{A}'' \in \mathbb{A}(\beta'')$ such that $\mathcal{A}'' = \mathcal{A} \uplus \mathcal{A}_1$. By the induction hypothesis, because $\beta'' \models \text{tr}(\phi_2, \mathbf{B}'')$, we obtain $\mathcal{A} \uplus \mathcal{A}_1 \models \phi_2$ and, since the choice of \mathcal{A}_1 was arbitrary, we obtain $\mathcal{A} \models \phi_1 \multimap \phi_2$. \square

We remind that, since there are at most 2^k sequences $1 \leq i_1 < \dots < i_\ell \leq k$, the size of each of the formulae $\mathcal{A}(\mathbf{B})$, $\#(\mathbf{B}, \mathbf{B}')$, $\uplus(\mathbf{B}, \mathbf{B}', \mathbf{B}'')$ and $\triangleleft(\mathbf{B}, \mathbf{B}')$ is $2^{O(k)}$. It is easy to check that, given any SIL^+ formula ϕ such that $\text{P}(\phi) \subseteq \{p_1, \dots, p_k\}$, its translation to QBF takes $|\phi| \cdot 2^{O(k)}$ time.

In the following, we provide a tight complexity result by bounding the number of ports that occur in a boolean term b from an atomic proposition $p \multimap b$, $p \overset{\exists}{\multimap} b$, $p \multimap b$

or $p \stackrel{\exists}{\dashv} b$, by a constant $n \geq 1$, independent of the input. We shall denote by SIL_n^+ the fragment of SIL^+ formulae that meets this condition.

Corollary 2. *The satisfiability and entailment problems for SIL^+ are in EXPSPACE. If $n \geq 1$ is a constant not part of the input, the satisfiability and entailment problems for SIL_n^+ are PSPACE-complete.*

Proof: The EXPSPACE upper bound for satisfiability is immediate, since the QBF translation of any SIL^+ formula ϕ , such that $\text{P}(\phi) \subseteq \{p_1, \dots, p_k\}$ takes time $|\phi| \cdot 2^{O(k)}$. For the entailment problem, let ϕ and ψ be two SIL^+ formulae, such that $\text{P}(\phi) \cup \text{P}(\psi) \subseteq \{p_1, \dots, p_k\}$ and assume that there exists an architecture \mathcal{A} such that $\mathcal{A} \models \phi$ and $\mathcal{A} \not\models \psi$. By Theorem 2, there exists a boolean valuation $\beta : \mathbf{B} \rightarrow \{\perp, \top\}$, such that $\mathcal{A} \in \mathbb{A}(\beta)$ and $\beta \models \text{tr}(\phi, \mathbf{B})$. Moreover, since $\mathcal{A} \not\models \psi$, for every boolean valuation $\beta' : \mathbf{B} \rightarrow \{\perp, \top\}$, such that $\mathcal{A} \in \mathbb{A}(\beta')$, we have $\beta' \models \neg \text{tr}(\psi, \mathbf{B})$. By Fact 3, since $\mathcal{A} \in \mathbb{A}(\beta) \cap \mathbb{A}(\beta')$, for any such valuation β' , we have that β and β' are the same, thus $\beta \models \text{tr}(\phi, \mathbf{B}) \wedge \neg \text{tr}(\psi, \mathbf{B})$. Since EXPSPACE is closed under complement, by Savitch's Theorem, we obtain the EXPSPACE upper bound for entailment.

For the second point, the upper bound is established noticing that the number of sequences $1 \leq i_1 < \dots < i_\ell \leq k$, for $\ell \leq n$ is bounded by $\binom{k}{n}$, thus the translation of a SIL_n^+ formula in QBF takes polynomial time. For the PSPACE-hard lower bound, we reduce from the validity of QBF sentences $\forall x_1 \exists y_1 \dots \forall x_k \exists y_k . F$, where F is a propositional formula with free variables $x_1, y_1, \dots, x_k, y_k$, written in positive normal form (note that this is w.l.o.g.). To this end, we consider, for each variable $x \in \{x_1, y_1, \dots, x_k, y_k\}$ two ports x_t and x_f . Let **false** be a shorthand for $\text{emp} \wedge p \dashv p$, where p is a port which is not a member of $\{x_t, x_f \mid x \in \{x_1, y_1, \dots, x_k, y_k\}\}$. Intuitively, $\text{has}(x_t) \stackrel{\text{def}}{=} x_t \dashv x_t \dashv \text{false}$ (resp. $\text{has}(x_f) \stackrel{\text{def}}{=} x_f \dashv x_f \dashv \text{false}$) encodes the fact that x is true (resp. false). Given a set $S \subseteq \{x_1, y_1, \dots, x_k, y_k\}$, we write $A_S \stackrel{\text{def}}{=} \ast_{x \in S} x_t \dashv x_t \vee x_f \dashv x_f$. Considering the total order $x_1 < y_1 < \dots < x_k < y_k$, we write $A_{\{\leq x\}} \stackrel{\text{def}}{=} A_{\{x' \mid x' \leq x\}}$. The reduction from QBF to SIL_n^+ is implemented by the following recursive function:

$$\begin{aligned} \tau(x) &\stackrel{\text{def}}{=} \text{has}(x_t) \\ \tau(\neg x) &\stackrel{\text{def}}{=} \text{has}(x_f) \\ \tau(F_1 \wedge F_2) &\stackrel{\text{def}}{=} \tau(F_1) \wedge \tau(F_2) \\ \tau(F_1 \vee F_2) &\stackrel{\text{def}}{=} \tau(F_1) \vee \tau(F_2) \\ \tau(\forall x_i . G) &\stackrel{\text{def}}{=} A_{\{x_i\}} \dashv \tau(G) \\ \tau(\exists y_i . G) &\stackrel{\text{def}}{=} [A_{\{\leq x_i\}} \wedge ((A_{\{\leq y_i\}} \wedge \tau(G)) \dashv \text{false})] \dashv \text{false} \end{aligned}$$

We show that any QBF sentence $\forall x_1 \exists y_1 \dots \forall x_k \exists y_k . F$ is valid if and only if $\text{emp} \wedge \tau(\forall x_1 \exists y_1 \dots \forall x_k \exists y_k . F)$ is satisfiable, or equivalently, the entailment between emp and $\tau(\forall x_1 \exists y_1 \dots \forall x_k \exists y_k . F)$ holds. The encoding of the universal quantifier is directly via $A_{\{x\}} \dashv \tau(G)$ that asserts the validity of $\tau(G)$ under any extension of the model (architecture) with domain either $\{x_t\}$ or $\{x_f\}$. The existential quantifier is encoded using a double negation. If $\mathcal{A} \models (P \wedge Q) \dashv \text{false}$, then for any extension $\mathcal{A}' \models P$ we have $\mathcal{A}' \not\models Q$. Now assume that $\mathcal{A} \models [A_{\{\leq x_i\}} \wedge ((A_{\{\leq y_i\}} \wedge \tau(G)) \dashv \text{false})] \dashv \text{false}$. Then,

for any extension \mathcal{A}' of \mathcal{A} , such that $\mathcal{A}' \models A_{\{\leq x_i\}}$, we have $\mathcal{A}' \not\models (A_{\{\leq y_i\}} \wedge \tau(G)) \rightarrow *$ false. This means that there exists an extension \mathcal{A}'' such that $\mathcal{A} \uplus \mathcal{A}' \uplus \mathcal{A}'' \models \tau(G)$, which captures the fact that some valuation of y_i makes the sentence valid. A similar encoding is used in [?, Proposition 6]. \square

5.2 Decidability of SIL*

We recall that SIL* is the fragment of SIL in which negation is allowed, but not the magic wand. The proof of decidability for SIL* follows a very similar pattern to the decidability proof for SIL⁺ (§5.1). Just as before, we first define an equivalence relation on architectures, then we characterize the equivalence classes of this relation by test formulae. As a consequence, each formula of SIL* is equivalent to a boolean combination of test formulae from a finite set and, moreover, based on this fact, we obtain a small model property that implies the decidability of SIL*.

The main difficulty here is that SIL* has negation, which allows to describe architectures with invisible ports in the domain. For instance, the formula $\neg \text{emp} * \neg \text{emp}$ states the existence of at least two ports, none of them corresponding to a port symbol. When composing such architectures, these invisible ports can determine which interactions are kept and which are lost, based on their *visible interaction type*, which is formally defined next.

Example 6. Consider the architectures $\mathcal{A}_1 = \langle \{p, \alpha\}, \{\{p, \alpha, \beta\}\} \rangle$ and $\mathcal{A}_2 = \langle \{\beta\}, \{\{p, \beta\}\} \rangle$, where the set of visible ports is $P = \{p\}$. Because $\{p, \alpha, \beta\} \in \text{inter}(\mathcal{A}_1)$ has a non-empty intersection with $\text{dom}(\mathcal{A}_2) = \{\beta\}$, we obtain $\mathcal{A}_1 \uplus \mathcal{A}_2 = \langle \{p, \alpha, \beta\}, \emptyset \rangle$. \blacksquare

Let $P \subseteq \text{Ports}$ be a set of visible ports and $\mathcal{A} = \langle D, \mathcal{I} \rangle$ be an architecture. For an invisible port $x \in D \setminus P$, we define its visible interaction type as the set of interactions involving x , restricted to their visible ports: $\text{vt}_{\mathcal{A}, P}(x) \stackrel{\text{def}}{=} \mathcal{I}^{\{x\}} \sqcap P$. The function $\tau_{\mathcal{A}, P} : 2^{2^P} \rightarrow 2^{\text{Ports}}$ gives the set of invisible ports with a given visible interaction type from the domain of \mathcal{A} :

$$\tau_{\mathcal{A}, P}(\mathcal{S}) = \{x \in D \setminus P \mid \text{vt}_{\mathcal{A}, P}(x) = \mathcal{S}\}, \text{ for any } \mathcal{S} \in 2^{2^P}$$

Consider further the function $b_P : \mathbb{N} \times 2^{2^P} \rightarrow \mathbb{N}$, defined by the recurrence relation:

$$b_P(1, \mathcal{S}) \stackrel{\text{def}}{=} 1 \text{ and } \forall n > 1. b_P(n, \mathcal{S}) \stackrel{\text{def}}{=} 2 \cdot \sum_{\mathcal{S}' \subseteq \mathcal{S}} b_P(n-1, \mathcal{S}')$$

Definition 5. Given architectures $\mathcal{A}_i = \langle D_i, \mathcal{I}_i \rangle$, for $i = 1, 2$, a finite set of ports $P \subseteq \text{Ports}$ and an integer $n \geq 0$, we have $\mathcal{A}_1 \approx_P^n \mathcal{A}_2$ if and only if the following hold:

1. $D_1 \cap P = D_2 \cap P$,
2. $\mathcal{I}_1^{\cap(D_1 \cap P)} \sqcap P = \mathcal{I}_2^{\cap(D_2 \cap P)} \sqcap P$,
3. for all $\mathcal{S} \in 2^{2^P}$, we have:
 - (a) $\|\tau_{\mathcal{A}_1, P}(\mathcal{S})\| < b_P(n, \mathcal{S}) \Rightarrow \|\tau_{\mathcal{A}_2, P}(\mathcal{S})\| = \|\tau_{\mathcal{A}_1, P}(\mathcal{S})\|$,
 - (b) $\|\tau_{\mathcal{A}_1, P}(\mathcal{S})\| \geq b_P(n, \mathcal{S}) \Rightarrow \|\tau_{\mathcal{A}_2, P}(\mathcal{S})\| \geq b_P(n, \mathcal{S})$.

It is easy to prove that \approx_p^n is an equivalence relation, for any $P \subseteq \text{Ports}$ and any $n \geq 1$. Moreover, given any set of ports $P' \subseteq P$ and any integer $n' \leq n$, we have $\mathcal{A}_1 \approx_p^n \mathcal{A}_2 \Rightarrow \mathcal{A}_1 \approx_{p'}^{n'} \mathcal{A}_2$. The following lemma proves that \approx_p^n is compatible with the composition of architectures:

Lemma 10. *Let $\mathcal{A} = \langle D, I \rangle$ and $\mathcal{A}' = \langle D', I' \rangle$ be architectures and $P \subseteq \text{Ports}$ be a set of ports, such that $\mathcal{A} \approx_p^n \mathcal{A}'$, for some $n \geq 2$. Then for any architectures $\mathcal{A}_i = \langle D_i, I_i \rangle$, such that $\mathcal{A} = \mathcal{A}_1 \uplus \mathcal{A}_2$ there exist architectures $\mathcal{A}'_i = \langle D'_i, I'_i \rangle$, such that $\mathcal{A}' = \mathcal{A}'_1 \uplus \mathcal{A}'_2$ and $\mathcal{A}_i \approx_{p'}^{n-1} \mathcal{A}'_i$, for $i = 1, 2$.*

Proof: We define two mappings $\mu_i : 2^{2^P} \rightarrow 2^{\text{Ports}}$ describing how the ports from $D' \setminus P$ occur in the interactions of \mathcal{A}'_i , for $i = 1, 2$, respectively. The idea is to define the architectures \mathcal{A}'_i such that $\mu_i = \tau_{\mathcal{A}'_i, P}$, for $i = 1, 2$. Let $\mathcal{S} \in 2^{2^P}$ be an arbitrary set of interactions involving only visible ports. We distinguish the cases below:

1. If $\|\tau_{\mathcal{A}, P}(\mathcal{S})\| = \|\tau_{\mathcal{A}', P}(\mathcal{S})\|$ then there exists a bijection $\pi_{\mathcal{S}} : \tau_{\mathcal{A}', P}(\mathcal{S}) \rightarrow \tau_{\mathcal{A}, P}(\mathcal{S})$. In this case, for each $x \in \tau_{\mathcal{A}, P}(\mathcal{S})$ and each $\mathcal{S}' \supseteq \mathcal{S}$, we require that:

$$x \in \mu_i(\mathcal{S}') \iff \pi_{\mathcal{S}}(x) \in \tau_{\mathcal{A}_i, P}(\mathcal{S}'), \text{ for all } i = 1, 2 \quad (1)$$

2. Else $\|\tau_{\mathcal{A}, P}(\mathcal{S})\| \neq \|\tau_{\mathcal{A}', P}(\mathcal{S})\|$, thus necessarily $\|\tau_{\mathcal{A}, P}(\mathcal{S})\| \geq b_P(n, \mathcal{S})$ and $\|\tau_{\mathcal{A}', P}(\mathcal{S})\| \geq b_P(n, \mathcal{S})$, because we assumed that $\mathcal{A} \approx_p^n \mathcal{A}'$. Because $\mathcal{A} = \mathcal{A}_1 \uplus \mathcal{A}_2$, we have that (D_1, D_2) is a partition of D and define $E_i \stackrel{\text{def}}{=} \tau_{\mathcal{A}, P}(\mathcal{S}) \cap D_i$, a partition of $\tau_{\mathcal{A}, P}(\mathcal{S})$, for $i = 1, 2$. We distinguish the cases below:

- (a) if $\|E_1\| < \frac{b_P(n, \mathcal{S})}{2}$ and $\|E_2\| \geq \frac{b_P(n, \mathcal{S})}{2}$ let (E'_1, E'_2) be a partition of $\tau_{\mathcal{A}', P}(\mathcal{S})$ such that $\|E'_1\| = \|E_1\|$ and $\|E'_2\| \geq \frac{b_P(n, \mathcal{S})}{2}$. Such a partition exists because $\|\tau_{\mathcal{A}', P}(\mathcal{S})\| \geq b_P(n, \mathcal{S})$. Since $\|E'_1\| = \|E_1\|$, there exists a bijection $\rho_{\mathcal{S}} : E'_1 \rightarrow E_1$. Then for each $x \in E_1$ and each $\mathcal{S}' \supseteq \mathcal{S}$, we require:

$$x \in \mu_1(\mathcal{S}') \iff \rho_{\mathcal{S}}(x) \in \tau_{\mathcal{A}_1, P}(\mathcal{S}') \quad (2)$$

Further, we split E'_2 between the sets $\{\mu_2(\mathcal{S}') \mid \mathcal{S} \subseteq \mathcal{S}'\}$ such that, for each $\mathcal{S}' \supseteq \mathcal{S}$, the following hold:

$$\begin{aligned} \|\tau_{\mathcal{A}_2, P}(\mathcal{S}')\| < b_P(n-1, \mathcal{S}') &\Rightarrow \|\mu_2(\mathcal{S}') \cap E'_2\| = \|\tau_{\mathcal{A}_2, P}(\mathcal{S}') \cap E_2\| \\ \|\tau_{\mathcal{A}_2, P}(\mathcal{S}')\| \geq b_P(n-1, \mathcal{S}') &\Rightarrow \|\mu_2(\mathcal{S}') \cap E'_2\| \geq b_P(n-1, \mathcal{S}') \end{aligned} \quad (3)$$

Note that, since $\|E'_2\| \geq \sum_{\mathcal{S} \subseteq \mathcal{S}'} b_P(n-1, \mathcal{S}')$, such a partitioning of E'_2 is always possible.

- (b) else, if $\|E_1\| \geq \frac{b_P(n, \mathcal{S})}{2}$ and $\|E_2\| < \frac{b_P(n, \mathcal{S})}{2}$, we partition $\tau_{\mathcal{A}', P}(\mathcal{S})$ symmetrically.
- (c) otherwise, if $\|E_1\| \geq \frac{b_P(n, \mathcal{S})}{2}$ and $\|E_2\| \geq \frac{b_P(n, \mathcal{S})}{2}$, then let (E'_1, E'_2) be a partition of $\tau_{\mathcal{A}', P}(\mathcal{S})$ such that $\|E'_1\| \geq \frac{b_P(n, \mathcal{S})}{2}$ and $\|E'_2\| \geq \frac{b_P(n, \mathcal{S})}{2}$. Such a partitioning exists because $\|\tau_{\mathcal{A}', P}(\mathcal{S})\| \geq b_P(n, \mathcal{S})$. Then we split E'_i between the sets $\{\mu_i(\mathcal{S}') \mid \mathcal{S} \subseteq \mathcal{S}'\}$ such that, for each $\mathcal{S}' \subseteq \mathcal{S}$, the following hold, for $i = 1, 2$:

$$\begin{aligned} \|\tau_{\mathcal{A}_i, P}(\mathcal{S}')\| < b_P(n-1, \mathcal{S}') &\Rightarrow \|\mu_i(\mathcal{S}') \cap E'_i\| = \|\tau_{\mathcal{A}_i, P}(\mathcal{S}') \cap E_i\| \\ \|\tau_{\mathcal{A}_i, P}(\mathcal{S}')\| \geq b_P(n-1, \mathcal{S}') &\Rightarrow \|\mu_i(\mathcal{S}') \cap E'_i\| \geq b_P(n-1, \mathcal{S}') \end{aligned} \quad (4)$$

Note that, since $\|E'_i\| \geq \sum_{S \in S'} b_P(n-1, S')$, $i = 1, 2$, such a partitioning is always possible.

Moreover, nothing else is in $\mu_i(S')$, for any $S' \in 2^{2^p}$, for $i = 1, 2$. We define now the domains of \mathcal{A}'_1 and \mathcal{A}'_2 as follows:

$$D'_i \stackrel{\text{def}}{=} (D_i \cap P) \cup \bigcup_{S \in 2^{2^p}} \mu_i(S), \text{ for } i = 1, 2 \quad (5)$$

Because the sets $\{\tau_{\mathcal{A}_i, P}(S) \mid S \in 2^{2^p}\}$ form a partition of $D_i \setminus P$, by the definition of μ_i , the sets $\{\mu_i(S) \mid S \in 2^{2^p}\}$ form a partition of $D'_i \setminus P$, for $i = 1, 2$, respectively. Then we can define mappings $\lambda_i : D'_i \setminus P \rightarrow 2^{2^p}$ as $\lambda_i(x) \stackrel{\text{def}}{=} S \iff x \in \mu_i(S)$, for all $x \in D'_i \setminus P$, for $i = 1, 2$. Similarly, since $\{\tau_{\mathcal{A}', P}(S) \mid S \in 2^{2^p}\}$ is a partition of $D' \setminus P$, we can define the mapping $\lambda : D' \setminus P \rightarrow 2^{2^p}$ as $\lambda(x) \stackrel{\text{def}}{=} S \iff x \in \tau_{\mathcal{A}', P}(S)$. Next, we define the interaction sets of \mathcal{A}'_i as:

$$I'_i \stackrel{\text{def}}{=} I' \cap D'_i \cup X_i \cup Y_i$$

where:

$$X_i \stackrel{\text{def}}{=} \{I \cup \{x, \alpha_i\} \mid x \in D'_i \setminus P, I \in \lambda_i(x) \setminus \lambda(x), I \cap D'_{3-i} \neq \emptyset\} \cup \\ \{I \cup \{x, \alpha_i\} \cup (D'_{3-i} \setminus P) \mid x \in D'_i \setminus P, I \in \lambda_i(x) \setminus \lambda(x), I \cap D'_{3-i} = \emptyset\}$$

$$Y_i \stackrel{\text{def}}{=} \{I \cup \{\beta_i\} \mid I \in (I_i \cap D_i \cap P) \setminus I' \cap D'_i, I \cap D'_{3-i} \neq \emptyset\} \cup \\ \{I \cup (D'_{3-i} \setminus P) \cup \{\beta_i\} \mid I \in (I_i \cap D_i \cap P) \setminus I' \cap D'_i, I \cap D'_{3-i} = \emptyset\}$$

and $\alpha_i, \beta_i \in \text{Ports} \setminus (D' \cup \bigcup I')$ are pairwise distinct ports that do not occur in \mathcal{A}' , respectively, for $i = 1, 2$. Next, we prove the following facts:

Fact 4 For any $S \in 2^{2^p}$, we have $\tau_{\mathcal{A}'_i, P}(S) = \mu_i(S)$, for each $i = 1, 2$.

Proof: We prove the case $i = 1$, the case $i = 2$ being identical. Let $x \in D'_1 \setminus P$ be an arbitrary port and $S \in 2^{2^p}$ be a set of visible interactions. We have:

$$\begin{aligned} x \in \tau_{\mathcal{A}'_1, P}(S) &\iff I'_1 \cap \{x\} \cap P = S \\ &\iff \{I \cap P \mid I \in I'_1, x \in I\} = S \\ &\iff \{I \cap P \mid I \in I' \cap D'_1, x \in I\} \cup \{I \cap P \mid I \in X_1, x \in I\} \cup \{I \cap P \mid I \in Y_1, x \in I\} = S \\ &\iff \{I \cap P \mid I \in I', x \in I\} \cup \{I \cap P \mid I \in X_1, x \in I\} = S, \text{ since } x \in D'_1 \setminus P \\ &\iff \lambda(x) \cup (\lambda_1(x) \setminus \lambda(x)) = S \\ &\iff \lambda(x) \cup \lambda_1(x) = S \end{aligned}$$

It is sufficient to prove $\lambda(x) \subseteq \lambda_1(x)$ in order to obtain $x \in \tau_{\mathcal{A}'_1, P}(\mathcal{S}) \iff \lambda_1(x) = \mathcal{S} \iff x \in \mu_1(\mathcal{S})$, as required. Since $x \in D'_1$, by the definition of μ_1 , it must be the case that $x \in \tau_{\mathcal{A}', P}(\mathcal{S}')$, for some $\mathcal{S}' \subseteq \mathcal{S}$. Then $\lambda(x) = \mathcal{S}' \subseteq \mathcal{S} = \lambda_1(x)$ follows. \square

Next, we prove that $\mathcal{A}_i \approx_P^{n-1} \mathcal{A}'_i$, for $i = 1, 2$. Again, we consider only the case $i = 1$, the other case being identical. We consider the three points of Definition 5 below:

(1) $D'_1 \cap P = (D_1 \cap P) \cup \bigcup_{\mathcal{S} \in 2^{2^P}} (\mu_1(\mathcal{S}) \cap P) = D_1 \cap P$, because $\mu_1(\mathcal{S}) \subseteq D'_1 \setminus P$, and thus $\mu_1(\mathcal{S}) \cap P = \emptyset$, for any $\mathcal{S} \in 2^{2^P}$.

(2) We need to show that $I_1^{\cap(D_1 \cap P)} \sqcap P = I_1'^{\cap(D'_1 \cap P)} \sqcap P$. Note that $I_1 = I^{\cap D_1} \cup A_1 \cup B_1$, where:

$$\begin{aligned} A_1 &= \{I \mid I \in \mathcal{I}_1 \setminus \mathcal{I}, I \cap (D_1 \setminus P) \neq \emptyset\} \\ B_1 &= \{I \mid I \in \mathcal{I}_1 \setminus \mathcal{I}, I \cap D_1 \subseteq P\} \end{aligned}$$

This is because $\mathcal{I} = (\mathcal{I}_1 \cap \mathcal{I}_2) \cup (\mathcal{I}_1 \cap 2^{\bar{D}_2}) \cup (\mathcal{I}_2 \cap 2^{\bar{D}_1})$, hence $I^{\cap D_1} = I \cap \mathcal{I}_1$ and $I_1 = (I \cap \mathcal{I}_1) \cup (\mathcal{I}_1 \setminus \mathcal{I}) = I^{\cap D_1} \cup (A_1 \cup B_1)$.

Fact 5 $I^{\cap(D_1 \cap P)} \sqcap P = I'^{\cap(D'_1 \cap P)} \sqcap P$

Proof: “ \subseteq ” Let $I \in I^{\cap(D_1 \cap P)}$ be an interaction. Then $I \cap D_1 \cap P \neq \emptyset$ and consequently $I \cap D \cap P \neq \emptyset$. But then $I \cap P \in (I^{\cap(D \cap P)}) \sqcap P = (I'^{\cap(D' \cap P)}) \sqcap P$, because $\mathcal{A} \approx_P^n \mathcal{A}'$, by Definition 5 (2). Then there exists $I' \in I'^{\cap(D' \cap P)}$ such that $I \cap P = I' \cap P$. Hence $I' \cap P \cap D_1 \neq \emptyset$ and $I' \in I'^{\cap(D_1 \cap P)}$, which implies $I \cap P = I' \cap P \in I'^{\cap(D_1 \cap P)} \sqcap P$. The other direction is symmetric. \square

Fact 6 $(I \cup A_1)^{\cap(D_1 \cap P)} \sqcap P = (I' \cup X_1)^{\cap(D_1 \cap P)} \sqcap P$

Proof: “ \subseteq ” Let $I \in (I \cup A_1)^{\cap(D_1 \cap P)}$ be an interaction. If $I \in I^{\cap(D_1 \cap P)}$ then $I \cap P \in I'^{\cap(D_1 \cap P)} \sqcap P$, by Fact 5. Assume that $I \in A_1^{\cap(D_1 \cap P)}$, then $I \in \mathcal{I}_1 \setminus \mathcal{I}$, $I \cap (D_1 \setminus P) \neq \emptyset$ and $I \cap D_1 \cap P \neq \emptyset$. Since $I \cap (D_1 \setminus P) \neq \emptyset$, there exists $x \in I \cap (D_1 \setminus P)$ and let $\mathcal{S} \stackrel{\text{def}}{=} \text{vt}_{\mathcal{A}_1, P}(x)$. Then $x \in \tau_{\mathcal{A}_1, P}(\mathcal{S})$ and, by the definition of μ_1 , there exists $x' \in D'_1 \setminus P$ such that $x' \in \mu_1(\mathcal{S})$ and, consequently $\lambda_1(x') = \mathcal{S}$. Since $x \in I \cap (D_1 \setminus P)$ and $\mathcal{S} = \text{vt}_{\mathcal{A}_1, P}(x)$ we have $I \cap P \in \mathcal{S}$ and thus $I \cap P \in \lambda_1(x')$. We distinguish the following cases:

- if $I \cap P \notin \lambda(x')$ then $I \cap P \in X_1^{\cap(D_1 \cap P)} \sqcap P$.
- else, $I \cap P \in \lambda(x')$ and, because $x' \in D'_1 \setminus P \subseteq D' \setminus P$, there exists $I' \in \mathcal{I}'$ such that $I \cap P = I' \cap P$. Moreover, since $(I \cap P) \cap (D_1 \cap P) \neq \emptyset$, we obtain $I \cap P = I' \cap P \in I'^{\cap(D_1 \cap P)} \sqcap P$.

“ \supseteq ” Let $I \in (I' \cup X_1)^{\cap(D_1 \cap P)}$ be an interaction. If $I \in I'^{\cap(D_1 \cap P)}$ then $I \cap P \in I^{\cap(D_1 \cap P)}$, by Fact 5. Assume that $I \in X_1^{\cap(D_1 \cap P)}$, then there exists $x' \in D'_1 \setminus P$, such that $I \cap P \in \lambda_1(x') \setminus \lambda(x')$. By the definition of λ_1 , there exists $x \in D_1 \setminus P$, such that $I \cap P \in \mathcal{I}_1^{\cap \{x\}} \sqcap P$. Then there exists an interaction $J \in \mathcal{I}_1$ such that $x \in J$ and $J \cap P = I \cap P$. Since, moreover, $(I \cap P) \cap (D_1 \cap P) \neq \emptyset$, we have $J \cap D_1 \cap P \neq \emptyset$, hence $J \in \mathcal{I}_1^{\cap(D_1 \cap P)}$. Since $x \in J$ and $x \in D_1 \setminus P$, we have $J \cap (D_1 \setminus P) \neq \emptyset$, hence $J \notin B_1$. Then $J \in (I \cup A_1)^{\cap(D_1 \cap P)}$ and $I \cap P = J \cap P \in (I \cup A_1)^{\cap(D_1 \cap P)} \sqcap P$. \square

Back to point (2) of Definition 5, it suffices to show the following points:

- $B_1^{\cap(D_1 \cap P)} \subseteq \mathcal{I}'_1^{\cap(D_1 \cap P)} \sqcap P$: Let $I \in B_1^{\cap(D_1 \cap P)}$ be an interaction. Then $I \cap P = I$ and $I \cap D_1 \cap P \neq \emptyset$. If $I \in (\mathcal{I} \cup A_1)^{\cap(D_1 \cap P)}$ then $I = I \cap P \in (\mathcal{I} \cup A_1)^{\cap(D_1 \cap P)} \sqcap P = (\mathcal{I}' \cup X_1)^{\cap(D_1 \cap P)} \sqcap P$, by Fact 6. Then $I \in \mathcal{I}'_1^{\cap(D_1 \cap P)} \sqcap P$. Otherwise, $I \notin (\mathcal{I} \cup A_1)^{\cap(D_1 \cap P)}$ then $I \notin (\mathcal{I} \cup A_1)^{\cap(D_1 \cap P)} \sqcap P = (\mathcal{I}' \cup X_1)^{\cap(D_1 \cap P)} \sqcap P$, by Fact 5, thus $I \notin (\mathcal{I}' \cup X_1)^{\cap(D_1 \cap P)}$. But because $I \in B_1^{\cap(D_1 \cap P)}$, we have $I \in \mathcal{I}_1^{\cap(D_1 \cap P)}$, thus $I \in Y_1 \sqcap P \subseteq \mathcal{I}'_1^{\cap(D_1 \cap P)} \sqcap P$.
- $Y_1^{\cap(D_1 \cap P)} \sqcap P \subseteq \mathcal{I}_1^{\cap(D_1 \cap P)} \sqcap P$: If $I \in Y_1^{\cap(D_1 \cap P)}$, then $I \cap P \in (\mathcal{I}_1^{\cap(D_1 \cap P)}) \sqcap P$, by the definition of Y_1 .

(3) by Fact 4, it is sufficient to prove that, for all $\mathcal{S} \in 2^{2^P}$:

1. $\|\tau_{\mathcal{A}_1, P}(\mathcal{S})\| < b_P(n-1, \mathcal{S}) \Rightarrow \|\mu_1(\mathcal{S})\| = \|\tau_{\mathcal{A}_1, P}(\mathcal{S})\|$:

Let $\mathcal{S} \in 2^{2^P}$ be an arbitrary set of interactions such that $\|\tau_{\mathcal{A}_1, P}(\mathcal{S})\| < b_P(n-1, \mathcal{S})$ and let $x \in \mu_1(\mathcal{S})$ be a port. We shall exhibit a unique port $y \in \tau_{\mathcal{A}_1, P}(\mathcal{S})$ in order to prove that $\|\mu_1(\mathcal{S})\| \leq \|\tau_{\mathcal{A}_1, P}(\mathcal{S})\|$. By the definition of μ_1 , there exists a set $\mathcal{S}' \subseteq \mathcal{S}$ such that $x \in \tau_{\mathcal{A}', P}(\mathcal{S}')$. We distinguish the following cases:

- if $\|\tau_{\mathcal{A}, P}(\mathcal{S}')\| = \|\tau_{\mathcal{A}', P}(\mathcal{S}')\|$ then let $y = \pi_{\mathcal{S}'}(x) \in \tau_{\mathcal{A}_1, P}(\mathcal{S})$, where $\pi_{\mathcal{S}'} : \tau_{\mathcal{A}', P}(\mathcal{S}') \rightarrow \tau_{\mathcal{A}, P}(\mathcal{S}')$ is the bijection from (1).
- else, if $\|\tau_{\mathcal{A}, P}(\mathcal{S}')\| \neq \|\tau_{\mathcal{A}', P}(\mathcal{S}')\|$ and $\|\tau_{\mathcal{A}, P}(\mathcal{S}') \cap D_1\| < \frac{b_P(n, \mathcal{S}')}{2}$ then let $y = \rho_{\mathcal{S}'}(x) \in \tau_{\mathcal{A}_1, P}(\mathcal{S})$, where $\rho_{\mathcal{S}'}$ is the bijection from (2).
- otherwise, if $\|\tau_{\mathcal{A}, P}(\mathcal{S}')\| \neq \|\tau_{\mathcal{A}', P}(\mathcal{S}')\|$ and $\|\tau_{\mathcal{A}, P}(\mathcal{S}') \cap D_1\| \geq \frac{b_P(n, \mathcal{S}')}{2}$ then, because $\|\tau_{\mathcal{A}_1, P}(\mathcal{S})\| < b_P(n-1, \mathcal{S})$, we obtain, by (4) that:

$$\|\mu_1(\mathcal{S}) \cap E'_1\| = \|\tau_{\mathcal{A}_1, P}(\mathcal{S}) \cap \tau_{\mathcal{A}, P}(\mathcal{S}') \cap D_1\|$$

where $E'_1 \subseteq \tau_{\mathcal{A}', P}(\mathcal{S}')$ is such that $x \in E'_1$. Then there exists a bijection $\xi : \mu_1(\mathcal{S}) \cap E'_1 \rightarrow \tau_{\mathcal{A}_1, P}(\mathcal{S}) \cap \tau_{\mathcal{A}, P}(\mathcal{S}') \cap D_1$ and let $y = \xi(x) \in \tau_{\mathcal{A}_1, P}(\mathcal{S})$.

The unique $y \in \tau_{\mathcal{A}_1, P}(\mathcal{S})$ is defined as the image of x via a bijection that chooses among the above disjoint cases. Moreover, since these are the only cases that explain why $x \in \mu_1(\mathcal{S})$, i.e. nothing else is in $\mu_1(\mathcal{S})$, we obtain that $\|\mu_1(\mathcal{S})\| = \|\tau_{\mathcal{A}_1, P}(\mathcal{S})\|$, as required.

2. $\|\tau_{\mathcal{A}_1, P}(\mathcal{S})\| \geq b_P(n-1, \mathcal{S}) \Rightarrow \|\mu_1(\mathcal{S})\| \geq b_P(n-1, \mathcal{S})$:

Let $\mathcal{S} \in 2^{2^P}$ be a set of interactions such that $\|\tau_{\mathcal{A}_1, P}(\mathcal{S})\| \geq b_P(n, \mathcal{S})$. By the definition of μ_1 , for each $x \in \mu_1(\mathcal{S})$ there exists $\mathcal{S}' \subseteq \mathcal{S}$ such that $x \in \tau_{\mathcal{A}', P}(\mathcal{S}')$ and let $\mathcal{S}_1, \dots, \mathcal{S}_k \subseteq \mathcal{S}$ be all the sets of interactions such that $\tau_{\mathcal{A}', P}(\mathcal{S}_i) \cap \mu_1(\mathcal{S}) \neq \emptyset$. Moreover, the sets $\tau_{\mathcal{A}', P}(\mathcal{S}_i)$ are pairwise disjoint and $\mu_1(\mathcal{S}) = \bigcup_{i=1}^k \mu_1(\mathcal{S}) \cap \tau_{\mathcal{A}', P}(\mathcal{S}_i)$, leading to $\|\mu_1(\mathcal{S})\| = \sum_{i=1}^k \|\mu_1(\mathcal{S}) \cap \tau_{\mathcal{A}', P}(\mathcal{S}_i)\|$. For an arbitrary $1 \leq i \leq k$, we distinguish the following cases:

- if $\|\tau_{\mathcal{A}, P}(\mathcal{S}_i)\| = \|\tau_{\mathcal{A}', P}(\mathcal{S}_i)\|$, then by (1), we have:

$$\|\mu_1(\mathcal{S}) \cap \tau_{\mathcal{A}', P}(\mathcal{S}_i)\| = \|\tau_{\mathcal{A}_1, P}(\mathcal{S}) \cap \tau_{\mathcal{A}, P}(\mathcal{S}_i)\|$$

- else, if $\|\tau_{\mathcal{A},P}(\mathcal{S}_i)\| \neq \|\tau_{\mathcal{A}',P}(\mathcal{S}_i)\|$ and $\|\tau_{\mathcal{A},P}(\mathcal{S}_i) \cap D_1\| < \frac{b_P(n, \mathcal{S}_i)}{2}$, then by (2), we have:

$$\|\mu_1(\mathcal{S}) \cap \tau_{\mathcal{A}',P}(\mathcal{S}_i)\| = \|\tau_{\mathcal{A}_1,P}(\mathcal{S}) \cap \tau_{\mathcal{A},P}(\mathcal{S}_i)\|$$

- otherwise, if $\|\tau_{\mathcal{A},P}(\mathcal{S}_i)\| \neq \|\tau_{\mathcal{A}',P}(\mathcal{S}_i)\|$ and $\|\tau_{\mathcal{A},P}(\mathcal{S}_i) \cap D_1\| \geq \frac{b_P(n, \mathcal{S}_i)}{2}$, since $\|\tau_{\mathcal{A}_1,P}(\mathcal{S})\| \geq b_P(n-1, \mathcal{S})$, by (4) we obtain:

$$\|\mu_1(\mathcal{S}) \cap \tau_{\mathcal{A}',P}(\mathcal{S}_i)\| \geq b_P(n-1, \mathcal{S})$$

If $\|\mu_1(\mathcal{S}) \cap \tau_{\mathcal{A}',P}(\mathcal{S}_i)\| \geq b_P(n-1, \mathcal{S})$ for some $1 \leq i \leq k$, then $\|\mu_1(\mathcal{S})\| \geq b_P(n-1, \mathcal{S})$ and we are done. Otherwise, we compute:

$$\|\mu_1(\mathcal{S})\| = \sum_{i=1}^k \|\mu_1(\mathcal{S}) \cap \tau_{\mathcal{A}',P}(\mathcal{S}_i)\| = \sum_{i=1}^k \|\tau_{\mathcal{A}_1,P}(\mathcal{S}) \cap \tau_{\mathcal{A},P}(\mathcal{S}_i)\| = \|\tau_{\mathcal{A}_1,P}(\mathcal{S})\| \geq b_P(n-1, \mathcal{S})$$

The latter equality is by definition of μ_1 and the assumption $\|\mu_1(\mathcal{S}) \cap \tau_{\mathcal{A}',P}(\mathcal{S}_i)\| < b_P(n-1, \mathcal{S})$, for all $1 \leq i \leq k$.

Finally, we must prove that $\mathcal{A}'_1 \uplus \mathcal{A}'_2 = \mathcal{A}'$. We compute:

$$\begin{aligned} D'_1 \cup D'_2 &= ((D_1 \cup D_2) \cap P) \cup \bigcup_{\mathcal{S} \in 2^{2^P}} \mu_1(\mathcal{S}) \cup \bigcup_{\mathcal{S} \in 2^{2^P}} \mu_2(\mathcal{S}) \\ &= (D \cap P) \cup (D' \setminus P), \text{ because } \{\mu_1(\mathcal{S})\}_{\mathcal{S} \in 2^{2^P}} \cup \{\mu_2(\mathcal{S})\}_{\mathcal{S} \in 2^{2^P}} \text{ partitions } D' \setminus P \\ &= (D' \cap P) \cup (D' \setminus P), \text{ since } \mathcal{A} \approx_n^P \mathcal{A}' \text{ thus } D' \cap P = D \cap P \\ &= D' \end{aligned}$$

By the definition of \mathcal{I}'_i , $i = 1, 2$, we have:

$$\mathcal{I}'_1 \cap \mathcal{I}'_2 = (\mathcal{I}'^{\cap(D'_1)} \cup X_1 \cup Y_1) \cap (\mathcal{I}'^{\cap(D'_2)} \cup X_2 \cup Y_2)$$

and prove that $\mathcal{I}'_1 \cap \mathcal{I}'_2 = \mathcal{I}'^{\cap(D'_1)} \cap \mathcal{I}'^{\cap(D'_2)}$, by showing the following:

- $\mathcal{I}'^{\cap(D'_1)} \cap X_2 = \emptyset$: if there exists an interaction $I \in \mathcal{I}'^{\cap(D'_1)} \cap X_2$, then $\alpha_2 \in I$, by the definition of X_2 , and $I \notin \mathcal{I}'$, because $\alpha_2 \notin \bigcup \mathcal{I}'$, contradiction.
- $\mathcal{I}'^{\cap(D'_1)} \cap Y_2 = \emptyset$: if there exists an interaction $I \in \mathcal{I}'^{\cap(D'_1)} \cap Y_2$, then $\beta_2 \in I$, by the definition of Y_2 and $I \notin \mathcal{I}'$, because $\beta_2 \notin \bigcup \mathcal{I}'$, contradiction.
- $X_1 \cap X_2 = \emptyset$: if there exists an interaction $I \in X_1 \cap X_2$, then $\alpha_1 \in I$, by the definition of X_1 and thus $I \notin X_2$, by the definition of X_2 , contradiction.
- $Y_1 \cap X_2 = \emptyset$: if there exists an interaction $I \in Y_1 \cap X_2$, then $\beta_1 \in I$, by the definition of Y_1 and thus $I \notin X_2$, by the definition of X_2 , contradiction.
- $Y_1 \cap Y_2 = \emptyset$: if there exists an interaction $I \in Y_1 \cap Y_2$, then $\beta_1 \in Y_1$, by the definition of Y_1 and thus $I \notin Y_2$, by the definition of Y_2 , contradiction.

Proving the emptiness of the remaining sets $X_1 \cap \mathcal{I}'^{\cap(D'_2)}$, $X_1 \cap Y_2$ and $Y_1 \cap \mathcal{I}'^{\cap(D'_2)}$ is done symmetrically. Also, by the definition of X_i and Y_i , for $i = 1, 2$, we have that $I \cap D'_1 \neq \emptyset$ and $I \cap D'_2 \neq \emptyset$, for all $I \in \bigcup_{i=1,2} X_i \cup Y_i$. Consequently, we obtain:

$$\mathcal{I}'_i \cap 2^{\overline{D'}_{3-i}} = \mathcal{I}'^{\cap(D'_i)}, \text{ for } i = 1, 2$$

and conclude the proof as follows:

$$(\mathcal{I}'_1 \cap \mathcal{I}'_2) \cup (\mathcal{I}'_1 \cap 2^{\overline{D}'_2}) \cup (\mathcal{I}'_2 \cap 2^{\overline{D}'_1}) = (\mathcal{I}'^{\cap(D'_1)} \cap \mathcal{I}'^{\cap(D'_2)}) \cup (\mathcal{I}'^{\cap(D'_1)} \cap 2^{\overline{D}'_2}) \cup (\mathcal{I}'^{\cap(D'_2)} \cap 2^{\overline{D}'_1}) = \mathcal{I}'$$

□

The next theorem proves that the architectures which are equivalent in the sense of Definition 5 cannot be distinguished by SIL* formulae up to a given bound, defined recursively on the structure of formulae:

$$\begin{aligned} \text{bnd}(\text{emp}) &\stackrel{\text{def}}{=} 1 & \text{bnd}(p \multimap b) &\stackrel{\text{def}}{=} 1 \\ \text{bnd}(p \multimap b) &\stackrel{\text{def}}{=} 1 & \text{bnd}(p \overset{\exists}{\multimap} b) &\stackrel{\text{def}}{=} 1 \\ \text{bnd}(\psi_1 \wedge \psi_2) &\stackrel{\text{def}}{=} \max(\text{bnd}(\psi_1), \text{bnd}(\psi_2)) & \text{bnd}(\neg\psi_1) &\stackrel{\text{def}}{=} \text{bnd}(\psi_1) \\ \text{bnd}(\psi_1 * \psi_2) &\stackrel{\text{def}}{=} \max(\text{bnd}(\psi_1), \text{bnd}(\psi_2)) + 1 & & \end{aligned}$$

Theorem 3. Let $\mathcal{A} = \langle D, \mathcal{I} \rangle$ and $\mathcal{A}' = \langle D', \mathcal{I}' \rangle$ be architectures, $P \in 2^{\text{Ports}}$ be a set of ports and $n \geq 1$ be an integer, such that $\mathcal{A} \approx_p^n \mathcal{A}'$. Then, for any formula ψ of SIL*, such that $\text{P}(\psi) \subseteq P$ and $\text{bnd}(\psi) \leq n$, we have $\mathcal{A} \models \psi$ if and only if $\mathcal{A}' \models \psi$.

Proof: By induction on the structure of ψ . We consider the cases:

- **emp:** if $\mathcal{A} \models \text{emp}$ then $D = \emptyset$ and $\mathcal{I} = \emptyset$. Since $\mathcal{A} \approx_p^n \mathcal{A}'$, we have $D \cap P = D' \cap P = \emptyset$. Suppose, for a contradiction, that there exists a port $x \in D' \setminus P$. Then there exists a set $\mathcal{S} \in 2^{2^P}$ such that $x \in \tau_{\mathcal{A}', P}(\mathcal{S})$, hence $\|\tau_{\mathcal{A}', P}(\mathcal{S})\| \geq 1$. Since $\mathcal{A} \approx_p^n \mathcal{A}'$, it must be that $\|\tau_{\mathcal{A}, P}(\mathcal{S})\| \geq 1$, which contradicts with $\|\tau_{\mathcal{A}, P}(\mathcal{S})\| = 0$, a consequence of $D = \emptyset$. Hence $D' = \emptyset$ and $\mathcal{I}' = \emptyset$ follows, since \mathcal{A}' is an architecture, thus $\mathcal{A}' \models \text{emp}$. The other direction is symmetrical.
- **$p \multimap b$:** if $\mathcal{A} \models p \multimap b$, we have $D = \{p\}$ and $I \vdash b$, for all $I \in \mathcal{I}$. Since $\mathcal{A} \approx_p^n \mathcal{A}'$ and $p \in \text{P}(p \multimap b) \subseteq P$, we obtain $D \cap P = D' \cap P = \{p\}$. Moreover, $D' \setminus P = \emptyset$ follows in the same way as above and thus $D' = \{p\}$. Since $\mathcal{A} \approx_p^n \mathcal{A}'$, we have $\mathcal{I}^{\cap P} \cap P = \mathcal{I}'^{\cap P} \cap P$. Let $I' \in \mathcal{I}'$ be an interaction. Then $I' = J \cup U'$, where $J \subseteq P$ and $U' \cap P = \emptyset$. Consequently, there exists an interaction $I = J \cap U$, for some $U \cap P = \emptyset$. Moreover, since $I \vdash b$ and $\text{P}(b) \subseteq P$, we have $J \vdash b$, thus $I' \vdash b$ and $\mathcal{A}' \models p \multimap b$ follows.
- **$p \overset{\exists}{\multimap} b$:** by an argument similar to the point above.
- **$p \multimap b$:** by an argument similar to the point above.
- **$\psi_1 \wedge \psi_2$:** if $\mathcal{A} \models \psi_1 \wedge \psi_2$ then $\mathcal{A} \models \psi_i$, for $i = 1, 2$. By the induction hypothesis, since $\text{bnd}(\psi_i) \leq \max(\text{bnd}(\psi_1), \text{bnd}(\psi_2)) = \text{bnd}(\psi_1 \wedge \psi_2)$, we obtain $\mathcal{A}' \models \psi_i$, for $i = 1, 2$, hence $\mathcal{A}' \models \psi_1 \wedge \psi_2$.

- $\neg\psi_1$: by a direct application of the induction hypothesis.
- $\psi_1 * \psi_2$: if $\mathcal{A} \models \phi_1 * \phi_2$ then there exist architectures $\mathcal{A}_i \models \phi_i$, for $i = 1, 2$, such that $\mathcal{A} = \mathcal{A}_1 \uplus \mathcal{A}_2$. By Lemma 10, because $\mathcal{A} \approx_p^n \mathcal{A}'$, there exist architectures \mathcal{A}'_i , such that $\mathcal{A}_i \approx_p^{n-1} \mathcal{A}'_i$, for $i = 1, 2$ and $\mathcal{A}' = \mathcal{A}'_1 \uplus \mathcal{A}'_2$. By the induction hypothesis, since $\text{bnd}(\psi_i) \leq \max(\text{bnd}(\psi_1), \text{bnd}(\psi_2)) \leq n - 1$, we obtain that $\mathcal{A}'_i \models \psi_i$, for $i = 1, 2$, and thus $\mathcal{A}' \models \psi_1 * \psi_2$. \square

Next, we move on to the definition of test formulae for SIL*:

Definition 6. Given a set of port symbols $P \subseteq \text{PSym}$ and an integer $n \geq 1$, we denote by $\text{TestForm}(P, n)$ the following set of formulae, for each $p, q_1, \dots, q_k \in P$ and each $1 \leq m \leq b_P(n, \emptyset)$:

$$\begin{aligned} \text{has}(p) &\stackrel{\text{def}}{=} p \multimap p \multimap * \perp \quad p \overset{\exists}{\multimap} q_1 \dots q_k \stackrel{\text{def}}{=} p \overset{\exists}{\multimap} q_1 \dots q_k * \top \\ \text{type}(q_1, \dots, q_k) \geq m &\stackrel{\text{def}}{=} \underbrace{(\exists x. x \overset{\exists}{\multimap} q_1 \dots q_k) * \dots * (\exists x. x \overset{\exists}{\multimap} q_1 \dots q_k)}_{m \text{ times}} \end{aligned}$$

Given architectures \mathcal{A}_1 and \mathcal{A}_2 , we write $\mathcal{A}_1 \cong_p^n \mathcal{A}_2$ for $\mathcal{A}_1 \models \psi \iff \mathcal{A}_2 \models \psi$, for all $\psi \in \text{TestForm}(P, n)$.

The following lemma proves that the equivalence of architectures via test formulae is a refinement of the equivalence relation introduced by Definition 5.

Lemma 11. Given a set of ports $P \in 2^{\text{Ports}}$ and an integer $n \geq 1$, for any two architectures $\mathcal{A}_i = \langle D_i, I_i \rangle$, for $i = 1, 2$, we have $\mathcal{A}_1 \approx_p^n \mathcal{A}_2$ if $\mathcal{A}_1 \cong_p^n \mathcal{A}_2$.

Proof: We prove the three points of Definition 5:

(1) Suppose, for a contradiction, that $D_1 \cap P \not\subseteq D_2 \cap P$, thus there exists a port $p \in D_1 \cap P$ such that $p \notin D_2$. Then $\mathcal{A}_1 \models \text{has}(p)$ and $\mathcal{A}_2 \not\models \text{has}(p)$. Since $p \in P$ thus $\text{has}(p) \in \text{TestForm}(P, n)$, we reached a contradiction with $\mathcal{A}_1 \cong_p^n \mathcal{A}_2$. Hence $D_1 \cap P \subseteq D_2 \cap P$ and the other direction is symmetrical.

(2) Suppose, for a contradiction, that $I_1 \cap^{(D_1 \cap P)} P \not\subseteq I_2 \cap^{(D_2 \cap P)} P$. Then, there exists an interaction $J \in I_1 \cap^{(D_1 \cap P)} P$ such that $J \notin I_2 \cap^{(D_2 \cap P)} P$. Let $p \in J \cap D_1$ be a port (we know that one exists because $J \in I_1 \cap^{(D_1 \cap P)} P$) and let $\{q_1, \dots, q_k\} \stackrel{\text{def}}{=} J \setminus \{p\}$. Since $p \in J \cap D_1$ and $J \subseteq P$, by the previous point, we have $p \in D_2 \cap P$. We have $\mathcal{A}_1 \models p \overset{\exists}{\multimap} q_1 \dots q_k$ and since $\mathcal{A}_1 \cong_p^n \mathcal{A}_2$, we obtain $\mathcal{A}_2 \models p \overset{\exists}{\multimap} q_1 \dots q_k$. But then we obtain $J \in I_2 \cap^{(D_2 \cap P)} P$, contradiction. Hence $I_1 \cap^{(D_1 \cap P)} P \subseteq I_2 \cap^{(D_2 \cap P)} P$ and the other direction is symmetrical.

(3) Let $S \in 2^{\text{Ports}}$ be a set of visible ports. We distinguish the following cases:

- if $\|\tau_{\mathcal{A}_1, P}(S)\| < b_P(n, S)$ and $\|\tau_{\mathcal{A}_2, P}(S)\| \neq \|\tau_{\mathcal{A}_1, P}(S)\|$ then let $m = \|\tau_{\mathcal{A}_1, P}(S)\|$. We have $\mathcal{A}_1 \models \text{type}(S) \geq m \wedge \neg \text{type}(S) \geq m + 1$ and $\mathcal{A}_2 \not\models \text{type}(S) \geq m \wedge \neg \text{type}(S) \geq m + 1$. Since $m + 1 \leq b_P(n, S) \leq b_P(n, \emptyset)$, we obtain that $\text{type}(S) \geq m, \text{type}(S) \geq m + 1 \in \text{TestForm}(P, n)$, thus $\mathcal{A}_1 \not\cong_p^n \mathcal{A}_2$, contradiction. Then $\|\tau_{\mathcal{A}_1, P}(S)\| < b_P(n, S) \Rightarrow \|\tau_{\mathcal{A}_2, P}(S)\| = \|\tau_{\mathcal{A}_1, P}(S)\|$.

- if $\|\tau_{\mathcal{A}_1, P}(\mathcal{S})\| \geq b_P(n, \mathcal{S})$ and $\|\tau_{\mathcal{A}_2, P}(\mathcal{S})\| < b_P(n, \mathcal{S})$ then let $\|\tau_{\mathcal{A}_2, P}(\mathcal{S})\| = m$. We have $\mathcal{A}_2 \models \text{type}(\mathcal{S}) \geq m \wedge \neg \text{type}(\mathcal{S}) \geq m + 1$ and $\mathcal{A}_1 \not\models \text{type}(\mathcal{S}) \geq m \wedge \neg \text{type}(\mathcal{S}) \geq m + 1$. Since $m + 1 \leq b_P(n, \mathcal{S}) \leq b_P(n, \emptyset)$, we obtain that $\mathcal{A}_1 \not\equiv_P^n \mathcal{A}_2$, contradiction. Then $\|\tau_{\mathcal{A}_1, P}(\mathcal{S})\| \geq b_P(n, \mathcal{S}) \Rightarrow \|\tau_{\mathcal{A}_2, P}(\mathcal{S})\| \geq b_P(n, \mathcal{S})$. \square

A first consequence of this result is that every formula of SIL^* is equivalent to a finite boolean combination of test formulae.

Corollary 3. *Each formula ψ of SIL^* is equivalent to a finite boolean combination of test formulae from $\text{TestForm}(P(\psi), \text{bnd}(\psi))$.*

Proof: The proof is the same as for Corollary 1. \square

The other consequence is a small model property for the SIL^* fragment, which entails the decidability of its satisfiability problem.

Corollary 4. *If ψ is a satisfiable SIL^* formula has a model $\mathcal{A} = \langle D, I \rangle$ such that $\|D\| \leq B$ and $\|I\| \leq B$, for each $I \in \mathcal{I}$, where $B = \mathcal{O}(2^{\|\mathbf{P}(\psi)\|}) \cdot b_{P(\psi)}(\text{bnd}(\psi), \emptyset)$.*

Proof: Let $\mathcal{A}' = \langle D', I' \rangle$ be the architecture obtained from \mathcal{A} as follows :

- remove from D and from each $I \in \mathcal{I}$ enough many ports $p \in D \setminus P(\psi)$, such that $\|\tau_{\mathcal{A}', P(\psi)}(\mathcal{S})\| \leq b_{P(\psi)}(\text{bnd}(\psi), \emptyset)$, for each $\mathcal{S} \in 2^{2^{\text{Ports}}}$, and
- remove from each from each $I \in \mathcal{I}$ all ports $p \in I \setminus (D \cup P(\psi))$.

It is easy to check that $\mathcal{A} \approx_{P(\psi)}^{b_{P(\psi)}(\text{bnd}(\psi), \emptyset)} \mathcal{A}'$ thus, by Theorem 3, we obtain that $\mathcal{A}' \models \psi$. Further, we compute:

$$\begin{aligned} \|D'\| &= \|D' \cap P(\psi)\| + \|D' \setminus P(\psi)\| \\ &= \|D' \cap P(\psi)\| + \sum_{\mathcal{S} \in 2^{2^{\text{Ports}}}} \|\tau_{\mathcal{A}', P(\psi)}(\mathcal{S})\| \\ &\leq \|P(\psi)\| + 2^{\|\mathbf{P}(\psi)\|} \cdot b_{P(\psi)}(\text{bnd}(\psi), \emptyset) = B \end{aligned}$$

Let $I \in \mathcal{I}'$ be an interaction. We compute:

$$\begin{aligned} \|I\| &= \|D \cap I\| + \|I \setminus D\| \\ &\leq \|D\| + \|P(\psi)\| = B \quad \square \end{aligned}$$

5.3 Decidability of Component-based Extensions of SIL

In this section we extend the decidability results from §5.2 and §5.1 to fragments of the logic SL_a obtained by considering variables $i, j \in \text{IVars}$ ranging over component identifiers and function symbols $p, q \in \text{PFun}$, interpreted as functions mapping component identifiers to ports. Moreover, we allow equality atoms $i = j$ and port terms $\rho(i)$ to occur anywhere a port symbol $p \in \text{PSym}$ is allowed to occur in SIL^* and SIL^+ , respectively.

6 Behaviors of Component-based Systems

In this section we define the *behaviors* of an architecture, which are the sequences of interaction events, ordered by the moment in time when the events occur. We consider systems consisting of finitely many components, with no à priori bound on their number, that are replicas of a small number of finite-state machines.

Formally, a *finite-state machine* (FSM) is a pair $M = (\mathbf{Q}, \rightarrow)$, where \mathbf{Q} is a finite set of states and $\rightarrow \subseteq \mathbf{Q} \times 2^{\text{Ports}} \times \mathbf{Q}$ is a transition relation, where $q \xrightarrow{I} q'$ stands for $(q, I, q') \in \rightarrow$. We denote by $\Sigma(M) \stackrel{\text{def}}{=} \{I \mid \exists q, q' \in \mathbf{Q} . q \xrightarrow{I} q'\}$ the set of transition labels of M . We write $(\mathbf{Q}_1, \rightarrow_1) \subseteq (\mathbf{Q}_2, \rightarrow_2)$ for $\mathbf{Q}_1 = \mathbf{Q}_2$ and $\rightarrow_1 \subseteq \rightarrow_2$.

The *asynchronous product* of two FSMs $M_i = (\mathbf{Q}_i, \rightarrow_i)$, for $i = 1, 2$, is the FSM $M_1 \otimes M_2 \stackrel{\text{def}}{=} (\mathbf{Q}_1 \times \mathbf{Q}_2, \rightarrow)$ where, for all $q_i, q'_i \in \mathbf{Q}_i$, $i = 1, 2$ and all $I \in 2^{\text{Ports}}$, $(q_1, q_2) \xrightarrow{I} (q'_1, q'_2)$ if and only if one of the following holds:

- $q_1 \xrightarrow{I} q'_1$ and $q_2 \xrightarrow{I} q'_2$,
- $q_1 \xrightarrow{I} q'_1$, $q_2 = q'_2$ and $I \notin \Sigma(M_2)$,
- $q_2 \xrightarrow{I} q'_2$, $q_1 = q'_1$ and $I \notin \Sigma(M_1)$.

Definition 7. Given an architecture $\mathcal{A} = \langle D, \mathcal{I} \rangle$ and a FSM $M = (\mathbf{Q}, \rightarrow)$, such that $I \cap D \neq \emptyset$, for all $I \in \Sigma(M)$. We define $[\mathcal{A}]_M \stackrel{\text{def}}{=} (\mathbf{Q}, \rightarrow_{\mathcal{A}})$ where, for all $q, q' \in \mathbf{Q}$ and all $I \in 2^{\text{Ports}}$ we have, $q \xrightarrow{I}_{\mathcal{A}} q'$ if and only if $q \xrightarrow{I} q'$ and $I \in \mathcal{I}$.

Note that $[\mathcal{A}]_M$ is undefined if $\Sigma(M)$ contains interactions that do not intersect with $\text{dom}(\mathcal{A})$. The following theorem relates the composition of architectures with the asynchronous product of their behaviors.

Theorem 4. Let $\mathcal{A}_i = \langle D_i, \mathcal{I}_i \rangle$ be architectures and $M = (\mathbf{Q}_i, \rightarrow_i)$ be FSMs, such that $[\mathcal{A}_i]_{M_i}$ is defined, for all $i = 1, 2$. Then the following hold:

1. $[\mathcal{A}_1 \uplus \mathcal{A}_2]_{M_1 \otimes M_2} \subseteq [\mathcal{A}_1]_{M_1} \otimes [\mathcal{A}_2]_{M_2}$,
2. $[\mathcal{A}_1 \uplus \mathcal{A}_2]_{M_1 \otimes M_2} = [\mathcal{A}_1]_{M_1} \otimes [\mathcal{A}_2]_{M_2}$ if, moreover, $I \cap D_{3-i} \neq \emptyset \Rightarrow I \in \mathcal{I}_{3-i}$, for all $I \in \mathcal{I}_i$, $i = 1, 2$.

Proof: In the following, we denote:

$$\begin{aligned} \mathcal{A}_1 \uplus \mathcal{A}_2 &\stackrel{\text{def}}{=} \langle D_1 \cup D_2, \mathcal{I}_{\uplus} \rangle \\ M_1 \otimes M_2 &\stackrel{\text{def}}{=} (\mathbf{Q}_1 \times \mathbf{Q}_2, \rightarrow_{12}) \\ [\mathcal{A}_1 \uplus \mathcal{A}_2]_{M_1 \otimes M_2} &\stackrel{\text{def}}{=} (\mathbf{Q}_1 \times \mathbf{Q}_2, \rightarrow) \\ [\mathcal{A}_i]_{M_i} &\stackrel{\text{def}}{=} (\mathbf{Q}_i, \rightarrow_{\mathcal{A}_i}), \text{ for } i = 1, 2 \\ [\mathcal{A}_1]_{M_1} \otimes [\mathcal{A}_2]_{M_2} &\stackrel{\text{def}}{=} (\mathbf{Q}_1 \times \mathbf{Q}_2, \rightarrow_{\otimes}) \end{aligned}$$

Note that, because $[\mathcal{A}_i]_{M_i}$ is defined, for $i = 1, 2$, for each $I \in \Sigma(M_1 \otimes M_2) = \Sigma(M_1) \cup \Sigma(M_2)$ we have $I \cap (D_1 \cup D_2) \neq \emptyset$, thus $[\mathcal{A}_1 \uplus \mathcal{A}_2]_{M_1 \otimes M_2}$ is defined.

1 Let $(q_1, q_2) \xrightarrow{I} (q'_1, q'_2)$ be a transition of $[\mathcal{A}_1 \uplus \mathcal{A}_2]_{M_1 \otimes M_2}$, for some $I \in 2^{\text{Ports}}$. Then $I \in \mathcal{I}_{\uplus}$ and $(q_1, q_2) \xrightarrow{I}_{12} (q'_1, q'_2)$. Since $\mathcal{I}_{\uplus} = (\mathcal{I}_1 \cap \mathcal{I}_2) \cup (\mathcal{I}_1 \cap 2^{\overline{D}_2}) \cup (\mathcal{I}_2 \cap 2^{\overline{D}_1})$, we distinguish the following cases: (1) $I \in \mathcal{I}_1 \cap \mathcal{I}_2$, (2) $I \in \mathcal{I}_1$ and $I \cap D_2 = \emptyset$

and (3) $I \in \mathcal{I}_2$ and $I \cap D_1 = \emptyset$. Moreover, based on the definition of $M_1 \otimes M_2$, we distinguish the following cases: (a) $q_i \xrightarrow{I} q'_i$, for $i = 1, 2$, (b) $q_1 \xrightarrow{I} q'_1$, $q_2 = q'_2$ and $I \notin \Sigma(M_2)$, (c) $I \notin \Sigma(M_1)$, $q_1 = q'_1$ and $q_2 \xrightarrow{I} q'_2$. We give the proof in the following composed cases:

(1a) Since $I \in \mathcal{I}_i$ and $q_i \xrightarrow{I} q'_i$, we obtain $q_i \xrightarrow{I} q'_i$, for both $i = 1, 2$, thus $(q_1, q_2) \xrightarrow{I} (q'_1, q'_2)$.

(1b) Since $I \in \mathcal{I}_1$ and $q_1 \xrightarrow{I} q'_1$, we obtain $q_1 \xrightarrow{I} q'_1$. Moreover, $q_2 = q'_2$ and $I \notin \Sigma(M_2) \supseteq \Sigma([\mathcal{A}_2]_{M_2})$, thus $(q_1, q_2) \xrightarrow{I} (q'_1, q_2)$.

(1c) This case is symmetrical to (1b).

(2a) Since $I \in \mathcal{I}_1$ and $q_1 \xrightarrow{I} q'_1$, we obtain $q_1 \xrightarrow{I} q'_1$. Moreover, because $I \cap D_2 = \emptyset$ and since $[\mathcal{A}_2]_{M_2}$ is defined, we obtain $I \notin \Sigma(M_2)$. If $q_2 = q'_2$, we obtain that $(q_1, q_2) \xrightarrow{I} (q'_1, q_2)$. Else, $q_2 \neq q'_2$ and $q_2 \xrightarrow{I} q'_2$ contradicts $I \notin \Sigma(M_2)$.

(2b) Similar to (2a), using directly that $q_2 = q'_2$ and $I \notin \Sigma(M_2)$.

(2c) Because $I \cap D_2 = \emptyset$ and since $[\mathcal{A}_2]_{M_2}$ is defined, we obtain $I \notin \Sigma(M_2)$, which contradicts $q_2 \xrightarrow{I} q'_2$.

The cases (3a), (3b) and (3c) are symmetrical to (2a), (2b) and (2c), respectively.

2 To show that $[\mathcal{A}_1 \uplus \mathcal{A}_2]_{M_1 \otimes M_2} \supseteq [\mathcal{A}_1]_{M_1} \otimes [\mathcal{A}_2]_{M_2}$, let $(q_1, q_2) \xrightarrow{I} (q'_1, q'_2)$ be a transition of $[\mathcal{A}_1 \uplus \mathcal{A}_2]_{M_1 \otimes M_2} = [\mathcal{A}_1]_{M_1} \otimes [\mathcal{A}_2]_{M_2}$. We distinguish the following cases:

(1) if $q_i \xrightarrow{I} q'_i$, then $q_i \xrightarrow{I} q'_i$ and $I \in \mathcal{I}_i$, for both $i = 1, 2$. We obtain $(q_1, q_2) \xrightarrow{I} (q'_1, q'_2)$ and $I \in \mathcal{I}_1 \cap \mathcal{I}_2 \subseteq \mathcal{I}_\omega$, thus $(q_1, q_2) \xrightarrow{I} (q'_1, q'_2)$.

(2) if $q_1 \xrightarrow{I} q'_1$, $q_2 = q'_2$ and $I \notin \Sigma([\mathcal{A}_2]_{M_2})$, we consider two cases:

(a) if $I \in \mathcal{I}_2 \setminus \Sigma(M_2)$ then $I \in \mathcal{I}_1 \cap \mathcal{I}_2 \subseteq \mathcal{I}_\omega$ and $(q_1, q_2) \xrightarrow{I} (q'_1, q_2)$, thus $(q_1, q_2) \xrightarrow{I} (q'_1, q_2)$.

(b) else $I \notin \mathcal{I}_2$ and since $I \in \mathcal{I}_1$, by the hypothesis $I \cap D_2 \neq \emptyset \Rightarrow I \in \mathcal{I}_2$, we deduce that $I \cap D_2 = \emptyset$. Then $I \in \mathcal{I}_1 \cap 2\bar{D}_2 \subseteq \mathcal{I}_\omega$. Moreover, since $[\mathcal{A}_2]_{M_2}$ is defined, we obtain $I \notin \Sigma(M_2)$, thus $(q_1, q_2) \xrightarrow{I} (q'_1, q_2)$ and $(q_1, q_2) \xrightarrow{I} (q'_1, q_2)$ follows.

(3) This case is symmetrical to the above one. □

7 Dynamic Reconfigurability

We extend architectures to capture reconfigurability, by distinguishing between the *architecture layer*, describing the components that are active in the system and their inter-

actions, and the *map layer*, which is the graph onto which the components are *deployed*. Formally, the component layer consists of:

- a countably infinite set Id of *component identifiers*, ranged over by the set of identifier variables $\text{IVar} = \{i, j, \dots\}$,
- a finite set of total *port functions* of type $\text{Id} \rightarrow \text{Ports}$, denoted by the set of function symbols $\text{PFun} = \{p, q, \dots\}$.

The map layer consists of:

- a countably infinite set Nodes of *map nodes*, with a designated element $\text{nil} \in \text{Nodes}$ and ranged over by the node variables $\text{NVar} = \{n, m, \dots\}$,
- a partial *map* $M : \text{Nodes} \rightarrow_{\text{fin}} \text{Nodes}^k$, with finite domain $\text{dom}(M)$, where $\text{nil} \notin \text{dom}(M)$. We assume that the image of each node $n \in \text{dom}(M)$ consists of exactly $k \geq 1$ nodes $M(n) = (n_1, \dots, n_k)$.

The link between the layers is established by a finite partial *deployment* function $\Delta : \text{Id} \rightarrow_{\text{fin}} \text{dom}(M) \cup \{\text{nil}\}$. By Arch we denote the set of architectures $\mathcal{A} = \langle D, \mathcal{I} \rangle$, with $D \subseteq \text{Ports}$ and $\mathcal{I} \subseteq 2^{\text{Ports}}$, such that $D \cap I \neq \emptyset$, for all $I \in \mathcal{I}$. Moreover, by Maps^k we denote the set of maps $M : \text{Nodes} \rightarrow_{\text{fin}} \text{Nodes}^k$.

We describe such systems using a combination of two resource logics, defined in the following. Given a constant $k \geq 1$, the formulae of the *Separation Logic of Maps* (SL_m^k) are defined by the following syntax:

$$\begin{aligned} t &:= \text{nil} \mid n \in \text{NVar} \\ \phi &:= t_1 = t_2 \mid \text{emp}_m \mid i \rightsquigarrow t, i \in \text{IVar} \mid t_0 \mapsto (t_1, \dots, t_k) \mid Q(t_1, \dots, t_{\#(Q)}) \mid \\ &\quad \phi_1 \wedge \phi_2 \mid \neg \phi_1 \mid \exists i. \phi_1, i \in \text{IVar} \mid \exists n. \phi_1, n \in \text{NVar} \mid \phi_1 *_m \phi_2 \mid \phi_1 \dashv*_m \phi_2 \end{aligned}$$

where $Q(t_1, \dots, t_{\#(Q)}) \in \text{NPred}$ is a predicate symbol of type $\text{Nodes}^{\#(Q)} \rightarrow \{\perp, \top\}$. By \top_m we denote the equality $n = n$, the choice of $n \in \text{NVar}$ being unimportant. A SL_m -sentence is a formula in which all variables occur within the scope of a quantifier.

Since SL_m^k formulae contain two types of variables, we consider extended valuations $\nu : \text{IVar} \cup \text{NVar} \rightarrow \text{Id} \cup \text{Nodes}$, such that $\nu(x) \in \text{Id}$, when $x \in \text{IVar}$ and $\nu(n) \in \text{Nodes}$, when $n \in \text{NVar}$. For a term t , we write $\nu(t)$ to denote the node nil if $t = \text{nil}$ and the node $\nu(n)$ if $t = n \in \text{NVar}$. The semantics of SL_m^k is defined by a satisfaction relation $\langle \Delta, M \rangle \models_\nu^X \phi$ between pairs of deployments and maps and formulae, parameterized by a valuation $\nu : \text{IVar} \cup \text{NVar} \rightarrow \text{Id} \cup \text{Nodes}$ and an interpretation of predicate symbols $\mathcal{X} : \text{NPred} \rightarrow \bigcup_{k \geq 1} 2^{\text{Nodes}^k \times \text{Maps}}$, such that $\mathcal{X}(Q) \subseteq \text{Nodes}^{\#(Q)} \times \text{Maps}$.

$$\begin{aligned} \langle \Delta, M \rangle \models_\nu^X t_1 = t_2 &\iff \nu(t_1) = \nu(t_2) \\ \langle \Delta, M \rangle \models_\nu^X \text{emp}_m &\iff M = \emptyset \\ \langle \Delta, M \rangle \models_\nu^X x \rightsquigarrow t &\iff \Delta(\nu(x)) = \nu(t) \\ \langle \Delta, M \rangle \models_\nu^X t_0 \mapsto (t_1, \dots, t_k) &\iff \text{dom}(M) = \{\nu(t_0)\} \text{ and } M(\nu(t_0))_\ell = (\nu(t_1), \dots, \nu(t_k)) \\ \langle \Delta, M \rangle \models_\nu^X Q(t_1, \dots, t_{\#(Q)}) &\iff \langle \langle \nu(t_1), \dots, \nu(t_{\#(Q)}) \rangle, M \rangle \in \mathcal{X}(Q) \\ \langle \Delta, M \rangle \models_\nu^X \exists x. \phi_1 &\iff \langle \Delta, M \rangle \models_{\nu[x \leftarrow i]}^X \phi_1, \text{ for some node } i \in \text{Id} \\ \langle \Delta, M \rangle \models_\nu^X \exists n. \phi_1 &\iff \langle \Delta, M \rangle \models_{\nu[n \leftarrow v]}^X \phi_1, \text{ for some node } v \in \text{Nodes} \\ \langle \Delta, M \rangle \models_\nu^X \phi_1 *_m \phi_2 &\iff \text{there exists maps } M_1, M_2 \text{ such that } \text{dom}(M_1) \cap \text{dom}(M_2) = \emptyset, \\ &\quad M = M_1 \cup M_2 \text{ and } \langle \Delta, M_i \rangle \models_\nu^X \phi_i, \text{ for } i = 1, 2 \\ \langle \Delta, M \rangle \models_\nu^X \phi_1 \dashv*_m \phi_2 &\iff \text{for all maps } M_1 \text{ such that } \text{dom}(M_1) \cap \text{dom}(M) = \emptyset \\ &\quad \text{and } \langle \Delta, M_1 \rangle \models_\nu^X \phi_1, \text{ we have } \langle \Delta, M_1 \cup M \rangle \models_\nu^X \phi_2 \end{aligned}$$

The semantics of the boolean connectives is standard, thus we omit it for brevity.

The combined *Separation Logic of Architectures and Maps* (SL_{am}) is the extension of SL_a which allows SL_m^k sentences to occur as atomic propositions. A SL_{am} formula is interpreted over structures (\mathcal{A}, Δ, M) , where $\mathcal{A} = \langle D, \mathcal{I} \rangle$ is an architecture, Δ is a deployment and M is a map, by a satisfaction relation $(\mathcal{A}, \Delta, M) \models_v^X \phi$ parameterized by a valuation $v : \text{IVar} \cup \text{NVar} \rightarrow \text{Id} \cup \text{Nodes}$, as before, and an interpretation $\mathcal{X} : \text{IPred} \cup \text{NPred} \rightarrow \bigcup_{k \geq 1} 2^{\text{Id}^k} \cup 2^{\text{Nodes}^k}$, such that $\mathcal{X}(P) \subseteq \text{Id}^{\#(P)}$, when $P \in \text{IPred}$, and $\mathcal{X}(P) \subseteq \text{Nodes}^{\#(P)}$, when $P \in \text{NPred}$.

Both SL_a and SL_m use predicate symbols, whose interpretation is the least solution of a system of inductive definitions of the form:

$$\begin{aligned} P(x_1, \dots, x_{\#P}) &\leftarrow \rho_a, \quad \text{where } P \in \text{IPred} \text{ and } x_1, \dots, x_{\#(P)} \in \text{IVar} \\ Q(n_1, \dots, n_{\#Q}) &\leftarrow \rho_m, \quad \text{where } Q \in \text{NPred} \text{ and } n_1, \dots, n_{\#(Q)} \in \text{NVar} \end{aligned}$$

where the logical fragments used for the rules of the system are given by the syntax:

$$\begin{aligned} \rho_a &:= x = y \mid x \neq y \mid \text{emp}_a \mid p(x) \multimap b \mid p(x) \overset{\exists}{\multimap} b \mid p(x) \multimap b \mid p(x) \overset{\exists}{\multimap} b \mid \\ &\quad P(x_1, \dots, x_{\#P}) \mid \rho'_a * \rho''_a \mid \exists x. \rho'_a \\ \rho_m &:= t_1 = t_2 \mid t_1 \neq t_2 \mid \text{emp}_m \mid t_1 \mapsto t_2 \mid Q(t_1, \dots, t_{\#Q}) \mid \rho'_m * \rho''_m \mid \exists n. \rho'_m \end{aligned}$$

The main restriction here is that all predicate symbols occur at positive polarity within the rules, which ensures the monotonicity of the rules and the existence of least solutions. The following example defines two common structures, used in many applications.

Example 7. A pipeline architecture, starting with x and ending with y , where x_p refers to the component previous to x and y_n to the component next to y :

$$\begin{aligned} \text{pipe}(x, x_p, y_n, y) &\leftarrow \text{emp} * x = y_n * x_p = y \\ \text{pipe}(x, x_p, y_n, y) &\leftarrow \exists z. \text{in}(x) \multimap \text{out}(x_p) * \text{out}(x) \overset{\exists}{\multimap} \text{in}(z) * \text{pipe}(z, x, y_n, y) \end{aligned}$$

An acyclic list map, stretching between nodes n and m :

$$\begin{aligned} \text{alist}(n, m) &\leftarrow \text{emp} * m = n \\ \text{alist}(n, m) &\leftarrow \exists n'. n \mapsto n' * m \neq n * m * \text{alist}(n', m) \end{aligned}$$

■

7.1 Synchronization and Deployment Rules

We consider universally quantified sentences that describe the interactions of the architecture, of the form $\forall \bar{x}. \phi_m(\bar{x}) \rightarrow \phi_a(\bar{x})$, where $\bar{x} = x_1, \dots, x_m \in \text{IVar}$, ϕ_m is a formula of SL_m and ϕ_a is a formula of SL_a . We call these sentences *synchronization rules* in the following.

Example 8. The synchronization rule below requires that each two components deployed on neighbouring map nodes interact via their *in* and *out* ports, respectively:

$$\begin{aligned} \forall x \forall y. (\exists n \exists m. n \neq m \wedge n \mapsto m * m \top_m \wedge x \rightsquigarrow n \wedge y \rightsquigarrow m) \rightarrow \\ \text{in}(x) \overset{\exists}{\multimap} \text{out}(y) * \text{out}(y) \overset{\exists}{\multimap} \text{in}(x) * \top_a \end{aligned}$$

■

The following formula states that an identifier belongs to an existing component, i.e. is allocated. We recall that the set PFun of port symbols is finite.

$$\text{alloc}_a(x) \stackrel{\text{def}}{=} \left(\bigvee_{p \in \text{PFun}} p(x) \multimap p(x) \right) \multimap \perp_a$$

We write $\exists_a x . \phi$ as a shorthand for $\exists x . \text{alloc}_a(x) \wedge \phi$ and $\forall_a x . \phi$ for $\forall x . \text{alloc}_a(x) \rightarrow \phi$.

In a similar way, we define the set of nodes that are part of the domain of the map:

$$\text{alloc}_m(n) \stackrel{\text{def}}{=} n \mapsto n \multimap \perp_m$$

and write $\exists_m n . \phi$ (resp. $\forall_m n . \phi$) for $\exists n . \text{alloc}_m(n) \wedge \phi$ (resp. $\forall n . \text{alloc}_m(n) \rightarrow \phi$).

We can now specify *deployment rules*, which are sentences such as:

$$\begin{aligned} & \forall_a x \exists_m n . x \rightsquigarrow n \quad (\text{every component is deployed}) \\ & \forall_m n \exists_a x . x \rightsquigarrow n \quad (\text{every node has a deployed component}) \\ & \forall_m n \forall_a x \forall_a y . x \rightsquigarrow n \wedge y \rightsquigarrow n \rightarrow x = y \quad (\text{at most one component is deployed on each node}) \end{aligned}$$

Note that we need SL_{am} to write synchronization rules, whereas deployment rules can be written using only SL_m . We state the following synthesis problem:

Definition 8 (Architecture Synthesis). *Given a SL_m sentence ϕ , synchronization rules Φ_1, \dots, Φ_k and deployment rules Ψ_1, \dots, Ψ_ℓ , find a SL_a sentence ψ such that the following SL_{am} formula is valid:*

$$\left(\phi \wedge \bigwedge_{i=1}^k \Phi_i \wedge \bigwedge_{i=1}^{\ell} \Psi_i \right) \rightarrow \psi$$

Example 9. Considering the formula $\exists n \exists m . \text{alist}(n, m)$ (Example 7), the synchronization rule:

$$(\Phi) \forall x \forall y . (\exists n \exists m . n \neq m \wedge n \mapsto m * \top_m \wedge x \rightsquigarrow n \wedge y \rightsquigarrow m) \rightarrow \\ \text{in}(x) \stackrel{\exists}{\multimap} \text{out}(y) * \text{out}(y) \stackrel{\exists}{\multimap} \text{in}(x) * \top_a$$

and the deployment rules:

$$\begin{aligned} (\Psi_1) & \forall_a x \exists_m n . x \rightsquigarrow n \\ (\Psi_2) & \forall_m n \exists_a x . x \rightsquigarrow n \\ (\Psi_3) & \forall_m n \forall_a x \forall_a y . x \rightsquigarrow n \wedge y \rightsquigarrow n \rightarrow x = y \end{aligned}$$

a solution to the architecture synthesis problem is $\exists x \exists x_p \exists y_n \exists y . \text{pipe}(x, x_p, y_n, y)$ (Example 7). ■

7.2 Reconfiguration Axioms

In this section we tackle the problem of defining the reconfiguration actions. First, we describe their operational semantics, in terms of updates of the map and the deployment and then we give their axiomatic semantics in terms of Hoare triples. The latter is

derived from a set of *local axioms*, describing the changes to the (small set of) cells affected by the reconfiguration and a *frame rule* enabling a general weakest pre- (strongest post-) condition calculus.

A *reconfiguration sequence* is a set of actions written in the following syntax:

$$\begin{aligned} i &\in \text{IVar}, n \in \text{NVar}, \ell \in \{1, \dots, k\} \\ \text{term} &:= n \mid \text{nil} \\ \text{action} &:= n.\ell = \text{term} \mid \text{deploy}(i, n) \mid \text{delete}(n) \mid n = \text{new} \mid n = \text{term} \mid n = m.\ell \end{aligned}$$

The operational semantics is given in terms of steps $(\sigma, \Delta, M) \rightsquigarrow (\sigma', \Delta', M')$ where $\sigma : \text{NVar} \rightarrow \text{Nodes}$ is a *store*, Δ and M are the deployment and map functions, and σ', Δ' and M' denote the next values of σ, Δ and M , respectively. Given a tuple $\tau \in \text{Nodes}^k$ and $1 \leq \ell \leq k$, we denote by τ_ℓ its ℓ -th element and by $\tau_{\{\ell \leftarrow \alpha\}}$ the tuple with the same elements as τ except for its ℓ -th element, who is set to α . The following rules define the reconfiguration steps:

$$\begin{aligned} &\frac{\sigma(n) \in \text{dom}(M) \quad \tau = M(\sigma(n))_{\{\ell \leftarrow \sigma(t)\}}}{(\sigma, \Delta, M) \rightsquigarrow (\sigma, \Delta, M[\sigma(n) \leftarrow \tau])} \quad n.\ell = t \\ &\frac{}{(\sigma, \Delta, M) \rightsquigarrow (\sigma, \Delta[i \leftarrow \sigma(n)], M)} \quad \text{deploy}(i, n) \\ &\frac{\sigma(n) \in \text{dom}(M) \quad M' = M \setminus \{\langle \sigma(n), M(n) \rangle\}}{(\sigma, \Delta, M) \rightsquigarrow (\sigma, \Delta, M')} \quad \text{free}(n) \\ &\frac{v \notin \text{dom}(M) \quad M' = M \cup \{\langle v, \overbrace{(\text{nil}, \dots, \text{nil})}^k \rangle\}}{(\sigma, \Delta, M) \rightsquigarrow (\sigma[n \leftarrow v], \Delta, M')} \quad n = \text{new} \\ &\frac{}{(\sigma, \Delta, M) \rightsquigarrow (\sigma[n \leftarrow \sigma(t)], \Delta, M)} \quad n = t \\ &\frac{\sigma(m) \in \text{dom}(M)}{(\sigma, \Delta, M) \rightsquigarrow (\sigma[n \leftarrow M(\sigma(m))_\ell], \Delta, M)} \quad n = m.\ell \end{aligned}$$

In order to carry out deductive verification of reconfiguration sequences, we define the semantics of the reconfiguration actions by the following set of local axioms, the encompass the principle of local reasoning:

$$\begin{aligned} &\{\exists m_1 \dots \exists m_k . n \mapsto (m_1, \dots, m_k)\} \quad n.\ell = t \quad \{n \mapsto (m_1, \dots, m_{\ell-1}, t, m_{\ell+1}, \dots, m_k)\} \\ &\quad \{\text{emp}_m\} \quad \text{deploy}(i, n) \quad \{i \rightsquigarrow n \wedge \text{emp}_m\} \\ &\quad \{\exists m . n \mapsto m\} \quad \text{free}(n) \quad \{\text{emp}_m\} \\ &\quad \{\text{emp}_m\} \quad n = \text{new} \quad \{n \mapsto \overbrace{(\text{nil}, \dots, \text{nil})}^k\} \\ &\quad \{n = m \wedge \text{emp}_m\} \quad n = t \quad \{n = t[m/n] \wedge \text{emp}_m\} \\ &\quad \{m \mapsto (m_1, \dots, m_k)\} \quad n := m.\ell \quad \{n = m_\ell \wedge m \mapsto (m_1, \dots, m_k)\} \end{aligned}$$

These small axioms define a full predicate transformer calculus by means of the following frame rule, that captures the idea of local reasoning:

$$\frac{\{\phi\}C\{\psi\}}{\{\phi *_m F\}C\{\psi *_m C\}} \text{modif}(C) \cap \text{fv}(F) = \emptyset$$

where $\text{modif}(n := \text{new}) = \text{modif}(n := t) = \text{modif}(n := [m]) = \{n\}$ and $\text{modif}([n] := t) = \text{modif}(\text{free}(n)) = \emptyset$ denotes the set of variables whose values are altered by the action.