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► **To cite this version:**

Gabriella Clemente. A technical remark on the Donaldson-Futaki invariant for Fano reductive group compactifications. 2019. hal-02267071

HAL Id: hal-02267071

<https://hal.science/hal-02267071>

Preprint submitted on 18 Aug 2019

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A technical remark on the Donaldson-Futaki invariant for Fano reductive group compactifications

Gabriella Clemente

Abstract

We present an elementary way of computing the Donaldson-Futaki invariant associated to a test-configuration of an anti-canonically polarized Fano reductive group compactification.

Reductive group compactifications from polytopes. Let G be a reductive group and $T \subset G$ be a maximal torus with character lattice M , Lie algebra \mathfrak{t} , and dual Lie algebra $\mathfrak{t}^* \simeq M_{\mathbb{R}} := M \otimes \mathbb{R}$. Let W be the Weyl group of (G, T) , and let Φ denote the root system of (G, T) with a fixed choice of positive roots Φ^+ . We declare 2ρ to be the sum of the positive roots. The positive Weyl chamber is $M_{\mathbb{R}}^+ := \{x \in M_{\mathbb{R}} \mid \langle \alpha, x \rangle \geq 0 \text{ for all } \alpha \in \Phi^+\}$. There is a one-to-one correspondence between lattice points $\lambda \in M_{\mathbb{R}}^+$ and irreducible G -representations E_{λ} . Furthermore, to a lattice point $\lambda \in M_{\mathbb{R}}^+$ corresponds a $G \times G$ -representation $\text{End}(E_{\lambda})$. The dimension of $\text{End}(E_{\lambda})$ is a polynomial

$$\dim(\text{End}(E_{\lambda})) = (\dim(E_{\lambda}))^2 = H_d(\lambda) + H_{d-1}(\lambda) + \dots$$

in λ , and here H_d stands for the degree d homogeneous part of the polynomial $\dim(\text{End}(E_{\lambda}))$, H_{d-1} stands for the degree $d-1$ part, and so on.

Let $P^+ := P \cap M_{\mathbb{R}}^+$, $C(P^+) \subseteq M_{\mathbb{R}} \times \mathbb{R}$ be the cone over $(P^+, 1)$, and consider the finitely generated algebra

$$R_P = \bigoplus_{\lambda \in C(P^+) \cap (M \times \mathbb{Z})} \text{End}(E_{\lambda}).$$

To any W -invariant lattice polytope $P \subseteq M_{\mathbb{R}}$, we can associate a polarized reductive group compactification (X_P, L_P) , where $X_P = Proj(R_P)$ and $L_P = \mathcal{O}(1)$.

The Fano condition. At the polytope level, Fano is the condition that the distance between 2ρ and any codimension one face of P^+ that does not meet the boundary of the positive Weyl chamber is equal to one. This is a result that can be found in [3], and which we recapitulate below.

Denote the Zariski closure of T in X_P by Z , which is a toric subvariety of X_P . When X_P is Fano, the support function v of P is of the form $v = v_{K_{\mathbb{C}}} + v_Z$, where $v_{K_{\mathbb{C}}}(x) = \langle 2\rho, x \rangle$ for all x in the positive Weyl chamber, $v_{K_{\mathbb{C}}}(wx) = v_{K_{\mathbb{C}}}(x)$ for all $w \in W$, and $v_Z(x) = -g_{-K_Z}(-x)$, where $-g_{-K_Z}$ is the support function of the anti-canonical line bundle of the toric subvariety $Z \subset X_P$. Since P is also the polytope of Z , the associated fan Σ_P gives rise to the toric subvariety Z . From the theory of toric varieties, $-K_Z = \sum_{\rho \in \Sigma(1)} D_{\rho}$, where $\Sigma(1)$ is the set of 1-dimensional cones of Σ_P and D_{ρ} is a prime torus invariant divisor on Z . The support function g_{-K_Z} has the property that $g_{-K_Z}(u_{\rho}) = -1$ for all $\rho \in \Sigma(1)$, where u_{ρ} is the minimal generator of the ray ρ . In particular, if a_i is the inward pointing normal to the i -th codimension one face of P , $g_{-K_Z}(a_i) = -g_{-K_Z}(-a_i) = -1$. Then, $v(a_i) = \langle a_i, 2\rho \rangle - 1$, and so the facet presentation of the polytope is

$$P = \{x \in M_{\mathbb{R}} \mid \langle a_i, x \rangle \geq \langle a_i, 2\rho \rangle - 1\}.$$

As a consequence, the equation that defines the i -th boundary face of P is $f_i(x) = \langle a_i, x - 2\rho \rangle + 1$ so that $f_i(2\rho) = 1$.

Calculation of the Donaldson-Futaki invariant. In the sequel, we obtain a number of identities that together with the Fano condition will allow us to simplify Alexeev's and Katzarkov's Donaldson-Futaki (DF) invariant:

Theorem. (*Theorem 3.3, [1]*) *Let f be a convex rational W -invariant piecewise linear function on P . Then the DF invariant of the corresponding test-configuration is given by the formula*

$$-F_1(f) = \frac{1}{2 \int_{P^+} H_d d\mu} \left(\int_{\partial P^+} f H_d d\sigma + 2 \int_{P^+} f H_{d-1} d\mu - a \int_{P^+} f H_d d\mu \right),$$

where

$$a = \frac{\int_{\partial P^+} H_d d\sigma + 2 \int_{P^+} H_{d-1} d\mu}{\int_{P^+} H_d d\mu}.$$

Here $d\mu$ is the Lebesgue measure restricted to P , and the boundary measure $d\sigma$ is a positive measure on ∂P that is normalized so that on each codimension one face, which is defined by an equation $l(x) := \langle a, x \rangle = c$, $d\sigma \wedge dl = \pm d\mu$ holds.

Choose once and for all an isomorphism $M_{\mathbb{R}} \simeq \mathbb{R}^n$ so that P can be viewed as though contained in \mathbb{R}^n .

Claim. Let $\Phi^+ = \{\alpha_1, \dots, \alpha_r\}$, $c = \prod_{i=1}^r \langle \alpha_i, \rho \rangle^2$, where $\rho = \frac{1}{2} \sum_{i=1}^r \alpha_i$, and let $\{e_j\}_{j=1}^n$ be the standard basis of \mathbb{R}^n . Then,

1.

$$H_d(x) = \frac{1}{c} \prod_{i=1}^r \langle \alpha_i, x \rangle^2,$$

2.

$$H_{d-1}(x) = \frac{1}{c} \sum_{j=1}^r 2 \langle \alpha_j, x \rangle \langle \alpha_j, \rho \rangle \langle \alpha_1, x \rangle^2 \dots \widehat{\langle \alpha_j, x \rangle}^2 \dots \langle \alpha_r, x \rangle^2,$$

3.

$$\nabla H_d(x) = \frac{1}{c} \sum_{j=1}^n \left(\sum_{i=1}^n 2 \langle \alpha_i, x \rangle \langle \alpha_i, e_j \rangle \langle \alpha_1, x \rangle^2 \dots \widehat{\langle \alpha_j, x \rangle}^2 \dots \langle \alpha_r, x \rangle^2 \right) e_j,$$

4. $\langle \nabla H_d(x), \rho \rangle = H_{d-1}(x),$

5. $\langle \nabla H_d(x), x \rangle = 2rH_d(x),$ and

6. for any smooth function $f : P \rightarrow \mathbb{R}$,

$$\operatorname{div}((x - 2\rho)fH_d) = \langle \nabla f, x - 2\rho \rangle H_d + (2r + n)fH_d - 2fH_{d-1}.$$

Proof. Let E_x be an irreducible representation with highest weight x . To prove 1. and 2., we make use of the Weyl dimension formula

$$\dim(E_x) = \frac{\prod_{i=1}^r \langle \alpha_i, x + \rho \rangle}{\prod_{i=1}^r \langle \alpha_i, \rho \rangle}.$$

From the expression

$$\dim(E_x)^2 = \frac{1}{c} \prod_{i=1}^r (\langle \alpha_i, x \rangle^2 + 2\langle \alpha_i, x \rangle \langle \alpha_i, \rho \rangle + \langle \alpha_i, \rho \rangle^2),$$

it follows that if d is the highest degree homogeneous part of the polynomial $\dim(E_x)^2$, then

$$H_d(x) = \frac{1}{c} \prod_{i=1}^r \langle \alpha_i, x \rangle^2,$$

and the $(d-1)$ -degree homogeneous part of $\dim(E_x)^2$ is

$$H_{d-1}(x) = \frac{1}{c} \sum_{j=1}^r 2\langle \alpha_j, x \rangle \langle \alpha_j, \rho \rangle \langle \alpha_1, x \rangle^2 \dots \langle \widehat{\alpha_j}, x \rangle^2 \dots \langle \alpha_r, x \rangle^2.$$

For 3., note that $\frac{\partial}{\partial x_j} \langle \alpha_i, x \rangle = \langle \alpha_i, e_j \rangle$ so that

$$\frac{\partial}{\partial x_j} H_d(x) = \frac{1}{c} \sum_{i=1}^r 2\langle \alpha_i, x \rangle \langle \alpha_i, e_j \rangle \langle \alpha_1, x \rangle^2 \dots \langle \widehat{\alpha_i}, x \rangle^2 \dots \langle \alpha_r, x \rangle^2,$$

and hence

$$\nabla H_d(x) = \sum_{j=1}^r \frac{\partial}{\partial x_j} H_d(x) e_j = \frac{1}{c} \sum_{j=1}^n \left(\sum_{i=1}^n 2\langle \alpha_i, x \rangle \langle \alpha_i, e_j \rangle \langle \alpha_1, x \rangle^2 \dots \langle \widehat{\alpha_j}, x \rangle^2 \dots \langle \alpha_r, x \rangle^2 \right) e_j.$$

For 4., notice that

$$\begin{aligned} \nabla H_d(x) &= \frac{1}{c} \sum_{j=1}^n \left(\sum_{i=1}^n 2\langle \alpha_i, x \rangle \langle \alpha_i, e_j \rangle \langle \alpha_1, x \rangle^2 \dots \langle \widehat{\alpha_j}, x \rangle^2 \dots \langle \alpha_r, x \rangle^2 \right) e_j \\ &= \sum_{i=1}^r \left(\frac{1}{c} 2\langle \alpha_i, x \rangle \langle \alpha_1, x \rangle^2 \dots \langle \widehat{\alpha_i}, x \rangle^2 \dots \langle \alpha_r, x \rangle^2 \right) \sum_{j=1}^n \langle \alpha_i, e_j \rangle e_j \\ &= \sum_{i=1}^r \left(\frac{1}{c} 2\langle \alpha_i, x \rangle \langle \alpha_1, x \rangle^2 \dots \langle \widehat{\alpha_i}, x \rangle^2 \dots \langle \alpha_r, x \rangle^2 \right) \alpha_i \end{aligned}$$

and then

$$\langle \nabla H_d(x), \rho \rangle = \sum_{i=1}^r \frac{1}{c} 2 \langle \alpha_i, x \rangle \langle \alpha_i, \rho \rangle \langle \alpha_1, x \rangle^2 \dots \langle \widehat{\alpha_i}, x \rangle^2 \dots \langle \alpha_r, x \rangle^2 = H_{d-1}(x).$$

For 5., observe that since

$$\nabla H_d(x) = \sum_{i=1}^r \left(\frac{1}{c} 2 \langle \alpha_i, x \rangle \langle \alpha_1, x \rangle^2 \dots \langle \widehat{\alpha_i}, x \rangle^2 \dots \langle \alpha_r, x \rangle^2 \right) \alpha_i,$$

and since for each i ,

$$\left\langle \frac{1}{c} 2 \langle \alpha_i, x \rangle \langle \alpha_1, x \rangle^2 \dots \langle \widehat{\alpha_i}, x \rangle^2 \dots \langle \alpha_r, x \rangle^2 \alpha_i, x \right\rangle = 2 \left(\frac{1}{c} \langle \alpha_1, x \rangle^2 \dots \langle \alpha_r, x \rangle^2 \right),$$

indeed we have that

$$\langle \nabla H_d(x), x \rangle = 2r \left(\frac{1}{c} \prod_{i=1}^r \langle \alpha_i, x \rangle^2 \right) = 2r H_d(x).$$

The above identities now imply the last point. Namely,

$$\begin{aligned} \operatorname{div}((x - 2\rho)f H_d) &= \langle \nabla(f H_d), x - 2\rho \rangle + \operatorname{div}(x - 2\rho)f H_d \\ &= \langle \nabla f, x - 2\rho \rangle H_d + \langle \nabla H_d, x - 2\rho \rangle f + n f H_d \\ &= \langle \nabla f, x - 2\rho \rangle H_d + \langle \nabla H_d, x \rangle f - 2 \langle \nabla H_d, \rho \rangle f + n f H_d \\ &= \langle \nabla f, x - 2\rho \rangle H_d + (2r + n) f H_d - 2 f H_{d-1}. \end{aligned}$$

□

The following is analogous to Theorem C in [2].

Proposition. *Suppose that P satisfies the Fano condition. Let $f : P \rightarrow \mathbb{R}$ be a function as in the theorem that is affine linear on P^+ . Then, the DF invariant of the test-configuration associated to f is given by*

$$-F_1(f) = \frac{1}{2 \operatorname{Vol}_{DH}(P^+)} \int_{P^+} \langle \nabla f, x - 2\rho \rangle H_d d\mu = \frac{1}{2} \langle \operatorname{bar}_{DH}(P^+) - 2\rho, \nabla f \rangle,$$

where $\operatorname{bar}_{DH}(P^+) = \frac{1}{\operatorname{Vol}_{DH}(P^+)} \int_{P^+} x H_d d\mu$ and $\operatorname{Vol}_{DH}(P^+) = \int_{P^+} H_d d\mu$ are the barycenter and respectively the volume of P^+ with respect to the Duistermaat-Heckman (DH) measure.

Proof. Suppose that ∂P^+ has k codimension one faces ∂P_i^+ . Let $\{\partial P_i^+ : i = 1, \dots, m\}$ be the set of all codimension one faces of P^+ that do not intersect the boundary of the positive Weyl chamber. Suppose that ∂P_i^+ is defined by $\langle a_i, x \rangle - c_i = 0$ and set $f_i(x) := \langle a_i, x \rangle - c_i$. The (inward) unit normal vector field to ∂P_i^+ is $-\frac{\nabla f_i}{\|\nabla f_i\|} = -\frac{a_i}{\|a_i\|}$. Since P^+ satisfies the Fano condition, for $x \in \partial P_i^+$, we have that

$$\begin{aligned} \langle (x - 2\rho)fH_d, -\frac{a_i}{\|a_i\|} \rangle &= -H_d f \left(\left\langle x - 2\rho, \frac{a_i}{\|a_i\|} \right\rangle \right) \\ &= \frac{-H_d(x)f}{\|a_i\|} \left(\langle x, a_i \rangle - \langle 2\rho, a_i \rangle \right) \\ &= \frac{H_d(x)f}{\|a_i\|} \left(\langle 2\rho, a_i \rangle - c_i \right) \\ &= \frac{H_d(x)f}{\|a_i\|}. \end{aligned}$$

The divergence theorem implies that

$$\int_{P^+} \operatorname{div}((x - 2\rho)fH_d) d\mu = \sum_{i=1}^m \int_{\partial P_i^+} \frac{H_d f}{\|a_i\|} d\sigma_i + \sum_{i=m+1}^k \int_{\partial P_i^+} \langle (x - 2\rho)fH_d, -\frac{a_i}{\|a_i\|} \rangle d\sigma_i,$$

where $d\sigma_i$ is the standard Lebesgue measure on ∂P with domain restricted to ∂P_i . When $i = m + 1, \dots, k$, ∂P_i^+ is in the boundary of the positive Weyl chamber, and

$$\int_{\partial P_i^+} \langle (x - 2\rho)fH_d, \frac{a_i}{\|a_i\|} \rangle d\sigma_i = 0.$$

Then

$$\int_{P^+} \operatorname{div}((x - 2\rho)fH_d) d\mu = \sum_{i=1}^m \int_{\partial P_i^+} \frac{H_d(x)f}{\|a_i\|} d\sigma_i$$

and the right hand side is the definition of

$$\int_{\partial P^+} fH_d d\sigma.$$

By 6. of the claim, taking $f = 1$, we obtain that

$$\operatorname{div}((x - 2\rho)H_d) = (2r + n)H_d - 2H_{d-1}.$$

Then, by the divergence theorem,

$$\int_{\partial P^+} H_d d\sigma = (2r + n) \int_{P^+} H_d d\mu - 2 \int_{P^+} H_{d-1} d\mu.$$

Hence,

$$a = \frac{\int_{\partial P^+} H_d d\sigma + 2 \int_{P^+} H_{d-1} d\mu}{\int_{P^+} H_d d\mu} = 2r + n.$$

Upon substituting the above calculations into Alexeev's and Katzarkov's DF invariant (cf. Theorem), again using 6. of the claim to rewrite the first integral, we find that

$$-F_1(f) = \frac{1}{2 \int_{P^+} H_d d\mu} \int_{P^+} \langle \nabla f, x - 2\rho \rangle H_d d\mu.$$

Suppose that f on P^+ is given as $f(x) = \sum_{j=1}^n b_j x_j + k$. Put $b = (b_1, \dots, b_n)$, $x = (x_1, \dots, x_n)$ and $2\rho = (2\rho_1, \dots, 2\rho_n)$, and let e_j be the j -th standard basis vector of \mathbb{R}^n . Then $\langle \nabla f, x - 2\rho \rangle = \sum_{j=1}^n b_j (x_j - 2\rho_j)$ and it follows that

$$\begin{aligned} -F_1(f) &= \frac{1}{2 \text{Vol}_{DH}(P^+)} \int_{P^+} \langle \nabla f, x - 2\rho \rangle H_d d\mu \\ &= \frac{1}{2 \text{Vol}_{DH}(P^+)} \left(\sum_{j=1}^n b_j \int_{P^+} x_j H_d d\mu - \sum_{j=1}^n b_j (2\rho_j) \text{Vol}_{DH}(P^+) \right) \\ &= \frac{1}{2} \left(\langle \text{bar}_{DH}(P^+), \sum_{j=1}^n b_j e_j \rangle - \sum_{j=1}^n b_j (2\rho_j) \right) \\ &= \frac{1}{2} \left(\langle \text{bar}_{DH}(P^+), b \rangle - \langle b, 2\rho \rangle \right) \\ &= \frac{1}{2} \langle \text{bar}_{DH}(P^+) - 2\rho, \nabla f \rangle. \end{aligned}$$

□

Acknowledgment. I thank Jean-Pierre Demailly, my PhD supervisor, for his encouragement to make this computation available. I thank the European Research Council for financial support in the form of a PhD grant from the project ‘‘Algebraic and Kähler geometry’’ (ALKAGE, no.

670846). The present work is an excerpt of my Master's degree project, which I completed in May of 2017, under the supervision of Richard Hind. I thank him and Mark Behrens, who funded me during the final stage of my Master's degree. I also thank those faculty members of the Notre Dame mathematics department who were supportive of my work.

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Gabriella Clemente
Université Grenoble Alpes, Institut Fourier, UMR 5582 du CNRS,
100 rue des Maths, 38610 Gières, France
email: gabriella.clemente@univ-grenoble-alpes.fr