



# Inference for continuous-time long memory randomly sampled processes

Mohamedou Ould Haye, Anne Philippe, Caroline Robet

## ► To cite this version:

Mohamedou Ould Haye, Anne Philippe, Caroline Robet. Inference for continuous-time long memory randomly sampled processes. 2021. hal-02266684v2

**HAL Id: hal-02266684**

**<https://hal.science/hal-02266684v2>**

Preprint submitted on 8 Oct 2021

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Inference for continuous-time long memory randomly sampled processes

Mohamedou Ould Haye<sup>1</sup>, Anne Philippe<sup>2\*</sup> and Caroline Robet<sup>2</sup>

<sup>1</sup>School of Mathematics and Statistics.

Carleton University, 1125 Colonel By Dr. Ottawa, ON, Canada, K1S 5B6

<sup>2</sup>Laboratoire de Mathématiques Jean Leray,

2 rue de la Houssinière, Université de Nantes, 44 322 Nantes France.

## Abstract

From a continuous-time long memory stochastic process, a discrete-time randomly sampled one is drawn using a renewal sampling process. We establish the existence of the spectral density of the sampled process, and we give its expression in terms of that of the initial process. We also investigate different aspects of the statistical inference on the sampled process. In particular, we obtain asymptotic results for the periodogram, the local Whittle estimator of the memory parameter and the long run variance of partial sums. We mainly focus on Gaussian continuous-time process. The challenge being that the randomly sampled process will no longer be jointly Gaussian.

**Keywords :** Long memory, sampled process, Whittle estimator, periodogram, spectral density, limit theorems, Poisson process, continuous-time Gaussian processes.

## 1 Introduction

Irregularly observed time series occur in many fields such as astronomy, finance, environmental, and biomedical sciences. Discretization of a continuous time process can produce unevenly time series. For example, physiological signals such as electromyography (EMG), electrocardiogram (ECG), as well as heartbeats (see e.g. Bardet and Bertrand (2010)) are measured at non regularly spaced times. In finance, market prices are tick-by-tick data; a tick being happening randomly, depending for instance on transaction prices. Such data constitute an other example of irregularly spaced time series (see e.g. Dacorogna (2001)).

Irregular sampling interval is also used whenever one has some uncertainty surrounding actual dates such as paleoclimatic time series (see e.g. Thomson (2009)), temperature and CO2 measurements data studied by Nieto-Barajas and Sinha (2015). In these instances, we do not control the way data are observed, as they are recorded at irregular time points. A common approach consists in fitting a continuous time process to discrete data (see for instance Jones (1985)).

Statistical tools available to handle unevenly time series are essentially developed for short range dependence (see e.g. Li (2014) and references therein). We can also refer to numerous papers in astronomy, that focus on spectrum estimation (see e.g. Thiebaut and Roques (2005)).

---

\*corresponding author : anne.philippe@univ-nantes.fr

To the best of our knowledge, few results are available when the continuous-time embedding process has a long memory. Actually, long memory statistical inference for continuous-time models is generally built upon a deterministically sampled process (see Tsai and Chan (2005a,b); Chambers (1996); Comte (1996)). However, as in the examples previously cited, in several applied contexts one has to deal with random sampling from a continuous process. Philippe et al. (2021) studied randomly-spaced observations, using a renewal process as a sampling tool. They showed that the intensity of the long memory is preserved when the distribution of sampling intervals has a finite moment, but there are also situations where a reduction of the long memory is observed. Consequently, the continuous time memory parameter cannot be estimated without a prior information on the sampling process. Bardet and Bertrand (2010) studied spectral density estimation of continuous-time Gaussian processes with stationary increments observed at random times.

In this paper, we describe the spectral properties of the resulting discrete-time-indexed randomly sampled process and we provide more explicit expressions for the spectral density of the sampled process. We mention that Philippe and Viano (2010) addressed resampling from a discrete-time process and obtained the existence of the spectral density. However, their spectral density expression is less explicit since it is expressed as a non explicit limit of an integral.

Most of existing long memory inferential techniques assume that the process is a subordinated Gaussian/linear one. Philippe et al. (2021) established a rather surprising characteristic consisting in the loss of the joint-Gaussianity of the sampled process when the original process was Gaussian. Therefore we cannot apply such results to our sampled processes that are neither Gaussian nor strongly linear. We study some aspects of the inference via spectral approaches. In particular, to establish the consistency of long memory parameter's local Whittle estimator using Dalla et al. (2006)'s assumptions for nonlinear long memory processes.

We now describe our sampling model. We start with  $X = (X_t)_{t \in \mathbb{R}^+}$ , a continuous time process and a renewal process  $(T_n)_{n \geq 0}$ . We study the discrete-time indexed process  $Y = (Y_n)_{n \geq 1}$  defined by

$$Y_n = X_{T_n} \quad n = 1, 2, \dots \quad (1)$$

We want to emphasise that the sampling process  $T_n$  is not observed. Throughout this paper, we will assume that, and refer to

$H_X$  :  $X$  is second-order stationary continuous time process with auto-covariance function  $\sigma_X$  and having a spectral density  $f_X$ : for all  $t \in \mathbb{R}$

$$\sigma_X(t) = \int_{-\infty}^{\infty} e^{i\lambda t} f_X(\lambda) d\lambda. \quad (2)$$

$H_T$  :  $(T_n)_{n \geq 0}$  independent of  $X$  and of i.i.d. increments  $T_{j+1} - T_j = \Delta_j \geq 0$  non degenerate with cumulative distribution function  $S$  and we let  $T_0 = 0$ .

We impose this specific initialization  $T_0 = 0$  only to simplify our notations since it implies that  $\Delta_j = T_{j+1} - T_j$  for all  $j \in \mathbb{N}$ . However, all the results remain true if we take  $T_0 = \Delta_0$  and  $\Delta_j = T_j - T_{j-1}$ , for  $j \geq 1$ .

The rest of the paper is organized as follows. Section 2 presents results on the existence of a spectral density for the process  $Y$  when the spectrum of  $X$  is absolutely continuous. We also provide an integral representation of such density. In Section 3, we establish the asymptotic distribution of the normalized periodogram of the sampled process. In Section 4, we show the consistency of  $Y$ -based local Whittle memory estimator. We also study the estimation of the so-called long-run variance.

## 2 Spectral density function of sampled process

Under the assumptions  $H_X$  and  $H_T$ , Philippe et al. (2021) show that if  $X$  is stationary then so is  $Y$ . Moreover, its covariance function is of the following form

$$\sigma_Y(j) = \text{Cov}(Y_1, Y_{j+1}) = \mathbb{E}(\sigma_X(T_j)). \quad (3)$$

Note that the independence of  $X$  and the renewal process imposed in  $H_T$ , is required to get (3). In the next proposition, we prove that the existence of the spectral density is preserved by random sampling and we establish the link between the spectral densities of processes  $X$  and  $Y$ .

**Proposition 1.** *Assume that the continuous-time process  $X$  satisfies  $H_X$  and that  $H_T$  holds. Then, the discrete-time process  $Y$  admits a spectral density and it is given by the following formula*

$$f_Y(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} p(x, \Psi_S(\lambda)) f_X(\lambda) d\lambda, \quad (4)$$

where  $\Psi_S$  is the characteristic function of the cumulative distribution function  $S$  defined in  $H_T$  and

$$p(x, z) = \frac{1 - |z|^2}{|1 - e^{-ix}z|^2}, \quad |z| < 1$$

is the well known Poisson kernel.

*Proof.* According to the stationarity property and (3) proved in Philippe et al. (2021) and the existence of the spectral density  $f_X$  in (2), the covariance function of  $Y$  can be computed via Fubini's theorem as follows:

$$\begin{aligned} \sigma_Y(j) &= \mathbb{E}(\sigma_X(T_j)) = \mathbb{E} \left( \int_{-\infty}^{\infty} e^{i\lambda T_j} f_X(\lambda) d\lambda \right) \\ &= \int_{-\infty}^{\infty} \left( \mathbb{E} \left( e^{i\lambda T_j} \right) \right) f_X(\lambda) d\lambda = \int_{-\infty}^{\infty} (\Psi_S(\lambda))^j f_X(\lambda) d\lambda. \end{aligned} \quad (5)$$

To prove (4), it will suffice to show that for every  $j \geq 0$ ,

$$\sigma_Y(j) = \int_{-\pi}^{\pi} e^{ijx} f_Y(x) dx, \quad (6)$$

as  $f_Y$  defined by (4) is clearly an even function. For this, we will use the following Poisson integral formula for the disk: if  $u$  is an analytic function on the disk  $|z| < 1$  and continuous on  $|z| = 1$  then its real and imaginary parts are harmonic and therefore for  $|z| < 1$ , we have

$$u(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(e^{ix}) p(x, z) dx.$$

Applying the above with  $u(z) = z^j$ , where  $j$  is a fixed nonnegative integer, we get

$$z^j = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ijx} p(x, z) dx, \quad \text{for all } |z| < 1, \quad (7)$$

and since for Lebesgue a.e.  $\lambda$ ,  $|\Psi_S(\lambda)| < 1$  ( $S$  being non degenerated), then for a.e.  $\lambda$ ,

$$(\Psi_S(\lambda))^j = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ijx} p(x, \Psi_S(\lambda)) dx. \quad (8)$$

Also taking  $j = 0$  in (7), we get

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} p(x, z) dx = 1 \quad \text{for all } |z| < 1.$$

Hence, by Fubini's theorem, we see that  $f_Y$ , as given in (4), is integrable on  $[-\pi, \pi]$ . Applying Fubini's theorem once again and substituting (8) in (5), we immediately get (6).  $\square$

The following corollary gives a precise expression of the spectral density of  $Y$  in the most common case of Poisson renewal process.

**Corollary 1.** *Assume that the continuous-time process  $X$  satisfies  $H_X$  and that  $(T_n)$  is a Poisson renewal process with rate 1, independent of  $X$ . If  $\lambda^2 f_X(\lambda)$  is bounded and continuous on the real line then*

$$f_Y(x) = \frac{u(\sin x, 1 - \cos x)}{2(1 - \cos x)}, \quad \text{for a.e. } x \text{ in the interval } (-\pi, \pi), \quad (9)$$

where  $u(x, y)$  is the harmonic function on the upper half plane with boundary condition  $u(x, 0) = x^2 f_X(x)$ . In particular, both spectral densities are equivalent near zero, i.e.,  $f_Y(x) \sim f_X(x)$  as  $x \rightarrow 0$ .

*Proof.* The exponential distribution has characteristic function  $(1 - i\lambda)^{-1}$  and hence from Proposition 1, we can easily derive that

$$f_Y(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \frac{\lambda^2}{(\lambda - \sin x)^2 + (1 - \cos x)^2} \right) f_X(\lambda) d\lambda \quad (10)$$

$$= \frac{1}{2(1 - \cos x)} \frac{1}{\pi} \int_{-\infty}^{\infty} \left( \frac{1 - \cos x}{(\lambda - \sin x)^2 + (1 - \cos x)^2} \right) \lambda^2 f_X(\lambda) d\lambda. \quad (11)$$

In the above we recognise the well known Poisson integral formula for the upper half plane for the function:  $x \mapsto x^2 f_X(x)$ : if  $g$  is continuous and bounded on the real line then the function defined by

$$u(x, y) := \frac{1}{\pi} \int_{-\infty}^{\infty} \left( \frac{y}{(x - \lambda)^2 + y^2} \right) g(\lambda) d\lambda \quad (12)$$

is harmonic on the upper half plane and satisfies  $u(x, 0) = g(x)$  (see for example the result 7.3 on page 147 of Axler et al. (2000)) and  $\frac{u(x, y)}{g(x)} \rightarrow 1$  uniformly in  $x$  as  $y \rightarrow 0$ . Combining (11) and (12) we get the stated result.  $\square$

The next proposition precises the behaviour of the spectral density of sampled process  $Y$  near zero, given in the previous corollary, under mild semi parametric conditions on the spectral density of the original process  $X$ .

**Proposition 2.** *Assume that  $T_n$  is a Poisson process independent of  $X$  with rate 1 and that  $X$  satisfies  $H_X$  with spectral density of the form*

$$f_X(\lambda) = |\lambda|^{-2d} \phi(\lambda), \quad (13)$$

with  $0 < d < 1/2$ ,  $\phi(0) \neq 0$  and  $\phi$  is continuous on  $[-1, 1]$  and differentiable on  $(-1, 1)$ . Then

$$f_Y(x) = |x|^{-2d} f_Y^*(x) \quad (14)$$

with  $f_Y^*$  is positive continuous on  $[-\pi, \pi]$  and

$$f_Y^*(x) = \phi(0) + \frac{\sigma_X(0)}{2\pi} |x|^{2d} + o(|x|^{2d}), \quad \text{as } x \rightarrow 0. \quad (15)$$

*Proof.* Since  $f_Y$  is even, we will consider  $x \in (0, \pi]$ . From (10), we have

$$f_Y(x) = \frac{1}{2\pi} \int_0^\infty \frac{\lambda^2}{(\lambda - \sin x)^2 + (1 - \cos x)^2} f_X(\lambda) d\lambda + \frac{1}{2\pi} \int_0^\infty \frac{\lambda^2}{(\lambda + \sin x)^2 + (1 - \cos x)^2} f_X(\lambda) d\lambda. \quad (16)$$

We study both integrals in (16) near  $x = 0$ .

$$\int_0^\infty \frac{\lambda^2}{(\lambda + \sin x)^2 + (1 - \cos x)^2} f_X(\lambda) d\lambda \xrightarrow{x \rightarrow 0} \int_0^\infty f_X(\lambda) d\lambda = \frac{\sigma_X(0)}{2} \quad (17)$$

since for fixed  $\lambda$ , as  $x \rightarrow 0$ , the integrand (in the left-hand side) clearly increases towards  $f_X(\lambda)$ . Let us deal with the first integral in (16).

$$\begin{aligned} \int_0^\infty \frac{\lambda^2}{(\lambda - \sin x)^2 + (1 - \cos x)^2} f_X(\lambda) d\lambda &= \int_0^{\sin x} \frac{\lambda^2}{(\lambda - \sin x)^2 + (1 - \cos x)^2} f_X(\lambda) d\lambda + \\ &\int_{\sin x}^{2\sin x} \frac{\lambda^2}{(\lambda - \sin x)^2 + (1 - \cos x)^2} f_X(\lambda) d\lambda + \int_{2\sin x}^\infty \frac{\lambda^2}{(\lambda - \sin x)^2 + (1 - \cos x)^2} f_X(\lambda) d\lambda. \end{aligned} \quad (18)$$

Using the fact that  $f_X(\lambda) = \lambda^{-2d} \phi(\lambda)$  and  $\sin^2(x/2) = (1 - \cos x)/2$  and putting  $\lambda = t \sin x$ , we obtain for the first integral in the right hand side above, with some  $u(t) \in (0, 1)$  and  $v(t) \in (0, 1)$ ,

$$\begin{aligned} \int_0^{\sin x} \frac{\lambda^2}{(\lambda - \sin x)^2 + 4(\sin(x/2))^4} f_X(\lambda) d\lambda &= (\sin x)^{-2d} \int_0^1 \frac{t^{2-2d} \sin x}{(1-t)^2 + \tan^2(x/2)} \phi(t \sin x) dt \\ &= (\sin x)^{-2d} \int_0^1 \frac{(1-t)^{2-2d} \sin x}{t^2 + \tan^2(x/2)} \phi((1-t) \sin x) dt \\ &= (\sin x)^{-2d} \int_0^1 (1 - (2-2d)(1-u(t))^{1-2d} t) \frac{\sin x}{t^2 + \tan^2(x/2)} (\phi(0) + \phi'(v(t)) \sin x (1-t)) dt \\ &= (\sin x)^{-2d} \left[ \int_0^1 \frac{\phi(0) \sin x}{t^2 + \tan^2(x/2)} dt + O \left( \sin x \int_0^1 \frac{t}{t^2 + \tan^2(x/2)} dt + \sin x \int_0^1 \frac{\sin x}{t^2 + \tan^2(x/2)} dt \right) \right]. \end{aligned}$$

Putting  $t = u \tan(x/2)$  the right-hand-side of the last equation is equal to

$$\begin{aligned} &(\sin x)^{-2d} \left[ 2\phi(0) \cos^2(x/2) \int_0^{1/\tan(x/2)} \frac{1}{u^2 + 1} du + \right. \\ &\quad \left. O \left( \sin x \int_0^1 \frac{t}{t^2 + \tan^2(x/2)} dt + \sin x \int_0^1 \frac{\sin x}{t^2 + \tan^2(x/2)} dt \right) \right] \\ &= (\sin x)^{-2d} [2\phi(0) \cos^2(x/2) \arctan(1/\tan(x/2)) + O(x(\log(1+x^2) - \log x) + 2x \arctan(2/x))] \end{aligned}$$

Then

$$\int_0^{\sin x} \frac{\lambda^2}{(\lambda - \sin x)^2 + 4(\sin(x/2))^4} f_X(\lambda) d\lambda = x^{-2d} (\phi(0)\pi + O(x \log x)), \quad \text{as } x \rightarrow 0. \quad (19)$$

Similarly, we have

$$\begin{aligned} \int_{\sin x}^{2 \sin x} \frac{\lambda^2}{(\lambda - \sin x)^2 + (\sin x/2)^4} f_X(\lambda) d\lambda &= (\sin x)^{-2d} \int_1^2 \frac{t^{2-2d} \sin x}{(1-t)^2 + \tan^2(x/2)} \phi(t \sin x) dt \\ &= (\sin x)^{-2d} \int_0^1 \frac{(1+t)^{2-2d} \sin x}{t^2 + \tan^2(x/2)} \phi((1+t) \sin x) dt \\ &= (\sin x)^{-2d} \int_0^1 (1 + (2-2d)(1+u(t))^{1-2d} t) \frac{\sin x}{t^2 + \tan^2(x/2)} (\phi(0) + \phi'(v(t)) \sin x(1+t)) dt \\ &= x^{-2d} (\phi(0)\pi + O(x \log x)). \end{aligned} \quad (20)$$

Then, we have as  $x \rightarrow 0$

$$\int_{2 \sin x}^{\infty} \frac{\lambda^2}{(\lambda - \sin x)^2 + \tan^2(x/2)} f_X(\lambda) d\lambda \rightarrow \int_0^{\infty} f_X(\lambda) d\lambda = \frac{\sigma_X(0)}{2}, \quad (21)$$

since the integrand is bounded uniformly in  $x$  by  $4f_X(\lambda)$  and converges (as  $x \rightarrow 0$ ) to  $f_X(\lambda)$  and hence we can apply Lebesgue's theorem. Combining (17) and (21) as well as (19) and (20), we obtain that

$$f_Y(x) = x^{-2d} f_Y^*(x), \quad f_Y^*(x) = \phi(0) + \frac{\sigma_X(0)}{2\pi} x^{2d} + o(x^{2d}) \quad \text{as } x \rightarrow 0.$$

Moreover,  $f_Y^*$  is continuous and positive on  $[-\pi, \pi]$ . Indeed, the continuity of  $f_Y^*$  follows from the fact that the 2nd integrand in the right hand side of (16) is continuous and uniformly bounded in  $x$  by  $4f_X(\lambda)$  which is integrable. As for the first integral in the right hand side of (16), after splitting it into three terms as in (18) and multiplying it by  $x^{2d}$ , we see that Lebesgue's dominated convergence theorem still applies. This completes the proof of Proposition 2.  $\square$

We now present a lemma that gives a quite precise expression of the covariance function of  $X$  from its spectral density. We will be imposing the following condition on  $f_X$ .

**Condition  $H_f$ :**  $f_X(\lambda) = c|\lambda|^{-2d}(1 - h(\lambda))$ ,  $0 < d < 1/2$ , where  $h$  is a nondecreasing function with  $h(0) = 0$  and  $h(x) \rightarrow 1$  as  $x \rightarrow \infty$  and  $h$  is differentiable at 0. We notice that condition  $H_f$  is not one of the usual slowly varying type conditions for Tauberian and Abelian theorems in the context of long range dependence (see Leonenko and Olenko (2013)). However, it guarantees a uniform control of the remainder  $g(x)$  in (22) rather than at infinity only.

**Remark 1.** If the spectral density  $f_X$  satisfies  $H_f$  instead of (13), then Proposition 2 still holds with  $c := c(d)$  instead of  $\phi(0)$ . The proof is essentially the same and is omitted.

**Lemma 1.** Assume that condition  $H_f$  is satisfied. Then, there exist positive constants  $C(d)$  and  $c(d)$  such that for all  $x > 0$ ,

$$\sigma_X(x) = c(d)x^{2d-1} + g(x), \quad (22)$$

with  $|g(x)| \leq \frac{C(d)}{|x|}$ .

*Proof.* Let  $x > 0$  be fixed. Since  $f_X$  is even we have,

$$\sigma_X(x) = 2 \int_0^\infty \cos(x\lambda) f_X(\lambda) d\lambda.$$

Without loss of generality, we take  $2c = 1$  in  $H_f$  and by the formula 3.761.9 of Gradshteyn and Ryzhik (2015)

$$\int_0^\infty \cos(x\lambda) \lambda^{-2d} d\lambda = \Gamma(1 - 2d) \sin(\pi d) x^{2d-1} =: c(d) x^{2d-1}.$$

Therefore, it remains to show that for some  $C(d) > 0$ ,

$$\left| \int_0^\infty \cos(\lambda x) \lambda^{-2d} h(\lambda) d\lambda \right| \leq C(d) x^{-1}.$$

The rest of the proof relies on applying integration by parts for Stieltjes integrals.

Let  $dU(\lambda) = \cos(\lambda x) \lambda^{-2d}$ . We have (by one integration by parts)

$$\begin{aligned} U(t) &= \int_0^t \cos(\lambda x) \lambda^{-2d} d\lambda \\ &= \frac{1}{x} \left[ \lambda^{-2d} \sin(\lambda x) \right]_{\lambda=0}^{\lambda=t} + \frac{2d}{x} \int_0^t \lambda^{-2d-1} \sin(\lambda x) d\lambda \\ &= \frac{1}{x} \left( t^{-2d} \sin(tx) + 2d \int_0^t \lambda^{-2d-1} \sin(\lambda x) d\lambda \right), \end{aligned}$$

clearly  $U$  is bounded and

$$\lim_{t \rightarrow \infty} U(t) = \frac{2d}{x} \int_0^\infty \lambda^{-2d-1} \sin(\lambda x) d\lambda.$$

Using the fact that  $h$  is nondecreasing,  $h(\lambda) \rightarrow 1$ , as  $\lambda \rightarrow \infty$ , and  $h(0) = 0$ , we obtain (via integration by parts at some steps in the calculation below

$$\int_a^b \cos(\lambda x) \lambda^{-2d} h(\lambda) d\lambda = \int_a^b h(\lambda) dU(\lambda) = [U(\lambda) h(\lambda)]_a^b - \int_a^b U(\lambda) dh(\lambda), \quad (23)$$

with

$$[U(\lambda) h(\lambda)]_a^b \xrightarrow[b \rightarrow \infty]{a \rightarrow 0} \frac{2d}{x} \int_0^\infty \lambda^{-2d-1} \sin(\lambda x) d\lambda,$$

$$\begin{aligned} - \int_a^b U(\lambda) dh(\lambda) &= - \frac{1}{x} \int_a^b \left( \lambda^{-2d} \sin(\lambda x) + 2d \int_0^\lambda u^{-2d-1} \sin(ux) du \right) dh(\lambda) \\ &= - \frac{1}{x} \int_a^b \lambda^{-2d} \sin(\lambda x) dh(\lambda) - \frac{2d}{x} \int_a^b \left( \int_0^\lambda u^{-2d-1} \sin(ux) du \right) dh(\lambda), \end{aligned}$$

also

$$\begin{aligned} & - \frac{2d}{x} \int_a^b \left( \int_0^\lambda u^{-2d-1} \sin(ux) du \right) dh(\lambda) \\ &= - \frac{2d}{x} h(b) \int_0^b \lambda^{-2d-1} \sin(\lambda x) d\lambda + \frac{2d}{x} h(a) \int_0^a \lambda^{-2d-1} \sin(\lambda x) d\lambda + \frac{2d}{x} \int_a^b \lambda^{-2d-1} \sin(\lambda x) h(\lambda) d\lambda. \end{aligned}$$



Since

$$-\frac{2d}{x}h(b)\int_0^b\lambda^{-2d-1}\sin(\lambda x)d\lambda + \frac{2d}{x}h(a)\int_0^a\lambda^{-2d-1}\sin(\lambda x)d\lambda \xrightarrow[b\rightarrow\infty]{a\rightarrow 0} -\frac{2d}{x}\int_0^\infty\lambda^{-2d-1}\sin(\lambda x)d\lambda,$$

$\int_a^b U(\lambda)dh(\lambda)$  has the same limit as

$$-\frac{1}{x}\int_a^b\lambda^{-2d}\sin(\lambda x)dh(\lambda) + \frac{2d}{x}\int_a^b\lambda^{-2d-1}\sin(\lambda x)h(\lambda)d\lambda,$$

as  $a \rightarrow 0$  and  $b \rightarrow \infty$ .

$$\begin{aligned} & \left| -\frac{1}{x}\int_a^b\lambda^{-2d}\sin(\lambda x)dh(\lambda) + \frac{2d}{x}\int_a^b\lambda^{-2d-1}\sin(\lambda x)h(\lambda)d\lambda \right| \\ & \leq \frac{1}{x}\int_a^b\lambda^{-2d}dh(\lambda) + \frac{2d}{x}\int_a^b\lambda^{-2d-1}h(\lambda)d\lambda \\ & = \frac{1}{x}\left(h(b)b^{-2d} - h(a)a^{-2d}\right) + \frac{2d}{x}\int_a^b\lambda^{-2d-1}h(\lambda)d\lambda + \frac{2d}{x}\int_a^b\lambda^{-2d-1}h(\lambda)d\lambda \\ & = \frac{1}{x}\left(h(b)b^{-2d} - h(a)a^{-2d}\right) + \frac{4d}{x}\int_a^b\lambda^{-2d-1}h(\lambda)d\lambda \xrightarrow[b\rightarrow 0]{a\rightarrow 0} \frac{4d}{x}\int_0^\infty\lambda^{-2d-1}h(\lambda)d\lambda := \frac{C(d)}{x}. \end{aligned}$$

We note that the integral above is indeed finite since  $h$  is a bounded function,  $h(0) = 0$ , and is differentiable at zero.

The proof of Lemma 1 is now complete.  $\square$

**Corollary 2.** *If  $T_n$  is a Poisson process and  $f_X$  satisfies condition  $H_f$  then*

$$\text{Var}(\sigma_X(T_r)) = O(r^{-\alpha}), \quad \text{as } r \rightarrow \infty,$$

where  $\alpha = \min(2, 3 - 4d)$ .

*Proof.* We have from the previous lemma,

$$\begin{aligned} \text{Var}(\sigma_X(T_r)) &= \text{Var}\left(c(d)T_r^{2d-1} + g(T_r)\right) \\ &= c^2(d)\text{Var}\left(T_r^{2d-1}\right) + \text{Var}(g(T_r)) + 2c(d)\text{Cov}\left(T_r^{2d-1}, g(T_r)\right) \\ &\leq c^2(d)\text{Var}\left(T_r^{2d-1}\right) + C^2(d)\mathbb{E}(T_r^{-2}) + 2c(d)C(d)\left[\text{Var}\left(T_r^{2d-1}\right)\right]^{1/2}\left[\mathbb{E}(T_r^{-2})\right]^{1/2}. \end{aligned}$$

For  $r \geq 3$ , as  $T_r$  has Gamma distribution with parameters  $(r, 1)$ , we have

$$\mathbb{E}(T_r^{-2}) = \int_0^\infty \frac{x^{r-2-1}}{\Gamma(r)}e^{-x}dx = \frac{\Gamma(r-2)}{\Gamma(r)} = \frac{1}{(r-1)(r-2)} = O(r^{-2}).$$

Also,

$$\text{Var}\left(T_r^{2d-1}\right) = \mathbb{E}\left(T_r^{4d-2}\right) - \left(\mathbb{E}\left(T_r^{2d-1}\right)\right)^2 = \frac{\Gamma(r-2+4d)}{\Gamma(r)} - \left(\frac{\Gamma(r-1+2d)}{\Gamma(r)}\right)^2$$

We know that as  $r \rightarrow \infty$ ,

$$\frac{\Gamma(r-a)}{\Gamma(r)} = r^{-a} \left( 1 - \frac{a(-a+1)}{2r} + O\left(\frac{1}{r^2}\right) \right),$$

and therefore we obtain that

$$\text{Var}\left(T_r^{2d-1}\right) = (1-2d)^2 r^{-2(1-2d)-1} + o\left(r^{-2(1-2d)-1}\right) = O\left(n^{-\alpha}\right),$$

which completes the proof of the corollary.  $\square$

### 3 Asymptotic theory of the periodogram

We consider in this section a stationary long memory zero-mean Gaussian process  $X = (X_t)_{t \in \mathbb{R}^+}$  having a spectral density of the form (13). Let  $Y = (X_{T_n})_{n \in \mathbb{N}}$ , where  $(T_n)_{n \in \mathbb{N}}$  is a Poisson process with rate equal 1 (actually any rate will do). As shown in Philippe et al. (2021) and in contrast with the original process  $X$ , while  $Y$  remains marginally normally distributed, it is no longer jointly Gaussian and, as a result,  $Y$  is not a linear process.

In this section, we extend some well-known facts about periodogram properties to the randomly sampled processes  $Y$ . In particular, our main result will be to establish that the normalized periodogram of  $Y$  will asymptotically converge to a weighted  $\chi^2$  distribution.

**Theorem 1.** *Assume that  $X$  is a stationary Gaussian process satisfying  $H_f$  and let  $Y = (X_{T_n})_{n \in \mathbb{N}}$  where  $(T_n)_{n \in \mathbb{N}}$  is a Poisson process with rate equal 1. Let*

$$I_n(\lambda_j) = \frac{1}{2\pi n} \left| \sum_{k=1}^n Y_k e^{ik\lambda_j} \right|^2,$$

*be the periodogram of  $Y_1, \dots, Y_n$  at Fourier frequency  $\lambda_j = 2\pi j/n$  for  $j \in \{1, \dots, \lfloor n/2 \rfloor\}$ . Then, we have for any fixed number of Fourier frequencies  $\nu \geq 1$ , and any  $j_1, \dots, j_\nu \in \{1, \dots, \lfloor n/2 \rfloor\}$  all distinct integers*

$$\left( \frac{I_n(\lambda_{j_1})}{f_Y(\lambda_{j_1})}, \dots, \frac{I_n(\lambda_{j_\nu})}{f_Y(\lambda_{j_\nu})} \right) \xrightarrow{\mathcal{D}} (L_{j_1}(d)[Z_1^2(j_1) + Z_2^2(j_1)], \dots, L_{j_\nu}(d)[Z_1^2(j_\nu) + Z_2^2(j_\nu)]), \quad (24)$$

*where  $(Z_1(1), Z_2(1), \dots, Z_1(\lfloor n/2 \rfloor), Z_2(\lfloor n/2 \rfloor))$  is a zero-mean Gaussian vector, with  $Z_1(j), Z_2(k)$  are independent for all  $j, k = 1, \dots, \lfloor n/2 \rfloor$  and*

$$\text{Var}(Z_1(j)) = \frac{1}{2} - \frac{R_j(d)}{L_j(d)} \quad (25)$$

*and*

$$\text{Var}(Z_2(j)) = \frac{1}{2} + \frac{R_j(d)}{L_j(d)}, \quad (26)$$

*and for  $j \neq k$ ,*

$$\text{Cov}(Z_1(j), Z_1(k)) = \frac{L_{j,k}(d) - R_{j,k}(d)}{\sqrt{L_j(d)L_k(d)}} \quad (27)$$

$$\text{Cov}(Z_2(j), Z_2(k)) = \frac{L_{j,k}(d) + R_{j,k}(d)}{\sqrt{L_j(d)L_k(d)}}, \quad (28)$$

with

$$L_j(d) = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\sin^2(\lambda/2)}{(2\pi j - \lambda)^2} \left| \frac{\lambda}{2\pi j} \right|^{-2d} d\lambda, \quad (29)$$

$$R_j(d) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin^2(\lambda/2)}{(2\pi j - \lambda)(2\pi j + \lambda)} \left| \frac{\lambda}{2\pi j} \right|^{-2d} d\lambda, \quad (30)$$

$$L_{j,k}(d) = \frac{(jk)^d}{\pi} \int_{-\infty}^{\infty} \frac{\sin^2(\lambda/2)}{(2\pi k - \lambda)(2\pi j - \lambda)} \left| \frac{\lambda}{2\pi} \right|^{-2d} d\lambda, \quad (31)$$

and

$$R_{j,k}(d) = \frac{(jk)^d}{\pi} \int_{-\infty}^{\infty} \frac{\sin^2(\lambda/2)}{(2\pi k + \lambda)(2\pi j - \lambda)} \left| \frac{\lambda}{2\pi} \right|^{-2d} d\lambda. \quad (32)$$

*Proof.* We will prove the broader result

$$\begin{aligned} Z_n := & \left( \frac{1}{\sqrt{2\pi n f_Y(\lambda_{j_1})}} \sum_{r=1}^n \cos(r\lambda_{j_1}) X_{T_r}, \frac{1}{\sqrt{2\pi n f_Y(\lambda_{j_1})}} \sum_{r=1}^n \sin(r\lambda_{j_1}) X_{T_r}, \dots, \right. \\ & \left. \frac{1}{\sqrt{2\pi n f_Y(\lambda_{j_\nu})}} \sum_{r=1}^n \cos(r\lambda_{j_\nu}) X_{T_r}, \frac{1}{\sqrt{2\pi n f_Y(\lambda_{j_\nu})}} \sum_{r=1}^n \sin(r\lambda_{j_\nu}) X_{T_r} \right) \\ & \xrightarrow{\mathcal{D}} \left( \sqrt{L_{j_1}(d)}(Z_1(j_1), Z_2(j_1)), \dots, \sqrt{L_{j_\nu}(d)}(Z_1(j_\nu), Z_2(j_\nu)) \right). \end{aligned} \quad (33)$$

Conditionally on  $T_1, \dots, T_n$ , the vector  $(X_{T_1}, \dots, X_{T_n})$  is Gaussian, and hence so is  $Z_n$ . Its covariance matrix  $\Sigma_T = \text{Var}(Z_n | T_1, \dots, T_n)$  has  $(i, k)$  entry of the form

$$\frac{1}{2\pi n \sqrt{f_Y(\lambda_{j_i}) f_Y(\lambda_{j_k})}} \sum_{r=1}^n \sum_{s=1}^n \sigma_X(T_r - T_s) h_{i,k}(r, s)$$

where

$$h_{i,k}(r, s) = \cos(r\lambda_{j_i}) \cos(s\lambda_{j_k}), \text{ or } \cos(r\lambda_{j_i}) \sin(s\lambda_{j_k}), \text{ or } \sin(r\lambda_{j_i}) \sin(s\lambda_{j_k}).$$

We prove (33) using the characteristic function: since  $X$  and  $T$  are independent, for  $u \in \mathbb{R}^{2\nu}$ , and with  $u'$  being the transpose of  $u$ ,

$$\mathbb{E}(e^{iu'Z_n}) = \mathbb{E} \left( \mathbb{E} \left( e^{iu'Z_n} \middle| T_1, \dots, T_n \right) \right) = \mathbb{E} \left( \exp \left( -\frac{1}{2} u' \Sigma_T u \right) \right).$$

As the characteristic function is bounded, it will suffice to show that

$$\Sigma_T \xrightarrow{P} \Sigma, \quad (34)$$

where  $\Sigma$  is the variance-covariance matrix of  $(\sqrt{L_1(j_1)}(Z_1(j_1), Z_2(j_1)), \dots, \sqrt{L_\nu(j_\nu)}(Z_1(j_\nu), Z_2(j_\nu)))$ .

When  $i$  and  $k$  are fixed, the form of  $h_{i,k}(r, s)$  is the same for all  $r$  and  $s$  and hence  $\mathbb{E}(\Sigma_T)$  will have entries of the form

$$\frac{1}{2\pi n \sqrt{f_Y(\lambda_{j_i}) f_Y(\lambda_{j_k})}} \sum_{r=1}^n \sum_{s=1}^n \mathbb{E}(\sigma_X(T_r - T_s)) h_{i,k}(r, s) = \frac{1}{2\pi n \sqrt{f_Y(\lambda_{j_i}) f_Y(\lambda_{j_k})}} \sum_{r=1}^n \sum_{s=1}^n \sigma_Y(r - s) h_{i,k}(r, s)$$

by (3), and therefore  $\mathbb{E}(\Sigma_T) \rightarrow \Sigma$  by virtue of Theorem 5 of Hurvich and Beltrão (1993) (the only condition required is second order stationarity of the process  $Y_i$  and the behaviour (14) of its spectral density). To complete the proof of (34), it will then suffice to show that

$$\text{Var}(\Sigma_T) \rightarrow 0, \quad (35)$$

i.e. the variances of the entries of  $\Sigma_T$  converge to zero. By Cauchy-Schwarz inequality, it will be enough to focus on the diagonals. We will treat those diagonals with cosine, as those with sine treat the same way. For some constant  $C$  (that may change from one expression to another), we obtain

$$\begin{aligned} & \text{Var} \left( \frac{1}{2\pi n f_Y(\lambda_j)} \sum_{r=1}^n \sum_{s=1}^n \sigma_X(T_r - T_s) \cos(r\lambda_j) \cos(s\lambda_j) \right) \\ & \sim \frac{C}{n^{2+4d}} \sum_{r,s,r',s'=1}^n \text{Cov}(\sigma_X(T_r - T_s) \cos(r\lambda_j) \cos(s\lambda_j), \sigma_X(T_{r'} - T_{s'}) \cos(r'\lambda_j) \cos(s'\lambda_j)) \\ & \leq \frac{C}{n^{2+4d}} \left( \sum_{r=1}^n \sum_{s=1}^n \sqrt{\text{Var}(\sigma_X(T_r - T_s))} \right)^2 \leq \frac{C}{n^{4d}} \left( \sum_{h=1}^n \sqrt{\text{Var}(\sigma_X(T_h))} \right)^2 \leq C \frac{n^{2d}}{n^{4d}} \rightarrow 0, \end{aligned} \quad (36)$$

using Corollary 2.  $\square$

## 4 Inference for the long-memory parameter

We still assume in this section that  $X = (X_t)_{t \in \mathbb{R}^+}$  is a stationary long memory zero-mean Gaussian process having a spectral density satisfying  $H_f$  condition. Periodogram-based approaches to estimate the long memory parameter  $d$  are very popular. Often one requires that the underlying process is linear or at least is built on martingale difference innovations. The reader is referred to Beran et al. (2013); Giraitis et al. (2012) for reviews of some recent works on this issue, as well as the book edited by Doukhan et al. (2003). The next lemma and its proof show that although it is not a linear process with i.i.d. innovations, the sampled process still satisfies important long memory 4th cumulant conditions. These 4th cumulant conditions will allow us to show both the convergence of an estimator of the memory parameter  $d$  and the estimation of the asymptotic variance, necessary for example in the inference about the mean of the original continuous time process  $X$ .

**Lemma 2.** *Assume that  $X$  is a zero mean stationary Gaussian process satisfying  $H_f$  and let  $Y = (X_{T_n})_{n \in \mathbb{N}}$ , where  $(T_n)_{n \in \mathbb{N}}$  is a Poisson process. Then for all  $d \in (0, 1/2)$ , we have*

$$\sup_{h \in \mathbb{N}} \sum_{r,s=0}^n |\text{cum}(Y_0, Y_h, Y_r, Y_s)| = O(n^{2d}). \quad (37)$$

and

$$\sum_{h,r,s=0}^n |\text{cum}(Y_0, Y_h, Y_r, Y_s)| = O(n^{4d} \log(n)). \quad (38)$$

*Proof.* The proof is postponed in Appendix. Note that the term  $\log(n)$  in the right hand side of (38) is needed only in the particular case  $d = 1/4$ , known to be borderline between weak long memory and strong long memory, as will be seen in the proof.  $\square$

#### 4.1 Consistency of Local Whittle estimator

We consider local Whittle estimator of the memory parameter  $d$  defined by

$$\hat{d}_n = \underset{\beta \in [-1/2, 1/2]}{\operatorname{argmin}} U_n(\beta)$$

where the contrast function  $U_n$  is defined by

$$U_n(\beta) = \log \left( \frac{1}{m_n} \sum_{j=1}^{m_n} \lambda_j^{2\beta} I_n(\lambda_j) \right) - \frac{2\beta}{m_n} \sum_{j=1}^{m_n} \log \lambda_j,$$

and the bandwidth parameter  $m = m_n$  satisfies  $m_n \rightarrow \infty$  and  $m_n = o(n)$ .

**Theorem 2.** *Suppose  $X$  is a stationary Gaussian process satisfying condition  $H_f$  and that  $Y_n = X_{T_n}$ , where  $T_n$  is a Poisson process with rate 1. Then,*

$$\hat{d}_n \xrightarrow[n \rightarrow \infty]{P} d. \quad (39)$$

In addition, for  $m_n = n^a$ ,  $0 < a < 1$ , we have

$$\hat{d}_n - d = o_P \left( \frac{1}{\log n} \right). \quad (40)$$

*Proof.* According to our result (15) and Dalla et al. (2006) (Corollary 1), we have

$$\hat{d} - d = O_P \left( m^{-1/2} \log m + \left( \frac{m}{n} \right)^{2d} + r_n \right),$$

for some remainder  $r_n$ , which we will be controlling as in what follows, depending on the convergence rate of  $m/n$  to zero. To prove (39) it will suffice to show that  $r_n \rightarrow 0$ .

**Case 1 :**  $\sqrt{n}(\log n)^{4/(1-2d)} = O(m)$ . From part (iv) of Corollary 1 of Dalla et al. (2006), the remainder  $r_n$  can be written as

$$r_n = \left( \frac{D_n^{**}}{n} \right)^{1/2} \left( \frac{n}{m} \right)^{1-2d} \log^3 n \rightarrow 0,$$

where

$$D_n^{**} = \sup_{h, r \in \mathbb{N}} \sum_{s=1}^n |\operatorname{cum}(Y_0, Y_h, Y_r, Y_s)|.$$

We have

$$D_n^{**} \leq \sup_{h \in \mathbb{N}} \sum_{r, s=0}^n |\operatorname{cum}(Y_0, Y_h, Y_r, Y_s)|,$$

so that by (37) we get  $D_n^{**} = O(n^{2d})$  and hence  $r_n = O(1/\log n)$ .

**Case 2 :**  $m = O(\sqrt{n}(\log n)^{4/(1-2d)})$ . We use (iii) of Corollary 1 of Dalla et al. (2006),

$$r_n = \left(\frac{D_n^*}{n}\right)^{1/2} \left(\frac{m}{n}\right)^{2d} \log^2 n \rightarrow 0,$$

where

$$D_n^* = \sum_{h,r,s=0}^n |\text{cum}(Y_0, Y_h, Y_r, Y_s)|.$$

According to (38),  $D_n^* = O(n^{4d} \log n)$ , and therefore

$$r_n = O\left(n^{d-1/2} (\log n)^{2(1+2d)/(1-2d)}\right) \rightarrow 0.$$

This concludes proof of (39).

To prove (40), we show that  $r_n = o(1/\log n)$ . This is immediate in case (2) above. Since  $m = n^a$ ,  $0 < a < 1$ , we will be in case (1) if  $a > 1/2$  and then

$$r_n = O\left(n^{1/2-d-a(1-2d)}\right) / \log n = o(1/\log n).$$

□

## 4.2 Long run variance

The 4th cumulant condition (37) is needed to estimate the long run variance of the sampled process. Such estimation plays a crucial role in many aspects of statistical inference. For example, when it comes to estimating the mean  $\mu$  of the original process  $X$ , as we have from Philippe et al. (2021)

$$\left(\text{Var}\left(n^{1/2-d}\bar{Y}_n\right)\right)^{-1/2} n^{1/2-d}(\bar{Y}_n - \mu) \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1)$$

and hence, it is important to obtain a consistent estimator of the variance above. Also such estimator is important in testing for short memory versus long memory or for stationarity versus unit root as such tests involve V/S type statistics and require estimating the long run variance (see Giraitis et al. (2006) and references therein for details). Let us write the spectral density of  $Y_i$  under the form  $f(\lambda) \sim c|\lambda|^{-2d}$  as  $\lambda \rightarrow 0$ . Let

$$\hat{\sigma}(h) = \frac{1}{n} \sum_{j=1}^{n-h} (Y_j - \bar{Y})(Y_{j+h} - \bar{Y})$$

be the sample covariance function of  $Y_i$ . Let the asymptotic variance of the normalized sum be

$$S^2(d) = \lim_{n \rightarrow \infty} \left(\text{Var}(n^{1/2-d}\bar{Y})\right) = 4c \int_{-\infty}^{\infty} \left(\frac{\sin(\lambda/2)}{\lambda}\right)^2 |\lambda|^{-2d} d\lambda.$$

Let

$$\hat{S}^2(d) = q^{-2d} \left( \hat{\sigma}(0) + 2 \sum_{h=1}^q \left(1 - \frac{h}{q}\right) \hat{\sigma}(h) \right).$$

**Proposition 3.** *Let  $\hat{d}$  be a consistent estimator for memory parameter  $d$  such that  $\log(n)(\hat{d}-d) = o_P(1)$ . Let  $q \rightarrow \infty$  as  $n \rightarrow \infty$  such that  $q = O(\sqrt{n})$ . Then we have*

$$\widehat{S}^2(\hat{d}) \xrightarrow{P} S^2(d).$$

*Proof.* Referring to Theorem 2.2. of Abadir et al. (2009) we just need to verify the cumulant condition

$$\sup_h \sum_{r,s=1}^n |\text{Cum}(Y_0, Y_h, Y_r, Y_s)| \leq \tilde{c}n^{2d},$$

for some positive constant  $\tilde{c}$ . This is the case according to Lemma 2.  $\square$

**Remark 2.** *A readily available candidate for  $\hat{d}$  above is the Whittle estimator for which the  $\log(n)$  consistency was established in Theorem 2.*

## Appendix : Proof of Lemma 2

*Proof.* The proof is essentially based on Corollary 2 and a well known cumulant formula.

Without loss of generality, we can assume that the Poisson rate is 1. The process  $Y$  is 4th order stationary as the conditional joint distribution of  $(Y_k, Y_{k+h}, Y_{k+r}, Y_{k+s})$  given  $(T_1, \dots, T_{k+\max(h,r,s)})$  is a multivariate normal with variance-covariance matrix  $M(T_k, T_{k+h}, T_{k+r}, T_{k+s})$  given by

$$M(T_k, T_{k+h}, T_{k+r}, T_{k+s}) := \begin{pmatrix} \sigma_X(0) & \sigma_X(T_{k+h}-T_k) & \sigma_X(T_{k+r}-T_k) & \sigma_X(T_{k+s}-T_k) \\ \sigma_X(T_{k+h}-T_k) & \sigma_X(0) & \sigma_X(T_{k+r}-T_{k+h}) & \sigma_X(T_{k+s}-T_{k+h}) \\ \sigma_X(T_{k+r}-T_k) & \sigma_X(T_{k+r}-T_{k+h}) & \sigma_X(0) & \sigma_X(T_{k+s}-T_{k+r}) \\ \sigma_X(T_{k+s}-T_k) & \sigma_X(T_{k+s}-T_{k+h}) & \sigma_X(T_{k+s}-T_{k+r}) & \sigma_X(0) \end{pmatrix} \quad (41)$$

which is  $k$  free. Hence it is enough to establish the lemma when  $k = 0$ . We apply the total law of cumulance formula, (Brillinger (1969)), which for the sake of clarity, we remind here: for all random vectors  $Z = (Z_1, \dots, Z_n)'$  and  $W$ , we have

$$\text{cum}(Z) = \sum_{\pi} \text{cum}[\text{cum}(X_{\pi_1}|W), \dots, \text{cum}(X_{\pi_b}|W)] \quad (42)$$

where  $X_{\pi_j} = (X_i, i \in \pi_j)$ , and  $\pi_1, \dots, \pi_b$ , ( $b = 1, \dots, n$ ) are the blocks of the permutation  $\pi$ , and the sum is over all permutations  $\pi$  of the set  $\{1, 2, \dots, n\}$ .

But condition on  $T$ , the process  $Y_t$  is jointly zero-mean Gaussian and therefore  $\mathbb{E}(Y_t|T) = 0$  as well as  $\text{cum}(Y_i, Y_j, Y_k, Y_\ell|T) = \text{cum}(Y_i, Y_j, Y_k|T) = 0$  for all  $i, j, k, \ell$ . Hence applying (42) to  $Y_t$  with  $W = T$ , only the two-by-two partitions of  $\{0, h, r, s\}$  will survive. and since  $\text{cum}(U, V) = \text{Cov}(U, V)$ , we get from (41)

$$\begin{aligned} \text{cum}(Y_0, Y_h, Y_r, Y_s) &= \text{Cov}(\sigma_X(T_h), \sigma_X(T_r - T_s)) + \text{Cov}(\sigma_X(T_r), \sigma_X(T_h - T_s)) \\ &\quad + \text{Cov}(\sigma_X(T_s), \sigma_X(T_r - T_h)). \end{aligned} \quad (43)$$

Note that for  $h < \min(r, s)$ ,  $\text{Cov}(\sigma_X(T_h), \sigma_X(T_r - T_s)) = 0$ . Moreover

$$\begin{aligned}
\sum_{1 \leq r \leq h \leq s \leq n} |\text{Cov}(\sigma_X(T_h), \sigma_X(T_r - T_s))| &\leq \sum_{1 \leq r \leq h \leq s \leq n} \text{Var}(\sigma_X(T_h))^{1/2} \text{Var}(\sigma_X(T_s - T_r))^{1/2} \\
&\leq \sum_{1 \leq r \leq h \leq s \leq n} h^{-\alpha/2} (1 + s - r)^{-\alpha/2} \\
&\leq h^{-\alpha/2} \sum_{1 \leq r \leq h} \sum_{t=1}^n t^{-\alpha/2} \\
&\leq h^{1-\alpha/2} \begin{cases} n^{1-\alpha/2} = n^{2d-1/2} & \text{if } d < 1/4 \\ \log(n) & \text{if } d \geq 1/4 \end{cases} \\
&\leq \begin{cases} n^{4d-1} & \text{if } d < 1/4 \\ \log(n) & \text{if } d \geq 1/4 \end{cases} \\
&\leq Cn^{2d} \text{ for all } 0 < d < 1/2.
\end{aligned}$$

The last configuration is

$$\begin{aligned}
\sum_{r,s=1}^h |\text{Cov}(\sigma_X(T_h), \sigma_X(T_r - T_s))| &= \sum_{r,s=1}^h |\text{Var}(\sigma_X(T_h))^{1/2} \text{Var}(\sigma_X(T_s - T_r))^{1/2}| \\
&\leq h^{-\alpha/2} \sum_{t=1}^h (h-t) t^{-\alpha/2} \\
&\leq h^{-\alpha/2} \begin{cases} Ch^{1-\alpha/2} = Ch^{2d-1/2} & \text{if } d < 1/4 \\ \log(h) & \text{if } d \geq 1/4 \end{cases} \\
&\leq Cn^{2d} \text{ for all } 0 < d < 1/2.
\end{aligned}$$

Therefore uniformly in  $h$  we have

$$\sum_{r,s=1}^n |\text{Cov}(\sigma_X(T_h), \sigma_X(T_r - T_s))| \leq Cn^{2d}.$$

For the remaining two terms in the right hand side of (43) we have, for fixed  $h$ ,

$$\begin{aligned}
\sum_{r,s=1}^n |\text{Cov}(\sigma_X(T_r), \sigma_X(T_h - T_s))| &= \sum_{r,s=1}^n |\text{Cov}(\sigma_X(T_s), \sigma_X(T_h - T_r))| \\
&\leq \sum_{r,s=1}^n \text{Var}(\sigma_X(T_s))^{1/2} \text{Var}(\sigma_X(T_h - T_r))^{1/2} \\
&\leq \sum_{r,s=1}^n s^{-\alpha/2} (1 + |h - r|)^{-\alpha/2} \\
&\leq C \begin{cases} n^{2-\alpha} = n^{4d-1} & \text{if } d < 1/4 \\ \log(n)^2 & \text{if } d \geq 1/4 \end{cases} \\
&\leq Cn^{2d} \text{ for all } 0 < d < 1/2.
\end{aligned}$$



This concludes the proof of (37).  
Let us now prove (38). Note that

$$\begin{aligned} \sum_{h,r,s=0}^n \text{cum}(Y_0, Y_h, Y_r, Y_s) &= 3 \sum_{h,r,s=1}^n \text{Cov}(\sigma_X(T_h), \sigma_X(T_r - T_s)) \\ &= 6 \sum_{h=1}^n \sum_{r < s=1}^n \text{Cov}(\sigma_X(T_h), \sigma_X(T_r - T_s)). \end{aligned} \quad (44)$$

Moreover, we have

$$\begin{aligned} \sum_{h,r,s=1}^n |\text{Cov}(\sigma_X(T_h), \sigma_X(T_r - T_s))| &\leq C \sum_{h,r,s=1}^n h^{-\alpha/2} (1 + |r - s|)^{-\alpha/2} \\ &\leq C \sum_{h=1}^n h^{-\alpha/2} \sum_{t=1}^n (n - t) t^{-\alpha/2} \\ &\leq C \begin{cases} nn^{2-\alpha} = n^{4d} & \text{if } d < 1/4 \\ n \log(n)^2 & \text{if } d > 1/4 \end{cases} \\ &\leq C n^{4d} \end{aligned}$$

In the particular case  $d = 1/4$  (where we still have  $\alpha = 2$ ), a supplementary term  $\log(n)$  is needed in the bound. Indeed we split the sum in the right hand side of (44) into 3 configurations. when  $1 \leq h \leq r < s \leq n$  the covariance  $\text{Cov}(\sigma_X(T_h), \sigma_X(T_r - T_s))$  is zero. When the sum is over  $1 \leq r < h \leq s \leq n$ , we get

$$\begin{aligned} \sum_{1 \leq r < h \leq s \leq n} |\text{Cov}(\sigma_X(T_h), \sigma_X(T_r - T_s))| &\leq C \sum_{s=1}^n \sum_{h=1}^s h^{-1} \sum_{r=1}^{h-1} (s - h + h - r)^{-1} \\ &\sim C \sum_{s=1}^n \sum_{h=1}^s h^{-1} (\log(s) - \log(s - h)) \\ &= -C \sum_{s=1}^n \sum_{h=1}^s (h/s)^{-1} \log(1 - h/s) (1/s) \\ &\sim -C \sum_{s=1}^n \left( \int_0^1 \frac{\log(1-x)}{x} dx \right) = C \frac{\pi^2}{6} n. \end{aligned}$$

For the last sum over  $1 \leq r < s \leq h \leq n$  (where we will need the log term) we have

$$\begin{aligned}
\sum_{1 \leq r < s \leq h \leq n} |\text{Cov}(\sigma_X(T_h), \sigma_X(T_r - T_s))| &\leq C \sum_{h=1}^n h^{-1} \sum_{s=1}^h \sum_{r=1}^{s-1} (s-r)^{-1} \\
&= \sum_{h=1}^n h^{-1} \sum_{t=1}^h (h-t)t^{-1} \\
&= \sum_{h=1}^n \sum_{t=1}^h (1-t/h)t^{-1} \\
&\sim C \sum_{h=1}^n (\log(h) - 1) \sim Cn \log(n).
\end{aligned}$$

This completes the proof of (38) in Lemma 2. □

## References

- Abadir, K. M., Distaso, W., and Giraitis, L. (2009). Two estimators of the long-run variance: beyond short memory. *J. Econometrics*, 150(1):56–70.
- Axler, S., Bourdon, P., and Ramey, W. (2000). *Harmonic function theory*. Graduate Texts in Mathematics 137. Springer.
- Bardet, J.-M. and Bertrand, P. (2010). A non-parametric estimator of the spectral density of a continuous-time gaussian process observed at random times. *Scandinavian Journal of Statistics*, 37(3):458–476.
- Beran, J., Feng, Y., Ghosh, S., and Kulik, R. (2013). *Long-memory processes*. Springer, Heidelberg. Probabilistic properties and statistical methods.
- Brillinger, D. R. (1969). The calculation of cumulants via conditioning. *Annals of the Institute of Statistical Mathematics*, 21(1):215–218.
- Chambers, M. J. (1996). The estimation of continuous parameter long-memory time series models. *Econometric Theory*, 12(2):374–390.
- Comte, F. (1996). Simulation and estimation of long memory continuous time models. *J. Time Ser. Anal.*, 17(1):19–36.
- Dacorogna, M. M. (2001). *An Introduction to High-Frequency Finance*. Academic Press.
- Dalla, V., Giraitis, L., and Hidalgo, J. (2006). Consistent estimation of the memory parameter for nonlinear time series. *J. Time Ser. Anal.*, 27(2):211–251.
- Doukhan, P., Oppenheim, G., and Taqqu, M. (2003). *Theory and Applications of Long-Range Dependence*. Birkhäuser, Boston.

- Giraitis, L., Koul, H. L., and Surgailis, D. (2012). *Large Sample Inference for Long Memory Processes*. Imperial College Press, London.
- Giraitis, L., Leipus, R., and Philippe, A. (2006). A test for stationarity *versus* trends and unit roots for a wide class of dependent errors. *Econometric Theory*, 22:989–1029.
- Gradshteyn, I. S. and Ryzhik, I. M. (2015). *Table of integrals, series, and products*. Elsevier/Academic Press, Amsterdam, eighth edition. Translated from the Russian, Translation edited and with a preface by Daniel Zwillinger and Victor Moll, Revised from the seventh edition [MR2360010].
- Hurvich, C. M. and Beltrão, K. I. (1993). Asymptotics for the low-frequency ordinates of the periodogram of a long-memory time series. *J. Time Ser. Anal.*, 14(5):455–472.
- Jones, R. H. (1985). 5 time series analysis with unequally spaced data. *Handbook of statistics*, 5:157–177.
- Leonenko, N. and Olenko, A. (2013). Tauberian and abelian theorems for long-range dependent random fields. *Methodology and Computing in Applied Probability*, 15(4):715–742.
- Li, Z. (2014). *Methods for irregularly sampled continuous time processes*. PhD thesis, University College of London.
- Nieto-Barajas, L. E. and Sinha, T. (2015). Bayesian interpolation of unequally spaced time series. *Stochastic Environmental Research and Risk Assessment*, 29(2):577–587.
- Philippe, A., Robet, C., and Viano, M.-C. (2021). Random discretization of stationary continuous time processes. *Metrika*, 84(3):375–400.
- Philippe, A. and Viano, M.-C. (2010). Random sampling of long-memory stationary processes. *Journal of Statistical Planning and Inference*, 140(5):1110 – 1124.
- Thiebaud, C. and Roques, S. (2005). Time-scale and time-frequency analyses of irregularly sampled astronomical time series. *EURASIP Journal on Advances in Signal Processing*, 2005(15):852587.
- Thomson, D. J. (2009). *Time-Series Analysis of Paleoclimate Data*, pages 949–959. Springer Netherlands, Dordrecht.
- Tsai, H. and Chan, K. S. (2005a). Maximum likelihood estimation of linear continuous time long memory processes with discrete time data. *J. R. Stat. Soc. Ser. B Stat. Methodol.*, 67(5):703–716.
- Tsai, H. and Chan, K. S. (2005b). Quasi-maximum likelihood estimation for a class of continuous-time long-memory processes. *J. Time Ser. Anal.*, 26(5):691–713.