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Abstract

We present an elementary way of computing the Donaldson-Futaki invariant associated to a test-configuration of an anti-canonically polarized Fano reductive group compactification.

Reductive group compactifications from polytopes. Let G be a reductive group and $T \subset G$ be a maximal torus with character lattice M, Lie algebra $\mathfrak{t}$, and dual Lie algebra $\mathfrak{t}^* \simeq M_{\mathbb{R}} := M \otimes \mathbb{R}$. Let W be the Weyl group of (G,T), and let Φ denote the root system of (G,T) with a fixed choice of positive roots Φ^+. We declare 2ρ to be the sum of the positive roots. The positive Weyl chamber is $M^+_{\mathbb{R}} := \{x \in M_{\mathbb{R}} | \langle \alpha, x \rangle \geq 0 \text{ for all } \alpha \in \Phi^+\}$. There is a one-to-one correspondence between lattice points $\lambda \in M^+_{\mathbb{R}}$ and irreducible G–representations E_λ. Furthermore, to a lattice point $\lambda \in M^+_{\mathbb{R}}$ corresponds a $G \times G$–representation $\mathrm{End}(E_\lambda)$. The dimension of $\mathrm{End}(E_\lambda)$ is a polynomial

$$\dim(\mathrm{End}(E_\lambda)) = (\dim(E_\lambda))^2 = H_d(\lambda) + H_{d-1}(\lambda) + \dots$$

in λ, and here H_d stands for the degree d homogeneous part of the polynomial $\dim(\mathrm{End}(E_\lambda))$, H_{d-1} stands for the degree $d-1$ part, and so on.

Let $P^+ := P \cap M^+_{\mathbb{R}}$, $C(P^+) \subseteq M_{\mathbb{R}} \times \mathbb{R}$ be the cone over $(P^+, 1)$, and consider the finitely generated algebra

$$R_P = \bigoplus_{\lambda \in C(P^+) \cap (M \times \mathbb{Z})} \mathrm{End}(E_\lambda).$$

To any $W-$invariant lattice polytope $P \subseteq M_{\mathbb{R}}$, we can associate a polarized reductive group compactification (X_P, L_P), where $X_P = Proj(R_P)$ and $L_P = \mathcal{O}(1)$.

The Fano condition. At the polytope level, Fano is the condition that the distance between 2ρ and any codimension one face of P^+ that does not meet the boundary of the positive Weyl chamber is equal to one. This is a result that can be found in [3], and which we recapitulate below.

Denote the Zariski closure of T in X_P by Z, which is a toric subvariety of X_P. When X_P is Fano, the support function v of P is of the form $v = v_{K_{\mathbb{C}}} + v_Z$, where $v_{K_{\mathbb{C}}}(x) = \langle 2\rho, x\rangle$ for all x in the positive Weyl chamber, $v_{K_{\mathbb{C}}}(wx) = v_{K_{\mathbb{C}}}(x)$ for all $w \in W$, and $v_Z(x) = -g_{-K_Z}(-x)$, where $-g_{-K_Z}$ is the support function of the anti-canonical line bundle of the toric subvariety $Z \subset X_P$. Since P is also the polytope of Z, the associated fan Σ_P gives rise to the toric subvariety Z. From the theory of toric varieties, $-K_Z = \sum_{\rho\in\Sigma(1)} D_\rho$, where $\Sigma(1)$ is the set of 1-dimensional cones of Σ_P and D_ρ is a prime torus invariant divisor on Z. The support function g_{-K_Z} has the property that $g_{-K_Z}(u_\rho) = -1$ for all $\rho \in \Sigma(1)$, where u_ρ is the minimal generator of the ray ρ. In particular, if a_i is the inward pointing normal to the i-th codimension one face of P, $g_{-K_Z}(a_i) = -g_{-K_Z}(-a_i) = -1$. Then, $v(a_i) = \langle a_i, 2\rho\rangle - 1$, and so the facet presentation of the polytope is

$$P = \{x \in M_{\mathbb{R}} | \langle a_i, x\rangle \geq \langle a_i, 2\rho\rangle - 1\}.$$

As a consequence, the equation that defines the i-th boundary face of P is $f_i(x) = \langle a_i, x - 2\rho\rangle + 1$ so that $f_i(2\rho) = 1$.

Calculation of the Donaldson-Futaki invariant. In the sequel, we obtain a number of identities that together with the Fano condition will allow us to simplify Alexeev's and Katzarkov's Donaldson-Futaki (DF) invariant:

Theorem. *(Theorem 3.3, [1]) Let f be a convex rational $W-$invariant piecewise linear function on P. Then the DF invariant of the corresponding test-configuration is given by the formula*

$$-F_1(f) = \frac{1}{2\int_{P^+} H_d d\mu}\Big(\int_{\partial P^+} f H_d d\sigma + 2\int_{P^+} f H_{d-1} d\mu - a\int_{P^+} f H_d d\mu\Big),$$

where

$$a = \frac{\int_{\partial P^+} H_d d\sigma + 2\int_{P^+} H_{d-1} d\mu}{\int_{P^+} H_d d\mu}.$$

Here $d\mu$ is the Lebesgue measure restricted to P, and the boundary measure $d\sigma$ is a positive measure on ∂P that is normalized so that on each codimension one face, which is defined by an equation $l(x) := \langle a, x\rangle = c$, $d\sigma \wedge dl = \pm d\mu$ holds.

Choose once and for all an isomorphism $M_{\mathbb{R}} \simeq \mathbb{R}^n$ so that P can be viewed as though contained in $\mathbb{R}^n$.

Claim. *Let $\Phi^+ = \{\alpha_1, \dots, \alpha_r\}$, $c = \prod_{i=1}^r \langle \alpha_i, \rho\rangle^2$, where $\rho = \frac{1}{2}\sum_{i=1}^r \alpha_i$, and let $\{e_j\}_{j=1}^n$ be the standard basis of $\mathbb{R}^n$. Then,*

1.

$$H_d(x) = \frac{1}{c}\prod_{i=1}^r \langle \alpha_i, x\rangle^2,$$

2.

$$H_{d-1}(x) = \frac{1}{c}\sum_{j=1}^r 2\langle \alpha_j, x\rangle\langle \alpha_j, \rho\rangle\langle \alpha_1, x\rangle^2 \dots \widehat{\langle \alpha_j, x\rangle}^2 \dots \langle \alpha_r, x\rangle^2,$$

3.

$$\nabla H_d(x) = \frac{1}{c}\sum_{j=1}^n \Big(\sum_{i=1}^n 2\langle \alpha_i, x\rangle\langle \alpha_i, e_j\rangle\langle \alpha_1, x\rangle^2 \dots \widehat{\langle \alpha_j, x\rangle}^2 \dots \langle \alpha_r, x\rangle^2\Big) e_j,$$

4. $\langle \nabla H_d(x), \rho\rangle = H_{d-1}(x)$,

5. $\langle \nabla H_d(x), x\rangle = 2rH_d(x)$, *and*

6. for any smooth function $f : P \to \mathbb{R}$,

$$div((x-2\rho)fH_d) = \langle \nabla f, x - 2\rho\rangle H_d + (2r+n)fH_d - 2fH_{d-1}.$$

Proof. Let E_x be an irreducible representation with highest weight x. To prove 1. and 2., we make use of the Weyl dimension formula

$$\dim(E_x) = \frac{\prod_{i=1}^r \langle \alpha_i, x+\rho\rangle}{\prod_{i=1}^r \langle \alpha_i, \rho\rangle}.$$

From the expression

$$\dim(E_x)^2 = \frac{1}{c}\prod_{i=1}^{r}(\langle\alpha_i, x\rangle^2 + 2\langle\alpha_i, x\rangle\langle\alpha_i, \rho\rangle + \langle\alpha_i, \rho\rangle^2),$$

it follows that if d is the highest degree homogeneous part of the polynomial $\dim(E_x)^2$, then

$$H_d(x) = \frac{1}{c}\prod_{i=1}^{r}\langle\alpha_i, x\rangle^2,$$

and the $(d-1)$–degree homogeneous part of $\dim(E_x)^2$ is

$$H_{d-1}(x) = \frac{1}{c}\sum_{j=1}^{r} 2\langle\alpha_j, x\rangle\langle\alpha_j, \rho\rangle\langle\alpha_1, x\rangle^2 \ldots \widehat{\langle\alpha_j, x\rangle}^2 \ldots \langle\alpha_r, x\rangle^2.$$

For 3., note that $\frac{\partial}{\partial x_j}\langle\alpha_i, x\rangle = \langle\alpha_i, e_j\rangle$ so that

$$\frac{\partial}{\partial x_j}H_d(x) = \frac{1}{c}\sum_{i=1}^{r} 2\langle\alpha_i, x\rangle\langle\alpha_i, e_j\rangle\langle\alpha_1, x\rangle^2 \ldots \widehat{\langle\alpha_i, x\rangle}^2 \ldots \langle\alpha_r, x\rangle^2,$$

and hence

$$\nabla H_d(x) = \sum_{j=1}^{r}\frac{\partial}{\partial x_j}H_d(x)e_j = \frac{1}{c}\sum_{j=1}^{n}\Big(\sum_{i=1}^{n} 2\langle\alpha_i, x\rangle\langle\alpha_i, e_j\rangle\alpha_1, x\rangle^2 \ldots \widehat{\langle\alpha_j, x\rangle}^2 \ldots \langle\alpha_r, x\rangle^2\Big)e_j.$$

For 4., notice that

$$\begin{aligned}
\nabla H_d(x) &= \frac{1}{c}\sum_{j=1}^{n}\Big(\sum_{i=1}^{n} 2\langle\alpha_i, x\rangle\langle\alpha_i, e_j\rangle\langle\alpha_1, x\rangle^2 \ldots \widehat{\langle\alpha_j, x\rangle}^2 \ldots \langle\alpha_r, x\rangle^2\Big)e_j \\
&= \sum_{i=1}^{r}\Big(\frac{1}{c}2\langle\alpha_i, x\rangle\langle\alpha_1, x\rangle^2 \ldots \widehat{\langle\alpha_i, x\rangle}^2 \ldots \langle\alpha_r, x\rangle^2\Big)\sum_{j=1}^{n}\langle\alpha_i, e_j\rangle e_j \\
&= \sum_{i=1}^{r}\Big(\frac{1}{c}2\langle\alpha_i, x\rangle\langle\alpha_1, x\rangle^2 \ldots \widehat{\langle\alpha_i, x\rangle}^2 \ldots \langle\alpha_r, x\rangle^2\Big)\alpha_i
\end{aligned}$$

and then

$$\langle \nabla H_d(x), \rho\rangle = \sum_{i=1}^{r} \frac{1}{c} 2\langle \alpha_i, x\rangle\langle \alpha_i, \rho\rangle\langle \alpha_1, x\rangle^2 \dots \widehat{\langle \alpha_i, x\rangle}^2 \dots \langle \alpha_r, x\rangle^2 = H_{d-1}(x).$$

For 5., observe that since

$$\nabla H_d(x) = \sum_{i=1}^{r} \Big(\frac{1}{c} 2\langle \alpha_i, x\rangle\langle \alpha_1, x\rangle^2 \dots \widehat{\langle \alpha_i, x\rangle}^2 \dots \langle \alpha_r, x\rangle^2 \Big) \alpha_i,$$

and since for each i,

$$\Big\langle \frac{1}{c} 2\langle \alpha_i, x\rangle\langle \alpha_1, x\rangle^2 \dots \widehat{\langle \alpha_i, x\rangle}^2 \dots \langle \alpha_r, x\rangle^2 \alpha_i, x \Big\rangle = 2\Big(\frac{1}{c}\langle \alpha_1, x\rangle^2 \dots\dots \langle \alpha_r, x\rangle^2 \Big),$$

indeed we have that

$$\langle \nabla H_d(x), x\rangle = 2r\Big(\frac{1}{c} \prod_{i=1}^{r} \langle \alpha_i, x\rangle^2 \Big) = 2r H_d(x).$$

The above identities now imply the last point. Namely,

$$\begin{aligned} div((x-2\rho) f H_d) &= \langle \nabla(f H_d), x - 2\rho\rangle + div(x-2\rho) f H_d \\ &= \langle \nabla f, x-2\rho\rangle H_d + \langle \nabla H_d, x - 2\rho\rangle f + n f H_d \\ &= \langle \nabla f, x-2\rho\rangle H_d + \langle \nabla H_d, x\rangle f - 2\langle \nabla H_d, \rho\rangle f + n f H_d \\ &= \langle \nabla f, x-2\rho\rangle H_d + (2r+n) f H_d - 2 f H_{d-1}. \end{aligned}$$

□

The following is analogous to Theorem C in [2].

Proposition. *Suppose that P satisfies the Fano condition. Let $f : P \to \mathbb{R}$ be a function as in the theorem that is affine linear on P^+. Then, the DF invariant of the test-configuration associated to f is given by*

$$-F_1(f) = \frac{1}{2Vol_{DH}(P^+)} \int_{P^+} \langle \nabla f, x - 2\rho\rangle H_d d\mu = \frac{1}{2}\langle bar_{DH}(P^+) - 2\rho, \nabla f\rangle,$$

where $bar_{DH}(P^+) = \frac{1}{Vol_{DH}(P^+)} \int_{P^+} x H_d d\mu$ and $Vol_{DH}(P^+) = \int_{P^+} H_d d\mu$ are the barycenter and respectively the volume of P^+ with respect to the Duistermaat-Heckman (DH) measure.

Proof. Suppose that ∂P^+ has k codimension one faces ∂P_i^+. Let $\{\partial P_i^+ : i = 1,\dots,m\}$ be the set of all codimension one faces of P^+ that do not intersect the boundary of the positive Weyl chamber. Suppose that ∂P_i^+ is defined by $\langle a_i, x\rangle - c_i = 0$ and set $f_i(x) := \langle a_i, x\rangle - c_i$. The (inward) unit normal vector field to ∂P_i^+ is $-\frac{\nabla f_i}{\|\nabla f_i\|} = -\frac{a_i}{\|a_i\|}$. Since P^+ satisfies the Fano condition, for $x \in \partial P_i^+$, we have that

$$\begin{aligned}\langle (x-2\rho) f H_d, -\frac{a_i}{\|a_i\|}\rangle &= -H_d f\Big(\Big\langle x - 2\rho, \frac{a_i}{\|a_i\|}\Big\rangle\Big)\\ &= \frac{-H_d(x) f}{\|a_i\|}\Big(\langle x, a_i\rangle - \langle 2\rho, a_i\rangle\Big)\\ &= \frac{H_d(x) f}{\|a_i\|}\Big(\langle 2\rho, a_i\rangle - c_i\Big)\\ &= \frac{H_d(x) f}{\|a_i\|}.\end{aligned}$$

The divergence theorem implies that

$$\int_{P^+} div((x-2\rho) f H_d) d\mu = \sum_{i=1}^{m} \int_{\partial P_i^+} \frac{H_d f}{\|a_i\|} d\sigma_i + \sum_{i=m+1}^{k} \int_{\partial P_i^+} \langle (x-2\rho) f H_d, -\frac{a_i}{\|a_i\|}\rangle d\sigma_i,$$

where $d\sigma_i$ is the standard Lebesgue measure on ∂P with domain restricted to ∂P_i. When $i = m+1,\dots,k$, ∂P_i^+ is in the boundary of the positive Weyl chamber, and

$$\int_{\partial P_i^+} \langle (x-2\rho) f H_d, \frac{a_i}{\|a_i\|}\rangle d\sigma_i = 0.$$

Then

$$\int_{P^+} div((x-2\rho) f H_d) d\mu = \sum_{i=1}^{m} \int_{\partial P_i^+} \frac{H_d(x) f}{\|a_i\|} d\sigma_i$$

and the right hand side is the definition of

$$\int_{\partial P^+} f H_d d\sigma.$$

By 6. of the claim, taking $f = 1$, we obtain that

$$div((x-2\rho) H_d) = (2r+n) H_d - 2H_{d-1}.$$

Then, by the divergence theorem,

$$\int_{\partial P^+} H_d d\sigma = (2r+n)\int_{P^+} H_d d\mu - 2\int_{P^+} H_{d-1} d\mu.$$

Hence,

$$a = \frac{\int_{\partial P^+} H_d d\sigma + 2\int_{P^+} H_{d-1} d\mu}{\int_{P^+} H_d d\mu} = 2r+n.$$

Upon substituting the above calculations into Alexeev's and Katzarkov's DF invariant (cf. Theorem), again using 6. of the claim to rewrite the first integral, we find that

$$-F_1(f) = \frac{1}{2\int_{P^+} H_d d\mu}\int_{P^+} \langle \nabla f, x - 2\rho\rangle H_d d\mu.$$

Suppose that f on P^+ is given as $f(x) = \sum_{j=1}^n b_j x_j + k$. Put $b = (b_1, \dots, b_n)$, $x = (x_1, \dots, x_n)$ and $2\rho = (2\rho_1, \dots, 2\rho_n)$, and let e_j be the j–th standard basis vector of $\mathbb{R}^n$. Then $\langle \nabla f, x - 2\rho\rangle = \sum_{j=1}^n b_j(x_j - 2\rho_j)$ and it follows that

$$\begin{aligned}
-F_1(f) &= \frac{1}{2Vol_{DH}(P^+)}\int_{P^+} \langle \nabla f, x - 2\rho\rangle H_d d\mu \\
&= \frac{1}{2Vol_{DH}(P^+)}\Big(\sum_{j=1}^n b_j \int_{P^+} x_j H_d d\mu - \sum_{j=1}^n b_j(2\rho_j) Vol_{DH}(P^+)\Big) \\
&= \frac{1}{2}\Big(\langle bar_{DH}(P^+), \sum_{j=1}^n b_j e_j\rangle - \sum_{j=1}^n b_j(2\rho_j)\Big) \\
&= \frac{1}{2}\Big(\langle bar_{DH}(P^+), b\rangle - \langle b, 2\rho\rangle\Big) \\
&= \frac{1}{2}\langle bar_{DH}(P^+) - 2\rho, \nabla f\rangle.
\end{aligned}$$

□

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