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A technical remark on the Donaldson-Futaki invariant for Fano reductive group compactifications

Gabriella Clemente

Abstract

We present an elementary way of computing the Donaldson-Futaki invariant associated to a test-configuration of an anti-canonically polarized Fano reductive group compactification.

Reductive group compactifications from polytopes. Let *G* be a reductive group and $T \subset G$ be a maximal torus with character lattice *M*, Lie algebra t, and dual Lie algebra $t^* \simeq M_{\mathbb{R}} := M \otimes \mathbb{R}$. Let *W* be the Weyl group of (G, T), and let Φ denote the root system of (G, T) with a fixed choice of positive roots Φ^+ . We declare 2ρ to be the sum of the positive roots. The positive Weyl chamber is $M_{\mathbb{R}}^+ := \{x \in M_{\mathbb{R}} | \langle \alpha, x \rangle \ge 0 \text{ for all } \alpha \in \Phi^+\}$. There is a one-to-one correspondence between lattice points $\lambda \in M_{\mathbb{R}}^+$ and irreducible *G*-representations E_{λ} . Furthermore, to a lattice point $\lambda \in M_{\mathbb{R}}^+$ corresponds a $G \times G$ -representation $\text{End}(E_{\lambda})$. The dimension of $\text{End}(E_{\lambda})$ is a polynomial

$$\dim(\operatorname{End}(E_{\lambda})) = (\dim(E_{\lambda}))^{2} = H_{d}(\lambda) + H_{d-1}(\lambda) + \dots$$

in λ , and here H_d stands for the degree d homogeneous part of the polynomial dim(End(E_λ)), H_{d-1} stands for the degree d-1 part, and so on.

Let $P^+ := P \cap M^+_{\mathbb{R}}$, $C(P^+) \subseteq M_{\mathbb{R}} \times \mathbb{R}$ be the cone over $(P^+, 1)$, and consider the finitely generated algebra

$$R_P = \bigoplus_{\lambda \in C(P^+) \cap (M \times \mathbb{Z})} \operatorname{End}(E_{\lambda}).$$

To any *W*-invariant lattice polytope $P \subseteq M_{\mathbb{R}}$, we can associate a polarized reductive group compactification (X_P, L_P) , where $X_P = Proj(R_P)$ and $L_P = O(1)$.

The Fano condition. At the polytope level, Fano is the condition that the distance between 2ρ and any codimension one face of P^+ that does not meet the boundary of the positive Weyl chamber is equal to one. This is a result that can be found in [3], and which we recapitulate below.

Denote the Zariski closure of *T* in X_P by *Z*, which is a toric subvariety of X_P . When X_P is Fano, the support function *v* of *P* is of the form $v = v_{K_{\mathbb{C}}} + v_Z$, where $v_{K_{\mathbb{C}}}(x) = \langle 2\rho, x \rangle$ for all *x* in the positive Weyl chamber, $v_{K_{\mathbb{C}}}(wx) = v_{K_{\mathbb{C}}}(x)$ for all $w \in W$, and $v_Z(x) = -g_{-K_Z}(-x)$, where $-g_{-K_Z}$ is the support function of the anti-canonical line bundle of the toric subvariety $Z \subset X_P$. Since *P* is also the polytope of *Z*, the associated fan Σ_P gives rise to the toric subvariety *Z*. From the theory of toric varieties, $-K_Z = \sum_{\rho \in \Sigma(1)} D_\rho$, where $\Sigma(1)$ is the set of 1-dimensional cones of Σ_P and D_ρ is a prime torus invariant divisor on *Z*. The support function g_{-K_Z} has the property that $g_{-K_Z}(u_\rho) = -1$ for all $\rho \in \Sigma(1)$, where u_ρ is the minimal generator of the ray ρ . In particular, if a_i is the inward pointing normal to the *i*-th codimension one face of *P*, $g_{-K_Z}(a_i) = -g_{-K_Z}(-a_i) = -1$. Then, $v(a_i) = \langle a_i, 2\rho \rangle - 1$, and so the facet presentation of the polytope is

$$P = \{ x \in M_{\mathbb{R}} | \langle a_i, x \rangle \ge \langle a_i, 2\rho \rangle - 1 \}.$$

As a consequence, the equation that defines the *i*-th boundary face of *P* is $f_i(x) = \langle a_i, x - 2\rho \rangle + 1$ so that $f_i(2\rho) = 1$.

Calculation of the Donaldson-Futaki invariant. In the sequel, we obtain a number of identities that together with the Fano condition will allow us to simplify Alexeev's and Katzarkov's Donaldson-Futaki (DF) invariant:

Theorem. (Theorem 3.3, [1]) Let f be a convex rational W-invariant piecewise linear function on P. Then the DF invariant of the corresponding test-configuration is given by the formula

$$-F_{1}(f) = \frac{1}{2\int_{P^{+}}H_{d}d\mu} \Big(\int_{\partial P^{+}}fH_{d}d\sigma + 2\int_{P^{+}}fH_{d-1}d\mu - a\int_{P^{+}}fH_{d}d\mu\Big),$$

where

$$a = \frac{\int_{\partial P^+} H_d d\sigma + 2 \int_{P^+} H_{d-1} d\mu}{\int_{P^+} H_d d\mu}$$

Here $d\mu$ is the Lebesgue measure restricted to *P*, and the boundary measure $d\sigma$ is a positive measure on ∂P that is normalized so that on each codimension one face, which is defined by an equation $l(x) := \langle a, x \rangle = c$, $d\sigma \wedge dl = \pm d\mu$ holds.

Choose once and for all an isomorphism $M_{\mathbb{R}} \simeq \mathbb{R}^n$ so that P can be viewed as though contained in \mathbb{R}^n .

Claim. Let $\Phi^+ = \{\alpha_1, \dots, \alpha_r\}$, $c = \prod_{i=1}^r \langle \alpha_i, \rho \rangle^2$, where $\rho = \frac{1}{2} \sum_{i=1}^r \alpha_i$, and let $\{e_j\}_{j=1}^n$ be the standard basis of \mathbb{R}^n . Then,

$$H_d(x) = \frac{1}{c} \prod_{i=1}^r \langle \alpha_i, x \rangle^2$$

2.

1.

$$H_{d-1}(x) = \frac{1}{c} \sum_{j=1}^{r} 2\langle \alpha_j, x \rangle \langle \alpha_j, \rho \rangle \langle \alpha_1, x \rangle^2 \dots \langle \widehat{\alpha_j, x} \rangle^2 \dots \langle \alpha_r, x \rangle^2,$$

3.

$$\nabla H_d(x) = \frac{1}{c} \sum_{j=1}^n \Big(\sum_{i=1}^n 2\langle \alpha_i, x \rangle \langle \alpha_i, e_j \rangle \langle \alpha_1, x \rangle^2 \dots \langle \widehat{\alpha_j, x} \rangle^2 \dots \langle \alpha_r, x \rangle^2 \Big) e_j,$$

4.
$$\langle \nabla H_d(x), \rho \rangle = H_{d-1}(x),$$

- 5. $\langle \nabla H_d(x), x \rangle = 2rH_d(x)$, and
- 6. for any smooth function $f : P \to \mathbb{R}$,

$$div((x-2\rho)fH_d) = \langle \nabla f, x-2\rho \rangle H_d + (2r+n)fH_d - 2fH_{d-1}$$

Proof. Let E_x be an irreducible representation with highest weight x. To prove 1. and 2., we make use of the Weyl dimension formula

$$\dim(E_x) = \frac{\prod_{i=1}^r \langle \alpha_i, x + \rho \rangle}{\prod_{i=1}^r \langle \alpha_i, \rho \rangle}$$

From the expression

$$\dim(E_x)^2 = \frac{1}{c} \prod_{i=1}^r (\langle \alpha_i, x \rangle^2 + 2 \langle \alpha_i, x \rangle \langle \alpha_i, \rho \rangle + \langle \alpha_i, \rho \rangle^2),$$

it follows that if *d* is the highest degree homogeneous part of the polynomial $\dim(E_x)^2$, then

$$H_d(x) = \frac{1}{c} \prod_{i=1}^r \langle \alpha_i, x \rangle^2,$$

and the (d-1)-degree homogeneous part of dim $(E_x)^2$ is

$$H_{d-1}(x) = \frac{1}{c} \sum_{j=1}^{r} 2\langle \alpha_j, x \rangle \langle \alpha_j, \rho \rangle \langle \alpha_1, x \rangle^2 \dots \langle \alpha_j, x \rangle^2 \dots \langle \alpha_r, x \rangle^2.$$

For 3., note that $\frac{\partial}{\partial x_j} \langle \alpha_i, x \rangle = \langle \alpha_i, e_j \rangle$ so that

$$\frac{\partial}{\partial x_j} H_d(x) = \frac{1}{c} \sum_{i=1}^r 2\langle \alpha_i, x \rangle \langle \alpha_i, e_j \rangle \langle \alpha_1, x \rangle^2 \dots \langle \widehat{\alpha_i, x} \rangle^2 \dots \langle \alpha_r, x \rangle^2,$$

and hence

$$\nabla H_d(x) = \sum_{j=1}^r \frac{\partial}{\partial x_j} H_d(x) e_j = \frac{1}{c} \sum_{j=1}^n \Big(\sum_{i=1}^n 2\langle \alpha_i, x \rangle \langle \alpha_i, e_j \rangle \alpha_1, x \rangle^2 \dots \langle \widehat{\alpha_j, x} \rangle^2 \dots \langle \alpha_r, x \rangle^2 \Big) e_j.$$

For 4., notice that

$$\nabla H_d(x) = \frac{1}{c} \sum_{j=1}^n \left(\sum_{i=1}^n 2\langle \alpha_i, x \rangle \langle \alpha_i, e_j \rangle \langle \alpha_1, x \rangle^2 \dots \langle \widehat{\alpha_j, x} \rangle^2 \dots \langle \alpha_r, x \rangle^2 \right) e_j$$
$$= \sum_{i=1}^r \left(\frac{1}{c} 2\langle \alpha_i, x \rangle \langle \alpha_1, x \rangle^2 \dots \langle \widehat{\alpha_i, x} \rangle^2 \dots \langle \alpha_r, x \rangle^2 \right) \sum_{j=1}^n \langle \alpha_i, e_j \rangle e_j$$
$$= \sum_{i=1}^r \left(\frac{1}{c} 2\langle \alpha_i, x \rangle \langle \alpha_1, x \rangle^2 \dots \langle \widehat{\alpha_i, x} \rangle^2 \dots \langle \alpha_r, x \rangle^2 \right) \alpha_i$$

and then

$$\langle \nabla H_d(x), \rho \rangle = \sum_{i=1}^r \frac{1}{c} 2 \langle \alpha_i, x \rangle \langle \alpha_i, \rho \rangle \langle \alpha_1, x \rangle^2 \dots \langle \widehat{\alpha_i, x} \rangle^2 \dots \langle \alpha_r, x \rangle^2 = H_{d-1}(x).$$

For 5., observe that since

$$\nabla H_d(x) = \sum_{i=1}^r \left(\frac{1}{c} 2\langle \alpha_i, x \rangle \langle \alpha_1, x \rangle^2 \dots \langle \widehat{\alpha_i, x} \rangle^2 \dots \langle \alpha_r, x \rangle^2\right) \alpha_i,$$

and since for each *i*,

$$\left\langle \frac{1}{c} 2\langle \alpha_i, x \rangle \langle \alpha_1, x \rangle^2 \dots \langle \widehat{\alpha_i, x} \rangle^2 \dots \langle \alpha_r, x \rangle^2 \alpha_i, x \right\rangle = 2 \left(\frac{1}{c} \langle \alpha_1, x \rangle^2 \dots \langle \alpha_r, x \rangle^2 \right),$$

indeed we have that

$$\langle \nabla H_d(x), x \rangle = 2r \left(\frac{1}{c} \prod_{i=1}^r \langle \alpha_i, x \rangle^2 \right) = 2r H_d(x).$$

The above identities now imply the last point. Namely,

$$\begin{aligned} div((x-2\rho)fH_d) &= \langle \nabla(fH_d), x-2\rho \rangle + div(x-2\rho)fH_d \\ &= \langle \nabla f, x-2\rho \rangle H_d + \langle \nabla H_d, x-2\rho \rangle f + nfH_d \\ &= \langle \nabla f, x-2\rho \rangle H_d + \langle \nabla H_d, x \rangle f - 2 \langle \nabla H_d, \rho \rangle f + nfH_d \\ &= \langle \nabla f, x-2\rho \rangle H_d + (2r+n)fH_d - 2fH_{d-1}. \end{aligned}$$

The following is analogous to Theorem C in [2].

Proposition. Suppose that P satisfies the Fano condition. Let $f : P \to \mathbb{R}$ be a function as in the theorem that is affine linear on P^+ . Then, the DF invariant of the test-configuration associated to f is given by

$$-F_1(f) = \frac{1}{2Vol_{DH}(P^+)} \int_{P^+} \langle \nabla f, x - 2\rho \rangle H_d d\mu = \frac{1}{2} \langle bar_{DH}(P^+) - 2\rho, \nabla f \rangle,$$

where $bar_{DH}(P^+) = \frac{1}{Vol_{DH}(P^+)} \int_{P^+} xH_d d\mu$ and $Vol_{DH}(P^+) = \int_{P^+} H_d d\mu$ are the barycenter and respectively the volume of P^+ with respect to the Duistermaat-Heckman (DH) measure.

Proof. Suppose that ∂P^+ has k codimension one faces ∂P_i^+ . Let $\{\partial P_i^+ : i = 1, \dots, m\}$ be the set of all codimension one faces of P^+ that do not intersect the boundary of the positive Weyl chamber. Suppose that ∂P_i^+ is defined by $\langle a_i, x \rangle - c_i = 0$ and set $f_i(x) := \langle a_i, x \rangle - c_i$. The (inward) unit normal vector field to ∂P_i^+ is $-\frac{\nabla f_i}{\|\nabla f_i\|} = -\frac{a_i}{\|a_i\|}$. Since P^+ satisfies the Fano condition, for $x \in \partial P_i^+$, we have that

$$\begin{aligned} \langle (x-2\rho)fH_d, -\frac{a_i}{\|a_i\|} \rangle &= -H_d f\left(\left\langle x-2\rho, \frac{a_i}{\|a_i\|} \right\rangle\right) \\ &= \frac{-H_d(x)f}{\|a_i\|} \left(\langle x, a_i \rangle - \langle 2\rho, a_i \rangle\right) \\ &= \frac{H_d(x)f}{\|a_i\|} \left(\langle 2\rho, a_i \rangle - c_i\right) \\ &= \frac{H_d(x)f}{\|a_i\|}. \end{aligned}$$

The divergence theorem implies that

$$\int_{P^+} div((x-2\rho)fH_d)d\mu = \sum_{i=1}^m \int_{\partial P_i^+} \frac{H_d f}{\|a_i\|} d\sigma_i + \sum_{i=m+1}^k \int_{\partial P_i^+} \langle (x-2\rho)fH_d, -\frac{a_i}{\|a_i\|} \rangle d\sigma_i,$$

where $d\sigma_i$ is the standard Lebesgue measure on ∂P with domain restricted to ∂P_i . When i = m + 1, ..., k, ∂P_i^+ is in the boundary of the positive Weyl chamber, and

$$\int_{\partial P_i^+} \langle (x-2\rho) f H_d, \frac{a_i}{\|a_i\|} \rangle d\sigma_i = 0.$$

Then

$$\int_{P^+} div((x-2\rho)fH_d)d\mu = \sum_{i=1}^m \int_{\partial P_i^+} \frac{H_d(x)f}{\|a_i\|} d\sigma_i$$

and the right hand side is the definition of

$$\int_{\partial P^+} f H_d d\sigma.$$

By 6. of the claim, taking f = 1, we obtain that

$$div((x-2\rho)H_d) = (2r+n)H_d - 2H_{d-1}.$$

Then, by the divergence theorem,

$$\int_{\partial P^+} H_d d\sigma = (2r+n) \int_{P^+} H_d d\mu - 2 \int_{P^+} H_{d-1} d\mu.$$

Hence,

$$a = \frac{\int_{\partial P^+} H_d d\sigma + 2 \int_{P^+} H_{d-1} d\mu}{\int_{P^+} H_d d\mu} = 2r + n.$$

Upon substituting the above calculations into Alexeev's and Katzarkov's DF invariant (cf. Theorem), again using 6. of the claim to rewrite the first integral, we find that

$$-F_1(f) = \frac{1}{2\int_{P^+} H_d d\mu} \int_{P^+} \langle \nabla f, x - 2\rho \rangle H_d d\mu.$$

Suppose that f on P^+ is given as $f(x) = \sum_{j=1}^n b_j x_j + k$. Put $b = (b_1, \dots, b_n)$, $x = (x_1, \dots, x_n)$ and $2\rho = (2\rho_1, \dots, 2\rho_n)$, and let e_j be the j-th standard basis vector of \mathbb{R}^n . Then $\langle \nabla f, x - 2\rho \rangle = \sum_{j=1}^n b_j (x_j - 2\rho_j)$ and it follows that

$$\begin{split} -F_{1}(f) &= \frac{1}{2Vol_{DH}(P^{+})} \int_{P^{+}} \langle \nabla f, x - 2\rho \rangle H_{d} d\mu \\ &= \frac{1}{2Vol_{DH}(P^{+})} \Big(\sum_{j=1}^{n} b_{j} \int_{P^{+}} x_{j} H_{d} d\mu - \sum_{j=1}^{n} b_{j} (2\rho_{j}) Vol_{DH}(P^{+}) \Big) \\ &= \frac{1}{2} \Big(\langle bar_{DH}(P^{+}), \sum_{j=1}^{n} b_{j} e_{j} \rangle - \sum_{j=1}^{n} b_{j} (2\rho_{j}) \Big) \\ &= \frac{1}{2} \Big(\langle bar_{DH}(P^{+}), b \rangle - \langle b, 2\rho \rangle \Big) \\ &= \frac{1}{2} \langle bar_{DH}(P^{+}) - 2\rho, \nabla f \rangle. \end{split}$$

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