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► **To cite this version:**

Dietrich Burde, Friedrich Wagemann. PIRASHVILI'S CONJECTURE ON THE LEIBNIZ HOMOLOGY FOR LIE ALGEBRAS. 2019. hal-02266972

HAL Id: hal-02266972

<https://hal.science/hal-02266972>

Preprint submitted on 17 Aug 2019

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PIRASHVILI'S CONJECTURE ON THE LEIBNIZ HOMOLOGY FOR LIE ALGEBRAS

DIETRICH BURDE AND FRIEDRICH WAGEMANN

ABSTRACT. We prove a conjecture of Pirashvili, which says that a non-trivial finite-dimensional complex Lie algebra \mathfrak{g} is semisimple if and only if its Leibniz (co)homology with trivial coefficients vanishes. We also prove several results on the Lie and Leibniz cohomology of perfect and complete Lie algebras.

1. INTRODUCTION

It is well-known that one can characterize finite-dimensional semisimple Lie algebras \mathfrak{g} over a field K of characteristic zero by the vanishing of certain Lie algebra cohomology groups. For example, by Whitehead's first lemma, we have $H^1(\mathfrak{g}, M) = 0$ for every finite-dimensional \mathfrak{g} -module M . The converse statement is also true - any Lie algebra whose first cohomology with coefficients in any finite-dimensional module vanishes is semisimple. This cannot be generalized to higher cohomology groups. In fact, the converse to the second Whitehead Lemma is no longer true, see [16]. There has been interest in a different vanishing condition, namely only for some canonical modules, like the trivial module or the adjoint module, and at the same time also involving the vanishing of higher cohomology groups. However, the cohomology of semisimple Lie algebras with trivial coefficients does not vanish in all degrees. It is well-known that $H^3(\mathfrak{g}, K)$ is nonzero. However, it is known that the adjoint cohomology groups $H^p(\mathfrak{g}, \mathfrak{g})$ vanish for all $p \geq 0$ for a semisimple Lie algebra, see [6]. But here the converse need not be true. There are many counterexamples, see the family of non-perfect reductive Lie algebras given in Example 2.10. This suggests that we should add the condition $H^1(\mathfrak{g}, K) = 0$ for the trivial module. Note that this is a strong condition on \mathfrak{g} , which is equivalent to $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, i.e., to \mathfrak{g} being perfect.

So the question is, whether \mathfrak{g} is semisimple if and only if \mathfrak{g} is perfect and $H^p(\mathfrak{g}, \mathfrak{g}) = 0$ for all $p \geq 0$. It turns out that this question is well known and has already a long history. In 1988 Angelopoulos states in [1] that the question goes back to M. Flato some decades ago, who asked whether semisimple Lie algebras \mathfrak{g} are characterized by the vanishing conditions $H^1(\mathfrak{g}) = H^1(\mathfrak{g}, \mathfrak{g}) = 0$. Afterwards several authors constructed non-semisimple Lie algebras \mathfrak{g} satisfying

$$H^1(\mathfrak{g}) = H^2(\mathfrak{g}) = H^0(\mathfrak{g}, \mathfrak{g}) = H^1(\mathfrak{g}, \mathfrak{g}) = H^2(\mathfrak{g}, \mathfrak{g}) = 0$$

see [2, 3, 4, 5]. With the development of Leibniz cohomology for Lie algebras the question became more precise and it turned out that Leibniz cohomology is the right tool for studying this question. Pirashvili [15] proved that the vanishing conditions $H^1(\mathfrak{g}) = H^p(\mathfrak{g}, \mathfrak{g}) = 0$ for

Date: August 17, 2019.

2000 Mathematics Subject Classification. Primary 17A32, Secondary 17B56.

Key words and phrases. Lie algebra cohomology, Leibniz algebra cohomology, complete Lie algebra, perfect Lie algebra.

all $p \geq 0$ are equivalent to the fact that all Leibniz homology groups with trivial coefficients vanish, i.e., that $HL_p(\mathfrak{g}, K) = 0$ for all $p \geq 1$. He then conjectured that these conditions are necessary and sufficient for \mathfrak{g} being semisimple and he formulated an even stronger conjecture.

In this article we prove his “weak” conjecture. This becomes possible by the work on Leibniz (co)homology by several authors and in particular by the results obtained in [9]. For simplicity we assume throughout this article that all algebras are defined over the complex numbers. In the first section we recall the definitions and results needed to state Pirashvili’s conjecture. We translate the so-called Pirashvili conditions for a Lie algebra \mathfrak{g} , namely the vanishing $HL^p(\mathfrak{g}, \mathbb{C}) = 0$ for all $p \geq 1$ to several other vanishing conditions.

In the third section we derive further important consequences for the Lie algebra cohomology of \mathfrak{g} satisfying the Pirashvili conditions. In particular, we prove that $H^p(\mathfrak{g}, \mathfrak{r}) = 0$ holds for all $p \geq 0$ for such a Lie algebra \mathfrak{g} , with solvable radical \mathfrak{r} . We show that the solvable radical of a non-trivial non-semisimple Lie algebra satisfying the Pirashvili conditions is nilpotent and non-abelian. For Lie algebras \mathfrak{g} whose solvable radical is nonzero and abelian we show that $H^1(\mathfrak{g}, \mathfrak{g})$ is non-zero. Here we use the Hochschild-Serre spectral sequence. This way we can even show a more precise result on $H^1(\mathfrak{g}, \mathfrak{g})$.

In the last section we prove Pirashvili’s conjecture, using the results of the previous sections and using spectral sequences for \mathfrak{g} with Levi decomposition $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{n}$, where the solvable radical of \mathfrak{g} is nilpotent, applied to the non-zero abelian ideal $Z(\mathfrak{n})$ of \mathfrak{g} .

2. STATEMENT OF PIRASHVILI’S CONJECTURE

In this section we will recall the definitions and results, which are needed to state Pirashvili’s so-called *weak conjecture* from [15]. The conjecture says that a non-trivial Lie algebra is semisimple if and only if its Leibniz homology with trivial coefficients vanishes. We find several equivalent conditions to these vanishing conditions, which enable us to restate Pirashvili’s conjecture in terms of Leibniz cohomology and Lie algebra cohomology.

We will assume that all algebras are finite-dimensional. Let \mathfrak{g} be a Lie algebra, M be a \mathfrak{g} -module and p be a non-negative integer. We denote by $H^p(\mathfrak{g}, M)$ respectively $H_p(\mathfrak{g}, M)$ the p -th Lie algebra cohomology group, respectively the p -th Lie algebra homology group in the sense of Chevalley-Eilenberg. For a left Leibniz algebra \mathfrak{L} and an \mathfrak{L} -bimodule M we denote by $HL^p(\mathfrak{L}, M)$ respectively $HL_p(\mathfrak{L}, M)$ the p -th Leibniz cohomology group respectively the p -th Leibniz homology group. As usual, let us drop the trivial module as coefficient from the notation. For further background on Leibniz algebras and their (co)homology see [12, 13, 8, 9]. Since every Lie algebra \mathfrak{g} is also a left Leibniz algebra, it makes sense to consider Leibniz (co)homology for Lie algebras, where the \mathfrak{g} -module M is viewed as a symmetric Leibniz bimodule. The following result is known as *direct duality*, see [10], page 16.

Lemma 2.1. *We have the following duality isomorphism*

$$(HL_p(\mathfrak{g}, M^*))^* \cong HL^p(\mathfrak{g}, M)$$

for all $p \geq 0$.

We also recall the Poincaré duality for unimodular Lie algebras. A Lie algebra \mathfrak{g} is called *unimodular*, if the trace of all adjoint operators $\text{ad}(x)$ is zero.

Lemma 2.2. *Let \mathfrak{g} be a unimodular Lie algebra of dimension d and M be a \mathfrak{g} -module. Then we have an isomorphism*

$$H_{d-k}(\mathfrak{g}, M) \cong H^k(\mathfrak{g}, M)$$

for all $0 \leq k \leq d$.

Definition 2.3. A Lie algebra \mathfrak{g} is called *perfect*, if $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. It is called *complete*, if $Z(\mathfrak{g}) = 0$ and $\text{Der}(\mathfrak{g}) = \text{ad}(\mathfrak{g})$. It is called *sympathetic* if it is both perfect and complete.

Note that a perfect Lie algebra is unimodular.

Lemma 2.4. *A Lie algebra \mathfrak{g} is perfect if and only if $H^1(\mathfrak{g}) = 0$. It is complete if and only if $H^0(\mathfrak{g}, \mathfrak{g}) = H^1(\mathfrak{g}, \mathfrak{g}) = 0$.*

Proof. The first claim follows from $H^1(\mathfrak{g}) \cong (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^*$ and the second claim follows from $H^0(\mathfrak{g}, \mathfrak{g}) \cong Z(\mathfrak{g})$ and $H^1(\mathfrak{g}, \mathfrak{g}) \cong \text{Der}(\mathfrak{g})/\text{ad}(\mathfrak{g})$. \square

Pirashvili shows in [15] the following equivalence between (co)homological conditions for a Lie algebra.

Proposition 2.5. *Let \mathfrak{g} be a non-trivial Lie algebra. Then the following conditions for \mathfrak{g} are equivalent.*

- (1) *We have $HL_p(\mathfrak{g}) = 0$ for all $p \geq 1$.*
- (2) *We have $HL^p(\mathfrak{g}) = 0$ for all $p \geq 1$.*
- (3) *We have $H_p(\mathfrak{g}, \mathfrak{g}) = 0$ for all $p \geq 0$.*
- (4) *We have $H^p(\mathfrak{g}, \mathfrak{g}) = 0$ for all $p \geq 0$ and \mathfrak{g} is perfect.*

Proof. By Lemma 2.1 the vanishing of all Leibniz homology groups is equivalent to the vanishing of all Leibniz cohomology groups. Now the proof follows from Lemma 4.2 of [15]. \square

As Leibniz and Lie algebra cohomology of a Lie algebra coincide in degree one, we have $H^1(\mathfrak{g}) = HL^1(\mathfrak{g})$.

Corollary 2.6. *Let \mathfrak{g} be a Lie algebra satisfying the Pirashvili conditions. Then \mathfrak{g} is perfect and complete, hence also unimodular and sympathetic.*

Note that such a Lie algebra is also rigid, because of $H^2(\mathfrak{g}, \mathfrak{g}) = 0$, see [7].

Definition 2.7. Let \mathfrak{g} be a non-trivial Lie algebra. We say that \mathfrak{g} satisfies the *Pirashvili conditions*, if it satisfies one of the conditions in Proposition 2.5.

We can now restate Pirashvili's conjecture as follows.

Conjecture (Pirashvili) 2.8. *A non-trivial finite-dimensional complex Lie algebra is semisimple if and only if it satisfies the Pirashvili conditions.*

One direction of Pirashvili's conjecture is clear. Every semisimple Lie algebra \mathfrak{g} satisfies the Pirashvili conditions, see [14]. In particular, a semisimple Lie algebra is perfect and all adjoint cohomology groups vanish. This follows from the first Whitehead Lemma and the following result of Carles [6].

Proposition 2.9. *Let \mathfrak{g} be a complete Lie algebra, whose nilradical is abelian. Then we have $H^p(\mathfrak{g}, \mathfrak{g}) = 0$ for all $p \geq 0$.*

Hence, to prove Pirashvili's conjecture we need to show that every Lie algebra satisfying the Pirashvili conditions is semisimple. Note that we cannot drop the assumption that \mathfrak{g} is perfect in (4) here. Carles' result shows that the following family of non-perfect Lie algebras has vanishing adjoint cohomology, but is not semisimple.

Example 2.10. *Let $\mathfrak{g} = \mathfrak{aff}(\mathbb{C}^n) \cong \mathfrak{gl}_n(\mathbb{C}) \ltimes \mathbb{C}^n$ be the affine Lie algebra with $n \geq 1$. We have $H^p(\mathfrak{g}, \mathfrak{g}) = 0$ for all $p \geq 0$, but \mathfrak{g} is not semisimple.*

3. FURTHER CONSEQUENCES OF THE PIRASHVILI CONDITIONS

For a Lie algebra \mathfrak{g} denote by \mathfrak{r} the solvable radical of \mathfrak{g} and by \mathfrak{n} its nilradical. By the Levi decomposition we have $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{r}$ with a semisimple subalgebra \mathfrak{s} of \mathfrak{g} . We can derive further consequences from the Pirashvili conditions by using spectral sequences, namely the Hochschild-Serre spectral sequence [11], applied to the Levi decomposition of \mathfrak{g} . In [9] several results on the Leibniz cohomology of \mathfrak{g} have been shown, which we can apply now. In Corollary 4.8 of [9] the following is stated.

Proposition 3.1. *Let \mathfrak{g} be a finite-dimensional complex Lie algebra. Then for every integer $p \geq 1$ there are isomorphisms*

$$HL^p(\mathfrak{g}) \cong HL^{p-1}(\mathfrak{g}, \mathfrak{r}^*)$$

of vector spaces.

The vanishing of all Leibniz (co)homology groups is equivalent to the vanishing of all Lie algebra (co)homology groups.

Proposition 3.2. *Let \mathfrak{g} be a Lie algebra and M be a \mathfrak{g} -module. Consider M also as a symmetric Leibniz \mathfrak{g} -bimodule. Then we have $H^p(\mathfrak{g}, M) = 0$ for every integer p with $0 \leq p \leq n$ if and only if $HL^p(\mathfrak{g}, M) = 0$ for every integer p with $0 \leq p \leq n$. The same result holds for homology groups.*

Proof. The first part is Theorem 2.6 in [9]. The proof is analogous for homology groups. \square

In general, the vanishing of one single Leibniz cohomology group $HL^p(\mathfrak{g}, M) = 0$ is not equivalent to the vanishing of the Lie algebra cohomology group $H^p(\mathfrak{g}, M)$. We obtain an important consequence of the Pirashvili conditions.

Proposition 3.3. *Let \mathfrak{g} be a Lie algebra satisfying the Pirashvili conditions. Then we have*

$$H^p(\mathfrak{g}, \mathfrak{r}) = 0$$

for all $p \geq 0$.

Proof. By using Proposition 3.2 and Lemma 2.1 we have for all $p \geq 1$

$$0 = HL^p(\mathfrak{g}) \cong HL^{p-1}(\mathfrak{g}, \mathfrak{r}^*) \cong (HL_{p-1}(\mathfrak{g}, \mathfrak{r}))^*.$$

By Proposition 3.2 this is equivalent to the vanishing the Lie algebra homology, i.e., we have $H_{p-1}(\mathfrak{g}, \mathfrak{r}) = 0$ for all $p \geq 1$. By Lemma 2.2, the Poincaré duality, all cohomology groups vanish, since \mathfrak{g} is unimodular by Corollary 2.6. \square

We can reformulate this consequence as follows.

Proposition 3.4. *Let \mathfrak{g} be a non-trivial Lie algebra. Then we have $H^p(\mathfrak{g}, \mathfrak{r}) = 0$ for all $p \geq 0$ if and only if $H^p(\mathfrak{r}, \mathfrak{r})^s = 0$ for all $p \geq 0$.*

Proof. By the Hochschild-Serre formula for $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{r}$ we have

$$H^p(\mathfrak{g}, \mathfrak{r}) = \bigoplus_{p=k+\ell} H^k(\mathfrak{s}) \otimes H^\ell(\mathfrak{r}, \mathfrak{r})^{\mathfrak{s}}.$$

Hence $H^\ell(\mathfrak{r}, \mathfrak{r})^{\mathfrak{s}} = 0$ for all $\ell \geq 0$ implies that $H^p(\mathfrak{g}, \mathfrak{r}) = 0$ for all $p \geq 0$. Conversely, if $H^p(\mathfrak{g}, \mathfrak{r}) = 0$ for all $p \geq 0$, then all tensor products $H^k(\mathfrak{s}) \otimes H^\ell(\mathfrak{r}, \mathfrak{r})^{\mathfrak{s}}$ in the direct sum are zero. For all $\ell \geq 3$ we obtain $H^3(\mathfrak{s}) \otimes H^{\ell-3}(\mathfrak{r}, \mathfrak{r})^{\mathfrak{s}} = 0$. Since \mathfrak{s} is semisimple, we have $H^3(\mathfrak{s}) \neq 0$, so that $H^{\ell-3}(\mathfrak{r}, \mathfrak{r})^{\mathfrak{s}} = 0$ for all $\ell \geq 3$. It remains to discuss the cases, where no factor $H^3(\mathfrak{s})$ appears in the formula, i.e., for $p = 0, 1, 2$. Since \mathfrak{s} is semisimple we have $H^1(\mathfrak{s}) = H^2(\mathfrak{s}) = 0$ by the Whitehead lemmas and $H^0(\mathfrak{s}) = \mathbb{C}$. Then the Hochschild-Serre formula implies that

$$0 = H^2(\mathfrak{g}, \mathfrak{r}) = H^0(\mathfrak{s}) \otimes H^2(\mathfrak{r}, \mathfrak{r})^{\mathfrak{s}} \cong H^2(\mathfrak{r}, \mathfrak{r})^{\mathfrak{s}}$$

In the same way we obtain $H^1(\mathfrak{r}, \mathfrak{r})^{\mathfrak{s}} = 0$ and $H^0(\mathfrak{r}, \mathfrak{r})^{\mathfrak{s}} = 0$. □

We can derive further consequences from the Pirashvili conditions by studying the Levi decomposition $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{r}$ with respect to algebraic properties of the solvable radical \mathfrak{r} .

Lemma 3.5. *Let \mathfrak{g} be a perfect Lie algebra. Then its solvable radical \mathfrak{r} is nilpotent.*

Proof. Since \mathfrak{g} is perfect, its Levi decomposition implies that

$$\mathfrak{s} \oplus \mathfrak{r} = \mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] = [\mathfrak{s} \oplus \mathfrak{r}, \mathfrak{s} \oplus \mathfrak{r}] = \mathfrak{s} \oplus ([\mathfrak{s}, \mathfrak{r}] + [\mathfrak{r}, \mathfrak{r}]).$$

Because the vector space sum is direct it follows that

$$\mathfrak{r} = [\mathfrak{s}, \mathfrak{r}] + [\mathfrak{r}, \mathfrak{r}] \subseteq \text{ad}(\mathfrak{g})(\mathfrak{r}) \subseteq \mathfrak{n}.$$

The last inclusion follows from the fact that $D(\mathfrak{r}) \subseteq \mathfrak{n}$ holds for all derivations $D \in \text{Der}(\mathfrak{g})$, hence in particular for inner derivations $D = \text{ad}(x)$. It follows that $\mathfrak{r} = \mathfrak{n}$ is nilpotent. □

Lemma 3.6. *Let \mathfrak{g} be a symplectic Lie algebra, whose solvable radical \mathfrak{r} is abelian. Then \mathfrak{g} is semisimple.*

Proof. Assume that \mathfrak{r} is abelian. Then $V = \mathfrak{r}$ is an m -dimensional vector space. We have the Levi decomposition $\mathfrak{g} = \mathfrak{s} \ltimes V$ with a Levi subalgebra \mathfrak{s} and V is an \mathfrak{s} -module. Denote by D the linear map on \mathfrak{g} which is zero on \mathfrak{s} and the identity on V . We claim that it is a derivation of \mathfrak{g} with $\text{tr}(D) = m$. The Lie bracket on $\mathfrak{s} \oplus V$ is given by

$$[(x, v), (y, u)] = ([x, y], x \cdot u - y \cdot v)$$

for all $x, y \in \mathfrak{s}$ and $u, v \in V$. Then $D([(x, v), (y, u)]) = (0, x \cdot u - y \cdot v)$ and

$$[D((x, v)), (y, u)] = [(0, v), (y, u)] = (0, -y \cdot v),$$

$$[(x, v), D((y, u))] = [(x, v), (0, u)] = (0, x \cdot u).$$

Since \mathfrak{g} is complete, it is an inner derivation. However, since \mathfrak{g} is perfect, all adjoint operators $\text{ad}(x)$ have zero trace. Hence $m = 0$ and \mathfrak{g} is semisimple. □

Both lemmas together imply the following result.

Proposition 3.7. *Let \mathfrak{g} be a non-trivial Lie algebra, which is not semisimple and satisfies the Pirashvili conditions. Then the solvable radical of \mathfrak{g} is nilpotent and non-abelian.*

In fact, we can say more for Lie algebras whose solvable radical is abelian.

Proposition 3.8. *Let $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{r}$ be a Lie algebra, whose solvable radical \mathfrak{r} is nonzero and abelian. Then $H^1(\mathfrak{g}, \mathfrak{g}) \neq 0$, i.e., \mathfrak{g} is not complete. In case \mathfrak{r} does not contain, in its decomposition as an \mathfrak{s} -module, any direct factor isomorphic to a proper ideal of \mathfrak{s} we have*

$$H^1(\mathfrak{r}, \mathfrak{r})^{\mathfrak{s}} \cong H^1(\mathfrak{r}, \mathfrak{g})^{\mathfrak{s}} \cong H^1(\mathfrak{g}, \mathfrak{g}).$$

Proof. The Hochschild-Serre spectral sequence has

$$E_2^{p,q} = H^p(\mathfrak{s}, H^q(\mathfrak{r}, \mathfrak{g}))$$

and converges to $H^\bullet(\mathfrak{g}, \mathfrak{g})$. We have $H^1(\mathfrak{s}, H^q(\mathfrak{r}, \mathfrak{g})) = 0$ for all $q \geq 0$ by Whitehead's first lemma, thus the only space contributing to $H^1(\mathfrak{g}, \mathfrak{g})$ is $H^1(\mathfrak{r}, \mathfrak{g})^{\mathfrak{s}}$. We will show that this space is nonzero. For this we use the long exact sequence in the cohomology of the abelian Lie algebra \mathfrak{r} induced from the short exact sequence of coefficients

$$0 \rightarrow \mathfrak{r} \rightarrow \mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{r} \rightarrow 0.$$

This short exact sequence of coefficients is first an exact sequence of \mathfrak{g} -modules, and then by restriction to $\mathfrak{r} \subset \mathfrak{g}$ an exact sequence of V -modules. Observe that \mathfrak{r} and the quotient module $\mathfrak{g}/\mathfrak{r}$ are trivial \mathfrak{r} -modules. Part of the long exact sequence reads

$$\dots \rightarrow H^0(\mathfrak{r}, \mathfrak{g}/\mathfrak{r}) \rightarrow H^1(\mathfrak{r}, \mathfrak{r}) \rightarrow H^1(\mathfrak{r}, \mathfrak{g}) \rightarrow \dots$$

Since the functor of \mathfrak{s} -invariants is exact on the subcategory of finite dimensional \mathfrak{s} -modules, we deduce an exact sequence

$$\dots \rightarrow H^0(\mathfrak{r}, \mathfrak{g}/V)^{\mathfrak{s}} \rightarrow H^1(\mathfrak{r}, \mathfrak{r})^{\mathfrak{s}} \rightarrow H^1(\mathfrak{r}, \mathfrak{g})^{\mathfrak{s}} \rightarrow \dots$$

But we have $H^0(\mathfrak{r}, \mathfrak{g}/\mathfrak{r})^{\mathfrak{s}} = 0$, because $H^0(\mathfrak{r}, \mathfrak{g}/\mathfrak{r})$ is the space of \mathfrak{r} -invariants of the trivial module $\mathfrak{g}/\mathfrak{r}$, thus $H^0(\mathfrak{r}, \mathfrak{g}/\mathfrak{r}) = \mathfrak{g}/V$, and $(\mathfrak{g}/\mathfrak{r})^{\mathfrak{s}} = 0$ as the quotient module $\mathfrak{g}/\mathfrak{r} \cong \mathfrak{s}$ does not contain nonzero \mathfrak{s} -invariants. In order to conclude that $H^1(\mathfrak{r}, \mathfrak{g})^{\mathfrak{s}} \neq 0$, we note that $H^1(\mathfrak{r}, \mathfrak{r})^{\mathfrak{s}} \neq 0$, because $H^1(\mathfrak{r}, \mathfrak{r})^{\mathfrak{s}} = \text{Hom}_{\mathfrak{s}}(\mathfrak{r}, \mathfrak{r})$ and $\text{id}_{\mathfrak{r}}$ is a nonzero morphism of \mathfrak{s} -modules.

For the second claim note that the term following $H^1(\mathfrak{r}, \mathfrak{g})^{\mathfrak{s}}$ in the long exact sequence is $H^1(\mathfrak{r}, \mathfrak{g}/\mathfrak{r})^{\mathfrak{s}}$ and we have $H^1(\mathfrak{r}, \mathfrak{g}/\mathfrak{r})^{\mathfrak{s}} = \text{Hom}_{\mathfrak{s}}(\mathfrak{r}, \mathfrak{g}/\mathfrak{r})$, because the Lie algebra \mathfrak{r} is abelian and the \mathfrak{r} -module $\mathfrak{g}/\mathfrak{r}$ is trivial. Both \mathfrak{r} and $\mathfrak{g}/\mathfrak{r} \cong \mathfrak{s}$ decompose into direct factors and none of them is isomorphic, thus the space of \mathfrak{s} -morphisms is zero. \square

Corollary 3.9. *Let $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{r}$ be a Lie algebra, whose solvable radical \mathfrak{r} is nonzero, abelian and irreducible as \mathfrak{s} -module. Then $H^1(\mathfrak{g}, \mathfrak{g})$ is 1-dimensional.*

4. PROOF OF PIRASHVILI'S CONJECTURE

We now complete the proof of Pirashvili's conjecture by showing the following theorem.

Theorem 4.1. *Let \mathfrak{g} be a non-trivial Lie algebra satisfying the Pirashvili conditions. Then \mathfrak{g} is semisimple.*

Proof. Suppose that \mathfrak{g} satisfies the Pirashvili conditions, but is not semisimple. Then its solvable radical must be nilpotent and non-abelian by Proposition 3.7. Hence it coincides with the nilradical \mathfrak{n} of \mathfrak{g} , which is a characteristic ideal of \mathfrak{g} . Therefore its center $Z(\mathfrak{n})$ is a nonzero abelian ideal of \mathfrak{g} . We apply now the same reasoning as in the proof of Proposition 3.8, to the abelian ideal $Z(\mathfrak{n})$ of \mathfrak{g} and the Levi decomposition $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{n}$. For this consider the abelian Lie algebra $Z(\mathfrak{n})$ and the split short exact sequence of $Z(\mathfrak{n})$ -modules

$$0 \rightarrow Z(\mathfrak{n}) \rightarrow \mathfrak{n} \rightarrow \mathfrak{n}/Z(\mathfrak{n}) \rightarrow 0.$$

All three modules here are trivial $Z(\mathfrak{n})$ -modules and the resulting sequence of complex vector spaces splits. Observe that the induced long exact sequence

$$\dots \rightarrow H^p(Z(\mathfrak{n}), Z(\mathfrak{n})) \rightarrow H^p(Z(\mathfrak{n}), \mathfrak{n}) \rightarrow H^p(Z(\mathfrak{n}), \mathfrak{n}/Z(\mathfrak{n})) \rightarrow H^{p+1}(Z(\mathfrak{n}), Z(\mathfrak{n})) \rightarrow \dots$$

therefore also splits into short exact sequences and starts with the above split short exact sequence, which yields, because all $Z(\mathfrak{n})$ -modules are trivial and finite dimensional

$$0 \rightarrow Z(\mathfrak{n})^* \otimes Z(\mathfrak{n}) \rightarrow Z(\mathfrak{n})^* \otimes \mathfrak{n} \rightarrow Z(\mathfrak{n})^* \otimes (\mathfrak{n}/Z(\mathfrak{n})) \rightarrow 0,$$

and then

$$0 \rightarrow \Lambda^2 Z(\mathfrak{n})^* \otimes Z(\mathfrak{n}) \rightarrow \Lambda^2 Z(\mathfrak{n})^* \otimes \mathfrak{n} \rightarrow \Lambda^2 Z(\mathfrak{n})^* \otimes (\mathfrak{n}/Z(\mathfrak{n})) \rightarrow 0$$

and so on.

We want now to determine the $\mathfrak{g}/Z(\mathfrak{n})$ -invariants of the terms of (the second of) these sequences. As $Z(\mathfrak{n})$ acts trivially, we can equally well determine the \mathfrak{g} -invariants. We will determine them by first taking the \mathfrak{n} -invariants and then the \mathfrak{s} -invariants, as for any \mathfrak{g} -module V we have $(V^{\mathfrak{n}})^{\mathfrak{s}} = V^{\mathfrak{g}}$. For the \mathfrak{n} -invariants, we obtain

$$0 \rightarrow Z(\mathfrak{n})^* \otimes Z(\mathfrak{n}) \rightarrow Z(\mathfrak{n})^* \otimes \mathfrak{n}^{\mathfrak{n}} \rightarrow Z(\mathfrak{n})^* \otimes \mathfrak{n}^{\mathfrak{n}}/Z(\mathfrak{n}) \rightarrow \dots$$

This gives in fact an isomorphism

$$Z(\mathfrak{n})^* \otimes Z(\mathfrak{n}) \cong Z(\mathfrak{n})^* \otimes Z(\mathfrak{n}) = H^1(Z(\mathfrak{n}), \mathfrak{n})^{\mathfrak{n}}.$$

Now we obtain for the $\mathfrak{g}/Z(\mathfrak{n})$ -invariants simply the \mathfrak{s} -invariants of the previous sequence, i.e., an isomorphism

$$\mathrm{Hom}_{\mathfrak{s}}(Z(\mathfrak{n}), Z(\mathfrak{n})) \cong \mathrm{Hom}_{\mathfrak{s}}(Z(\mathfrak{n}), Z(\mathfrak{n})) = H^1(Z(\mathfrak{n}), \mathfrak{n})^{\mathfrak{g}/Z(\mathfrak{n})}.$$

Therefore the space $H^0(\mathfrak{g}/Z(\mathfrak{n}), H^1(Z(\mathfrak{n}), \mathfrak{n})) = \mathrm{Hom}_{\mathfrak{s}}(Z(\mathfrak{n}), Z(\mathfrak{n}))$ is nonzero as it contains $\mathrm{id}_{Z(\mathfrak{n})} \in \mathrm{Hom}_{\mathfrak{s}}(Z(\mathfrak{n}), Z(\mathfrak{n}))$. But this space occurs as a direct factor in $H^1(\mathfrak{g}, \mathfrak{n})$ according to the Hochschild-Serre spectral sequence of the Lie algebra \mathfrak{g} with respect to the ideal $Z(\mathfrak{n})$ and with respect to the \mathfrak{g} -module \mathfrak{n} . Hence we have $H^1(\mathfrak{g}, \mathfrak{r}) \neq 0$, because the solvable radical \mathfrak{r} coincides here with the nilradical \mathfrak{n} . This contradicts Proposition 3.3. \square

ACKNOWLEDGMENTS

Dietrich Burde is supported by the Austrian Science Foundation FWF, grant I3248. Friedrich Wagemann thanks the university of Vienna for a stay in Vienna in late spring 2019. We thank Wolfgang Moens for helpful discussions.

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