



HAL
open science

HYPERBOLICITY OF BASES OF LOG CALABI-YAU FAMILIES

Ya Deng

► **To cite this version:**

| Ya Deng. HYPERBOLICITY OF BASES OF LOG CALABI-YAU FAMILIES. 2019. hal-02266744

HAL Id: hal-02266744

<https://hal.science/hal-02266744>

Preprint submitted on 16 Aug 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

HYPERBOLICITY OF BASES OF LOG CALABI-YAU FAMILIES

YA DENG

ABSTRACT. We prove that for any maximally varying, log smooth family of Calabi-Yau klt pairs, its quasi-projective base is both of log general type, and pseudo Kobayashi hyperbolic. Moreover, such a base is Brody hyperbolic if the family is effectively parametrized.

CONTENTS

0. Introduction	1
1. Preliminary	3
2. Positivity of direct image sheaves	4
3. Construction of Viehweg-Zuo Higgs bundles	8
References	16

0. INTRODUCTION

0.1. **Set-up.** Following [Kob05], a complex space X is *pseudo Kobayashi hyperbolic*, if X is hyperbolic modulo a proper Zariski closed subset $\Delta \subsetneq X$, that is, the Kobayashi pseudo distance $d_X : X \times X \rightarrow [0, +\infty[$ of X satisfies that $d_X(p, q) > 0$ for every pair of distinct points $p, q \in X$ not both contained in Δ . In particular, any non-constant holomorphic map $\gamma : \mathbb{C} \rightarrow X$ has image $\gamma(\mathbb{C}) \subset \Delta$. When such Δ is an empty set, this definition reduces to the usual definition of *Kobayashi hyperbolicity*, and the Kobayashi pseudo distance d_X is a distance. A Kobayashi hyperbolic complex manifold X is *Brody hyperbolic*, that is, there are no non-constant holomorphic maps $f : \mathbb{C} \rightarrow X$.

Let X° and Y° be complex quasi-projective manifolds, and let $D^\circ = \sum_{i=1}^m a_i D_i^\circ$ be a Kawamata log terminal (klt for short) \mathbb{Q} -divisor on X° with simple normal crossing support. The morphism $f^\circ : (X^\circ, D^\circ) \rightarrow Y^\circ$ is a *log smooth family* if $f^\circ : X^\circ \rightarrow Y^\circ$ is a smooth projective morphism with connected fibers, and D° is *relatively normal crossing over Y°* , namely each stratum $D_{i_1}^\circ \cap \cdots \cap D_{i_k}^\circ$ of D° is dominant onto and smooth over Y° under f° . Such log smooth family is *of maximal variation* if the logarithmic Kodaira-Spencer map $\rho : T_{Y^\circ} \rightarrow R^1 f_{*}^{\circ}(T_{X^\circ/Y^\circ}(-\log D^\circ))$ is generically injective. Here ρ is defined to be the edge morphism of Rf_* for the short exact sequence

$$0 \rightarrow T_{X^\circ/Y^\circ}(-\log D^\circ) \rightarrow T_{X^\circ}(-\log D^\circ) \xrightarrow{df^\circ} (f^\circ)^* T_{Y^\circ} \rightarrow 0.$$

0.2. **Main results.** The goal of this paper is to prove the hyperbolicity for bases of log smooth families of Calabi-Yau pairs with maximal variation.

Date: Friday 16th August, 2019.

2010 Mathematics Subject Classification. 32Q45, 14C30, 32J25, 14H15.

Key words and phrases. pseudo Kobayashi hyperbolicity, Viehweg hyperbolicity, Brody hyperbolicity, log Calabi-Yau family, Viehweg-Zuo Higgs bundles, Campana-Păun criterion, Finsler metric.

The author is supported by the Knut and Alice Wallenberg Foundation postdoctoral scholarship.

Theorem A. *Let $f^\circ : (X^\circ, D^\circ) \rightarrow Y^\circ$ be a log smooth family over a quasi-projective manifold Y° . Assume that the family is of maximal variation, each fiber $(X_y, D_y) := (f^\circ)^{-1}(y)$ is a klt pair, and $K_{X_y} + D_y \equiv_{\mathbb{Q}} 0$. Then the base Y°*

- (i) *is of log general type, and*
- (ii) *is pseudo Kobayashi hyperbolic.*

Theorem A.(i) is often referred to *Viehweg hyperbolicity* in the literature. Although Theorem A.(i) and Theorem A.(ii) are conjecturally to be equivalent by the tantalizing Lang's conjecture [Lan91, Chapter VIII. Conjecture 1.4], we cannot conclude one from the other directly at the present time. Theorem A can be seen as some sort of *Shafarevich hyperbolicity conjecture* for log smooth families of Calabi-Yau pairs. As first formulated by Viehweg and Kovács, Shafarevich's conjecture for higher dimensional fibers and parametrizing spaces states that a family of canonically polarized manifolds of maximal variation has as its base a variety of log general type. The Shafarevich hyperbolicity conjecture as well as its generalized formulations drew a lot of attention for a long time, and much progress has been achieved during the last two decades. We refer the reader to the survey [Keb13, CKT16] for more details, including references to earlier results that are not mentioned here for lack of space.

The log smooth family $f^\circ : (X^\circ, D^\circ) \rightarrow Y^\circ$ is *effectively parametrized* if the logarithmic Kodaira-Spencer map

$$T_{Y^\circ, y} \rightarrow H^1(X_y, T_{X_y}(-\log D_y))$$

is injective for any $y \in Y^\circ$, where $(X_y, D_y) := (f^\circ)^{-1}(y)$ is the fiber. As a consequence of Theorem A, one has the following

Theorem B. *In the setting of Theorem A, assume additionally that the log smooth family $f^\circ : (X^\circ, D^\circ) \rightarrow Y^\circ$ is effectively parametrized, then*

- (i) *the base Y° is Brody hyperbolic, and*
- (ii) *every irreducible subvariety of Y° is of log general type.*

According to another conjecture of Lang [Lan86, Conjecture 5.6], Theorem B.(ii) and Theorem B.(i) are also expected to be equivalent. It is quite natural to ask whether the base Y° in Theorem B is Kobayashi hyperbolic, which we cannot prove at this moment. However, let us mention that we know the Kobayashi hyperbolicity for the bases of effectively parametrized smooth families of projective manifolds with big and nef canonical bundle by our work [DA19, Theorem C].

The proofs of Theorems A and B are discussed in § 3.2.

0.3. Outline of the proof. Based on the work by Campana-Păun [CP19] and the author [DA19], the proof of Theorem A is reduced to the construction of certain negatively twisted Higgs bundles (which is called *Viehweg-Zuo Higgs bundles* in Definition 3.1) over the base Y° (see Theorem 3.2). Indeed, once the Viehweg-Zuo (VZ for short) Higgs bundle on the base is established, Theorem A.(i) follows from the celebrated theorem of Campana-Păun [CP19] on the vast generalization of generic semipositivity result of Miyaoka; and Theorem A.(ii) can be deduced from [DA19, Theorem F and Theorem 2.12] on the construction of a *generically non-degenerate* smooth Finsler metric over the base (up to a birational model) with holomorphic sectional curvature bounded above by a negative constant.

Let us briefly explain the construction of VZ Higgs bundles over the base space Y° in Theorem A. The starting point is the strong positivity of direct images of log Calabi-Yau families by Ambro [Amb05] and Cao-Guenancia-Păun [CGP17] (see Theorem 1.6 below). Based on this theorem, we apply Abramovich's \mathbb{Q} -mild reduction for semi log canonical (slc for short) families (see Theorem 1.5) as in [DA19], to find a smooth projective compactification of Y°

so that after replacing the original family by Viehweg’s fiber product, one gains enough positivity for the relative dualizing sheaves. This enables us to perform Viehweg’s cyclic cover technique. To construct the desired negatively twisted Higgs bundles, we mainly follow the general strategy by Viehweg-Zuo [VZ02, VZ03] to extend their Hodge theoretical methods to the logarithmic setting. However, different Higgs bundles defined in the proof of Theorem 3.2 are related in a more direct manner inspired by the recent work of Popa-Schnell [PS17] on the alternative construction via *tautological sections of cyclic coverings*. Our work is also influenced by the work [PTW18]. Indeed, we have to perform some “a priori” birational modification of the base so that certain desingularization of the cyclic cover is smooth over an open set of the base whose complement is a simple normal crossing divisor, by applying an important technique in [PTW18, Proposition 4.4].

Convention. Throughout this paper we will work over complex number field \mathbb{C} . An algebraic fiber space $f : X \rightarrow Y$ is a surjective projective morphism with connected fibers between smooth projective manifolds. SNC is the abbreviation for *simple normal crossing*. For a normal crossing divisor D on a complex manifold X , we write $\Omega_X(\log D)$ instead of $\Omega_X(\log D_{\text{red}})$ for simplicity.

1. PRELIMINARY

For the reader’s convenience, we recall some notions and definitions in [AT16, KP17] which are frequently used in this paper.

Definition 1.1. A pair (X, D) consists of an equidimensional demi-normal variety X and an effective \mathbb{Q} -divisor $D = \sum_{i=1}^m a_i D_i$ on X , such that the \mathbb{Q} -divisor $K_X + D$ is \mathbb{Q} -Cartier on X . A pair (X, D) is *log smooth* if X is smooth, and D has SNC support.

The *slc pair* is an extension of the pair with log canonical singularities to non-normal varieties.

Definition 1.2 (slc pair). An *slc pair* (X, D) consists of a reduced variety X and a Weil \mathbb{R} -divisor $D = \sum_{i=1}^m a_i D_i$ on X with $K_X + D$ a \mathbb{Q} -Cartier divisor so that

- (i) The variety X satisfies Serre’s condition S2,
- (ii) X is Gorenstein in codimension 1, and
- (iii) if $\nu : X^\vee \rightarrow X$ is the normalization, then the pair $(X^\vee, \sum_{i=1}^m a_i \nu^{-1}(D_i) + D^\vee)$ is log canonical, where D^\vee denotes the preimage of the double locus on D .

The *slc family* arises naturally in the moduli theory of higher dimensional varieties.

Definition 1.3 (slc family). An *slc family* over a normal variety consists of a pair (X, D) and a flat proper surjective morphism $f : (X, D) \rightarrow Y$ such that

- (i) each fiber $(X_y, D_y) := f^{-1}(y)$ is an slc pair.
- (ii) The m -th reflexive power $\omega_{X/Y}^{[m]}(D) := \mathcal{O}_X(mK_{X/Y} + mD)$ is flat for any $m \in \mathbb{N}$.
- (iii) The family $f : (X, D) \rightarrow Y$ satisfies the *Kollár condition*, that is, for any $m \in \mathbb{N}$, the reflexive power $\omega_{X/Y}^{[m]}(D)$ commutes with arbitrary base change.

It is easy to see that a log smooth family $f^\circ : (X^\circ, D^\circ) \rightarrow Y^\circ$ on a quasi-projective manifold is an slc family.

Notation 1.4. For a log smooth family $f^\circ : (X^\circ, D^\circ) \rightarrow Y^\circ$, let $X^{or} := X^\circ \times_{Y^\circ} \cdots \times_{Y^\circ} X^\circ$ be the r -fold fiber product of $f^\circ : X^\circ \rightarrow Y^\circ$, and $D^{or} := \sum_{j=1}^r \text{pr}_j^* D^\circ$, where $\text{pr}_j : X^{or} \rightarrow X^\circ$ denotes the projection to the j -th factor. The induced morphism $f^{or} : (X^{or}, D^{or}) \rightarrow Y^\circ$ is also a log smooth family, which we say *r -fold fiber product of $f^\circ : (X^\circ, D^\circ) \rightarrow Y^\circ$* .

The following \mathbb{Q} -mild reduction by Abramovich [DA19, Corollary A.2] is applied to find a “good” compactification of the log smooth family $f^\circ : (X^\circ, D^\circ) \rightarrow Y^\circ$ in Theorem A.

Theorem 1.5 (\mathbb{Q} -mild reduction, Abramovich). *Let $f^\circ : (X^\circ, D^\circ) \rightarrow Y^\circ$ be a log smooth family with Y° a quasi-projective manifold. Assume that (X°, D°) is a klt pair. Then for any given finite subset $S \subset Y^\circ$, there exist*

- (i) a smooth projective compactification $Y^\circ \subset Y$ with $B := Y \setminus Y^\circ$ an SNC divisor,
- (ii) an SNC divisor $T \subset Y$ containing B and disjoint from S ,
- (iii) a finite morphism $W \rightarrow Y$ unramified outside T with W a smooth projective manifold, and
- (iv) an slc family $g : (X_W, D_W) \rightarrow W$ extending the given log smooth family $(X^\circ, D^\circ) \times_Y W$.

While the \mathbb{Q} -mild reduction is only stated for the compact setting ($D^\circ = \emptyset$) in [DA19, Corollary A.2], the proof can be repeated here verbatim, and let us explain it briefly.

Sketch of the proof of Theorem 1.5. In the same spirit as [DA19, Theorem A.1], one first applies the work of [DR18, Theorem 1.5] on moduli of Alexeev stable maps of slc pairs, to compatify the slc family $f^\circ : (X^\circ, D^\circ) \rightarrow Y^\circ$ in Theorem 1.5 into a projective morphism $f : (\mathcal{X}, \mathcal{D}) \rightarrow \mathcal{Y}$. Here \mathcal{X} and \mathcal{Y} are Deligne-Mumford stacks with projective coarse moduli spaces, and \mathcal{D} is a \mathbb{Q} -divisor on \mathcal{X} , such that the fibers of f are slc pairs. The rest of the proof is the destackification of f , which completely follows from [DA19, Proof of Corollary 1.2]. \square

The following result on the strong positivity of direct images can be thought as a logarithmic extension of Viehweg’s $Q_{n,m}$ -conjecture [Vie83, p. 330-331].

Theorem 1.6 ([Amb05], [CGP17, Corollary D]). *In the setting of Theorem A, let $f : (X, D) \rightarrow Y$ be any projective compactification of $f^\circ : (X^\circ, D^\circ) \rightarrow Y^\circ$, so that (X, D) is a smooth klt pair, and Y is smooth. Then $f_*(mK_{X/Y} + mD)^{\star\star}$ is a big line bundle for m sufficiently large and divisible.* \square

2. POSITIVITY OF DIRECT IMAGE SHEAVES

We will replace the log smooth family $f^\circ : (X^\circ, D^\circ) \rightarrow Y^\circ$ by its r -fold fiber product which is still log smooth over Y° , so that its projective compactification possesses the strong positivity for direct image sheaves.

Theorem 2.1. *For the log Calabi-Yau family $f^\circ : (X^\circ, D^\circ) \rightarrow Y^\circ$ in Theorem A, there exist a smooth projective compactification $Y \supset Y^\circ$, a smooth klt pair $(X, D = \sum_{i=1}^m a_i D_i)$, an ample line bundle A on Y , and an algebraic fiber space $f : X \rightarrow Y$ so that*

- (i) each irreducible component D_i of D is dominant onto Y under f ;
- (ii) the restriction $(X, D)|_{f^{-1}(Y^\circ)} \rightarrow Y^\circ$ is the r -fold fiber product of $f^\circ : (X^\circ, D^\circ) \rightarrow Y^\circ$. In particular, $f : (X, D) \rightarrow Y$ is log smooth over Y° ;
- (iii) the complement $B := Y \setminus Y^\circ$ is an SNC divisor, and $\Delta := f^*B$ has SNC support, and
- (iv) the linear system $|mK_{X/Y} + mD - mf^*A|$ is non-empty for m sufficiently large and divisible.

Proof. By the \mathbb{Q} -mild reduction in Theorem 1.5, we can take a projective compactification $Y \supset Y^\circ$ with $B := Y \setminus Y^\circ$ simple normal crossing, and a finite morphism $\tau : W \rightarrow Y$ so that the log smooth family $(X^\circ, D^\circ) \times_Y W \rightarrow W_0$ extends to an slc family $g : (Z, D_Z) \rightarrow W$:

$$(2.0.1) \quad \begin{array}{ccc} (X^\circ, D^\circ) \times_Y W & \hookrightarrow & (Z, D_Z) \\ \downarrow & & \downarrow g \\ W^\circ & \hookrightarrow & W \end{array}$$

Here $W^\circ := \tau^{-1}(Y^\circ)$. We denote by $Z^r := Z \times_W \cdots \times_W Z$ the r -th fiber product of $Z \rightarrow W$, and set $\text{pr}_j : Z^r \rightarrow Z$ to be the projection to the j -th factor. Write $D_Z^r := \sum_{j=1}^r \text{pr}_j^* D_Z$. As is well-known (see e.g. [BHPS13, Proposition 2.12] or [AT16, Proposition 4.5]), this fiber product family $g^r : (Z^r, D_Z^r) \rightarrow W$ is still an slc family.

Claim 2.2. *For some sufficiently large and divisible m , $\omega_{Z/W}^{[m]}(mD_Z)$ is invertible, $g_*(\omega_{Z/W}^{[m]}(mD_Z))$ is a big line bundle, and for any $r \in \mathbb{Z}_{>0}$, one has*

$$(2.0.2) \quad (g^r)_*(\omega_{Z^r/W}^{[m]}(mD_Z^r)) = g_*(\omega_{Z/W}^{[m]}(mD_Z))^{\otimes r}.$$

Proof of Claim 2.2. By [AT16, Proposition 4.4], (Z, D_Z) is a log canonical pair, and in particular $K_Z + D_Z$ is \mathbb{Q} -Cartier. Moreover, it was proved in [WW18, Proposition 4.1] that (Z, D_Z) is even klt. Take a log resolution $\mu : \tilde{Z} \rightarrow Z$ of (Z, D_Z) with

$$(2.0.3) \quad \mu^*(K_Z + D_Z) = K_{\tilde{Z}} + \tilde{D}_Z - \tilde{E},$$

where \tilde{D}_Z and \tilde{E} are both effective \mathbb{Q} -divisors, such that $(\tilde{D}_Z + \tilde{E})_{\text{red}}$ is simple normal crossing, \tilde{E} is exceptional and the coefficients of irreducible components of \tilde{D}_Z are all in $(0, 1)$. Set $Z^\circ := (g \circ \mu)^{-1}(W^\circ)$, and $D_Z^\circ := Z^\circ \cap \tilde{D}_Z$. Then $(Z^\circ, D_Z^\circ) \rightarrow W^\circ$ is isomorphic to $(X^\circ, D^\circ) \times_Y W \rightarrow W^\circ$. Set $\tau^\circ := \tau|_{W^\circ}$. Then the logarithmic Kodaira-Spencer map associated to $(Z^\circ, D_Z^\circ) \rightarrow W^\circ$ is equal to

$$\rho' : T_{W^\circ} \xrightarrow{d\tau^\circ} (\tau^\circ)^* T_{Y^\circ} \xrightarrow{(\tau^\circ)^* \rho} (\tau^\circ)^* R^1 f_* (T_{X^\circ/Y^\circ}(-\log D^\circ))$$

where $\rho : T_{Y^\circ} \rightarrow R^1 f_* (T_{X^\circ/Y^\circ}(-\log D^\circ))$ is the logarithmic Kodaira-Spencer map of $(X^\circ, D^\circ) \rightarrow Y^\circ$, which is generically injective by the assumption. Since $\tau : W \rightarrow Y$ is finite to one, ρ' is also generically injective. By Theorem 1.6, for $m \gg 0$ large and divisible enough, $\tilde{g}_*(mK_{\tilde{Z}/W} + m\tilde{D}_Z)^{**}$ is a big line bundle. On the other hand, by (2.0.3) one has

$$\tilde{g}_*(mK_{\tilde{Z}/W} + m\tilde{D}_Z) = g_*(mK_{Z/W} + mD_Z),$$

which is a reflexive sheaf of rank 1, and thus invertible. Therefore, $g_*(mK_{Z/W} + mD_Z)$ is a big line bundle as well.

Finally, note that if $m(K_{Z/W} + D_Z)$ is Cartier, then using Definition 1.3 the line bundle $\mathcal{O}_Z(m(K_{Z/W} + D_Z)) = \omega_{Z/W}^{[m]}(mD_Z)$ is compatible with base-change, and thus

$$\omega_{Z^r/W}^{[m]}(mD_Z^r) = (\text{pr}_1)^* \omega_{Z/W}^{[m]}(mD_Z) \otimes \cdots \otimes (\text{pr}_r)^* \omega_{Z/W}^{[m]}(mD_Z).$$

(2.0.2) then follows from the projection formula. \square

By the above claim, $g_*(\omega_{Z/W}^{[m]}(mD_Z))^{\otimes r} \otimes \tau^*(\mathcal{O}_W(-2mA - mB))$ is effective for some $r \gg 0$. Combining (2.0.2), there is a morphism

$$(2.0.4) \quad \tau^* A^m \rightarrow (g^r)_*(\omega_{Z^r/W}^{[m]}(mD_Z^r)) \otimes \tau^*(\mathcal{O}_W(-mA - mB))$$

which is generically isomorphic. To keep notation as simple as possible, we replace $(X^\circ, D^\circ) \rightarrow Y^\circ$ by its r -folded fiber product $(X^{or}, D^{or}) \rightarrow Y^\circ$, which is still a log smooth family. Then $(Z, D_Z) \rightarrow W$ is replaced by its r -folded fiber product $(Z^r, D_Z^r) \rightarrow W$ so that (2.0.1) still holds. Hence (2.0.4) is renamed as

$$(2.0.5) \quad \tau^* A^m \rightarrow g_*(\omega_{Z/W}^{[m]}(mD_Z)) \otimes \tau^*(\mathcal{O}_W(-mA - mB))$$

which is generically isomorphic. Pushing forward (2.0.4) by τ_* , by the projection formula (2.0.5) induces a morphism

$$(2.0.6) \quad A^m \otimes \tau_* \mathcal{O}_W \rightarrow \tau_* g_*(\omega_{Z/W}^{[m]}(mD_Z)) \otimes \mathcal{O}_Y(-mA - mB)$$

which is also generically isomorphic. For $m \gg 0$ sufficiently large such that $A^m \otimes \tau_* \mathcal{O}_W$ is generated by global sections everywhere, (2.0.6) shows that $\tau_* g_*(\omega_{Z/W}^{[m]}(mD_Z)) \otimes \mathcal{O}_Y(-mA - mB)$ is generated by global sections at general points of Y .

Take a smooth projective compactification $X \supset X^\circ$, and define D to be the closure of D° in X . After passing to a blow-up of X with the center in $X \setminus X^\circ$, we can assume that $f^\circ : (X^\circ, D^\circ) \rightarrow Y^\circ$ can be compactified into a surjective morphism $f : (X, D) \rightarrow Y$

$$(2.0.7) \quad \begin{array}{ccc} (X^\circ, D^\circ) & \hookrightarrow & (X, D) \\ \downarrow f^\circ & & \downarrow f \\ Y^\circ & \hookrightarrow & Y \end{array}$$

so that $f^*B + D$ has SNC support. In particular, (X, D) is a log smooth klt pair, which is log smooth over Y° . Set $Z_1 := X \times_Y W$, and Z_2 denotes to be its normalization. Since Z_2 is birational to Z , one can take a log resolution $\tilde{Z} \rightarrow Z$ of (Z, D_Z) , so that it also resolves the birational map $Z \dashrightarrow Z_2$, and $\tilde{g}^*(W \setminus W^\circ)$ is normal crossing.

$$(2.0.8) \quad \begin{array}{ccccccc} & & \phi & & & & \\ & & \curvearrowright & & & & \\ X & \xleftarrow{h} & Z_1 & \xleftarrow{v} & \tilde{Z} & \xrightarrow{\mu} & Z \\ \downarrow f & & g_1 \downarrow & & \downarrow \tilde{g} & & \downarrow g \\ Y & \xleftarrow{\tau} & W & \xleftarrow{=} & W & \xrightarrow{=} & W \end{array}$$

Since $\tau : W \rightarrow Y$ is flat, by [Vie83, Proof of Lemma 3.3] or [Mor87, (4.10)], Z_1 is irreducible Gorenstein, $h^* \omega_{X/Y} = \omega_{Z_1/W}$ and $v_* \omega_{\tilde{Z}/W}^m \subset \omega_{Z_1/W}^m$. By flat base change and the projection formula, one has

$$\begin{aligned} & \tilde{g}_* \left(\mathcal{O}_{\tilde{Z}}(mK_{\tilde{Z}/W} + m\phi^*(D + f^*B)) \right) \hookrightarrow (g_1)_* (\omega_{Z_1/W}^m \otimes h^* \mathcal{O}_X(mD + mf^*B)) \\ & = (g_1)_* \left(h^* (\omega_{X/Y}^m \otimes \mathcal{O}_X(mD + mf^*B)) \right) \xrightarrow{\cong} \tau^* f_* (\mathcal{O}_X(mK_{X/Y} + mD + mf^*B)), \end{aligned}$$

which is an isomorphism over W° . Note that $\tilde{g}^{-1}(W^\circ) \simeq X_0 \times_Y W$ for $\mu : \tilde{g}^{-1}(W^\circ) \rightarrow Z_0$ is an isomorphism, and thus $\phi^*(D + f^*B) - \tilde{D}_Z$ is supported on $\tilde{g}^*(W \setminus W^\circ)$. Since \tilde{D}_Z is klt, and $\phi^* f^* B = \tilde{g}^* \tau^* B \geq \tilde{g}^*(W \setminus W^\circ)$, $\phi^*(D + f^*B) - \tilde{D}_Z$ is thus effective. One has the following morphism

$$\begin{aligned} & g_* (\mathcal{O}_Z(mK_{Z/W} + mD_Z)) = \tilde{g}_* \left(\mathcal{O}_{\tilde{Z}}(mK_{\tilde{Z}/W} + m\tilde{D}_Z) \right) \rightarrow \\ & \tilde{g}_* \left(\mathcal{O}_{\tilde{Z}}(mK_{\tilde{Z}/W} + m\phi^*(D + f^*B)) \right) \rightarrow \tau^* f_* (\mathcal{O}_X(mK_{X/Y} + mD + mf^*B)) \end{aligned}$$

which is an isomorphism over $W^\circ = \tau^{-1}(Y^\circ)$. Hence the morphism

$$(2.0.9) \quad \begin{aligned} \tau_* g_* (\omega_{Z/W}^{[m]}(mD_Z)) & \rightarrow \tau_* \tau^* f_* (\mathcal{O}_X(mK_{X/Y} + mD + mf^*B)) \\ & = f_* (\mathcal{O}_X(mK_{X/Y} + mD + mf^*B)) \otimes \tau_* \mathcal{O}_W. \end{aligned}$$

is isomorphic over Y° . The (surjective) trace map $\tau_* \mathcal{O}_W \rightarrow \mathcal{O}_Y$ as well as (2.0.9) induces a generically surjective morphism

$$(2.0.10) \quad \Psi : \tau_* g_* (\omega_{Z/W}^{[m]}(mD_Z)) \otimes \mathcal{O}_Y(-mA - mB) \rightarrow f_* (\mathcal{O}_X(mK_{X/Y} + mD)) \otimes \mathcal{O}_Y(-mA).$$

We denote by

$$(2.0.11) \quad \Phi : H^0 \left(Z, \mathcal{O}_Z(mK_{Z/W} + mD_Z - g^* \tau^*(mA + mB)) \right) \rightarrow H^0 \left(X, \mathcal{O}_X(mK_{X/Y} + mD - mf^*A) \right)$$

the injective \mathbb{C} -linear map induced by Ψ , which indeed does not depend on the choice of the intermediate birational model \tilde{Z} of Z in (2.0.8). Recall that for $m \gg 0$, the left hand side of (2.0.11) is non-empty. Hence the linear system $|mK_{X/Y} + mD - mf^*A|$ is non-empty. \square

As we mentioned in the end of § 0, since we require the discriminant locus of the the new family obtained by the desingularization of certain cyclic cover to be simple normal crossing, we need some sort of “base change property” of direct images. The following proposition follows the ideas in [PTW18, Proposition 4.4], and it can be seen as the log version of [DA19, Theorem 1.24].

Proposition 2.3. *Let $f : (X, D) \rightarrow Y$ be the surjective morphism in Theorem 2.1, which is log smooth over Y° . Then there exists a birational morphism $\mu : \tilde{Y} \rightarrow Y$ from a projective manifold \tilde{Y} , a new birational model $\tilde{f} : (\tilde{X}, \tilde{D}) \rightarrow \tilde{Y}$*

$$\begin{array}{ccc} (\tilde{X}, \tilde{D}) & \xrightarrow{\phi} & (X, D) \\ \downarrow \tilde{f} & & \downarrow f \\ \tilde{Y} & \xrightarrow{\mu} & Y \end{array}$$

where (\tilde{X}, \tilde{D}) is a log smooth klt pair, and a Zariski open set $V \subset Y^\circ$ so that one has the following:

- (1) every irreducible component of \tilde{D} is dominant onto \tilde{Y} ;
- (2) the morphism $\tilde{V} := \mu^{-1}(V) \xrightarrow{\mu} V$ is an isomorphism;
- (3) the complement $T := \tilde{Y} \setminus \tilde{V}$ is an SNC divisor;
- (4) the restriction $(\tilde{X}, \tilde{D})|_{\mu^{-1}(Y^\circ)} \rightarrow \mu^{-1}(Y^\circ)$ is isomorphic to $(X^\circ, D^\circ) \times_Y \tilde{Y}$. In particular, $\phi|_{\tilde{f}^{-1}(\tilde{V})} : (\tilde{X}, \tilde{D})|_{\tilde{f}^{-1}(\tilde{V})} \rightarrow (X, D)|_{f^{-1}(V)}$ is an isomorphism;
- (5) the divisor $\tilde{f}^*(T) + \tilde{D}$ has SNC support, and
- (6) there exists a divisor $\Gamma \in |mK_{\tilde{X}/\tilde{Y}} + m\tilde{D} + m\tilde{E} - m\mu^*A|$ supported on $\tilde{f}^{-1}(T)$ for m sufficiently large and divisible, where A is an ample line bundle on Y , and \tilde{E} is an effective and \tilde{f} -exceptional divisor with $\tilde{f}(\tilde{E}) \subset T$. Moreover, Γ contains one irreducible component P with multiplicity one, and $\Gamma \geq \tilde{f}^*(\tilde{B})$, where $\tilde{B} := \mu^{-1}(Y \setminus Y^\circ)$.

Proof. By Theorem 2.1 one can pick a $\sigma \in H^0(Z, mK_{Z/W} + mD_Z - g^*\tau^*(2mA + mB))$ so that $s := \Phi(\sigma) \in H^0(X, \mathcal{O}_X(mK_{X/Y} + mD - 2mf^*A))$ is non-zero, where Φ is defined in (2.0.11). Let $U \subset Y^\circ$ be the Zariski open set so that the zero divisor of s is supported on $f^{-1}(Y \setminus U)$. Take a birational morphism $\mu : \tilde{Y} \rightarrow Y$ so that $\tilde{U} := \mu^{-1}(U) \xrightarrow{\mu} U$ is an isomorphism, and $S := \tilde{Y} \setminus \tilde{U}$ is an SNC divisor. Write $\tilde{Y}^\circ := \mu^{-1}(Y^\circ)$, $B := Y \setminus Y^\circ$, and $\tilde{B} = \tilde{Y} \setminus \tilde{Y}^\circ$. Let $(X \times_Y \tilde{Y})^\sim$ be the normalization of main component of fiber product $X \times_Y \tilde{Y}$ dominating \tilde{Y} , and let $\tilde{X} \rightarrow (X \times_Y \tilde{Y})^\sim$ be a blow-up so that $\tilde{f}^*\tilde{B} + \phi^*D$ has SNC support. Set \tilde{D} to be the strict transform of D under ϕ , and (1) is satisfied. Hence (\tilde{X}, \tilde{D}) is also a log smooth klt pair, which is log smooth over \tilde{Y}° . Write $\tilde{X}^\circ := \tilde{f}^{-1}(\tilde{Y}^\circ)$, and $\tilde{D}^\circ := \tilde{D} \cap \tilde{X}^\circ$. One can easily see that (4) is satisfied automatically, and $\tilde{X}^\circ \cap \tilde{f}^*(S)$ is SNC.

Let $\tau : W \rightarrow Y$ be the surjective finite morphism in the proof of Theorem 2.1 so that there is an slc family $(Z, D_Z) \rightarrow W$ extends the log smooth family $(X^\circ, D^\circ) \times_Y W \rightarrow W_0$ as in (2.0.1). Take a desingularization $\tilde{W} \rightarrow W \times_Y \tilde{Y}$, and $\tilde{\tau} : \tilde{W} \rightarrow \tilde{Y}$ is a generically finite to one surjective morphism, which is finite over \tilde{V} . Hence we can leave out a codimension at least two closed subvariety on $\tilde{Y} \setminus \tilde{V}$ so that $\tilde{\tau}$ is finite (and thus flat). Set $(\tilde{Z}, \tilde{D}_{\tilde{Z}}) :=$

$(Z, D_Z) \times_W \tilde{W} \xrightarrow{\tilde{g}} \tilde{W}$, which is also an slc family, and extends $(\tilde{X}^\circ, \tilde{D}^\circ) \times_{\tilde{Y}} \tilde{W}$:

$$(2.0.12) \quad \begin{array}{ccc} (\tilde{X}^\circ, \tilde{D}^\circ) \times_{\tilde{Y}} \tilde{W} & \hookrightarrow & (\tilde{Z}, \tilde{D}_{\tilde{Z}}) \\ \downarrow & & \downarrow \tilde{g} \\ \tilde{W}^\circ & \hookrightarrow & \tilde{W} \end{array}$$

where $\tilde{W}^\circ := \tilde{\tau}^{-1}(\tilde{Y}^\circ)$. By the base change property in Definition 1.3, one has

$$\delta^* \sigma \in H^0\left(\tilde{Z}, \mathcal{O}_{\tilde{Z}}(mK_{\tilde{Z}/\tilde{W}} + m\tilde{D}_{\tilde{Z}} - (\tau \circ \delta \circ \tilde{g})^*(2mA + mB))\right)$$

where $\delta : \tilde{W} \rightarrow W$. Since $\mu^*B \geq \tilde{B}$, and $\tau \circ \delta = \mu \circ \tilde{\tau}$, then $\delta^* \sigma$ induces a section

$$\tilde{\sigma} \in H^0\left(\tilde{Z}, \mathcal{O}_{\tilde{Z}}(mK_{\tilde{Z}/\tilde{W}} + m\tilde{D}_{\tilde{Z}} - 2m\tilde{g}^*\tilde{\tau}^*(\mu^*A) - m\tilde{g}^*\tilde{\tau}^*\tilde{B})\right)$$

which is isomorphic to itself when restricted to $\tilde{\tau}^{-1}(\tilde{U})$. Being similar to the \mathbb{C} -linear map Φ defined in (2.0.11), $\tilde{\sigma}$ gives rise to a section

$$\tilde{s} \in H^0\left(\tilde{X}, \mathcal{O}_{\tilde{X}}((mK_{\tilde{X}/\tilde{Y}} + m\tilde{D}) \otimes \tilde{f}^*\mu^*A^{-2m})\right),$$

which is isomorphic to s when restricted on $\tilde{U} \simeq U$. Note that \tilde{s} is only defined over a big open set of \tilde{Y} for we leave out a codimension at least two subvariety contained in $T = \tilde{Y} \setminus \tilde{U}$. It therefore gives rise to the zero divisor

$$\Gamma_0 \in |mK_{\tilde{X}/\tilde{Y}} + m\tilde{D} + m\tilde{E} - 2m\tilde{f}^*\mu^*A|$$

where \tilde{E} is an effective and \tilde{f} -exceptional divisor with $\tilde{f}(\tilde{E}) \subset S$. Note that Γ_0 is supported on $\tilde{f}^{-1}(S)$ for the zero divisor of s is supported on $f^{-1}(Y \setminus U)$.

Now we take a general smooth hypersurface $H \in |m\mu^*A|$ so that $H \cap \tilde{U} \neq \emptyset$ and $T := H + S$ has SNC support. Then for $\tilde{V} := \tilde{Y} \setminus T$ and $V := \mu(\tilde{V})$, one has $\tilde{V} = \mu^{-1}(V)$ and $\mu : \tilde{V} \rightarrow V$ is an isomorphism. Hence (2) and (3) are satisfied. Blowing up a bit more with centers in $\text{Supp}(\tilde{f}^*\tilde{B})$, one can assume that $\tilde{f}^*T + \tilde{D}$ has SNC support and (5) is satisfied. Set

$$\Gamma := \Gamma_0 + \tilde{f}^*H \in |mK_{\tilde{X}/\tilde{Y}} + m\tilde{D} + m\tilde{E} - m\tilde{f}^*\mu^*A|.$$

By this construction, Γ is supported on $\tilde{f}^{-1}(T)$. Since $\tilde{f} : \tilde{X} \rightarrow \tilde{Y}$ is smooth over $\tilde{Y}^\circ \supset \tilde{U}$, $\Gamma|_{\tilde{U}} = \tilde{f}^*H|_{\tilde{U}}$ is smooth, and thus Γ contains one irreducible component P with multiplicity one, where P is the closure of $\tilde{f}^*H|_{\tilde{U}}$ in \tilde{X} . To show the last statement, one might assume that $A - B$ is also ample. Then we replace Γ by

$$\Gamma + m\tilde{f}^*\mu^*B := \Gamma_0 + \tilde{f}^*H \in |mK_{\tilde{X}/\tilde{Y}} + m\tilde{D} + m\tilde{E} - m\tilde{f}^*\mu^*(A - B)|$$

and the claim follows from the fact $\tilde{f}^*\mu^*B \geq \tilde{f}^*\tilde{B}$. \square

3. CONSTRUCTION OF VIEHWEG-ZUO HIGGS BUNDLES

3.1. Definition of Viehweg-Zuo Higgs bundles. Following Hitchin and Simpson, a *logarithmic Higgs bundle (resp. sheaf) over a log pair* (Y, B) consists of a vector bundle (resp. torsion free sheaf) \mathcal{E} and a morphism $\theta : \mathcal{E} \rightarrow \mathcal{E} \otimes \Omega_Y(\log B)$ so that $\theta \wedge \theta = 0$. This section is devoted to the construction of certain negatively curved Higgs bundles on the base in Theorem A. This type of Higgs bundles, first introduced by Viehweg-Zuo [VZ02, VZ03] and later developed by Popa-Taji-Wu [PTW18], has proven to be a powerful tool in studying the hyperbolicity of moduli spaces. Let us give the definition in an abstract way, following [DA19, Definition 2.1].

Definition 3.1 (Abstract Viehweg-Zuo Higgs bundles). Let Y° be a quasi-projective manifold, and let $Y \supset Y^\circ$ be a projective compactification of Y° with the boundary $B := Y \setminus Y^\circ$ simple normal crossing. A Viehweg-Zuo Higgs bundle on Y° is a logarithmic Higgs bundle $(\tilde{\mathcal{E}}, \tilde{\theta})$ on Y consisting of the following data:

- (i) an SNC divisor T on Y so that $T \geq B$,
- (ii) a big and nef line bundle \mathcal{B} over Y with the augmented base locus $\mathbf{B}_+(\mathcal{B}) \subset T$,
- (iii) a logarithmic Higgs bundle $(\mathcal{E}, \theta) := (\bigoplus_{q=0}^n E^{n-q,q}, \bigoplus_{q=0}^n \theta_{n-q,q})$ induced by the grading of Deligne extension of a \mathbb{Z} -variation of polarized Hodge structure defined over $Y \setminus T$ with eigenvalues of residues lying in $[0, 1) \cap \mathbb{Q}$, and
- (iv) a sub-Higgs sheaf $(\mathcal{F}, \eta) \subset (\tilde{\mathcal{E}}, \tilde{\theta})$

so that one has the following:

- (1) $(\tilde{\mathcal{E}}, \tilde{\theta}) := (\mathcal{B}^{-1} \otimes \mathcal{E}, 1 \otimes \theta)$. In particular, $\tilde{\theta} : \tilde{\mathcal{E}} \rightarrow \tilde{\mathcal{E}} \otimes \Omega_Y(\log T)$, and $\tilde{\theta} \wedge \tilde{\theta} = 0$.
- (2) (\mathcal{F}, η) has log poles only on the boundary B , that is, $\eta : \mathcal{F} \rightarrow \mathcal{F} \otimes \Omega_Y(\log B)$.
- (3) Write $\tilde{\mathcal{E}}_k := \mathcal{B}^{-1} \otimes E^{n-k,k}$, and denote by $\mathcal{F}_k := \tilde{\mathcal{E}}_k \cap \mathcal{F}$. Then the first stage \mathcal{F}_0 of \mathcal{F} is an *effective line bundle*. In other words, there exists a non-trivial morphism $\mathcal{O}_Y \rightarrow \mathcal{F}_0$.

Let us recall that, the augmented base locus of a big line bundle L on a projective manifold Y , denoted by $\mathbf{B}_+(L)$, is defined to be

$$\mathbf{B}_+(L) := \bigcap_{m=1}^{\infty} \mathbf{B}(mL - A),$$

where A is any ample line bundle on Y , and $\mathbf{B}(\bullet)$ is the *stable base locus* for a line bundle.

3.2. Hyperbolicity via VZ Higgs bundles. We are able to state our main result in this section.

Theorem 3.2. *For the log smooth family $(X^\circ, D^\circ) \rightarrow Y^\circ$ in Theorem A, after replacing Y° by a birational model, there exists a VZ Higgs bundle over some smooth projective compactification $Y \supset Y^\circ$.*

Let us first show how the above theorem implies Theorem A.

Proof of Theorem A. Once the VZ Higgs bundle is constructed, the proof of Theorem A.(i) should be well-known to the experts, and we briefly recall the proof for completeness sake. The first step is the construction of Viehweg-Zuo (big) sheaf, which is due to Viehweg-Zuo in [VZ02]. Since $(\bigoplus_{q=0}^n \mathcal{F}_q, \bigoplus_{q=0}^n \eta_q)$ is a *sub-Higgs sheaf* of $(\tilde{\mathcal{E}}, \tilde{\theta})$, as initiated in [VZ02], for any $q = 1, \dots, n$, the morphism $\eta^q : \mathcal{F} \rightarrow \mathcal{F} \otimes \bigotimes^q \Omega_Y(\log B)$ defined by iterating $\eta : \mathcal{F} \rightarrow \mathcal{F} \otimes \Omega_Y(\log B)$ q -times induces

$$(3.2.1) \quad \mathcal{F}_0 \rightarrow \mathcal{F}_q \otimes \bigotimes^q \Omega_Y(\log B).$$

(3.2.1) factors through $\mathcal{F}_q \otimes S^q \Omega_Y(\log B)$ by $\eta \wedge \eta = 0$. Recall that \mathcal{F}_0 is effective, one thus has a morphism

$$(3.2.2) \quad \mathcal{O}_Y \rightarrow \mathcal{F}_0 \rightarrow \mathcal{F}_q \otimes S^q \Omega_Y(\log B).$$

We denote by N_q and K_q the kernels of $\eta_q : \mathcal{F}_q \rightarrow \mathcal{F}_{q+1} \otimes \Omega_Y(\log B)$ and $\theta_{n-q,q} : E_{n-q,q} \rightarrow E_{n-q+1,q+1} \otimes \Omega_Y(\log T)$ respectively, which are both torsion free sheaves. Then $N_q = (\mathcal{B}^{-1} \otimes K_q) \cap \mathcal{F}_q$. By the work of Zuo [Zuo00] (see also [PW16, Bru17, FF17, Bru18] for various generalizations) on the negativity of kernels of Kodaira-Spencer maps of Hodge bundles, K_q^* is *weakly positive* in the sense of Viehweg¹, cf. [VZ02, Lemma 4.4.(v)]. Hence there exists

¹By [Bru17], one can even prove that the Hodge metric induces a semi-negatively curved singular hermitian metric for K_q in the sense of Raufi [Rau15] and Păun-Takayama [PT18] (cf. also [HPS16]).

a morphism

$$\mathcal{B} \otimes K_q^* \rightarrow N_q^*$$

which is generically surjective. Let $k \in \mathbb{Z}_{\geq 0}$ the minimal non-negative integer so that $\eta^k \neq 0$ and $\eta^{k+1} = 0$. As proved in [VZ02, Corollary 4.5], k must be positive. Indeed, if this is not the case, one has $\mathcal{O}_Y \subset K_0 \otimes \mathcal{B}^{-1}$, which is not possible. Hence there exists a non-trivial morphism

$$\mathcal{O}_Y \rightarrow \mathcal{F}_0 \rightarrow N_k \otimes S^k \Omega_Y(\log B).$$

In other words, there exists a non-trivial morphism

$$(3.2.3) \quad \mathcal{B} \otimes K_k^* \rightarrow N_k^* \rightarrow S^k \Omega_Y(\log B).$$

Since \mathcal{B} is big and nef, $\mathcal{B} \otimes K_k^*$ is big in the sense of Viehweg [VZ02, Definition 1.1.(c)], and thus for any ample line bundle \mathcal{A} on Y there exists $\alpha \gg 0$ so that $S^\alpha(\mathcal{B} \otimes K_k^*) \otimes \mathcal{A}^{-1}$ is generated by global sections over a Zariski open set. By (3.2.3) there is a non-zero morphism

$$\mathcal{A} \rightarrow S^{\alpha k} \Omega_Y(\log B).$$

Such \mathcal{A} is called the Viehweg-Zuo big sheaf in literatures.

Once the Viehweg-Zuo sheaf \mathcal{A} is constructed, it follows from [CP15, Theorem 4.1] that $K_Y + B$ is big. Theorem A.(i) is thus proved.

The proof of Theorem A.(ii) is exactly the same as [DA19, Proof of Theorem A]. In [DA19, §2] we establish an algorithm to construct a Finsler metric whose holomorphic sectional curvature is bounded above by a negative constant via VZ Higgs bundles. By proving some sort of *infinitesimal generic Torelli theorem* (cf. [DA19, Theorem F]) for VZ Higgs bundles, in [DA19, Theorem B] we show that this Finsler metric is *generically non-degenerate*. The pseudo Kobayashi hyperbolicity of Y° follows from a bimeromorphic criteria in [DA19, Lemma 2.10]. \square

Remark 3.3. Let us mention that although the Lang conjecture [Lan91, Chapter VIII. Conjecture 1.4] on the equivalence between pseudo Kobayashi hyperbolicity and being of log general type is quite open at the present time, we know that it holds for Hilbert modular varieties by [Rou16, CRT17] and subvarieties of Abelian varieties [Yam19].

We show how Theorem A implies Theorem B, whose proof is quite standard.

Proof of Theorem B. We first prove (ii). Pick any positively dimensional irreducible closed subvariety $Z \subset Y^\circ$. We take a desingularization $\tilde{Z} \rightarrow Z$, which is positively dimensional, connected quasi-projective manifold. Define a base change $f_{\tilde{Z}} : (X^\circ, D^\circ) \times_{Y^\circ} \tilde{Z} \rightarrow \tilde{Z}$, which is also a log smooth family of Calabi-Yau klt pairs. Since the logarithmic Kodaira-Spencer map is functorial under the base change, by the assumption of effective parametrization for f° , the logarithmic Kodaira-Spencer map of $f_{\tilde{Z}}$ is generically injective. Hence \tilde{Z} is of log general type by Theorem A.(i), and (i) is proved.

We will prove (i) by contradiction. Suppose that there exists a non-constant holomorphic map $\gamma : \mathbb{C} \rightarrow Y^\circ$. Set Z to be the Zariski closure of the image $\gamma(\mathbb{C})$, and take a desingularization $\tilde{Z} \rightarrow Z$, which is positively dimensional, connected quasi-projective manifold. Then $\gamma : \mathbb{C} \rightarrow Z$ lifts to a Zariski dense entire curve $\tilde{\gamma} : \mathbb{C} \rightarrow \tilde{Z}$. By the above arguments, \tilde{Z} can be realized as the base of a log smooth family of Calabi-Yau klt pairs with maximal variation. This is a contradiction, since by Theorem A.(ii), \tilde{Z} cannot have any Zariski dense entire curve. \square

3.3. Construction of VZ Higgs bundles. In this subsection we construct the VZ Higgs bundles on some birational modification of the base Y° in Theorem A.

Proof of Theorem 3.2. We first apply Proposition 2.3 to make the following set-up. Changing the smooth birational model, we may replace Y° by $\tilde{Y}^\circ := \mu^{-1}(Y^\circ)$ in Proposition 2.3, and by abuse of notation we rename \tilde{Y}° as Y° . There exist a smooth projective compactification $Y \supset Y^\circ$, a log smooth klt pair (X, D) and a surjective morphism $f : (X, D) \rightarrow Y$ which is log smooth over Y° . Set n to be the relative dimension of $X \rightarrow Y$. For m sufficiently large and divisible one can find an effective divisor

$$\Gamma \in |mK_{X/Y} + mD + mE - mf^*A|$$

which is supported on $f^{-1}(T)$, contains one irreducible component with multiplicity one and $\Gamma \geq \Delta := f^*B$ by (6). Here T is an SNC divisor on Y containing $B := Y \setminus Y^\circ$ by (3), E is an effective and exceptional divisor with $f(E) \subset T$, and the line bundles A and $A - B$ on Y are both big and nef with the augmented base locus $\mathbf{B}_+(A) = \mathbf{B}_+(A - B) \subset B$. $\Gamma + D$ has SNC support by (5), and each irreducible component of D dominates onto Y by (1).

Set

$$(3.3.1) \quad H := \Gamma + m[D] - mD \in |mK_{X/Y} + m[D] + mE - mf^*A|$$

which has SNC support, and contains an irreducible component with multiplicity one. Let Z_{cyc} be the cyclic cover of X obtained by taking the m -th roots along H , and let Z_{nor} be the normalization of Z_{cyc} , which is *irreducible* by [EV92, Lemma 3.15.(a)]. Write $\pi : Z_{\text{nor}} \rightarrow Y$. Since H is relatively normal crossing over $V := Y \setminus T$, $Z_{\text{nor}}|_{\pi^{-1}(V)} \rightarrow V$ is a *locally trivial family*. Let $Z \rightarrow Z_{\text{nor}}$ be a *functorial resolution of singularities*, which is a *simultaneous resolution* for $Z_{\text{nor}}|_{\pi^{-1}(V)} \rightarrow V$. Write $g : Z \rightarrow Y$. Then $Z|_{g^{-1}(V)} \rightarrow V$ is smooth. Blowing up Z a bit more with centers in $g^{-1}(T)$, one can assume that $\Pi := g^{-1}(T)$ is an SNC divisor.

Write $\Delta := f^*B$ and $\Sigma := f^*T$. Leaving out a codimension at least two subvariety of Y , one can assume that

- (1) both B and T are smooth;
- (2) both $f : X \rightarrow Y$ and $g : Z \rightarrow Y$ are flat; in particular, the f -exceptional divisor E in (3.3.1) disappears;
- (3) Δ (resp. Π) is relatively normal crossing over B (resp. T);
- (4) for any $I = \{i_1, \dots, i_\ell\} \subset \{1, \dots, r\}$, the surjective morphism $D_I \rightarrow Y$ is also flat, and $D_I \cap \Delta$ is relatively normal crossing over B . Here we denote $D_I := D_{i_1} \cap \dots \cap D_{i_\ell}$.

Claim 3.4 (good partial compactification). *Write $\Omega_{X/Y}(\log(\Delta + [D])) := \Omega_X(\log(\Delta + [D]))/f^*\Omega_Y(\log B)$, $\Omega_{Z/Y}(\log \Pi) := \Omega_Z(\log \Pi)/g^*\Omega_Y(\log T)$. Then they are all locally free. In other words, one has the following short exact sequences of locally free sheaves*

$$(3.3.2) \quad 0 \rightarrow f^*\Omega_Y(\log B) \rightarrow \Omega_X(\log(\Delta + [D])) \rightarrow \Omega_{X/Y}(\log(\Delta + [D])) \rightarrow 0$$

$$(3.3.3) \quad 0 \rightarrow g^*\Omega_Y(\log T) \rightarrow \Omega_Z(\log \Pi) \rightarrow \Omega_{Z/Y}(\log \Pi) \rightarrow 0.$$

Proof of Claim 3.4. As is well-known, the morphism $h : (Z, \Pi) \rightarrow (Y, T)$ is a “good partial compactification” of the smooth morphism $Z \setminus \Pi \rightarrow Y \setminus T$ in the sense of [VZ02, Definition 2.1.(c)], and one can easily show that $\Omega_{Z/Y}(\log \Pi)$ is locally free.

To prove the local freeness of $\Omega_{X/Y}(\log(\Delta + [D]))$, it suffices to show that for any $y_0 \in Y$ and $\sigma \in \Omega_Y(\log B)(U)$, where $U \ni y_0$ is some open set with $\sigma(y_0) \neq 0$, $f^*\sigma(x_0) \neq 0$ for any $x_0 \in f^{-1}(y_0)$.

Note that there is a unique $I \subset \{1, \dots, r\}$ so that $x_0 \in D_I$ and $x_0 \notin D_J$ for any other $J \supsetneq I$. After reordering, we can assume that $I = \{\ell + 1, \dots, p\}$. Take a local coordinate system (x_1, \dots, x_m) centering at x_0 so that $\Delta_{\text{red}} = (x_1 \cdots x_\ell = 0)$ and $D_i = (x_i = 0)$ for $i \in I$. Since $\Delta = f^*B$, the morphism $f^*\Omega_Y(\log B) \rightarrow \Omega_X(\log(\Delta + [D]))$ factors through $\Omega_X(\log \Delta)$. Note

that $d \log x_1, \dots, d \log x_p, dx_{p+1}, \dots, dx_m$ and $d \log x_1, \dots, d \log x_\ell, dx_{\ell+1}, \dots, dx_m$ form the local basis for $\Omega_X(\log(\Delta + [D]))$ and $\Omega_X(\log \Delta)$. Hence $f^* \sigma = \sum_{j=1}^{\ell} a_j(x) d \log x_j + \sum_{i=\ell+1}^m a_i(x) dx_i$, where $a_i(x)$ are local holomorphic functions. When we write $f^* \sigma$ in terms of the basis of $\Omega_X(\log(\Delta + [D]))$, one has

$$f^* \sigma = \sum_{j=1}^{\ell} a_j(x) d \log x_j + \sum_{i=\ell+1}^p a_i(x) x_i d \log x_i + \sum_{k=p+1}^m a_k(x) dx_k.$$

Since $x_0 \in D_I = (x_{\ell+1} = \dots = x_p = 0)$, one has

$$(3.3.4) \quad f^* \sigma(x_0) = \sum_{j=1}^{\ell} a_j(x_0) d \log x_j + \sum_{k=p+1}^m a_k(x_0) dx_k.$$

When $f^* \sigma$ is seen as the local section in $\Omega_X(\log(\Delta + [D]))$, $f^* \sigma(x_0) = 0$ if and only if $a_1(x_0) = \dots = a_\ell(x_0) = a_{p+1}(x_0) = \dots = a_m(x_0) = 0$. On the other hand, since the projective morphism $f_I : (D_I, D_I \cap \Delta) \rightarrow (Y, B)$ satisfies that f_I is flat, $D_I \cap \Delta \rightarrow B$ is relatively normal crossing, we thus conclude that $f_I^* \sigma(x_0) \neq 0$. Since $d \log x_1, \dots, d \log x_\ell, dx_{p+1}, \dots, dx_m$ form the local basis for $\Omega_{D_I}(\log D_I \cap \Delta)$, by (3.3.4), $f^* \sigma(x_0) = f_I^* \sigma(x_0) \neq 0$, and we conclude that $\Omega_{X/Y}(\log(\Delta + [D]))$ is also locally free. \square

By the above claim, such a partial compactification of the log smooth family $f^\circ : (X^\circ, D^\circ) \rightarrow Y^\circ$ can thus be seen as the “good partial compactification” in the log setting.

For any $p \in \mathbb{Z}_{>0}$, let

$$(3.3.5) \quad \Omega_X^p(\log(\Delta + [D])) \otimes \mathcal{L}^{-1} = \mathcal{F}^0 \supset \mathcal{F}^1 \supset \dots \supset \mathcal{F}^p \supset \mathcal{F}^{p+1} = 0$$

be the Koszul filtration associated to (3.3.2) twisted with $\mathcal{L}^{-1} := -K_{X/Y} - [D] + f^* A$, defined by

$$(3.3.6) \quad \mathcal{F}^i := \text{Im} \left(f^* \Omega_Y^i(\log B) \otimes \Omega_X^{p-i}(\log(\Delta + [D])) \otimes \mathcal{L}^{-1} \rightarrow \Omega_X^p(\log(\Delta + [D])) \otimes \mathcal{L}^{-1} \right)$$

so that the associated graded objects are given by

$$\text{gr}^i \mathcal{F}^\bullet := \mathcal{F}^i / \mathcal{F}^{i+1} = f^* \Omega_Y^i(\log B) \otimes \Omega_{X/Y}^{p-i}(\log(\Delta + [D])) \otimes \mathcal{L}^{-1}$$

The tautological exact sequence associated to (3.3.5) is defined by

$$(3.3.7) \quad \begin{array}{ccccccc} f^* \Omega_Y(\log B) \otimes \Omega_{X/Y}^{p-1}(\log(\Delta + [D])) \otimes \mathcal{L}^{-1} & & \Omega_{X/Y}^p(\log(\Delta + [D])) \otimes \mathcal{L}^{-1} & & & & \\ & \parallel & & \parallel & & & \\ 0 & \longrightarrow & \text{gr}^1 \mathcal{F}^\bullet & \longrightarrow & \mathcal{F}^0 / \mathcal{F}^2 & \longrightarrow & \text{gr}^0 \mathcal{F}^\bullet \longrightarrow 0 \end{array}$$

By taking higher direct images Rf_* of (3.3.7), the connecting morphisms of the associated long exact sequences induce $F^{p,q} \xrightarrow{\tau_{p,q}} F^{p-1,q+1} \otimes \Omega_Y(\log B)$, where we denote

$$F^{p,q} := R^q f_* \left(\Omega_{X/Y}^p(\log(\Delta + [D])) \otimes \mathcal{L}^{-1} \right) / \text{torsion}.$$

Recall that Z_{cyc} is the cyclic cover of X by taking m -th root along a normal crossing divisor $H \in |m\mathcal{L}|$ on X , and $\delta : Z \rightarrow Z_{\text{nor}}$ is a desingularization of the normalization Z_{nor} of Z_{cyc} .

Denote by $\psi : Z \xrightarrow{\delta} Z_{\text{nor}} \xrightarrow{\phi} X$ the composition map.

Claim 3.5. For any $i = 1, \dots, n$, one has a natural morphism

$$(3.3.8) \quad \Xi : \psi^* \left(\Omega_X^i(\log(\Delta + [D])) \otimes \mathcal{L}^{-1} \right) \rightarrow \Omega_Z^i(\log \Pi).$$

Proof of Claim 3.5. Note that $\phi : Z_{\text{nor}} \rightarrow X$ is a surjective and finite morphism, which is étale over $X \setminus H_{\text{red}}$, and non-singular over $X_0 := X \setminus \text{Sing}(H_{\text{red}})$. For the $Z_{\text{nor}}^\circ := \phi^{-1}(X_0)$, $Z_{\text{nor}} \setminus Z_{\text{nor}}^\circ$ has codimension at least two for $\text{Sing}(H_{\text{red}})$ is a subvariety in X of codimension at least two. Write $H_0 := H|_{X_0}$, and $\phi_0 : Z_{\text{nor}}^\circ \rightarrow X_0$ for the restriction of ϕ on Z_{nor}° . As observed by Popa-Schnell [PS17] and Wei-Wu [WW18], there is a natural morphism

$$\phi_0^*(\Omega_{X_0}^i(\log H_0) \otimes \mathcal{L}_{|X_0}^{-1}) \rightarrow \Omega_{Z_{\text{nor}}^\circ}^i,$$

and after taking the reflexive hull, one has

$$(3.3.9) \quad \phi^*(\Omega_X^i(\log H) \otimes \mathcal{L}^{-1}) \rightarrow \Omega_{Z_{\text{nor}}}^{[i]},$$

where $\Omega_{Z_{\text{nor}}}^{[i]}$ is the reflexive differential form defined in [GKKP11, Notation 2.17]. Note that Z_{nor} is normal and has quotient singularities, it thus has klt singularities. It then follows from [GKKP11, Theorem 4.3] that, there is a sheaf morphism $\delta^* \Omega_{Z_{\text{nor}}}^{[i]} \rightarrow \Omega_Z^i$, and combining (3.3.9) one has

$$\psi^*(\Omega_X^i(\log H) \otimes \mathcal{L}^{-1}) \rightarrow \Omega_Z^i.$$

Note that $\Gamma \geq \Delta$, and $(m[D] - mD)_{\text{red}} = D_{\text{red}}$ for D is klt. Hence we have the inclusion

$$\Omega_X^i(\log(\Delta + \lceil D \rceil)) \subset \Omega_X^i(\log H)$$

and the lemma follows from the obvious inclusion $\Omega_Z^i \subset \Omega_Z^i(\log \Pi)$. \square

Pulling back (3.3.2) by ψ^* , we have a short exact sequence of locally free sheaves

$$(3.3.10) \quad 0 \rightarrow g^* \Omega_Y(\log B) \rightarrow \psi^* \Omega_X(\log(\Delta + \lceil D \rceil)) \rightarrow \psi^* \Omega_{X/Y}(\log(\Delta + \lceil D \rceil)) \rightarrow 0$$

In a similar way as (3.3.5), we associate (3.3.3) and (3.3.10) with two filtrations

$$(3.3.11) \quad \Omega_Z^p(\log \Pi) = \mathcal{G}^0 \supset \mathcal{G}^1 \supset \dots \supset \mathcal{G}^p \supset \mathcal{G}^{p+1} = 0$$

$$(3.3.12) \quad \psi^*(\Omega_X^p(\log(\Delta + \lceil D \rceil)) \otimes \mathcal{L}^{-1}) = \tilde{\mathcal{F}}^0 \supset \tilde{\mathcal{F}}^1 \supset \dots \supset \tilde{\mathcal{F}}^p \supset \tilde{\mathcal{F}}^{p+1} = 0$$

defined by

$$\mathcal{G}^i := \text{Im} \left(g^* \Omega_Y^i(\log T) \otimes \Omega_Z^{p-i}(\log \Pi) \rightarrow \Omega_Z^p(\log \Pi) \right),$$

$$\tilde{\mathcal{F}}^i := \text{Im} \left(g^* \Omega_Y^i(\log B) \otimes \psi^* \left(\Omega_X^{p-i}(\log(\Delta + \lceil D \rceil)) \otimes \mathcal{L}^{-1} \right) \rightarrow \psi^* \left(\Omega_X^p(\log(\Delta + \lceil D \rceil)) \otimes \mathcal{L}^{-1} \right) \right).$$

Their associated graded objects are thus given by

$$(3.3.13) \quad \text{gr}^i \mathcal{G}^\bullet := \mathcal{G}^i / \mathcal{G}^{i+1} = g^* \Omega_Y^i(\log T) \otimes \Omega_{Z/Y}^{p-i}(\log \Pi)$$

$$\text{gr}^i \tilde{\mathcal{F}}^\bullet := \tilde{\mathcal{F}}^i / \tilde{\mathcal{F}}^{i+1} = g^* \Omega_Y^i(\log B) \otimes \psi^* \left(\Omega_{X/Y}^{p-i}(\log(\Delta + \lceil D \rceil)) \otimes \mathcal{L}^{-1} \right) = \psi^* \text{gr}^i \tilde{\mathcal{F}}^\bullet.$$

One can easily show that Ξ defined in (3.3.8) is compatible with the filtration structures $\Xi : \tilde{\mathcal{F}}^\bullet \rightarrow \mathcal{G}^\bullet$ in (3.3.11) and (3.3.12). It thus induces a morphism between their graded terms $\text{gr}^i \tilde{\mathcal{F}}^\bullet \rightarrow \text{gr}^i \mathcal{G}^\bullet$, and in particular, a morphism between the following short exact

sequences

(3.3.14)

$$\begin{array}{ccccccc}
g^* \Omega_Y(\log B) \otimes \psi^* \left(\Omega_{X/Y}^{p-1}(\log(\Delta + [D])) \otimes \mathcal{L}^{-1} \right) & & \psi^* \left(\Omega_{X/Y}^p(\log(\Delta + [D])) \otimes \mathcal{L}^{-1} \right) & & & & \\
\parallel & & \parallel & & & & \\
0 \longrightarrow \mathrm{gr}^1 \tilde{\mathcal{F}}^\bullet & \longrightarrow & \tilde{\mathcal{F}}^0 / \tilde{\mathcal{F}}^2 & \longrightarrow & \mathrm{gr}^0 \tilde{\mathcal{F}}^\bullet & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 \longrightarrow \mathrm{gr}^1 \mathcal{G}^\bullet & \longrightarrow & \mathcal{G}^0 / \mathcal{G}^2 & \longrightarrow & \mathrm{gr}^0 \mathcal{G}^\bullet & \longrightarrow & 0 \\
\parallel & & \parallel & & \parallel & & \\
g^* \Omega_Y(\log T) \otimes \Omega_{Z/Y}^{p-1}(\log \Pi) & & & & \Omega_{Z/Y}^p(\log \Pi) & &
\end{array}$$

Pushing forward (3.3.14) by Rg_* , the edge morphisms induce

$$(3.3.15) \quad \begin{array}{ccc}
\tilde{F}^{p,q} & \xrightarrow{\varphi_{p,q}} & \tilde{F}^{p-1,q+1} \otimes \Omega_Y(\log B) \\
\downarrow \rho_{p,q} & & \downarrow \rho_{p-1,q+1} \otimes \iota \\
E_0^{p,q} & \xrightarrow{\theta'_{p,q}} & E_0^{p-1,q+1} \otimes \Omega_Y(\log T)
\end{array}$$

where $\iota : \Omega_Y(\log B) \hookrightarrow \Omega_Y(\log T)$ denotes the natural inclusion, $E_0^{p,q} := R^q g_* (\Omega_{Z/Y}^p(\log \Pi))$, which is locally free by a theorem of Steenbrink [Ste77] (see also [Zuc84, Kol86, Kaw02, KMN02] for various generalizations), and

$$\tilde{F}^{p,q} := R^q g_* \left(\psi^* \left(\Omega_{X/Y}^p(\log(\Delta + [D])) \otimes \mathcal{L}^{-1} \right) \right) / \text{torsion}.$$

Let us mention that the similar construction as $(\bigoplus_{p+q=\ell} \tilde{F}^{p,q}, \bigoplus_{p+q=\ell} \varphi_{p,q})$ is made by Taji in his work [Taj18] on a conjecture of Kebekus-Kovács [KK08, Conjecture 1.6].

Recall that $\psi : Z \xrightarrow{\delta} Z_{\mathrm{nor}} \xrightarrow{\phi} X$. As is well-known, Z_{nor} has rational singularities (see e.g. [EV92, §3]), and one thus has $R^q \delta_* \mathcal{O}_Z = 0$ for any $q > 0$. By the projection formula and the degeneration of relative Leray spectral sequences, for any locally free sheaf \mathcal{E} on X , one has

$$(3.3.16) \quad R^q \psi_*(\psi^* \mathcal{E}) = \mathcal{E} \otimes R^q \psi_* \mathcal{O}_Z = \mathcal{E} \otimes R^q \phi_*(\delta_* \mathcal{O}_Z) = 0, \quad \forall q > 0$$

thanks to the finiteness of ϕ . Applying (3.3.16) to (3.3.13), for any $q > 0$, we have $R^q \psi_*(\mathrm{gr}^i \tilde{\mathcal{F}}^\bullet) = 0$, and therefore, the exactness of the tautological short exact sequence of $\tilde{\mathcal{F}}^\bullet$ is preserved under the direct images ψ_* as follows:

(3.3.17)

$$\begin{array}{ccccccc}
0 \longrightarrow \psi_*(\mathrm{gr}^1 \tilde{\mathcal{F}}^\bullet) & \longrightarrow & \psi_*(\tilde{\mathcal{F}}^0 / \tilde{\mathcal{F}}^2) & \longrightarrow & \psi_*(\mathrm{gr}^0 \tilde{\mathcal{F}}^\bullet) & \longrightarrow & 0 \\
\parallel & & \parallel & & \parallel & & \\
0 \longrightarrow \mathrm{gr}^1 \mathcal{F}^\bullet \otimes \psi_* \mathcal{O}_Z & \longrightarrow & \mathcal{F}^0 / \mathcal{F}^2 \otimes \psi_* \mathcal{O}_Z & \longrightarrow & \mathrm{gr}^0 \mathcal{F}^\bullet \otimes \psi_* \mathcal{O}_Z & \longrightarrow & 0
\end{array}$$

By the collapse of relative Leray spectral sequences, one has

$$\begin{aligned}
R^q g_*(\mathrm{gr}^i \tilde{\mathcal{F}}^\bullet) &\stackrel{(3.3.13)}{=} R^q g_*(\psi^* \mathrm{gr}^i \mathcal{F}^\bullet) \stackrel{(3.3.16)}{=} R^q f_*(\psi_*(\psi^* \mathrm{gr}^i \mathcal{F}^\bullet)) = R^q f_*(\mathrm{gr}^i \mathcal{F}^\bullet \otimes \psi_* \mathcal{O}_Z) \\
&= \Omega_Y^i(\log B) \otimes R^q f_* \left(\Omega_{X/Y}^{p-i}(\log(\Delta + [D])) \otimes \mathcal{L}^{-1} \otimes \psi_* \mathcal{O}_Z \right).
\end{aligned}$$

Therefore, $(\bigoplus_{p+q=\ell} \tilde{F}^{p,q}, \bigoplus_{p+q=\ell} \varphi_{p,q})$ can also be defined alternatively by pushing forward (3.3.17) via Rf_* , with $\varphi_{p,q}$ the edge morphisms.

By [EV92, Corollary 3.11], the cyclic group $G := \mathbb{Z}/m\mathbb{Z}$ acts on $\psi_* \mathcal{O}_Z = \phi_* \mathcal{O}_{Z_{\text{nor}}}$, and one has the decomposition

$$\psi_* \mathcal{O}_Z = \mathcal{O}_X \oplus \bigoplus_{i=1}^{m-1} (\mathcal{L}^{(i)})^{-1}, \quad \text{where } \mathcal{L}^{(i)} := \mathcal{L}^i \otimes \mathcal{O}_X(-\lfloor \frac{iH}{m} \rfloor).$$

In particular, the G -invariant part $(\psi_* \mathcal{O}_Z)^G = \mathcal{O}_X$. Hence one can easily show that the cyclic group G acts on (3.3.17), whose G -invariant part is

$$\begin{array}{ccccccc} 0 & \longrightarrow & \psi_*(\text{gr}^1 \tilde{\mathcal{F}}^\bullet)^G & \longrightarrow & \psi_*(\tilde{\mathcal{F}}^0/\tilde{\mathcal{F}}^2)^G & \longrightarrow & \psi_*(\text{gr}^0 \tilde{\mathcal{F}}^\bullet)^G \longrightarrow 0 \\ & & \parallel & & \parallel & & \parallel \\ 0 & \longrightarrow & \text{gr}^1 \mathcal{F}^\bullet & \longrightarrow & \mathcal{F}^0/\mathcal{F}^2 & \longrightarrow & \text{gr}^0 \mathcal{F}^\bullet \longrightarrow 0 \end{array}$$

Therefore, $(\bigoplus_{p+q=\ell} F^{p,q}, \bigoplus_{p+q=\ell} \tau_{p,q})$ is a direct factor of $(\bigoplus_{p+q=\ell} \tilde{F}^{p,q}, \bigoplus_{p+q=\ell} \varphi_{p,q})$. Combining (3.3.15), we have

$$(3.3.18) \quad \begin{array}{ccc} E_0^{p,q} & \xrightarrow{\theta'_{p,q}} & E_0^{p-1,q+1} \otimes \Omega_Y(\log T) \\ \rho_{p,q} \uparrow & & \uparrow \rho_{p-1,q+1} \otimes \iota \\ \tilde{F}^{p,q} & \xrightarrow{\varphi_{p,q}} & \tilde{F}^{p-1,q+1} \otimes \Omega_Y(\log B) \\ \uparrow & & \uparrow \\ F^{p,q} & \xrightarrow{\tau_{p,q}} & F^{p-1,q+1} \otimes \Omega_Y(\log B) \end{array}$$

Note that $f_* \mathcal{O}_X = \mathcal{O}_Y$ for $f : X \rightarrow Y$ is an algebraic fiber space of relative dimension n . Hence

$$(3.3.19) \quad F_0^{n,0} = f_*(K_{X/Y} - \Delta + \Delta_{\text{red}} + [D] + \mathcal{L}^{-1}) = f_*(\Delta_{\text{red}} + f^*(A - B)) \supset \mathcal{B},$$

where $\mathcal{B} := A - B$ is a big and nef line bundle with $\mathbf{B}_+(\mathcal{B}) \subset B$. Define $F_0^{n-q,q} := \mathcal{B}^{-1} \otimes F^{n-q,q}$, and

$$\tau'_{n-q,q} : \mathcal{B}^{-1} \otimes F^{n-q,q} \xrightarrow{1 \otimes \tau_{n-q,q}} \mathcal{B}^{-1} \otimes F^{n-q-1,q+1} \otimes \Omega_Y(\log B)$$

By (3.3.18), one has the following diagram:

$$(3.3.20) \quad \begin{array}{ccc} \mathcal{B}^{-1} \otimes E_0^{n-q,q} & \xrightarrow{1 \otimes \theta'_{n-q,q}} & \mathcal{B}^{-1} \otimes E_0^{n-q-1,q+1} \otimes \Omega_Y(\log T) \\ \rho'_{n-q,q} \uparrow & & \uparrow \rho'_{n-q-1,q+1} \otimes \iota \\ F_0^{n-q,q} & \xrightarrow{\tau'_{n-q,q}} & F_0^{n-q-1,q+1} \otimes \Omega_Y(\log B) \end{array}$$

where $F_0^{n,0}$ is an effective line bundle by (3.3.19). Note that all the objects in (3.3.20) are only defined over a big open set Y' of Y .

Write $Z_0 := g^{-1}(V)$, which is smooth over $V = Y \setminus T$. The local system $R^n g_* \mathbb{C}|_{Z_0}$ extends to a locally free sheaf \mathcal{V} on Y (here Y is projective rather than the big open set!) equipped with the logarithmic connection

$$\nabla : \mathcal{V} \rightarrow \mathcal{V} \otimes \Omega_Y(\log T),$$

whose eigenvalues of the residues lie in $[0, 1) \cap \mathbb{Q}$ (the so-called *lower canonical extension* in [Kol86]). By [Sch73, CKS86, Kol86], the Hodge filtration of $R^n g_* \mathbb{C}|_{Z_0}$ extends to a filtration

$\mathcal{V} := \mathcal{F}^0 \supset \mathcal{F}^1 \supset \dots \supset \mathcal{F}^n \supset \mathcal{F}^{n+1} = 0$ of vector bundles so that their graded sheaves $E^{n-q,q} := \mathcal{F}^{n-q}/\mathcal{F}^{n-q+1}$ are also locally free, and there exists

$$\theta_{n-q,q} : E^{n-q,q} \rightarrow E^{n-q-1,q+1} \otimes \Omega_Y(\log T)$$

for each $q = 0, \dots, n$. As mentioned above, $E_0^{n-q,q}$ is locally free by Steenbrink's theorem. By a theorem of Steenbrink-Zucker [Zuc84], we know that $(\bigoplus_{q=0}^n E_0^{n-q,q}, \bigoplus_{q=0}^n \theta'_{n-q,q}) = (\bigoplus_{q=0}^n E^{n-q,q}, \bigoplus_{q=0}^n \theta_{n-q,q})|_{Y'}$, hence it can be extended to the whole projective manifold Y defined by $(\bigoplus_{q=0}^n E^{n-q,q}, \bigoplus_{q=0}^n \theta_{n-q,q})$. For every $q = 0, \dots, n$, we replace $F_0^{n-q,q}$ by its reflexive hull and thus the morphisms $\tau'_{n-q,q}, \rho'_{n-q,q}$ and the diagram (3.3.20) extends to the whole Y .

To finish the construction, we have to introduce the sub-Higgs sheaf of $(\bigoplus_{q=0}^n \mathcal{B}^{-1} \otimes E^{n-q,q}, \bigoplus_{q=0}^n \mathbb{1} \otimes \theta_{n-q,q})$ in Definition 3.1. For each $q = 0, \dots, n$, we define a coherent torsion-free sheaf $\mathcal{F}_q := \rho'_{n-q,q}(F_0^{n-q,q}) \subset \mathcal{B}^{-1} \otimes E^{n-q,q}$, and denote η_q by the restriction of $\mathbb{1} \otimes \theta_{n-q,q}$ to \mathcal{F}_q . By (3.3.20), one has

$$\eta_q : \mathcal{F}_q \rightarrow \mathcal{F}_{q+1} \otimes \Omega_Y(\log B).$$

Then $(\bigoplus_{q=0}^n \mathcal{F}_q, \bigoplus_{q=0}^n \eta_q)$ is a sub-Higgs sheaf of $(\bigoplus_{q=0}^n \mathcal{B}^{-1} \otimes E^{n-q,q}, \bigoplus_{q=0}^n \mathbb{1} \otimes \theta_{n-q,q})$. By (3.3.19), there exists a morphism $\mathcal{O}_Y \rightarrow \mathcal{F}_0$ which is an isomorphism over V . The VZ Higgs bundle is therefore constructed. \square

Remark 3.6. The morphism Ξ defined in (3.3.8) was first observed by Popa-Schnell [PS17], and was later generalized to the log setting in [Wei17, WW18]. This morphism inspires us to construct an intermediate Higgs bundle $(\bigoplus_{q=0}^n \tilde{F}^{n-q,q}, \bigoplus_{q=0}^n \varphi_{n-q,q})$, which relates $(\bigoplus_{q=0}^n F^{n-q,q}, \bigoplus_{q=0}^n \tau_{n-q,q})$ with $(\bigoplus_{q=0}^n E_0^{n-q,q}, \bigoplus_{q=0}^n \theta'_{n-q,q})$ in a more direct manner. In the above proof, we do not require the divisor H for cyclic cover to be *generically smooth* over the base², which is more flexible than the original construction in [VZ02, VZ03].

Acknowledgements. I would like to thank Professors Dan Abramovich, Bo Berndtsson, Junyan Cao and Yoshinori Gongyo for very helpful discussions. I also thank Professor Kang Zuo for pointing out the reference [VZ06]. I sincerely thank the anonymous referee for his careful reading of the manuscript and very helpful suggestions for improvement on writing the paper.

REFERENCES

- [Amb05] Florin Ambro, *The moduli b -divisor of an lc-trivial fibration*, Compos. Math. **141** (2005), no. 2, 385–403, <https://doi.org/10.1112/S0010437X04001071>. \uparrow 2, 4
- [AT16] Kenneth Ascher and Amos Turchet, *A fibered power theorem for pairs of log general type*, Algebra Number Theory **10** (2016), no. 7, 1581–1600, <https://doi.org/10.2140/ant.2016.10.1581>. \uparrow 3, 5
- [BHPS13] Bhargav Bhatt, Wei Ho, Zsolt Patakfalvi, and Christian Schnell, *Moduli of products of stable varieties*, Compos. Math. **149** (2013), no. 12, 2036–2070, <https://doi.org/10.1112/S0010437X13007288>. \uparrow 5
- [Bru17] Yohan Brunebarbe, *Semi-positivity from Higgs bundles*, arXiv e-prints (2017), arXiv:1707.08495. \uparrow 9
- [Bru18] ———, *Symmetric differentials and variations of Hodge structures*, J. Reine Angew. Math. **743** (2018), 133–161, <https://doi.org/10.1515/crelle-2015-0109>. \uparrow 9
- [CGP17] Junyan Cao, Henri Guenancia, and Mihai Păun, *Variation of singular Kähler-Einstein metrics*, arXiv e-prints (2017), arXiv:1710.01825. \uparrow 2, 4
- [CKS86] Eduardo Cattani, Aroldo Kaplan, and Wilfried Schmid, *Degeneration of Hodge structures*, Ann. of Math. (2) **123** (1986), no. 3, 457–535, <https://doi.org/10.2307/1971333>. \uparrow 15

²As pointed out by Zuo, when the base is a curve, it has already been studied in [VZ06, §3].

- [CKT16] Benoît Claudon, Stefan Kebekus, and Behrouz Taji, *Generic positivity and applications to hyperbolicity of moduli spaces*, arXiv e-prints (2016), arXiv:1610.09832. ↑ 2
- [CP15] Frédéric Campana and Mihai Păun, *Orbifold generic semi-positivity: an application to families of canonically polarized manifolds*, Ann. Inst. Fourier (Grenoble) **65** (2015), no. 2, 835–861, http://aif.cedram.org/item?id=AIF_2015__65_2_835_0. ↑ 10
- [CP19] ———, *Foliations with positive slopes and birational stability of orbifold co-tangent bundles*, Publ. Math. Inst. Hautes Études Sci. **129** (2019), 1–49, <https://doi.org/10.1007/s10240-019-00105-w>. ↑ 2
- [CRT17] Benoit Cadorel, Erwan Rousseau, and Behrouz Taji, *Hyperbolicity of singular spaces*, arXiv e-prints (2017), arXiv:1710.08832. ↑ 10
- [DA19] Ya Deng and Dan Abramovich, *On the hyperbolicity of base spaces for maximally variational families of smooth projective varieties*, <https://hal.archives-ouvertes.fr/hal-01874489>, HAL pré-publication, hal-01874489, version 2. ↑ 2, 4, 7, 8, 10
- [DR18] Ruadhaí Dervan and Julius Ross, *Stable maps in higher dimensions*, Mathematische Annalen (2018), <https://doi.org/10.1007/s00208-018-1706-8>. ↑ 4
- [EV92] Hélène Esnault and Eckart Viehweg, *Lectures on vanishing theorems*, DMV Seminar, vol. 20, Birkhäuser Verlag, Basel, 1992, <https://doi.org/10.1007/978-3-0348-8600-0>. ↑ 11, 14, 15
- [FF17] Osamu Fujino and Taro Fujisawa, *On semipositivity theorems*, arXiv e-prints (2017), arXiv:1701.02039. ↑ 9
- [GKKP11] Daniel Greb, Stefan Kebekus, Sándor J. Kovács, and Thomas Peternell, *Differential forms on log canonical spaces*, Publications mathématiques de l’IHÉS **114** (2011), no. 1, 87–169, <https://doi.org/10.1007/s10240-011-0036-0>. ↑ 13
- [HPS16] Christopher Hacon, Mihnea Popa, and Christian Schnell, *Algebraic fiber spaces over abelian varieties: around a recent theorem by Cao and Paun*, arXiv e-prints (2016), arXiv:1611.08768. ↑ 9
- [Kaw02] Yujiro Kawamata, *On algebraic fiber spaces*, Contemporary trends in algebraic geometry and algebraic topology (Tianjin, 2000), Nankai Tracts Math., vol. 5, World Sci. Publ., River Edge, NJ, 2002, pp. 135–154. ↑ 14
- [Keb13] Stefan Kebekus, *Differential forms on singular spaces, the minimal model program, and hyperbolicity of moduli stacks*, Handbook of moduli. Vol. II, Adv. Lect. Math. (ALM), vol. 25, Int. Press, Somerville, MA, 2013, pp. 71–113. ↑ 2
- [KK08] Stefan Kebekus and Sándor J. Kovács, *Families of canonically polarized varieties over surfaces*, Invent. Math. **172** (2008), no. 3, 657–682, <https://doi.org/10.1007/s00222-008-0128-8>. ↑ 14
- [KMN02] Kazuya Kato, Toshiharu Matsubara, and Chikara Nakayama, *Log C^∞ -functions and degenerations of Hodge structures*, Algebraic geometry 2000, Azumino (Hotaka), Adv. Stud. Pure Math., vol. 36, Math. Soc. Japan, Tokyo, 2002, pp. 269–320. ↑ 14
- [Kob05] Shoshichi Kobayashi, *Hyperbolic manifolds and holomorphic mappings*, second ed., World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2005, <https://doi.org/10.1142/5936>, An introduction. ↑ 1
- [Kol86] János Kollár, *Higher direct images of dualizing sheaves. II*, Ann. of Math. (2) **124** (1986), no. 1, 171–202, <https://doi.org/10.2307/1971390>. ↑ 14, 15
- [KP17] Sándor J. Kovács and Zsolt Patakfalvi, *Projectivity of the moduli space of stable log-varieties and subadditivity of log-Kodaira dimension*, J. Amer. Math. Soc. **30** (2017), no. 4, 959–1021, <https://doi.org/10.1090/jams/871>. ↑ 3
- [Lan86] Serge Lang, *Hyperbolic and Diophantine analysis*, Bull. Amer. Math. Soc. (N.S.) **14** (1986), no. 2, 159–205, <https://doi.org/10.1090/S0273-0979-1986-15426-1>. ↑ 2
- [Lan91] ———, *Number theory. III*, Encyclopaedia of Mathematical Sciences, vol. 60, Springer-Verlag, Berlin, 1991, <https://doi.org/10.1007/978-3-642-58227-1>, Diophantine geometry. ↑ 2, 10
- [Mor87] Shigefumi Mori, *Classification of higher-dimensional varieties*, Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), Proc. Sympos. Pure Math., vol. 46, Amer. Math. Soc., Providence, RI, 1987, pp. 269–331. ↑ 6
- [PS17] Mihnea Popa and Christian Schnell, *Viehweg’s hyperbolicity conjecture for families with maximal variation*, Invent. Math. **208** (2017), no. 3, 677–713, <https://doi.org/10.1007/s00222-016-0698-9>. ↑ 3, 13, 16

- [PT18] Mihai Păun and Shigeharu Takayama, *Positivity of twisted relative pluricanonical bundles and their direct images*, J. Algebraic Geom. **27** (2018), no. 2, 211–272, <https://doi.org/10.1090/jag/702>. ↑ 9
- [PTW18] Mihnea Popa, Behrouz Taji, and Lei Wu, *Brody hyperbolicity of base spaces of certain families of varieties*, arXiv:1801.05898 (2018). ↑ 3, 7, 8
- [PW16] Mihnea Popa and Lei Wu, *Weak positivity for Hodge modules*, Math. Res. Lett. **23** (2016), no. 4, 1139–1155, <https://doi.org/10.4310/MRL.2016.v23.n4.a8>. ↑ 9
- [Rau15] Hossein Raufi, *Singular hermitian metrics on holomorphic vector bundles*, Ark. Mat. **53** (2015), no. 2, 359–382, <https://doi.org/10.1007/s11512-015-0212-4>. ↑ 9
- [Rou16] Erwan Rousseau, *Hyperbolicity, automorphic forms and Siegel modular varieties*, Ann. Sci. Éc. Norm. Supér. (4) **49** (2016), no. 1, 249–255, <https://doi.org/10.24033/asens.2281>. ↑ 10
- [Sch73] Wilfried Schmid, *Variation of Hodge structure: the singularities of the period mapping*, Invent. Math. **22** (1973), 211–319, <https://doi.org/10.1007/BF01389674>. ↑ 15
- [Ste77] Joseph Steenbrink, *Mixed Hodge structure on the vanishing cohomology*, Real and complex singularities (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976), Sijthoff and Noordhoff, Alphen aan den Rijn, 1977, pp. 525–563. ↑ 14
- [Taj18] Behrouz Taji, *Remarks on the Kodaira dimension of base spaces of families of manifolds*, arXiv e-prints (2018), arXiv:1809.05616. ↑ 14
- [Vie83] Eckart Viehweg, *Weak positivity and the additivity of the Kodaira dimension for certain fibre spaces*, Algebraic varieties and analytic varieties (Tokyo, 1981), Adv. Stud. Pure Math., vol. 1, North-Holland, Amsterdam, 1983, pp. 329–353. ↑ 4, 6
- [VZ02] Eckart Viehweg and Kang Zuo, *Base spaces of non-isotrivial families of smooth minimal models*, Complex geometry (Göttingen, 2000), Springer, Berlin, 2002, pp. 279–328. ↑ 3, 8, 9, 10, 11, 16
- [VZ03] ———, *On the Brody hyperbolicity of moduli spaces for canonically polarized manifolds*, Duke Math. J. **118** (2003), no. 1, 103–150, <https://doi.org/10.1215/S0012-7094-03-11815-3>. ↑ 3, 8, 16
- [VZ06] ———, *Numerical bounds for semi-stable families of curves or of certain higher-dimensional manifolds*, J. Algebraic Geom. **15** (2006), no. 4, 771–791, <https://doi.org/10.1090/S1056-3911-05-00423-6>. ↑ 16
- [Wei17] Chuanhao Wei, *Logarithmic Kodaira dimension and zeros of holomorphic log-one-forms*, arXiv e-prints (2017), arXiv:1711.05854. ↑ 16
- [WW18] Chuanhao Wei and Lei Wu, *Hyperbolicity for log smooth families with maximal variation*, arXiv e-prints (2018), arXiv:1811.07466. ↑ 5, 13, 16
- [Yam19] Katsutoshi Yamanoi, *Pseudo Kobayashi hyperbolicity of subvarieties of general type on abelian varieties*, J. Math. Soc. Japan **71** (2019), no. 1. ↑ 10
- [Zuc84] Steven Zucker, *Degeneration of Hodge bundles (after Steenbrink)*, Topics in transcendental algebraic geometry (Princeton, N.J., 1981/1982), Ann. of Math. Stud., vol. 106, Princeton Univ. Press, Princeton, NJ, 1984, pp. 121–141. ↑ 14, 16
- [Zuo00] Kang Zuo, *On the negativity of kernels of Kodaira-Spencer maps on Hodge bundles and applications*, Asian J. Math. **4** (2000), no. 1, 279–301, <https://doi.org/10.4310/AJM.2000.v4.n1.a17>, Kodaira’s issue. ↑ 9

DEPARTMENT OF MATHEMATICAL SCIENCES, CHALMERS UNIVERSITY OF TECHNOLOGY AND UNIVERSITY OF GOTHENBURG, SWEDEN

E-mail address: dengya.math@gmail.com

URL: <https://sites.google.com/site/dengyamath>