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► **To cite this version:**

Titus Lupu. Topological expansion in isomorphisms with random walks for matrix valued fields. 2019. hal-02266669v2

HAL Id: hal-02266669

<https://hal.science/hal-02266669v2>

Preprint submitted on 25 Feb 2020 (v2), last revised 2 Aug 2021 (v3)

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TOPOLOGICAL EXPANSION IN ISOMORPHISMS WITH RANDOM WALKS FOR MATRIX VALUED FIELDS

TITUS LUPU

ABSTRACT. We consider Gaussian fields of real symmetric, complex Hermitian or quaternionic Hermitian matrices over an electrical network, and describe how the isomorphisms with random walks for these fields make appear topological expansions encoded by ribbon graphs. We further consider matrix valued Gaussian fields twisted by an orthogonal, unitary or symplectic connection. In this case the isomorphisms make appear traces of holonomies of the connection along random walk loops parametrized by border cycles of ribbon graphs.

1. INTRODUCTION

It is known since the works of Symanzik [Sym65, Sym66, Sym69] and Brydges, Fröhlich and Spencer [BFS82] that the Gaussian free field (GFF) has representations involving random walks, sometimes referred to as "isomorphisms". For a survey on the subject we refer to [MR06, Szn12]. Here we will be interested in a representation that appears in Brydges, Fröhlich, Spencer [BFS82] and Dynkin [Dyn84a, Dyn84b], that expresses

$$\mathbb{E} \left[\prod_{k=1}^{2r} \phi(x_k) F(\phi^2/2) \right],$$

for ϕ a GFF, in terms of pairings of vertices x_k -s and random walks joining the pairs. Following [BHS19], we will call it the BFS-Dynkin isomorphism.

Kassel and Lévy considered vector valued GFFs twisted by an orthogonal or unitary connection [KL16]. In this setting isomorphism theorems make appear the holonomy of the connection along the random walks. Holonomies along random walk or Brownian loops have been also studied in [LJ17, LJ16, CLJR20].

In this paper we will consider fields of random Gaussian matrices, real symmetric, complex Hermitian or quaternionic Hermitian, on an electrical network. These are matrix valued GFFs. The matrix above any vertex of the network is proportional to a GOE, GUE or GSE matrix. Here we will write an isomorphism for

$$(1.1) \quad \left\langle \left(\prod_{l=1}^{m(\nu)} \text{Tr} \left(\prod_{k=\nu_1+\dots+\nu_{l-1}+1}^{\nu_1+\dots+\nu_l} \Phi(x_k) \right) \right) F(\text{Tr}(\Phi^2)/2) \right\rangle_{\beta, n},$$

where Φ is the matrix valued GFF, $\beta \in \{1, 2, 4\}$, n is the size of the matrices, $\nu_1, \dots, \nu_{m(\nu)}$ are positive integers with $|\nu| := \nu_1 + \dots + \nu_{m(\nu)}$ even, and $x_1, \dots, x_{|\nu|}$ vertices on the network. By taking the x_k -s equal inside each of the trace, we get symmetric polynomials in the eigenvalues. By expanding the traces and the product above, one can write a BFS-Dynkin's isomorphism for each of the terms of the sum. However, one gets many different terms that give identical contributions, many terms with contributions that cancel out, being of opposite sign, and many terms that give zero contribution. By regrouping the terms surviving to cancellation into powers of n , one gets a combinatorial structure known as topological expansion. The terms of the expansion correspond to ribbon graphs with $m(\nu)$ vertices, obtained by pairing and gluing $|\nu|$ ribbon half-edges. Each gluing may be straight or twisted. The power of n is then given by

2010 *Mathematics Subject Classification.* 60G15, 81T18, 81T25 (primary), and 60B20, 60J55 (secondary).

Key words and phrases. discrete gauge theory, Gaussian free field, Gaussian orthogonal ensemble, Gaussian symplectic ensemble, Gaussian unitary ensemble, isomorphism theorems, holonomy, matrix integrals, matrix models, random matrices, random walks, ribbon graphs, topological expansion, Wilson loops.

the number of cycles formed by the borders of the ribbons. It can be also expressed in terms of genera of compact surfaces, orientable or not.

The topological expansion has been introduced by 't Hooft for the study of Quantum Chromodynamics [tH74], and further developed by Brézin, Itzykson, Parisi and Zuber [BIPZ78, IZ80]. Nowadays there is a broad, primarily physics literature on this topic. In particular, topological expansion of one matrix or several matrix integrals is used for the enumeration of maps on surfaces and other graphical objects [BIZ80, Zvo97, LZ04, Eyn16]. Compared to the case of one matrix integrals, where each ribbon edge comes only with a scalar weight, in our setting each ribbon edge will be associated to a measure on random walk paths between two vertices x_k and $x_{k'}$ on the network. For an introduction to the topological expansion we refer [Zvo97, EKR18].

We will further extend our framework and consider matrix valued free fields twisted by a connection of orthogonal ($\beta = 1$), unitary ($\beta = 2$) or symplectic ($\beta = 4$) matrices. We rely for this on results of Kassel and Lévy for twisted vector valued GFFs [KL16]. If Φ is the matrix valued GFF twisted by a connection U and $(\lambda_1, \dots, \lambda_n)$ its fields of eigenvalues, then the isomorphism for

$$\left\langle \prod_{l=1}^{m(\nu)} \left(\sum_{i=1}^n \lambda_i (x_{\nu_1 + \dots + \nu_l})^{\nu_l} \right) F \left(\frac{1}{2} \sum_{i=1}^n \lambda_i^2 \right) \right\rangle_{\beta, n}^U$$

involves a topological expansion where instead of n to the power the number of cycles in a ribbon graph appears a product of traces of holonomies of the connection, one per each border cycle in the ribbon graph. The holonomies are taken along loops made of concatenated random walk paths. Such traces of holonomies along loops are called Wilson loop observables.

2. PRELIMINARIES

2.1. Quaternions, symplectic matrices and quaternionic Hermitian matrices. \mathbb{H} will denote the skew (non-commutative) field of quaternions. Its elements are of form

$$q = q_r + q_i \mathbf{i} + q_j \mathbf{j} + q_k \mathbf{k},$$

where $q_r, q_i, q_j, q_k \in \mathbb{R}$, and \mathbf{i}, \mathbf{j} and \mathbf{k} satisfy the relations

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1,$$

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}.$$

q_r is the real part $\text{Re}(q)$ of the quaternion q . The algebra of quaternions has the usual representation over 2×2 complex matrices:

$$\mathbf{C}(q) = \begin{pmatrix} q_r + q_i \mathbf{i} & -q_j - q_k \mathbf{i} \\ q_j - q_k \mathbf{i} & q_r - q_i \mathbf{i} \end{pmatrix}.$$

The conjugate of a quaternion is given by

$$\bar{q} = q_r - q_i \mathbf{i} - q_j \mathbf{j} - q_k \mathbf{k}.$$

$\mathbf{C}(\bar{q})$ is the adjoint of $\mathbf{C}(q)$ for the Hermitian inner product:

$$\mathbf{C}(\bar{q}) = \mathbf{C}(q)^*.$$

If $q_1, q_2 \in \mathbb{H}$,

$$\overline{q_1 q_2} = \bar{q}_2 \bar{q}_1.$$

The absolute value $|q|$ is given by

$$|q|^2 = q_r^2 + q_i^2 + q_j^2 + q_k^2 = q \bar{q} = \bar{q} q = \det(\mathbf{C}(q)).$$

For more on quaternions, we refer to [MGS14].

$\mathcal{M}_n(\mathbb{H})$ will denote the ring of $n \times n$ matrices with quaternionic entries. The product of matrices is defined in the same way as for matrices over a commutative field:

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}, \quad i, j \in \{1, \dots, n\}.$$

One can associate to a matrix $M \in \mathcal{M}_n(\mathbb{H})$ a $2n \times 2n$ matrix with complex entries, by replacing each entry M_{ij} by a 2×2 block $\mathbf{C}(M_{ij})$. The resulting matrix in $\mathcal{M}_{2n}(\mathbb{C})$ will be again denoted $\mathbf{C}(M)$. \mathbf{C} is then a morphism of rings from $\mathcal{M}_n(\mathbb{H})$ to $\mathcal{M}_{2n}(\mathbb{C})$.

The trace of a matrix of quaternions is defined as usually:

$$\mathrm{Tr}(M) = \sum_{i=1}^n M_{ii}.$$

However, it is more convenient to deal with the real part of the trace, as

$$\mathrm{Re}(\mathrm{Tr}(M)) = \frac{1}{2} \mathrm{Tr}(\mathbf{C}(M)),$$

and for $A, B \in \mathcal{M}_n(\mathbb{H})$,

$$\mathrm{Re}(\mathrm{Tr}(AB)) = \mathrm{Re}(\mathrm{Tr}(BA)).$$

Note that in general there is no equality between $\mathrm{Tr}(AB)$ and $\mathrm{Tr}(BA)$, since the product of quaternions is not commutative.

The quaternion adjoint of a matrix $M \in \mathcal{M}_n(\mathbb{H})$, denoted M^* , extends the notion of adjoint for matrices with complex entries:

$$(M^*)_{ij} = \overline{M_{ji}}, \quad i, j \in \{1, \dots, n\},$$

where $\overline{M_{ji}}$ is the quaternion conjugate. We have that

$$\mathbf{C}(M^*) = \mathbf{C}(M)^*,$$

where on the left-hand side $*$ denotes the quaternion adjoint, and on the right-hand side, $*$ denotes the complex adjoint. If $A, B \in \mathcal{M}_n(\mathbb{H})$,

$$(AB)^* = B^* A^*.$$

The quaternionic unitary group, $U(n, \mathbb{H})$ is the set of matrices $U \in \mathcal{M}_n(\mathbb{H})$ satisfying

$$UU^* = I_n,$$

I_n being the $n \times n$ identity matrix. The relation above is equivalent to

$$U^*U = I_n.$$

If $U \in U(n, \mathbb{H})$,

$$\det(\mathbf{C}(U)) = 1,$$

and $\mathbf{C}(U) \in SU(2n)$. The image of $U(n, \mathbb{H})$ by \mathbf{C} is the compact symplectic group $Sp(n)$, a subgroup of $SU(2n)$.

The set of $n \times n$ quaternionic Hermitian matrices $\mathcal{H}_n(\mathbb{H})$ is composed of matrices $M \in \mathcal{M}_n(\mathbb{H})$ satisfying

$$M^* = M.$$

M is quaternionic Hermitian if and only if $\mathbf{C}(M)$ is complex Hermitian. The diagonal entries of a quaternionic Hermitian matrix are real. Given $M \in \mathcal{H}_n(\mathbb{H})$, there exists $U \in U(n, \mathbb{H})$ such that U^*MU is diagonal with real entries:

$$U^*MU = \mathrm{Diag}(\lambda_1, \lambda_2, \dots, \lambda_n),$$

with $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \in \mathbb{R}$. The family $\lambda_1, \lambda_2, \dots, \lambda_n$ is uniquely determined. This is also the family of ordered eigenvalues of $\mathbf{C}(M)$, but the multiplicities have to be doubled. $\lambda_1, \lambda_2, \dots, \lambda_n$ are the right eigenvalues of M , and form the right spectrum of M , i.e. the set of $\lambda \in \mathbb{H}$, for which the equation

$$Mx = x\lambda$$

has a non-zero solution. There is also a notion of left spectrum, corresponding to the equation $Mx = \lambda x$. But the right and the left spectra do not necessarily coincide, even for quaternionic

Hermitian matrices. For details on eigenvalues of quaternionic matrices we refer to [Zha97, MGS14]. The trace of M equals

$$\mathrm{Tr}(M) = \mathrm{Re}(\mathrm{Tr}(M)) = \sum_{i=1}^n \lambda_i.$$

2.2. BFS-Dynkin isomorphism. Let $\mathcal{G} = (V, E)$ be an undirected connected graph, with V finite, and all vertices $x \in V$ of finite degree. We do not allow multiple edges or self-loops. Edges $\{x, y\} \in E$ are endowed with conductances $C(x, y) = C(y, x) > 0$. There also a not uniformly zero killing measure $(\kappa(x))_{x \in V}$, with $\kappa(x) \geq 0$. \mathcal{G} will be further referred to as electrical network. Let X_t be the Markov jump process to nearest neighbors with jump rates given by the conductances. X_t is also killed by κ . Let $\zeta \in (0, +\infty]$ be the first time X_t gets killed by κ .

Let $(G(x, y))_{x, y \in V}$ be the Green's function:

$$G(x, y) = G(y, x) = \mathbb{E}_{X_0=x} \left[\int_0^\zeta \mathbf{1}_{X_t=y} dt \right].$$

Let $p_t(x, y)$ be the transition probabilities of $(X_t)_{0 \leq t < \zeta}$. Then $p_t(x, y) = p_t(y, x)$ and

$$G(x, y) = \int_0^{+\infty} p_t(x, y) dt.$$

Let $\mathbb{P}_t^{x,y}$ be the bridge probability measure from x to y , where one conditions by $t < \zeta$. Let $\mu_{x,y}$ be the following measure on paths from x to y in finite time:

$$(2.1) \quad \mu^{x,y}(d\gamma) = \int_0^{+\infty} \mathbb{P}_t^{x,y}(d\gamma) p_t(x, y) dt.$$

$\mu^{x,y}$ has total mass $G(x, y)$. The image of $\mu^{x,y}$ by time reversal is $\mu^{y,x}$.

Given $x_1, x_2, \dots, x_{2r} \in V$ and $\mathbf{p} = \{\{a_1, b_1\}, \dots, \{a_r, b_r\}\}$ a partition in pairs of $\{1, \dots, 2r\}$, $\mu_{\mathbf{p}}^{x_1, x_2, \dots, x_{2r}}(d\gamma_1, \dots, d\gamma_r)$ will denote the following product measure on r paths:

$$\mu_{\mathbf{p}}^{x_1, x_2, \dots, x_{2r}}(d\gamma_1, \dots, d\gamma_r) = \prod_{i=1}^r \mu^{x_{a_i}, x_{b_i}}(d\gamma_i).$$

We will sometimes, in particular in Section 3.2, use the convention that $a_i < b_i$. The order of the pairs in \mathbf{p} will not be important.

In general, for a path γ and $x \in V$, $L(\gamma)$ will denote the occupation field of γ ,

$$L(\gamma)(x) = \int_0^{T(\gamma)} \mathbf{1}_{\gamma(t)=x} dt,$$

where $T(\gamma)$ is the life-time of the path.

The Gaussian free field (GFF) $(\phi(x))_{x \in V}$ will denote here the centered Gaussian process with covariance

$$\mathbb{E}[\phi(x)\phi(y)] = G(x, y).$$

If V is finite, the distribution of $(\phi(x))_{x \in V}$ is given by

$$(2.2) \quad \frac{1}{((2\pi)^{\mathrm{Card}(V)} \det G)^{\frac{1}{2}}} \exp \left(-\frac{1}{2} \sum_{x \in V} \kappa(x) \varphi(x)^2 - \frac{1}{2} \sum_{\{x, y\} \in E} C(x, y) (\varphi(y) - \varphi(x))^2 \right) \prod_{x \in V} d\varphi(x).$$

The following isomorphism relates the square of the GFF $(\phi(x)^2)_{x \in V}$ and the occupation fields of paths under measures $\mu^{x,y}$. It first appears in the work of Brydges, Fröhlich and Spencer [BFS82] (see also [Frö82, BFS83b, BFS83a]) and then in that of Dynkin [Dyn84a, Dyn84b] (see also [Dyn84c]). It is also related to earlier works of Symanzik [Sym65, Sym66, Sym69]. For more on isomorphism theorems, see [MR06, Szn12].

Theorem 2.1 (Brydges-Fröhlich-Spencer [BFS82], Dynkin [Dyn84a, Dyn84b]). *Let $r \in \mathbb{N} \setminus \{0\}$, $x_1, x_2, \dots, x_{2r} \in V$ and F a bounded measurable function $\mathbb{R}^V \rightarrow \mathbb{R}$. Then*

$$\mathbb{E} \left[\prod_{k=1}^{2r} \phi(x_k) F(\phi^2/2) \right] = \sum_{\substack{\mathbf{p} \text{ partition} \\ \text{of } \{1, \dots, 2r\} \\ \text{in pairs}}} \int_{\gamma_1, \dots, \gamma_r} \mathbb{E} \left[F(\phi^2/2 + L(\gamma_1) + \dots + L(\gamma_r)) \right] \mu_{\mathbf{p}}^{x_1, x_2, \dots, x_{2r}}(d\gamma_1, \dots, d\gamma_r).$$

2.3. Connections, gauge equivalence and Wilson loops. Let $n \in \mathbb{N}$, $n \geq 2$. Let \mathbb{U} be the group of either $n \times n$ orthogonal matrices $O(n)$, or unitary matrices $U(n)$, or quaternionic unitary matrices $U(n, \mathbb{H})$. We consider that each undirected edge in E consists of two directed edges of opposite direction. We consider a family of matrices in \mathbb{U} , $(U(x, y))_{\{x, y\} \in E}$, with

$$U(y, x) = U(x, y)^* = U(x, y)^{-1}, \quad \forall \{x, y\} \in E.$$

$(U(x, y))_{\{x, y\} \in E}$ is our *connection* on the vector bundle with base space \mathcal{G} and fiber \mathbb{R}^n .

Given a nearest neighbor oriented discrete path $\gamma = (y_1, y_2, \dots, y_j)$, the *holonomy* of U along γ is the product

$$\text{hol}^U(\gamma) = U(y_1, y_2)U(y_2, y_3) \dots U(y_{j-1}, y_j).$$

If the path γ is a nearest neighbor path parametrized by continuous time, and does only a finite number of jumps, the holonomy $\text{hol}^U(\gamma)$ is defined as the holonomy along the discrete skeleton of γ . $\overleftarrow{\gamma}$ will denote the time-reversal of a path γ . We have that

$$(2.3) \quad \text{hol}^U(\overleftarrow{\gamma}) = \text{hol}^U(\gamma)^* = \text{hol}^U(\gamma)^{-1}.$$

Given a nearest neighbor oriented discrete closed path (i.e. a loop) $\gamma = (y_1, y_2, \dots, y_j)$, with $y_j = y_1$, we will consider the observable

$$\text{Tr}(\text{hol}^U(\gamma))$$

in the orthogonal and unitary case, and

$$\text{Re}(\text{Tr}(\text{hol}^U(\gamma))) = \frac{1}{2} \text{Tr}(\mathbb{C}(\text{hol}^U(\gamma)))$$

in the quaternionic unitary case. Such observables are called *Wilson loops* [Wil74]. Note that the Wilson loop observable does not depend on where the loop γ is rooted. Indeed, if $\tilde{\gamma}$ is the loop visiting $(y_i, \dots, y_j, y_1, \dots, y_{i-1}, y_i)$ ($i \in \{2, \dots, j\}$), and if γ' is the path visiting (y_1, \dots, y_i) then

$$\text{hol}^U(\tilde{\gamma}) = \text{hol}^U(\gamma')^{-1} \text{hol}^U(\gamma) \text{hol}^U(\gamma').$$

Given an other family of matrices in \mathbb{U} , $(\mathfrak{U}(x))_{x \in V}$, this time on top of vertices, it induces a *gauge transformation* on the connection U :

$$(U(x, y))_{\{x, y\} \in E} \longmapsto (\mathfrak{U}(x)^{-1} U(x, y) \mathfrak{U}(y))_{\{x, y\} \in E}.$$

Two connections related by a gauge transformation are said to be *gauge equivalent*. A connection is *flat* if it is gauge equivalent to the identity connection. A criterion for flatness is that along any nearest neighbor loop $\gamma = (y_1, y_2, \dots, y_j)$, with $y_j = y_1$,

$$\text{hol}^U(\gamma) = I_n.$$

In general, any two gauge equivalent connections have the same Wilson loop observables. The converse is also true (but less obvious): the collection of all possible Wilson loop observables characterizes a connection up to gauge transformations [Gil81, Sen94, Lé04].

2.4. BFS-Dynkin isomorphism for the Gaussian free field twisted by a connection.

In [KL16] Kassel and Lévy introduced the vector valued GFF twisted by an orthogonal/unitary connection, and generalized the isomorphisms with random walks to this case. Here we will do a less abstract, more computational presentation of the same object. Kassel and Lévy's isomorphisms rely on a covariant Feynman-Kac formula ([BFS79] and Theorem 4.1 in [KL16]).

Let us consider on top of the electrical network $\mathcal{G} = (V, E)$ an orthogonal connection $(U(x, y))_{\{x, y\} \in E}$, $U(x, y) \in O(n)$. The Green's function associated to the connection U , G^U is a function from $V \times V$ to $\mathcal{M}_n(\mathbb{R})$ (i.e. the $n \times n$ matrices with real entries), with the entries given by

$$G_{ij}^U(x, y) = \int_{\gamma} \text{hol}_{ij}^U(\gamma) \mu^{x, y}(d\gamma), \quad x, y \in V, \quad i, j \in \{1, \dots, n\},$$

where the measure on paths $\mu^{x, y}(d\gamma)$ is given by (2.1). Since the image of $\mu^{x, y}$ by time reversal is $\mu^{y, x}$, and because of (2.3), we have that

$$G_{ij}^U(x, y) = G_{ji}^U(y, x), \quad G_{ij}^U(x, x) = G_{ji}^U(x, x).$$

i.e. $G^U(x, y)^T = G^U(y, x)$ and $G^U(x, x)$ is symmetric. G^U can be seen as a symmetric linear operator on $(\mathbb{R}^n)^V$. It is positive definite (see Proposition 3.14 in [KL16]). $\det G^U$ we will denote the determinant of this operator.

The \mathbb{R}^n -valued Gaussian free field on \mathcal{G} twisted by the connection U is a random Gaussian function $\hat{\phi} : V \rightarrow \mathbb{R}^n$ ($\hat{\phi}(x) = (\hat{\phi}_1(x), \dots, \hat{\phi}_n(x))$) with the distribution given by

$$\frac{1}{Z_{\text{GFF}}^U} \exp \left(-\frac{1}{2} \sum_{x \in V} \kappa(x) \|\hat{\phi}(x)\|^2 - \frac{1}{2} \sum_{\{x, y\} \in E} C(x, y) \|\hat{\phi}(x) - U(x, y)\hat{\phi}(y)\|^2 \right) \prod_{x \in V} \prod_{i=1}^n d\hat{\phi}(x)_i,$$

where $\|\cdot\|$ is the usual \mathbb{L}^2 norm on \mathbb{R}^n and

$$Z_{\text{GFF}}^U = ((2\pi)^n \text{Card}(V) \det G^U)^{\frac{1}{2}}.$$

Note that if $\{x, y\} \in E$,

$$\|\hat{\phi}(x) - U(x, y)\hat{\phi}(y)\|^2 = \|\hat{\phi}(y) - U(y, x)\hat{\phi}(x)\|^2.$$

We have that $\mathbb{E}[\hat{\phi}] \equiv 0$. As for the covariance structure, we have (see Proposition 5.1 in [KL16]):

$$\mathbb{E}[\hat{\phi}_i(x)\hat{\phi}_j(y)] = G_{ij}^U(x, y).$$

If $(\mathfrak{U}(x))_{x \in V}$ is a gauge transformation, then $(\mathfrak{U}(x)\hat{\phi}(x))_{x \in V}$ is the Gaussian free field related to the connection $(\mathfrak{U}(x)^{-1}U(x, y)\mathfrak{U}(y))_{\{x, y\} \in E}$. In particular, if the connection U is flat, the field $\hat{\phi}$ can be reduced to n i.i.d. copies of the scalar GFF with distribution (2.2).

In [KL16], Theorems 6.1 and 8.3, Kassel and Lévy gave a BFS-Dynkin-type isomorphism for GFFs twisted by connections.

Theorem 2.2 (Kassel-Lévy [KL16]). *Let $r \in \mathbb{N} \setminus \{0\}$, $x_1, x_2, \dots, x_{2r} \in V$, $J(1), J(2), \dots, J(2r) \in \{1, \dots, n\}$ and F a bounded measurable function $\mathbb{R}^V \rightarrow \mathbb{R}$. Then*

$$\begin{aligned} & \mathbb{E} \left[\prod_{k=1}^{2r} \hat{\phi}_{J(k)}(x_k) F(\|\hat{\phi}\|^2/2) \right] \\ &= \sum_{\substack{\text{partitions of} \\ \{1, \dots, 2r\} \\ \text{in pairs} \\ \{a_1, b_1\}, \dots, \{a_r, b_r\}}} \int_{\gamma_1, \dots, \gamma_r} \mathbb{E} \left[F(\|\hat{\phi}\|^2/2 + L(\gamma_1) + \dots + L(\gamma_r)) \right] \prod_{i=1}^r \text{hol}_{J(a_i)J(b_i)}^U(\gamma_i) \mu^{x_{a_i}, x_{b_i}}(d\gamma_i), \end{aligned}$$

where the sum runs over the $(2r)!/(2^r r!)$ partitions of $\{1, \dots, 2r\}$ in pairs.

It will be useful for the sequel to rewrite the isomorphism above as follows. Let $(\hat{X}^{(i)})_{i \geq 1}$ be an i.i.d. family of random Gaussian vectors with n components, following the law $\mathcal{N}(0, I_n)$.

Lemma 2.3. Let $r \in \mathbb{N} \setminus \{0\}$, $x_1, x_2, \dots, x_{2r} \in V$, $J(1), J(2), \dots, J(2r) \in \{1, \dots, n\}$ and F a bounded measurable function $\mathbb{R}^V \rightarrow \mathbb{R}$. Then

$$\begin{aligned} & \mathbb{E} \left[\prod_{k=1}^{2r} \hat{\phi}_{J(k)}(x_k) F(\|\hat{\phi}\|^2/2) \right] \\ &= \sum_{\substack{\text{partitions of} \\ \{1, \dots, 2r\} \\ \text{in pairs} \\ \{\{a_1, b_1\}, \dots, \{a_r, b_r\}\}}} \int_{\gamma_1, \dots, \gamma_r} \mathbb{E} \left[F(\|\hat{\phi}\|^2/2 + L(\gamma_1) + \dots + L(\gamma_r)) \right] \times \\ & \quad \times \mathbb{E} \left[\prod_{i=1}^r \left(\text{hol}^U(\gamma_i) \hat{X}^{(i)} \right)_{J(a_i)} \hat{X}_{J(b_i)}^{(i)} \right] \prod_{i=1}^r \mu^{x_{a_i}, x_{b_i}}(d\gamma_i), \end{aligned}$$

where the sum runs over the $(2r)!/(2^r r!)$ partitions of $\{1, \dots, 2r\}$ in pairs.

Proof. Indeed,

$$\mathbb{E} \left[\prod_{i=1}^r \left(\text{hol}^U(\gamma_i) \hat{X}^{(i)} \right)_{J(a_i)} \hat{X}_{J(b_i)}^{(i)} \right] = \prod_{i=1}^r \text{hol}_{J(a_i)J(b_i)}^U(\gamma_i). \quad \square$$

2.5. Ribbon graphs and surfaces. Here we describe the ribbon graphs and the related two-dimensional surfaces. For more details, we refer to [Eyn16], Sections 2.2 and 2.3, [EKR18], Chapter 2, [LZ04], Sections 3.2 and 3.3, [Zvo97], and [MT01], Section 3.3.

Let be $\nu = (\nu_1, \nu_2, \dots, \nu_m)$, where $m \geq 1$, and for all $l \in \{1, 2, \dots, m\}$, $\nu_l \in \mathbb{N} \setminus \{0\}$. We will denote

$$m(\nu) = m, \quad |\nu| = \sum_{l=1}^{m(\nu)} \nu_l.$$

We will assume that $|\nu|$ is even.

Given ν as above, we consider $m(\nu)$ vertices, where each vertex has adjacent *ribbon half-edges*, ν_1 half-edges for the first vertex, ν_2 for the second, etc. A ribbon half-edge is a two-dimensional object and carries an orientation. Also, the ribbon half-edges around each vertex are ordered in a cyclic way. The ribbon half-edges are numbered from 1 to $|\nu|$. See Figure 1 for an illustration with $\nu = (4, 3, 1)$.

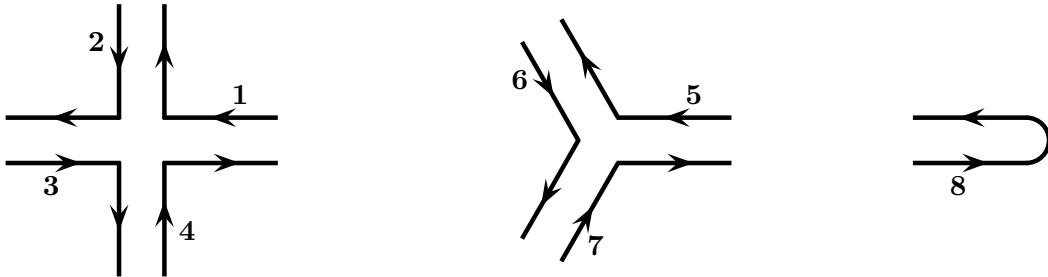


FIGURE 1. Ribbon half-edges in the case of $\nu = (4, 3, 1)$.

Since the total number of half-edges, $|\nu|$, is even, one can pair them to obtain a *ribbon graph* (not necessarily connected), with $m(\nu)$ vertices and $|\nu|/2$ *ribbon edges*. Each time we pair two half-edges, we can glue the corresponding ribbons in two different ways. Either the orientations of the two ribbon half-edges match, or are opposite. In the first case we get a *straight ribbon edge*, in the second a *twisted ribbon edge*. See Figure 2. We call such a pairing of ribbon half-edges that keeps straight or twists the ribbons a *ribbon pairing*. Let \mathcal{R}_ν be the set of all possible ribbon pairings associated to ν . The number of different ribbon pairings is

$$\text{Card}(\mathcal{R}_\nu) = \frac{|\nu|!}{2^{|\nu|/2} (|\nu|/2)!} 2^{|\nu|/2} = \frac{|\nu|!}{(|\nu|/2)!}.$$

Figure 3 displays an example of a ribbon pairing with only straight edges, and Figure 4 an example with both straight and twisted edges.



FIGURE 2. A straight ribbon edge on the left and a twisted ribbon edge on the right.

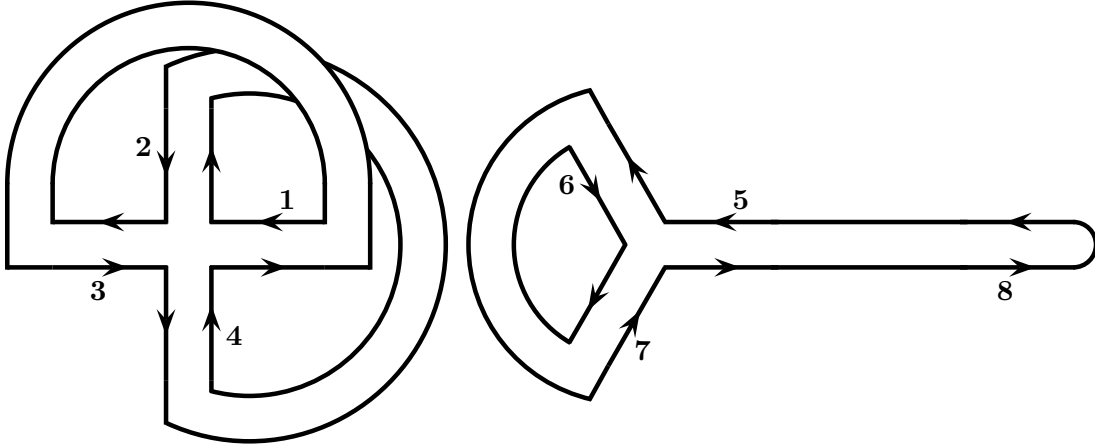


FIGURE 3. A ribbon pairing in the case of $\nu = (4, 3, 1)$ with only straight edges.

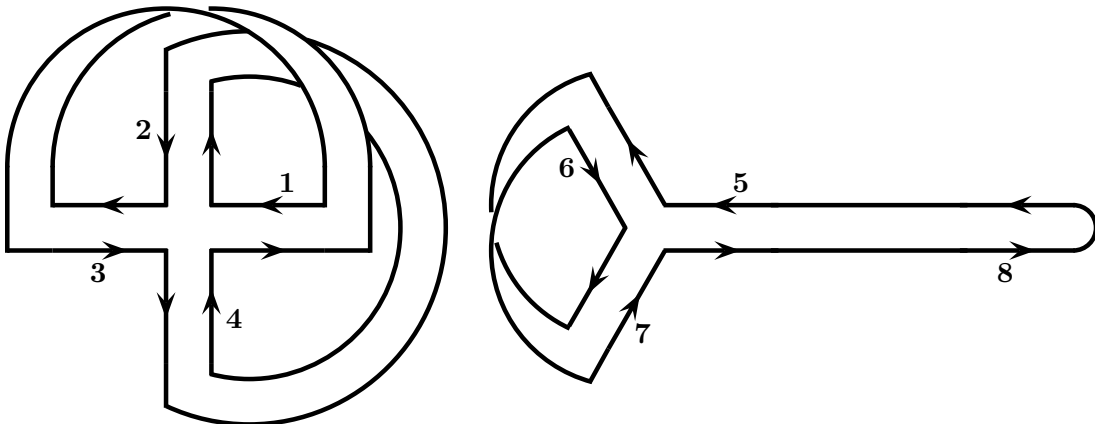


FIGURE 4. A ribbon pairing in the case of $\nu = (4, 3, 1)$ with straight and twisted edges.

A ribbon pairing $\rho \in \mathcal{R}_\nu$ induces a partition in pairs of $\{1, \dots, |\nu|\}$, denoted $\mathbf{p}_\nu(\rho)$. The pairs correspond to the labels of ribbon half-edges associated into an edge. Conversely, given \mathbf{p} a partition in pairs of $\{1, \dots, |\nu|\}$, $\mathcal{R}_{\nu, \mathbf{p}}$ will denote the subset of \mathcal{R}_ν made of all ribbon pairings ρ such that $\mathbf{p}_\nu(\rho) = \mathbf{p}$ ($\text{Card}(\mathcal{R}_{\nu, \mathbf{p}}) = 2^{|\nu|/2}$).

Given a ribbon pairing $\rho \in \mathcal{R}_\nu$, one can see the corresponding ribbon graph as a two-dimensional compact bordered surface (not necessarily connected). Let $f_\nu(\rho)$ denote the number of the connected components of the border, that is to say the number of distinct cycles formed by the borders of ribbons. On Figure 3, $f_\nu(\rho) = 3$, and on Figure 4, $f_\nu(\rho) = 2$. Then, one can glue along each connected component of the border a disk ($f_\nu(\rho)$ disks in total), and obtain in this way a two-dimensional compact surface (not necessarily connected) without border. We will denote it $\Sigma_\nu(\rho)$, and consider it up to diffeomorphisms. On the example of Figure 3, $\Sigma_\nu(\rho)$ has two connected components, a torus on the left and a sphere on the right. On the example of Figure 4, $\Sigma_\nu(\rho)$ has again two connected components, a Klein bottle on the left and a projective plane on the right. Observe that if all the edges are straight, the surfaces that appear are

orientable. Let $\chi_\nu(\rho)$ denote the Euler's characteristic of $\Sigma_\nu(\rho)$. According to Euler's formula,

$$\chi_\nu(\rho) = m(\nu) - \frac{|\nu|}{2} + f_\nu(\rho).$$

Next we introduce more combinatorial objects related to the ribbon pairings. We will consider tuples $(k_1, \mathbf{s}_1, k_2, \mathbf{s}_2, \dots, k_j, \mathbf{s}_j)$, where $j \in \mathbb{N} \setminus \{0\}$, each of the k_i is in $\mathbb{N} \setminus \{0\}$, and each of the \mathbf{s}_i is one of the three abstract symbols \rightarrow , \leftarrow or $\dot{=}$. We will endow such tuples by an equivalence relation \approx generated by the following rules.

- Cyclic permutation: for any $i \in \{2, \dots, j\}$, $(k_i, \mathbf{s}_i, \dots, k_j, \mathbf{s}_j, k_1, \mathbf{s}_1, \dots, k_{i-1}, \mathbf{s}_{i-1})$ is identified to $(k_1, \mathbf{s}_1, k_2, \mathbf{s}_2, \dots, k_j, \mathbf{s}_j)$.
- Reversal of the direction: $(k_j, \mathbf{r}(\mathbf{s}_j), \dots, k_2, \mathbf{r}(\mathbf{s}_2), k_1, \mathbf{r}(\mathbf{s}_1))$ is identified to $(k_1, \mathbf{s}_1, k_2, \mathbf{s}_2, \dots, k_j, \mathbf{s}_j)$, where $\mathbf{r}(\rightarrow)$ is \leftarrow , $\mathbf{r}(\leftarrow)$ is \rightarrow , and $\mathbf{r}(\dot{=})$ is $\dot{=}$.

For the lack of a better name, we will call *trails* the equivalence classes of \approx .

Given a ribbon pairing $\rho \in \mathcal{R}_\nu$, we will associate to ρ a set $\mathcal{T}_\nu(\rho)$ made of $f_\nu(\rho)$ trails, one per each border cycle in the ribbon pairing. One starts on such a border cycle in an arbitrary place, and travels along it in any of the two directions. Then one successively visits ribbon half-edges with labels k_1, k_2, \dots, k_j and then returns to the half-edge k_1 . One can go from the half-edge k_i to the half-edge k_{i+1} either by following a gluing, and we will denote this $k_i \dot{=} k_{i+1}$, or by going through a vertex. In the latter case, one either does a turn clockwise, and we will denote this $k_i \rightarrow k_{i+1}$, or counterclockwise, and we will denote this $k_i \leftarrow k_{i+1}$. A special rule is applied if the vertex has only one outgoing half-edge, one just makes the arrows in the trail and on the picture match, as in the example below. This is how a trail is obtained. Note that by construction, there is an alternation between on one hand $\dot{=}$, and on the other hand \rightarrow or \leftarrow . In the example of Figure 3, there are three trails:

$$(2.4) \quad (1, \rightarrow, 2, \dot{=}, 4, \rightarrow, 1, \dot{=}, 3, \rightarrow, 4, \dot{=}, 2, \rightarrow, 3, \dot{=}), \quad (5, \rightarrow, 6, \dot{=}, 7, \rightarrow, 5, \dot{=}, 8, \rightarrow, 8, \dot{=}), \quad (6, \rightarrow, 7, \dot{=}).$$

In the example of Figure 4, there are two trails:

$$(1, \rightarrow, 2, \dot{=}, 4, \rightarrow, 1, \dot{=}, 3, \leftarrow, 2, \dot{=}, 4, \leftarrow, 3, \dot{=}), \quad (5, \rightarrow, 6, \dot{=}, 7, \leftarrow, 6, \dot{=}, 7, \rightarrow, 5, \dot{=}, 8, \rightarrow, 8, \dot{=}).$$

We will also consider *oriented trails*. Like the (unoriented) trails, they can contain positive integer numbers and symbols \rightarrow and $\dot{=}$, but not the symbol \leftarrow . In the oriented trails we quotient by cyclic permutations, but not by the reversal of direction. We will associate oriented trails to ribbon pairing that contain only straight edges. Given $\rho \in \mathcal{R}_\nu$ with only straight edges, $\vec{\mathcal{T}}_\nu(\rho)$ will be a set of $f_\nu(\rho)$ oriented trails, one per each border cycle where one follows the cycle in the direction of the arrows (clockwise). The oriented trails corresponding to Figure 3 are given by (2.4).

2.6. One matrix integrals and topological expansion. $E_{\beta,n}$ will denote $\mathcal{S}_n(\mathbb{R})$, the space of real symmetric matrices, for $\beta = 1$, $\mathcal{H}_n(\mathbb{C})$, the space of complex Hermitian matrices, for $\beta = 2$, and $\mathcal{H}_n(\mathbb{H})$, the space of quaternionic Hermitian matrices, for $\beta = 4$.

$$\dim E_{\beta=1,n} = \frac{(n+1)n}{2}, \quad \dim E_{\beta=2,n} = n^2, \quad \dim E_{\beta=4,n} = 2n^2 - n.$$

$E_{\beta,n}$ is endowed with the real the inner product

$$(M, M') \mapsto \text{Re}(\text{Tr}(MM')).$$

The Gaussian Orthogonal Ensemble $\text{GOE}(n)$, the Gaussian Unitary Ensemble $\text{GUE}(n)$ and Gaussian Symplectic Ensemble $\text{GSE}(n)$ are Gaussian probability measures on $E_{\beta,n}$, with $\beta = 1$ for the $\text{GOE}(n)$, $\beta = 2$ for the $\text{GUE}(n)$ and $\beta = 4$ for the $\text{GSE}(n)$. We will use the usual notation $\text{G}\beta\text{E}(n)$. The density with respect to the Lebesgue measure on $E_{\beta,n}$ is given by

$$(2.5) \quad \frac{1}{Z_{\beta,n}} e^{-\frac{1}{2} \text{Tr}(M^2)}.$$

The distribution of the ordered family of eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ of $\text{G}\beta\text{E}(n)$ is given by

$$\frac{1}{Z_{\beta,n}^{\text{ev}}} \mathbf{1}_{\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n} \prod_{1 \leq i < j \leq n} (\lambda_i - \lambda_j)^\beta e^{-\frac{1}{2}(\lambda_1^2 + \dots + \lambda_n^2)} d\lambda_1 \dots d\lambda_n.$$

For more on random matrices see [Meh04].

Let be $\nu = (\nu_1, \nu_2, \dots, \nu_{m(\nu)})$, where for all $l \in \{1, 2, \dots, m(\nu)\}$, $\nu_l \in \mathbb{N} \setminus \{0\}$, and

$$|\nu| = \sum_{l=1}^{m(\nu)} \nu_l$$

is even. Next we recall the expressions for the matrix integrals

$$(2.6) \quad \left\langle \prod_{l=1}^{m(\nu)} \text{Tr}(M^{\nu_l}) \right\rangle_{\beta,n} = \frac{1}{Z_{\beta,n}} \int_{E_{\beta,n}} \left(\prod_{l=1}^{m(\nu)} \text{Tr}(M^{\nu_l}) \right) e^{-\frac{1}{2} \text{Tr}(M^2)} dM, \quad \beta \in \{1, 2, 4\}.$$

Note that if $|\nu|$ is odd, the above integrals are zero. The expression for (2.6) makes appear a polynomial in n , with powers $n^{f_\nu(\rho)}$, where $\rho \in \mathcal{R}_\nu$ are ribbon pairings associated to ν . Since $f_\nu(\rho)$ can be expressed using Euler's characteristic of surfaces, the expression for (2.6) is often referred to as *topological expansion*. The topological expansion has been introduced by 't Hooft [tH74]. The expansion for complex Hermitian and real symmetric matrices appears in [BIPZ78] (see also [BIZ80]). The combinatorics for quaternionic Hermitian matrices are given in [MW03] (see also [BP09]). For more on the topological expansion, we also refer to [Eyn16], Chapter 2, [EKR18], Chapter 2, [LZ04], Chapter 3, and [Zvo97].

Given a ribbon pairing $\rho \in \mathcal{R}_\nu$, we associate to it a weight $w_{\nu,\beta}(\rho)$ depending on β :

$$w_{\nu,\beta=1}(\rho) = \frac{1}{2^{|\nu|/2}}, \quad w_{\nu,\beta=2}(\rho) = 1_{\rho \text{ has only straight edges}}, \quad w_{\nu,\beta=4}(\rho) = (-2)^{\chi_\nu(\rho)} 2^{-2m(\nu)+|\nu|/2}.$$

In all three cases $\beta \in \{1, 2, 4\}$, for every \mathbf{p} partition in pairs of $\{1, \dots, |\nu|\}$,

$$\sum_{\rho \in \mathcal{R}_{\nu,\mathbf{p}}} w_{\nu,\beta}(\rho) = 1.$$

Theorem 2.4 (Brézin-Itzykson-Parisi-Zuber [BIPZ78], Mulase-Waldron [MW03]). *For $\beta \in \{1, 2, 4\}$ and $|\nu|$ even, the value of the matrix integral $\langle \prod_{l=1}^{m(\nu)} \text{Tr}(M^{\nu_l}) \rangle_{\beta,n}$ (2.6) is given by*

$$(2.7) \quad \left\langle \prod_{l=1}^{m(\nu)} \text{Tr}(M^{\nu_l}) \right\rangle_{\beta,n} = \sum_{\rho \in \mathcal{R}_\nu} w_{\nu,\beta}(\rho) n^{f_\nu(\rho)}.$$

For instance, $\langle \text{Tr}(M^2) \rangle_{\beta,n}$ equals $\dim E_{\beta,n}$. For $\nu = (4)$ and $\nu = (2, 2)$ one gets

$$\begin{aligned} \langle \text{Tr}(M^4) \rangle_{\beta=1,n} &= \frac{1}{2}n^3 + \frac{5}{4}n^2 + \frac{5}{4}n, & \langle (\text{Tr}(M^2))^2 \rangle_{\beta=1,n} &= \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{5}{4}n^2 + n, \\ \langle \text{Tr}(M^4) \rangle_{\beta=2,n} &= 2n^3 + n, & \langle (\text{Tr}(M^2))^2 \rangle_{\beta=2,n} &= n^4 + 2n^2, \\ \langle \text{Tr}(M^4) \rangle_{\beta=4,n} &= 8n^3 - 10n^2 + 5n, & \langle (\text{Tr}(M^2))^2 \rangle_{\beta=4,n} &= 4n^4 - 4n^3 + 5n^2 - 2n. \end{aligned}$$

3. MAIN STATEMENTS

3.1. Matrix valued free fields, isomorphisms and topological expansion. Let $\mathcal{G} = (V, E)$ be an electrical network as in Section 2.2. For $\beta \in \{1, 2, 4\}$, Φ will be a random Gaussian function from V to $E_{\beta,n}$. The distribution of Φ is

$$(3.1) \quad \frac{1}{Z_{\beta,n}^{\mathcal{G}}} \exp \left(-\frac{1}{2} \sum_{x \in V} \kappa(x) \text{Tr}(M(x)^2) - \frac{1}{2} \sum_{\{x,y\} \in E} C(x,y) \text{Tr}((M(y) - M(x))^2) \right) \prod_{x \in V} dM(x).$$

Φ can be obtained out of $\dim E_{\beta,n}$ i.i.d. copies of the scalar GFF 2.2, by considering the entries of the matrices. For any $x \in V$, $\Phi(x)/\sqrt{G(x,x)}$ is distributed as a $\text{G}\beta\text{E}(n)$ matrix. $\langle \cdot \rangle_{\beta,n}$ will denote the expectation with respect to the law of Φ for corresponding values of (β, n) .

Let be $\nu = (\nu_1, \nu_2, \dots, \nu_{m(\nu)})$, where for all $l \in \{1, 2, \dots, m(\nu)\}$, $\nu_l \in \mathbb{N} \setminus \{0\}$, and $|\nu|$ even. Let $x_1, x_2, \dots, x_{|\nu|}$ be vertices in V , not necessarily distinct and F a bounded measurable function $\mathbb{R}^V \rightarrow \mathbb{R}$. By applying Theorem 2.1, one can *a priori* write an isomorphism for

$$\left\langle \left(\prod_{l=1}^{m(\nu)} \text{Tr} \left(\prod_{k=\nu_1+\dots+\nu_{l-1}+1}^{\nu_1+\dots+\nu_l} \Phi(x_k) \right) \right) F(\text{Tr}(\Phi^2)/2) \right\rangle_{\beta, n}$$

However, if one expands the traces and the product, one gets many terms that give identical contributions, many terms with contributions that compensate, and many terms that do not contribute at all. Here we will be interested in the exact combinatorics that appear. What emerges is a topological expansion, generalizing that of Theorem 2.4.

Let be $\mu_{\nu, \beta, n}^{x_1, x_2, \dots, x_{|\nu|}}$ the following positive measure on families of $|\nu|/2$ nearest neighbor paths on \mathcal{G} :

$$\mu_{\nu, \beta, n}^{x_1, x_2, \dots, x_{|\nu|}}(d\gamma_1, \dots, d\gamma_{|\nu|/2}) = \sum_{\rho \in \mathcal{R}_\nu} w_{\nu, \beta}(\rho) n^{f_\nu(\rho)} \mu_{\mathbf{p}_\nu(\rho)}^{x_1, x_2, \dots, x_{|\nu|}}(d\gamma_1, \dots, d\gamma_{|\nu|/2}).$$

Next we give examples.

$$\begin{aligned} \mu_{\nu=(2), \beta=1, n}^{x_1, x_2} &= \left(\frac{1}{2}n^2 + \frac{1}{2}n \right) \mu^{x_1, x_2}, & \mu_{\nu=(1,1), \beta=1, n}^{x_1, x_2} &= n \mu^{x_1, x_2}, \\ \mu_{\nu=(4), \beta=1, n}^{x_1, x_2, x_3, x_4} &= \left(\frac{1}{4}n^3 + \frac{1}{2}n^2 + \frac{1}{4}n \right) \mu^{x_1, x_2} \otimes \mu^{x_3, x_4} + \left(\frac{1}{4}n^3 + \frac{1}{2}n^2 + \frac{1}{4}n \right) \mu^{x_1, x_4} \otimes \mu^{x_2, x_3} \\ &+ \left(\frac{1}{4}n^2 + \frac{3}{4}n \right) \mu^{x_1, x_3} \otimes \mu^{x_2, x_4}, \\ \mu_{\nu=(2,2), \beta=1, n}^{x_1, x_2, x_3, x_4} &= \left(\frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2 \right) \mu^{x_1, x_2} \otimes \mu^{x_3, x_4} + \left(\frac{1}{2}n^2 + \frac{1}{2}n \right) \mu^{x_1, x_4} \otimes \mu^{x_2, x_3} \\ &+ \left(\frac{1}{2}n^2 + \frac{1}{2}n \right) \mu^{x_1, x_3} \otimes \mu^{x_2, x_4}, \\ \mu_{\nu=(2), \beta=2, n}^{x_1, x_2} &= n^2 \mu^{x_1, x_2}, & \mu_{\nu=(1,1), \beta=2, n}^{x_1, x_2} &= n \mu^{x_1, x_2}, \\ \mu_{\nu=(4), \beta=2, n}^{x_1, x_2, x_3, x_4} &= n^3 \mu^{x_1, x_2} \otimes \mu^{x_3, x_4} + n^3 \mu^{x_1, x_4} \otimes \mu^{x_2, x_3} + n \mu^{x_1, x_3} \otimes \mu^{x_2, x_4}, \\ \mu_{\nu=(2,2), \beta=2, n}^{x_1, x_2, x_3, x_4} &= n^4 \mu^{x_1, x_2} \otimes \mu^{x_3, x_4} + n^2 \mu^{x_1, x_4} \otimes \mu^{x_2, x_3} + n^2 \mu^{x_1, x_3} \otimes \mu^{x_2, x_4}, \\ \mu_{\nu=(2), \beta=4, n}^{x_1, x_2} &= (2n^2 - n) \mu^{x_1, x_2}, & \mu_{\nu=(1,1), \beta=4, n}^{x_1, x_2} &= n \mu^{x_1, x_2}, \\ \mu_{\nu=(4), \beta=4, n}^{x_1, x_2, x_3, x_4} &= (4n^3 - 4n^2 + n) \mu^{x_1, x_2} \otimes \mu^{x_3, x_4} + (4n^3 - 4n^2 + n) \mu^{x_1, x_4} \otimes \mu^{x_2, x_3} \\ &+ (-2n^2 + 3n) \mu^{x_1, x_3} \otimes \mu^{x_2, x_4}, \\ \mu_{\nu=(2,2), \beta=4, n}^{x_1, x_2, x_3, x_4} &= (4n^4 - 4n^3 + n^2) \mu^{x_1, x_2} \otimes \mu^{x_3, x_4} + (2n^2 - n) \mu^{x_1, x_4} \otimes \mu^{x_2, x_3} \\ &+ (2n^2 - n) \mu^{x_1, x_3} \otimes \mu^{x_2, x_4}. \end{aligned}$$

In the examples of Section 2.5, the terms in $\mu_{\nu=(4,3,1), \beta, n}^{x_1, x_2, \dots, x_8}$ corresponding to the pairings displayed on Figures 3, respectively 4, are measures of form $\mu^{x_1, x_3} \otimes \mu^{x_2, x_4} \otimes \mu^{x_5, x_8} \otimes \mu^{x_6, x_7}$ with prefactors $w_{\nu=(4,3,1), \beta}(\rho) n^3$, respectively $w_{\nu=(4,3,1), \beta}(\rho) n^2$.

Theorem 3.1. *For $\beta \in \{1, 2\}$ and F a bounded measurable function $\mathbb{R}^V \rightarrow \mathbb{R}$, one has the following equality:*

$$\begin{aligned} &\left\langle \left(\prod_{l=1}^{m(\nu)} \text{Tr} \left(\prod_{k=\nu_1+\dots+\nu_{l-1}+1}^{\nu_1+\dots+\nu_l} \Phi(x_k) \right) \right) F(\text{Tr}(\Phi^2)/2) \right\rangle_{\beta, n} \\ &= \int_{\gamma_1, \dots, \gamma_{|\nu|/2}} \left\langle F(\text{Tr}(\Phi^2)/2 + L(\gamma_1) + \dots + L(\gamma_{|\nu|/2})) \right\rangle_{\beta, n} \mu_{\nu, \beta, n}^{x_1, x_2, \dots, x_{|\nu|}}(d\gamma_1, \dots, d\gamma_{|\nu|/2}), \end{aligned}$$

where $\langle \cdot \rangle_{\beta, n} \mu_{\nu, \beta, n}^{x_1, x_2, \dots, x_{|\nu|}}(\cdot)$ is a product measure. For $\beta = 4$,

$$\begin{aligned} & \left\langle \left(\prod_{l=1}^{m(\nu)} \operatorname{Re} \left(\operatorname{Tr} \left(\prod_{k=\nu_1+\dots+\nu_{l-1}+1}^{\nu_1+\dots+\nu_l} \Phi(x_k) \right) \right) \right) F(\operatorname{Tr}(\Phi^2)/2) \right\rangle_{\beta=4, n} \\ &= \int_{\gamma_1, \dots, \gamma_{|\nu|/2}} \left\langle F(\operatorname{Tr}(\Phi^2)/2 + L(\gamma_1) + \dots + L(\gamma_{|\nu|/2})) \right\rangle_{\beta=4, n} \mu_{\nu, \beta=4, n}^{x_1, x_2, \dots, x_{|\nu|}}(d\gamma_1, \dots, d\gamma_{|\nu|/2}). \end{aligned}$$

In particular, if $\lambda_1(x) \geq \lambda_2(x) \geq \dots \lambda_n(x)$ is the family of eigenvalues of $\Phi(x)$, $x \in V$, and

$$x_1 = \dots = x_{\nu_1}, \quad x_{\nu_1+1} = \dots = x_{\nu_1+\nu_2}, \quad \dots, \quad x_{|\nu|-\nu_{m(\nu)}+1} = \dots = x_{|\nu|},$$

then, for $\beta \in \{1, 2, 4\}$,

$$\begin{aligned} (3.2) \quad & \left\langle \prod_{l=1}^{m(\nu)} \left(\sum_{i=1}^n \lambda_i(x_{\nu_1+\dots+\nu_l})^{\nu_l} \right) F\left(\frac{1}{2} \sum_{i=1}^n \lambda_i^2\right) \right\rangle_{\beta, n} \\ &= \int_{\gamma_1, \dots, \gamma_{|\nu|/2}} \left\langle F\left(\frac{1}{2} \sum_{i=1}^n \lambda_i^2 + L(\gamma_1) + \dots + L(\gamma_{|\nu|/2})\right) \right\rangle_{\beta, n} \mu_{\nu, \beta, n}^{(x_{\nu_1, \nu_1}), \dots, (x_{|\nu|, \nu_{m(\nu)})}}(d\gamma_1, \dots, d\gamma_{|\nu|/2}), \end{aligned}$$

where the notation $(x_{\nu_1+\dots+\nu_l}, \nu_l)$ means that $x_{\nu_1+\dots+\nu_l}$ is repeated ν_l times.

Now, we consider a family of $|\nu|$ deterministic square matrices with complex entries of size $n \times n$:

$$\begin{aligned} & A(1, 2), \dots, A(\nu_1 - 1, \nu_1), A(\nu_1, 1), \\ & A(\nu_1 + 1, \nu_1 + 2), \dots, A(\nu_1 + \nu_2 - 1, \nu_1 + \nu_2), A(\nu_1 + \nu_2, \nu_1 + 1), \\ & \dots, A(|\nu| - \nu_{m(\nu)} + 1, |\nu| - \nu_{m(\nu)} + 2), \dots, A(|\nu| - 1, |\nu|), A(|\nu|, |\nu| - \nu_{m(\nu)} + 1). \end{aligned}$$

Note that by convention, for each $l \in \{1, \dots, m(\nu)\}$ such that $\nu_l = 1$, we have a single matrix $A(\nu_1 + \dots + \nu_l, \nu_1 + \dots + \nu_l)$. For $l \in \{1, \dots, m(\nu)\}$, $\Pi_{\nu, l}(\Phi, A)$ will denote the product

$$(3.3) \quad \Pi_{\nu, l}(\Phi, A) = \Phi(x_{\nu_1+\dots+\nu_{l-1}+1}) A(\nu_1 + \dots + \nu_{l-1} + 1, \nu_1 + \dots + \nu_{l-1} + 2) \Phi(x_{\nu_1+\dots+\nu_{l-1}+2}) \dots A(\nu_1 + \dots + \nu_l - 1, \nu_1 + \dots + \nu_l) \Phi(x_{\nu_1+\dots+\nu_l}) A(\nu_1 + \dots + \nu_l, \nu_1 + \dots + \nu_{l-1} + 1).$$

In case $\nu_l = 1$, $\Pi_{\nu, l}(\Phi, A) = \Phi(x_{\nu_1+\dots+\nu_l}) A(\nu_1 + \dots + \nu_l, \nu_1 + \dots + \nu_l)$.

Next, for $\beta \in \{1, 2\}$, we will write an isomorphism for

$$\left\langle \left(\prod_{l=1}^{m(\nu)} \operatorname{Tr} \left(\Pi_{\nu, l}(\Phi, A) \right) \right) F(\operatorname{Tr}(\Phi^2)/2) \right\rangle_{\beta, n}.$$

It will involve the following (complex valued) measure $\mu_{\nu, \beta, n, A}^{x_1, x_2, \dots, x_{|\nu|}}$ on $|\nu|/2$ nearest neighbor paths on \mathcal{G} :

$$\mu_{\nu, \beta, n, A}^{x_1, x_2, \dots, x_{|\nu|}}(d\gamma_1, \dots, d\gamma_{|\nu|/2}) = \sum_{\rho \in \mathcal{R}_\nu} w_{\nu, \beta}(\rho) \left(\prod_{\mathfrak{t} \in \mathcal{T}_\nu(\rho)} \operatorname{Tr}(\Pi_{\mathfrak{t}}(A)) \right) \mu_{\mathbf{P}_\nu(\rho)}^{x_1, x_2, \dots, x_{|\nu|}}(d\gamma_1, \dots, d\gamma_{|\nu|/2}).$$

$\Pi_{\mathfrak{t}}(A)$ is a product of matrices of form $A(k, k')$ or $A(k', k)^\top$, where \top denotes the transpose (and not the adjoint). For each sequence $k \rightarrow k'$ in the trail \mathfrak{t} we add the factor $A(k, k')$ to the product, and for each sequence $k \leftarrow k'$, we add the factor $A(k', k)^\top$, all by respecting cyclic order of the trail. For instance, if

$$(3.4) \quad \mathfrak{t} = (5, \rightarrow, 6, \dot{=} 7, \leftarrow, 6, \dot{=} 7, \rightarrow, 5, \dot{=} 8, \rightarrow, 8, \dot{=} 8),$$

which is one of the trails on Figure (4), then

$$\Pi_{\mathfrak{t}}(A) = A(5, 6)A(7, 6)^\top A(7, 5)A(8, 8).$$

Note that while the product $\Pi_{\mathfrak{t}}(A)$ depends on the particular representative of the equivalence class \mathfrak{t} , its trace does not. Indeed, the trace is invariant by a cyclic permutation of the factors.

Moreover, reversing the direction of a representative of \mathfrak{t} amounts to taking the transpose of the product, which has the same trace.

Next we give examples of measures $\mu_{\nu,\beta,n,A}^{x_1,x_2,\dots,x_{|\nu|}}$:

$$\begin{aligned}\mu_{\nu=(2),\beta=1,n,A}^{x_1,x_2} &= \left(\frac{1}{2} \operatorname{Tr}(A(1,2)) \operatorname{Tr}(A(2,1)) + \frac{1}{2} \operatorname{Tr}(A(1,2)A(2,1)^\top) \right) \mu^{x_1,x_2}, \\ \mu_{\nu=(1,1),\beta=1,n,A}^{x_1,x_2} &= \left(\frac{1}{2} \operatorname{Tr}(A(1,1)A(2,2)) + \frac{1}{2} \operatorname{Tr}(A(1,1)A(2,2)^\top) \right) \mu^{x_1,x_2}, \\ \mu_{\nu=(2),\beta=2,n,A}^{x_1,x_2} &= \operatorname{Tr}(A(1,2)) \operatorname{Tr}(A(2,1)) \mu^{x_1,x_2}, \\ \mu_{\nu=(1,1),\beta=2,n,A}^{x_1,x_2} &= \operatorname{Tr}(A(1,1)A(2,2)) \mu^{x_1,x_2}.\end{aligned}$$

Note that if all of the matrices $A(k,k')$ are equal to I_n , the $n \times n$ identity matrix, then $\mu_{\nu,\beta,n,A}^{x_1,x_2,\dots,x_{|\nu|}}$ is just $\mu_{\nu,\beta,n}^{x_1,x_2,\dots,x_{|\nu|}}$, because all of the traces $\operatorname{Tr}(\Pi_{\mathfrak{t}}(A))$ equal then n .

For $\beta = 4$, we will need a slightly different setting. We consider matrices $\tilde{A}(k,k')$, with the same indices (k,k') as for the matrices $A(k,k')$, but instead the $\tilde{A}(k,k')$'s are $n \times n$ quaternion valued. The (signed) measure $\mu_{\nu,\beta=4,n,\tilde{A}}^{x_1,x_2,\dots,x_{|\nu|}}$ will be defined as follows

$$\mu_{\nu,\beta=4,n,\tilde{A}}^{x_1,x_2,\dots,x_{|\nu|}}(d\gamma_1, \dots, d\gamma_{|\nu|/2}) = \sum_{\rho \in \mathcal{R}_\nu} w_{\nu,\beta=4}(\rho) \left(\prod_{\mathfrak{t} \in \mathcal{T}_\nu(\rho)} \operatorname{Re}(\operatorname{Tr}(\tilde{\Pi}_{\mathfrak{t}}(\tilde{A}))) \right) \mu_{\mathbf{p}_\nu(\rho)}^{x_1,x_2,\dots,x_{|\nu|}}(d\gamma_1, \dots, d\gamma_{|\nu|/2}).$$

$\tilde{\Pi}_{\mathfrak{t}}(\tilde{A})$ is a product of matrices of form $\tilde{A}(k,k')$ or $\tilde{A}(k',k)^*$. For each sequence $k \rightarrow k'$ in the trail \mathfrak{t} we add the factor $\tilde{A}(k,k')$ to the product, and for each sequence $k \leftarrow k'$, we add the factor $\tilde{A}(k',k)^*$, all by respecting cyclic order of the trail. Let us emphasize that for $\beta = 4$, we use the quaternion adjoint $*$, and not the transpose $^\top$. For the trail (3.4), one gets

$$\tilde{\Pi}_{\mathfrak{t}}(\tilde{A}) = \tilde{A}(5,6)\tilde{A}(7,6)^*\tilde{A}(7,5)\tilde{A}(8,8).$$

Next are two examples:

$$\begin{aligned}\mu_{\nu=(2),\beta=4,n,\tilde{A}}^{x_1,x_2} &= \left(2 \operatorname{Re}(\operatorname{Tr}(\tilde{A}(1,2)) \operatorname{Tr}(\tilde{A}(2,1))) - \operatorname{Re}(\operatorname{Tr}(\tilde{A}(1,2)\tilde{A}(2,1)^*)) \right) \mu^{x_1,x_2}, \\ \mu_{\nu=(1,1),\beta=4,n,\tilde{A}}^{x_1,x_2} &= \left(\frac{1}{2} \operatorname{Re}(\operatorname{Tr}(\tilde{A}(1,1)\tilde{A}(2,2))) + \frac{1}{2} \operatorname{Re}(\operatorname{Tr}(\tilde{A}(1,1)\tilde{A}(2,2)^*)) \right) \mu^{x_1,x_2}.\end{aligned}$$

Theorem 3.2. For $\beta \in \{1, 2\}$ and F a bounded measurable function $\mathbb{R}^V \rightarrow \mathbb{R}$, one has the following equality:

$$\begin{aligned}& \left\langle \left(\prod_{l=1}^{m(\nu)} \operatorname{Tr}(\Pi_{\nu,l}(\Phi, A)) \right) F(\operatorname{Tr}(\Phi^2)/2) \right\rangle_{\beta,n} \\ &= \int_{\gamma_1, \dots, \gamma_{|\nu|/2}} \left\langle F(\operatorname{Tr}(\Phi^2)/2 + L(\gamma_1) + \dots + L(\gamma_{|\nu|/2})) \right\rangle_{\beta,n} \mu_{\nu,\beta,n,A}^{x_1,x_2,\dots,x_{|\nu|}}(d\gamma_1, \dots, d\gamma_{|\nu|/2}),\end{aligned}$$

where $\langle \cdot \rangle_{\beta,n} \mu_{\nu,\beta,n,A}^{x_1,x_2,\dots,x_{|\nu|}}(\cdot)$ is a product measure. For $\beta = 4$,

$$\begin{aligned}& \left\langle \left(\prod_{l=1}^{m(\nu)} \operatorname{Re}(\operatorname{Tr}(\Pi_{\nu,l}(\Phi, \tilde{A}))) \right) F(\operatorname{Tr}(\Phi^2)/2) \right\rangle_{\beta=4,n} \\ &= \int_{\gamma_1, \dots, \gamma_{|\nu|/2}} \left\langle F(\operatorname{Tr}(\Phi^2)/2 + L(\gamma_1) + \dots + L(\gamma_{|\nu|/2})) \right\rangle_{\beta,n} \mu_{\nu,\beta=4,n,\tilde{A}}^{x_1,x_2,\dots,x_{|\nu|}}(d\gamma_1, \dots, d\gamma_{|\nu|/2}).\end{aligned}$$

3.2. Isomorphisms and topological expansion for matrix valued fields twisted by a connection. $\mathbb{U}_{\beta,n}$ will denote the group $O(n)$ if $\beta = 1$, $U(n)$ if $\beta = 2$, and $U(n, \mathbb{H})$ if $\beta = 4$. We consider a connection on \mathcal{G} , $(U(x,y))_{\{x,y\} \in E}$, with $U(x,y) \in \mathbb{U}_{\beta,n}$ and

$$U(y,x) = U(x,y)^* = U(x,y)^{-1}.$$

Let be $\langle \cdot \rangle_{\beta,n}^U$ the following probability measure on $(E_{\beta,n})^V$, defined by the density

$$(3.5) \quad \frac{1}{Z_{\beta,n}^{\mathcal{G},U}} \exp \left(-\frac{1}{2} \sum_{x \in V} \kappa(x) \text{Tr}(M(x)^2) - \frac{1}{2} \sum_{\{x,y\} \in E} C(x,y) \text{Tr}((M(y) - U(y,x)M(x)U(x,y))^2) \right).$$

Note that if $\{x,y\} \in E$, then

$$\text{Tr}((M(x) - U(x,y)M(y)U(y,x))^2) = \text{Tr}((M(y) - U(y,x)M(x)U(x,y))^2).$$

Φ will denote the field under the measure $\langle \cdot \rangle_{\beta,n}^U$. It is the matrix valued Gaussian free field twisted by the connection U . If $(\mathfrak{U}(x))_{x \in V}$ is a gauge transformation, then $(\mathfrak{U}(x)^{-1}\Phi(x)\mathfrak{U}(x))_{x \in V}$ is the field associated to the connection $(\mathfrak{U}(x)^{-1}U(x,y)\mathfrak{U}(y))_{\{x,y\} \in E}$. If the connection U is flat, then for all $x \in V$, $\Phi(x)/\sqrt{G(x,x)}$ is distributed as a $\text{G}\beta\text{E}(n)$ matrix. This is not necessarily the case if the connection is not flat.

As in Section 3.1, we take $\nu = (\nu_1, \nu_2, \dots, \nu_{m(\nu)})$, where for all $l \in \{1, 2, \dots, m(\nu)\}$, $\nu_l \in \mathbb{N} \setminus \{0\}$, and $|\nu|$ even. Let $x_1, x_2, \dots, x_{|\nu|}$ be vertices in V , not necessarily distinct. As in Section 3.1, the $A(k, k')$'s respectively $\tilde{A}(k, k')$'s are families of $|\nu|$ square matrices with complex, respectively quaternionic entries of size $n \times n$. The products $\Pi_{\nu,l}(\Phi, A)$ and $\Pi_{\nu,l}(\Phi, \tilde{A})$ are defined as in (3.3).

Next we will write an isomorphism for

$$\left\langle \left(\prod_{l=1}^{m(\nu)} \text{Tr} \left(\Pi_{\nu,l}(\Phi, A) \right) \right) F(\text{Tr}(\Phi^2)/2) \right\rangle_{\beta,n}^U, \quad \beta \in \{1, 2\},$$

and for

$$\left\langle \left(\prod_{l=1}^{m(\nu)} \text{Re} \left(\text{Tr} \left(\Pi_{\nu,l}(\Phi, \tilde{A}) \right) \right) \right) F(\text{Tr}(\Phi^2)/2) \right\rangle_{\beta,n}^U, \quad \beta = 4.$$

For $\beta = 1$, we introduce the following (signed) measure $\mu_{\nu, \beta=1, n, A, U}^{x_1, x_2, \dots, x_{|\nu|}}$ on $|\nu|/2$ nearest neighbor paths on \mathcal{G} :

$$\mu_{\nu, \beta=1, n, A, U}^{x_1, x_2, \dots, x_{|\nu|}}(d\gamma_1, \dots, d\gamma_{|\nu|/2}) = \sum_{\rho \in \mathcal{R}_\nu} w_{\nu, \beta=1}(\rho) \left(\prod_{\mathfrak{t} \in \mathcal{T}_\nu(\rho)} \text{Tr}(\Pi_{\mathfrak{t}}(U, A)(\gamma_1, \dots, \gamma_{|\nu|/2})) \right) \mu_{\mathbf{p}_\nu(\rho)}^{x_1, x_2, \dots, x_{|\nu|}}(d\gamma_1, \dots, d\gamma_{|\nu|/2}).$$

$\Pi_{\mathfrak{t}}(U, A)(\gamma_1, \dots, \gamma_{|\nu|/2})$ is a product of matrices of form $A(k, k')$ or $A(k', k)^\top$, and $\text{hol}^U(\gamma)$ or $\text{hol}^U(\gamma)^*$, γ being one of the paths. For each sequence $k \rightarrow k'$ in the trail \mathfrak{t} we add the factor $A(k, k')$ to the product, and for each sequence $k \leftarrow k'$, we add the factor $A(k', k)^\top$, as in the construction of $\mu_{\nu, \beta=1, n, A}^{x_1, x_2, \dots, x_{|\nu|}}$. Moreover, for each sequence $k \doteq k'$ with $k < k'$, a measure $\mu^{x_k, x_{k'}}(d\gamma_i)$ is present, and we add to the product the factor $\text{hol}^U(\gamma_i)$. For each sequence $k \doteq k'$ with $k > k'$, a measure $\mu^{x_{k'}, x_k}(d\gamma_i)$ is present, and we add to the product the factor $\text{hol}^U(\gamma_i)^* = \text{hol}^U(\gamma_i)^\top$. In the product the factors respect the cyclic order on the trail.

For $\beta = 2$, we will need oriented trails. The (complex) measure $\mu_{\nu, \beta=2, n, A, U}^{x_1, x_2, \dots, x_{|\nu|}}$ on $|\nu|/2$ nearest neighbor paths on \mathcal{G} is as follows:

$$\mu_{\nu, \beta=2, n, A, U}^{x_1, x_2, \dots, x_{|\nu|}}(d\gamma_1, \dots, d\gamma_{|\nu|/2}) = \sum_{\rho \in \mathcal{R}_\nu} w_{\nu, \beta=2}(\rho) \left(\prod_{\vec{\mathfrak{t}} \in \vec{\mathcal{T}}_\nu(\rho)} \text{Tr}(\Pi_{\vec{\mathfrak{t}}}(U, A)(\gamma_1, \dots, \gamma_{|\nu|/2})) \right) \mu_{\mathbf{p}_\nu(\rho)}^{x_1, x_2, \dots, x_{|\nu|}}(d\gamma_1, \dots, d\gamma_{|\nu|/2}).$$

$\Pi_{\vec{\mathfrak{t}}}(U, A)(\gamma_1, \dots, \gamma_{|\nu|/2})$ is a product of matrices of form $A(k, k')$ and $\text{hol}^U(\gamma)$ or $\text{hol}^U(\gamma)^*$, γ being one of the paths. For each sequence $k \rightarrow k'$ in the oriented trail $\vec{\mathfrak{t}}$ we add the factor $A(k, k')$ to the product. Moreover, for each sequence $k \doteq k'$ with $k < k'$, a measure $\mu^{x_k, x_{k'}}(d\gamma_i)$ is present, and we add to the product the factor $\text{hol}^U(\gamma_i)$. For each sequence $k \doteq k'$ with $k > k'$, a measure $\mu^{x_{k'}, x_k}(d\gamma_i)$ is present, and we add to the product the factor $\text{hol}^U(\gamma_i)^*$. In the product

the factors respect the cyclic order on the trail. The reason we use oriented trails is that the reversal of the orientation of the trail changes the trace, as complex adjoints of holonomies appear.

For $\beta = 4$, we return to unoriented trails. We introduce the (signed) measure $\mu_{\nu, \beta=4, n, \tilde{A}, U}^{x_1, x_2, \dots, x_{|\nu|}}$ on $|\nu|/2$ nearest neighbor paths in \mathcal{G} , constructed as follows:

$$\mu_{\nu, \beta=4, n, \tilde{A}, U}^{x_1, x_2, \dots, x_{|\nu|}}(d\gamma_1, \dots, d\gamma_{|\nu|/2}) = \sum_{\rho \in \mathcal{R}_\nu} w_{\nu, \beta=4}(\rho) \left(\prod_{\mathfrak{t} \in \mathcal{T}_\nu(\rho)} \text{Re}(\text{Tr}(\tilde{\Pi}_{\mathfrak{t}}(U, \tilde{A})(\gamma_1, \dots, \gamma_{|\nu|/2}))) \right) \mu_{\mathbf{p}_\nu(\rho)}^{x_1, x_2, \dots, x_{|\nu|}}(d\gamma_1, \dots, d\gamma_{|\nu|/2}).$$

$\tilde{\Pi}_{\mathfrak{t}}(U, \tilde{A})(\gamma_1, \dots, \gamma_{|\nu|/2})$ is a product of matrices of form $\tilde{A}(k, k')$ or $\tilde{A}(k', k)^*$, and $\text{hol}^U(\gamma)$ or $\text{hol}^U(\gamma)^*$, γ being one of the paths. For each sequence $k \rightarrow k'$ in the trail \mathfrak{t} we add the factor $\tilde{A}(k, k')$ to the product, and for each sequence $k \leftarrow k'$, we add the factor $\tilde{A}(k', k)^*$. Moreover, for each sequence $k \doteq k'$ with $k < k'$, a measure $\mu^{x_k, x_{k'}}(d\gamma_i)$ is present, and we add to the product the factor $\text{hol}^U(\gamma_i)$. For each sequence $k \doteq k'$ with $k > k'$, a measure $\mu^{x_{k'}, x_k}(d\gamma_i)$ is present, and we add to the product the factor $\text{hol}^U(\gamma_i)^*$. In the product the factors respect the cyclic order on the trail.

Next are some examples of measures $\mu_{\nu, \beta, n, A, U}^{x_1, x_2, \dots, x_{|\nu|}}$ and $\mu_{\nu, \beta=4, n, \tilde{A}, U}^{x_1, x_2, \dots, x_{|\nu|}}$:

$$\begin{aligned} \mu_{\nu=(2), \beta=1, n, A, U}^{x_1, x_2} &= \left(\frac{1}{2} \text{Tr}(A(1, 2)\text{hol}^U(\gamma)^*) \times \text{Tr}(A(2, 1)\text{hol}^U(\gamma)) \right. \\ &\quad \left. + \frac{1}{2} \text{Tr}(A(1, 2)\text{hol}^U(\gamma)^* A(2, 1)^\top \text{hol}^U(\gamma)) \right) \mu^{x_1, x_2}(d\gamma), \\ \mu_{\nu=(1,1), \beta=1, n, A, U}^{x_1, x_2} &= \left(\frac{1}{2} \text{Tr}(A(1, 1)\text{hol}^U(\gamma)A(2, 2)\text{hol}^U(\gamma)^*) \right. \\ &\quad \left. + \frac{1}{2} \text{Tr}(A(1, 1)\text{hol}^U(\gamma)A(2, 2)^\top \text{hol}^U(\gamma)^*) \right) \mu^{x_1, x_2}(d\gamma), \\ \mu_{\nu=(2), \beta=2, n, A, U}^{x_1, x_2} &= \text{Tr}(A(1, 2)\text{hol}^U(\gamma)^*) \times \text{Tr}(A(2, 1)\text{hol}^U(\gamma)) \mu^{x_1, x_2}(d\gamma), \\ \mu_{\nu=(1,1), \beta=2, n, A, U}^{x_1, x_2} &= \text{Tr}(A(1, 1)\text{hol}^U(\gamma)A(2, 2)\text{hol}^U(\gamma)^*) \mu^{x_1, x_2}(d\gamma) \\ \mu_{\nu=(2), \beta=4, n, \tilde{A}, U}^{x_1, x_2} &= \left(2 \text{Re}(\text{Tr}(\tilde{A}(1, 1)\text{hol}^U(\gamma)\tilde{A}(2, 2)\text{hol}^U(\gamma)^*)) \right. \\ &\quad \left. - \text{Re}(\text{Tr}(\tilde{A}(1, 1)\text{hol}^U(\gamma)\tilde{A}(2, 2)^\top \text{hol}^U(\gamma)^*)) \right) \mu^{x_1, x_2}(d\gamma), \\ \mu_{\nu=(1,1), \beta=4, n, \tilde{A}, U}^{x_1, x_2} &= \left(\frac{1}{2} \text{Re}(\text{Tr}(\tilde{A}(1, 1)\text{hol}^U(\gamma)\tilde{A}(2, 2)\text{hol}^U(\gamma)^*)) \right. \\ &\quad \left. + \frac{1}{2} \text{Re}(\text{Tr}(\tilde{A}(1, 1)\text{hol}^U(\gamma)\tilde{A}(2, 2)^\top \text{hol}^U(\gamma)^*)) \right) \mu^{x_1, x_2}(d\gamma) \end{aligned}$$

Theorem 3.3. For $\beta \in \{1, 2\}$ and F a bounded measurable function $\mathbb{R}^V \rightarrow \mathbb{R}$, one has the following equality:

$$(3.6) \quad \left\langle \left(\prod_{l=1}^{m(\nu)} \text{Tr}(\Pi_{\nu, l}(\Phi, A)) \right) F(\text{Tr}(\Phi^2)/2) \right\rangle_{\beta, n}^U \\ = \int_{\gamma_1, \dots, \gamma_{|\nu|/2}} \left\langle F(\text{Tr}(\Phi^2)/2 + L(\gamma_1) + \dots + L(\gamma_{|\nu|/2})) \right\rangle_{\beta, n}^U \mu_{\nu, \beta, n, A, U}^{x_1, x_2, \dots, x_{|\nu|}}(d\gamma_1, \dots, d\gamma_{|\nu|/2}),$$

where $\langle \cdot \rangle_{\beta,n}^U \mu_{\nu,\beta,n,A,U}^{x_1,x_2,\dots,x_{|\nu|}}(\cdot)$ is a product measure. For $\beta = 4$,

$$(3.7) \quad \left\langle \left(\prod_{l=1}^{m(\nu)} \operatorname{Re} \left(\operatorname{Tr} \left(\Pi_{\nu,l}(\Phi, \tilde{A}) \right) \right) \right) F(\operatorname{Tr}(\Phi^2)/2) \right\rangle_{\beta=4,n}^U \\ = \int_{\gamma_1, \dots, \gamma_{|\nu|/2}} \left\langle F(\operatorname{Tr}(\Phi^2)/2 + L(\gamma_1) + \dots + L(\gamma_{|\nu|/2})) \right\rangle_{\beta=4,n}^U \mu_{\nu,\beta=4,n,\tilde{A},U}^{x_1,x_2,\dots,x_{|\nu|}}(d\gamma_1, \dots, d\gamma_{|\nu|/2}).$$

Remark 3.4. Note that in the measures $\mu_{\nu,\beta,n,A,U}^{x_1,x_2,\dots,x_{|\nu|}}(d\gamma_1, \dots, d\gamma_{|\nu|/2})$ for $\beta \in \{1, 2\}$, and in $\mu_{\nu,\beta=4,n,\tilde{A},U}^{x_1,x_2,\dots,x_{|\nu|}}(d\gamma_1, \dots, d\gamma_{|\nu|/2})$ the holonomy along each path γ_i appears twice.

Remark 3.5. Consider the particular case when the connection U is flat, i.e. for any close path (loop) γ , $\operatorname{hol}^U(\gamma) = I_n$. There is a gauge transformation $\mathfrak{U} : V \rightarrow \mathbb{U}_{\beta,n}$

$$\forall x, y \in V \text{ such that } \{x, y\} \in E, \quad \mathfrak{U}(x)^{-1} \mathfrak{U}(y) = U(x, y).$$

Then for any $x, y \in V$ and γ nearest neighbor path from x to y ,

$$\operatorname{hol}^U(\gamma) = \mathfrak{U}(x)^{-1} \mathfrak{U}(y).$$

The field Φ under $\langle \cdot \rangle_{\beta,n}^U$ has same law as $(\mathfrak{U}(x)^{-1} \Phi(x) \mathfrak{U}(x))_{x \in V}$ under $\langle \cdot \rangle_{\beta,n}$. So, in the particular case of a flat connection, Theorem 3.3 follows directly from Theorem 3.2.

A special case of particular interest in Theorem 3.3 is when all the matrices $A(k, k')$ and $\tilde{A}(k, k')$ equal I_n and

$$x_1 = \dots = x_{\nu_1}, \quad x_{\nu_1+1} = \dots = x_{\nu_1+\nu_2}, \quad \dots, \quad x_{|\nu|-\nu_{m(\nu)}+1} = \dots = x_{|\nu|}.$$

Then, on the left-hand side of (3.6), we have

$$\forall l \in \{1, \dots, m(\nu)\}, \operatorname{Tr}(\Pi_{\nu,l}(\Phi, I_n)) = \sum_{i=1}^n \lambda_i(x_{\nu_1+\dots+\nu_l})^{\nu_l},$$

where $\lambda_1(x) \geq \lambda_2(x) \geq \dots \geq \lambda_n(x)$ is the family of eigenvalues of Φ . On the right-hand side of (3.6) appears a product of Wilson loops. Indeed, each $\Pi_{\mathfrak{T}}(U, I_n)(\gamma_1, \dots, \gamma_{|\nu|/2})$, $\Pi_{\overline{\mathfrak{T}}}(\tilde{A}, I_n)(\gamma_1, \dots, \gamma_{|\nu|/2})$ and $\tilde{\Pi}_{\mathfrak{T}}(U, I_n)(\gamma_1, \dots, \gamma_{|\nu|/2})$ is then a holonomy along a loop formed by concatenating some of the paths γ_k . In the example of Figure 3, the holonomies are

$$\operatorname{hol}^U(\gamma_1) \operatorname{hol}^U(\gamma_2) \operatorname{hol}^U(\overleftarrow{\gamma_1}) \operatorname{hol}^U(\overrightarrow{\gamma_2}), \quad \operatorname{hol}^U(\gamma_3) \operatorname{hol}^U(\gamma_4) \operatorname{hol}^U(\overleftarrow{\gamma_4}), \quad \text{and} \quad \operatorname{hol}^U(\overleftarrow{\gamma_3}).$$

where γ_1 and γ_2 are paths from x_4 to x_4 , γ_3 is a path from x_7 to x_7 and γ_4 a path from x_7 to x_8 . In the example of Figure 4, we get

$$\operatorname{hol}^U(\gamma_1) \operatorname{hol}^U(\overrightarrow{\gamma_2}) \operatorname{hol}^U(\overleftarrow{\gamma_1}) \operatorname{hol}^U(\overrightarrow{\gamma_2}), \quad \text{and} \quad \operatorname{hol}^U(\gamma_3) \operatorname{hol}^U(\gamma_3) \operatorname{hol}^U(\gamma_4) \operatorname{hol}^U(\overleftarrow{\gamma_4}).$$

Note that since the joint distribution of all eigenvalues above all vertices is invariant under gauge transformations, it is natural that expectations with respect to only these eigenvalues involve only Wilson loops. We summarize this paragraph in the following corollary.

Corollary 3.6. For $\beta \in \{1, 2, 4\}$ and U a non-flat connection, in the isomorphism for

$$\left\langle \prod_{l=1}^{m(\nu)} \left(\sum_{i=1}^n \lambda_i(x_{\nu_1+\dots+\nu_l})^{\nu_l} \right) F\left(\frac{1}{2} \sum_{i=1}^n \lambda_i^2\right) \right\rangle_{\beta,n}^U,$$

compared to the case of a flat connection (3.2), each power of n in the topological expansion, that is $n^{f_\nu(\rho)}$, is replaced by a product of $f_\nu(\rho)$ Wilson loops corresponding to holonomies along random walk loops, one for each border of the ribbon in the ribbon pairing ρ .

Next we observe that the law of $(\operatorname{Tr}(\Phi(x)))_{x \in V}$ is the same whatever the connection U .

Corollary 3.7. For $\beta \in \{1, 2, 4\}$, the field

$$\left(\frac{1}{\sqrt{n}} \operatorname{Tr}(\Phi(x))\right)_{x \in V} = \left(\frac{1}{\sqrt{n}} \sum_i^n \lambda_i(x)\right)_{x \in V}$$

is distributed under $\langle \cdot \rangle_{\beta, n}^U$ as a scalar GFF (2.2). In particular, this law is the same whatever the connection U .

Proof. The field $(\operatorname{Tr}(\Phi(x)))_{x \in V}$ is centered real-valued Gaussian under $\langle \cdot \rangle_{\beta, n}^U$, because so are the diagonal entries of matrices $(\Phi(x))_{x \in V}$. In the expression of two-point correlations

$$\langle \operatorname{Tr}(\Phi(x)) \operatorname{Tr}(\Phi(y)) \rangle_{\beta, n}^U$$

appear only ribbon graphs with one border component ($f_{\nu=(1,1)}(\rho) = 1$). Thus, the two-point correlations are expressed with

$$\operatorname{Tr}(\operatorname{hol}^U(\gamma) \operatorname{hol}^U(\gamma)^*) \mu^{x,y}(d\gamma) = n \mu^{x,y}(d\gamma).$$

Hence,

$$\langle \operatorname{Tr}(\Phi(x)) \operatorname{Tr}(\Phi(y)) \rangle_{\beta, n}^U = n G(x, y). \quad \square$$

4. PROOFS

4.1. Proof of Theorem 3.2. We give the proof of Theorem 3.2, which contains Theorem 3.1 as a special case. A possible approach would be to develop the product of traces on the left-hand side of the isomorphism identity, apply Theorem 2.1, and then recombine the terms to get the right-hand side of the isomorphism identity. This works well for $\beta \in \{1, 2\}$ (see [Zvo97] for one matrix integrals), but for $\beta = 4$ this approach is less tractable because of the non-commutativity of quaternions. Instead, we will rely on a recurrence over the numbers of edges $|\nu|/2$, similar to that in [BP09].

We use the notations of Section 3.1. For $\beta \in \{1, 2, 4\}$, let $(M^{(i)})_{i \geq 1}$ be an i.i.d. sequence of $G\beta E(n)$ matrices with distribution (2.5). Given $\mathbf{p} = \{\{a_1, b_1\}, \dots, \{a_{|\nu|/2}, b_{|\nu|/2}\}\}$ a partition of $\{1, \dots, |\nu|\}$ in pairs and $l \in \{1, \dots, m(\nu)\}$, $\Pi_{\nu, l, \mathbf{p}}(M, A)$ will denote the product of matrices

$$\begin{aligned} \Pi_{\nu, l, \mathbf{p}}(M, A) &= M^{(I_{\nu_1 + \dots + \nu_{l-1} + 1})} A(\nu_1 + \dots + \nu_{l-1} + 1, \nu_1 + \dots + \nu_{l-1} + 2) M^{(I_{\nu_1 + \dots + \nu_{l-1} + 2})} \\ &\dots A(\nu_1 + \dots + \nu_l - 1, \nu_1 + \dots + \nu_l) M^{(I_{\nu_1 + \dots + \nu_l})} A(\nu_1 + \dots + \nu_l, \nu_1 + \dots + \nu_{l-1} + 1), \end{aligned}$$

where the index I_k is such that

$$k \in \{a_{I_k}, b_{I_k}\}.$$

Note that how exactly the pairs $\{a_i, b_i\}$ in the partition \mathbf{p} are ordered will not be important. The product $\Pi_{\nu, l, \mathbf{p}}(M, \tilde{A})$ is defined similarly, with matrices $\tilde{A}(k, k')$ instead of $A(k, k')$. Note that in the products

$$\prod_{l=1}^{m(\nu)} \operatorname{Tr} \left(\Pi_{\nu, l, \mathbf{p}}(M, A) \right) \quad \text{and} \quad \prod_{l=1}^{m(\nu)} \operatorname{Re} \left(\operatorname{Tr} \left(\Pi_{\nu, l, \mathbf{p}}(M, \tilde{A}) \right) \right),$$

each $M^{(i)}$ for $i \in \{1, \dots, |\nu|/2\}$ appears exactly twice, at positions corresponding to a pair in the partition \mathbf{p} .

Lemma 4.1. *Then, for $\beta \in \{1, 2\}$ and F a bounded measurable function $\mathbb{R}^V \rightarrow \mathbb{R}$, one has the following equality:*

$$\begin{aligned} & \left\langle \left(\prod_{l=1}^{m(\nu)} \text{Tr} \left(\Pi_{\nu, l}(\Phi, A) \right) \right) F(\text{Tr}(\Phi^2)/2) \right\rangle_{\beta, n} \\ &= \sum_{\substack{\mathbf{p} \text{ partition} \\ \text{of } \{1, \dots, |\nu|\} \\ \text{in pairs}}} \int_{\gamma_1, \dots, \gamma_{|\nu|/2}} \left\langle \left(\prod_{l=1}^{m(\nu)} \text{Tr} \left(\Pi_{\nu, l, \mathbf{p}}(M, A) \right) \right) \right\rangle_{\beta, n} \\ & \times \left\langle F(\text{Tr}(\Phi^2)/2 + L(\gamma_1) + \dots + L(\gamma_{|\nu|/2})) \right\rangle_{\beta, n} \mu_{\mathbf{p}}^{x_1, x_2, \dots, x_{|\nu|}}(d\gamma_1, \dots, d\gamma_{|\nu|/2}). \end{aligned}$$

For $\beta = 4$,

$$\begin{aligned} & \left\langle \left(\prod_{l=1}^{m(\nu)} \text{Re} \left(\text{Tr} \left(\Pi_{\nu, l}(\Phi, \tilde{A}) \right) \right) \right) F(\text{Tr}(\Phi^2)/2) \right\rangle_{\beta=4, n} \\ &= \sum_{\substack{\mathbf{p} \text{ partition} \\ \text{of } \{1, \dots, |\nu|\} \\ \text{in pairs}}} \int_{\gamma_1, \dots, \gamma_{|\nu|/2}} \left\langle \left(\prod_{l=1}^{m(\nu)} \text{Re} \left(\text{Tr} \left(\Pi_{\nu, l, \mathbf{p}}(M, \tilde{A}) \right) \right) \right) \right\rangle_{\beta=4, n} \\ & \times \left\langle F(\text{Tr}(\Phi^2)/2 + L(\gamma_1) + \dots + L(\gamma_{|\nu|/2})) \right\rangle_{\beta=4, n} \mu_{\mathbf{p}}^{x_1, x_2, \dots, x_{|\nu|}}(d\gamma_1, \dots, d\gamma_{|\nu|/2}). \end{aligned}$$

Proof. On the left-hand side one can develop the product of traces

$$\left(\prod_{l=1}^{m(\nu)} \text{Tr} \left(\Pi_{\nu, l}(\Phi, A) \right) \right) \quad \text{and} \quad \text{Re} \left(\text{Tr} \left(\Pi_{\nu, l}(\Phi, \tilde{A}) \right) \right).$$

Then, one applies to each term in the sum Lemma 2.3 (in the particular case $U \equiv I_n$), so as to make appear the entries of $\text{G}\beta\text{E}(n)$ matrices $M^{(i)}$. Then one refactorizes these entries into

$$\left(\prod_{l=1}^{m(\nu)} \text{Tr} \left(\Pi_{\nu, l, \mathbf{p}}(M, A) \right) \right) \quad \text{and} \quad \left(\prod_{l=1}^{m(\nu)} \text{Re} \left(\text{Tr} \left(\Pi_{\nu, l, \mathbf{p}}(M, \tilde{A}) \right) \right) \right). \quad \square$$

It remains to compute the moments

$$\left\langle \left(\prod_{l=1}^{m(\nu)} \text{Tr} \left(\Pi_{\nu, l, \mathbf{p}}(M, A) \right) \right) \right\rangle_{\beta, n} \quad \text{and} \quad \left\langle \left(\prod_{l=1}^{m(\nu)} \text{Re} \left(\text{Tr} \left(\Pi_{\nu, l, \mathbf{p}}(M, \tilde{A}) \right) \right) \right) \right\rangle_{\beta=4, n}.$$

for $|\nu|/2$ i.i.d. $\text{G}\beta\text{E}(n)$ matrices. For $\beta = 4$ we will use expressions of moments of quaternionic Gaussian r.v. that appeared in [BP09].

Lemma 4.2 (Bryc-Pierce [BP09]). *Let $\xi \in \mathbb{H}$ be a quaternionic Gaussian r.v. with distribution*

$$(4.1) \quad \frac{1}{(2\pi)^2} e^{-\frac{1}{2}|q|^2} dq_r dq_i dq_j dq_k.$$

Then, for any $q_1, q_2 \in \mathbb{H}$,

$$\mathbb{E}[\text{Re}(\xi q_1 \bar{\xi} q_2)] = 4 \text{Re}(q_1) \text{Re}(q_2), \quad \mathbb{E}[\text{Re}(\xi q_1 \xi q_2)] = -2 \text{Re}(q_1 \bar{q}_2),$$

$$\mathbb{E}[\text{Re}(\xi q_1) \text{Re}(\bar{\xi} q_2)] = \text{Re}(q_1 q_2), \quad \mathbb{E}[\text{Re}(\xi q_1) \text{Re}(\xi q_2)] = \text{Re}(q_1 \bar{q}_2).$$

Lemma 4.3. Let M be a random $G\beta E(n)$ matrix (2.5), $\beta \in \{1, 2, 4\}$. Let $A^{(1)}, A^{(2)}$ be two deterministic matrices in $\mathcal{M}_n(\mathbb{C})$, and $\tilde{A}^{(1)}, \tilde{A}^{(2)}$ two deterministic matrices in $\mathcal{M}_n(\mathbb{H})$. Then

$$\begin{aligned} \langle \text{Tr}(MA^{(1)}MA^{(2)}) \rangle_{\beta=1,n} &= \frac{1}{2} \text{Tr}(A^{(1)}) \text{Tr}(A^{(2)}) + \frac{1}{2} \text{Tr}(A^{(1)}A^{(2)\top}), \\ \langle \text{Tr}(MA^{(1)}) \text{Tr}(MA^{(2)}) \rangle_{\beta=1,n} &= \frac{1}{2} \text{Tr}(A^{(1)}A^{(2)}) + \frac{1}{2} \text{Tr}(A^{(1)}A^{(2)\top}), \\ \langle \text{Tr}(MA^{(1)}MA^{(2)}) \rangle_{\beta=2,n} &= \text{Tr}(A^{(1)}) \text{Tr}(A^{(2)}), \quad \langle \text{Tr}(MA^{(1)}) \text{Tr}(MA^{(2)}) \rangle_{\beta=2,n} = \text{Tr}(A^{(1)}A^{(2)}), \\ \langle \text{Re}(\text{Tr}(M\tilde{A}^{(1)}M\tilde{A}^{(2)})) \rangle_{\beta=4,n} &= 2 \text{Re}(\text{Tr}(\tilde{A}^{(1)})) \text{Re}(\text{Tr}(\tilde{A}^{(2)})) - \text{Re}(\text{Tr}(\tilde{A}^{(1)}\tilde{A}^{(2)*})), \\ \langle \text{Re}(\text{Tr}(M\tilde{A}^{(1)}) \text{Re}(\text{Tr}(M\tilde{A}^{(2)}))) \rangle_{\beta=4,n} &= \frac{1}{2} \text{Re}(\text{Tr}(\tilde{A}^{(1)}\tilde{A}^{(2)})) + \frac{1}{2} \text{Re}(\text{Tr}(\tilde{A}^{(1)}\tilde{A}^{(2)*})). \end{aligned}$$

Proof. If $\beta \in \{1, 2\}$, the identities follow from the second moments of the entries:

$$\forall i \in \{1, \dots, n\}, \langle M_{ii}^2 \rangle_{\beta=1,n} = 1 = \frac{1}{2} + \frac{1}{2}, \quad \forall i \neq j \in \{1, \dots, n\}, \langle M_{ij}^2 \rangle_{\beta=1,n} = \langle M_{ij}M_{ji} \rangle_{\beta=1,n} = \frac{1}{2},$$

$$\forall i \in \{1, \dots, n\}, \langle M_{ii}^2 \rangle_{\beta=2,n} = 1, \quad \forall i \neq j \in \{1, \dots, n\}, \langle M_{ij}M_{ji} \rangle_{\beta=2,n} = 1,$$

all other second moments being zero.

If $\beta = 4$, the diagonal entries are still real and commute with quaternions, so for all $i \in \{1, \dots, n\}$,

$$\begin{aligned} \langle \text{Re}(\tilde{A}_{ii}^{(1)} M_{ii} \tilde{A}_{ii}^{(2)} M_{ii}) \rangle_{\beta=4,n} &= \text{Re}(\tilde{A}_{ii}^{(1)} \tilde{A}_{ii}^{(2)}) \\ &= 2 \text{Re}(\tilde{A}_{ii}^{(1)}) \text{Re}(\tilde{A}_{ii}^{(2)}) - \text{Re}(\tilde{A}_{ii}^{(1)} \overline{\tilde{A}_{ii}^{(2)}}), \end{aligned}$$

$$\begin{aligned} \langle \text{Re}(\tilde{A}_{ii}^{(1)} M_{ii}) \text{Re}(\tilde{A}_{ii}^{(2)} M_{ii}) \rangle_{\beta=4,n} &= \text{Re}(\tilde{A}_{ii}^{(1)}) \text{Re}(\tilde{A}_{ii}^{(2)}) \\ &= \frac{1}{2} \text{Re}(\tilde{A}_{ii}^{(1)} \tilde{A}_{ii}^{(2)}) + \frac{1}{2} \text{Re}(\tilde{A}_{ii}^{(1)} \overline{\tilde{A}_{ii}^{(2)}}). \end{aligned}$$

For the offdiagonal entries, $\sqrt{2}M_{ij}$ is distributed according to (4.1), so we apply Lemma 4.2. For all $i \neq j \in \{1, \dots, n\}$,

$$\langle \text{Re}(\tilde{A}_{ii}^{(1)} M_{ij} \tilde{A}_{jj}^{(2)} M_{ji}) \rangle_{\beta=4,n} = 2 \text{Re}(\tilde{A}_{ii}^{(1)}) \text{Re}(\tilde{A}_{jj}^{(2)}),$$

$$\langle \text{Re}(\tilde{A}_{ji}^{(1)} M_{ij} \tilde{A}_{ji}^{(2)} M_{ij}) \rangle_{\beta=4,n} = -\text{Re}(\tilde{A}_{ji}^{(1)} \overline{\tilde{A}_{ji}^{(2)}}),$$

$$\langle \text{Re}(\tilde{A}_{ji}^{(1)} M_{ij}) \text{Re}(\tilde{A}_{ij}^{(2)} M_{ji}) \rangle_{\beta=4,n} = \frac{1}{2} \text{Re}(\tilde{A}_{ji}^{(1)} \tilde{A}_{ij}^{(2)}),$$

$$\langle \text{Re}(\tilde{A}_{ji}^{(1)} M_{ij}) \text{Re}(\tilde{A}_{ji}^{(2)} M_{ij}) \rangle_{\beta=4,n} = \frac{1}{2} \text{Re}(\tilde{A}_{ji}^{(1)} \overline{\tilde{A}_{ji}^{(2)}}).$$

All other second moments are zero. □

The following lemma will conclude the proof of Theorem 3.2.

Lemma 4.4. For $\beta \in \{1, 2\}$, and \mathbf{p} a partition in pairs of $\{1, \dots, |\nu|\}$,

$$\left\langle \left(\prod_{l=1}^{m(\nu)} \text{Tr} \left(\Pi_{\nu, l, \mathbf{p}}(M, A) \right) \right) \right\rangle_{\beta, n} = \sum_{\rho \in \mathcal{R}_{\nu, \mathbf{p}}} w_{\nu, \beta}(\rho) \prod_{\mathbf{t} \in \mathcal{T}_{\nu}(\rho)} \text{Tr}(\Pi_{\mathbf{t}}(A)).$$

For $\beta = 4$,

$$\left\langle \left(\prod_{l=1}^{m(\nu)} \text{Re} \left(\text{Tr} \left(\Pi_{\nu, l, \mathbf{p}}(M, \tilde{A}) \right) \right) \right) \right\rangle_{\beta=4, n} = \sum_{\rho \in \mathcal{R}_{\nu, \mathbf{p}}} w_{\nu, \beta=4}(\rho) \prod_{\mathbf{t} \in \mathcal{T}_{\nu}(\rho)} \text{Re}(\text{Tr}(\tilde{\Pi}_{\mathbf{t}}(\tilde{A})).$$

Proof. The proof can be done by induction on $|\nu|/2$. If $|\nu|/2 = 1$, the identities above are exactly those of Lemma 4.3.

If $|\nu|/2 > 1$, we fix $\mathbf{p} = \{\{a_1, b_1\}, \dots, \{a_{|\nu|/2}, b_{|\nu|/2}\}\}$ a partition of $\{1, \dots, |\nu|\}$ in pairs. One can take conditional expectations with respect to $M^{(|\nu|/2)}$ that appears at positions $a_{|\nu|/2}$ and $b_{|\nu|/2}$ (i.e. conditioning on $(M^{(1)}, \dots, M^{(|\nu|/2-1)})$). For that one again applies Lemma 4.3. $\check{\mathbf{p}}$ will denote the partition of $\{1, \dots, |\nu|\} \setminus \{a_{|\nu|/2}, b_{|\nu|/2}\}$ induced by \mathbf{p} . To construct a ribbon pairing $\rho \in \mathcal{R}_{\nu, \mathbf{p}}$ one can proceed as follows.

- (1) First pair the ribbon half-edges $a_{|\nu|/2}$ and $b_{|\nu|/2}$ in an either straight or twisted way.
- (2) Then contract the ribbon edge $\{a_{|\nu|/2}, b_{|\nu|/2}\}$. After this operation one gets a new family of positive integers, $\check{\nu}_{\text{str}}$ if the edge $\{a_{|\nu|/2}, b_{|\nu|/2}\}$ was straight, or $\check{\nu}_{\text{tw}}$ if the edge $\{a_{|\nu|/2}, b_{|\nu|/2}\}$ was twisted, and in both cases $|\check{\nu}_{\text{str}}| = |\check{\nu}_{\text{tw}}| = |\nu| - 2$.
- (3) Finally take a ribbon pairing $\check{\rho}$ in $\mathcal{R}_{\check{\nu}_{\text{str}}, \check{\mathbf{p}}}$, respectively $\mathcal{R}_{\check{\nu}_{\text{tw}}, \check{\mathbf{p}}}$.

One can show that the steps (1) and (2) above correspond to taking the conditional expectation with respect to the law $M^{(|\nu|/2)}$, and the step (3) to further taking the expectations with respect to the law of $(M^{(1)}, \dots, M^{(|\nu|/2-1)})$. One needs to see how the weights $w_{\nu, \beta}(\rho)$ and $w_{\check{\nu}_{\text{str}}, \beta}(\check{\rho})$, respectively $w_{\check{\nu}_{\text{tw}}, \beta}(\check{\rho})$, are related, with a particular attention to the case $\beta = 4$.

We will not detail further the procedure, as it appears in the proof of a quaternionic Wick formula in Bryc and Pierce [BP09], Theorem 3.1. There the authors do in particular the verifications for the weights $w_{\nu, \beta=4}(\rho)$. Our setting is more general, and not limited to $\beta = 4$, but the recurrence on ribbon graphs is the same. It would be quite lengthy to fully reproduce the recurrence here. \square

4.2. Proof of Theorem 3.3. We give the proof of Theorem 3.3. We use the notations of Section 3.2. As in Section 4.1, for $\beta \in \{1, 2, 4\}$, $(M^{(i)})_{i \geq 1}$ is an i.i.d. sequence of $\text{G}\beta\text{E}(n)$ matrices with distribution (2.5). Consider $x_1, x_2, \dots, x_{|\nu|} \in V$, $\mathbf{p} = \{\{a_1, b_1\}, \dots, \{a_{|\nu|/2}, b_{|\nu|/2}\}\}$ a partition of $\{1, \dots, |\nu|\}$ in pairs, with $a_i < b_i$, and $\gamma_1, \dots, \gamma_{|\nu|/2}$ nearest neighbor paths on \mathcal{G} , with γ_i going from x_{a_i} to x_{b_i} . For $l \in \{1, \dots, m(\nu)\}$, $\Pi_{\nu, l, \mathbf{p}}(M, U, A)(\gamma_1, \dots, \gamma_{|\nu|/2})$ will denote the following product of matrices.

$$\begin{aligned} \Pi_{\nu, l, \mathbf{p}}(M, U, A)(\gamma_1, \dots, \gamma_{|\nu|/2}) = \\ \widehat{M}^{(\nu_1 + \dots + \nu_{l-1} + 1)} A(\nu_1 + \dots + \nu_{l-1} + 1, \nu_1 + \dots + \nu_{l-1} + 2) \widehat{M}^{(\nu_1 + \dots + \nu_{l-1} + 2)} \\ \dots A(\nu_1 + \dots + \nu_l - 1, \nu_1 + \dots + \nu_l) \widehat{M}^{(\nu_1 + \dots + \nu_l)} A(\nu_1 + \dots + \nu_l, \nu_1 + \dots + \nu_{l-1} + 1), \end{aligned}$$

where the matrix $\widehat{M}^{(k)}$ is given by

$$\widehat{M}^{(k)} = \text{hol}^U(\gamma_i) M^{(i)} \text{hol}^U(\gamma_i)^* \text{ if } k = a_i, \quad \widehat{M}^{(k)} = M^{(i)} \text{ if } k = b_i.$$

The product $\Pi_{\nu, l, \mathbf{p}}(M, U, \tilde{A})(\gamma_1, \dots, \gamma_{|\nu|/2})$ is defined similarly, with matrices $\tilde{A}(k, k')$ instead of $A(k, k')$.

Lemma 4.5. *Then, for $\beta \in \{1, 2\}$ and F a bounded measurable function $\mathbb{R}^V \rightarrow \mathbb{R}$, one has the following equality:*

$$\begin{aligned} & \left\langle \left(\prod_{l=1}^{m(\nu)} \text{Tr} \left(\Pi_{\nu, l}(\Phi, A) \right) \right) F(\text{Tr}(\Phi^2)/2) \right\rangle_{\beta, n}^U \\ &= \sum_{\substack{\mathbf{p} \text{ partition} \\ \text{of } \{1, \dots, |\nu|\} \\ \text{in pairs}}} \int_{\gamma_1, \dots, \gamma_{|\nu|/2}} \left\langle \left(\prod_{l=1}^{m(\nu)} \text{Tr} \left(\Pi_{\nu, l, \mathbf{p}}(M, U, A)(\gamma_1, \dots, \gamma_{|\nu|/2}) \right) \right) \right\rangle_{\beta, n} \\ & \quad \times \left\langle F \left(\text{Tr}(\Phi^2)/2 + L(\gamma_1) + \dots + L(\gamma_{|\nu|/2}) \right) \right\rangle_{\beta, n} \mu_{\mathbf{p}}^{x_1, x_2, \dots, x_{|\nu|}}(d\gamma_1, \dots, d\gamma_{|\nu|/2}). \end{aligned}$$

For $\beta = 4$,

$$\begin{aligned} & \left\langle \left(\prod_{l=1}^{m(\nu)} \operatorname{Re} \left(\operatorname{Tr} \left(\Pi_{\nu,l}(\Phi, \tilde{A}) \right) \right) \right) F(\operatorname{Tr}(\Phi^2)/2) \right\rangle_{\beta=4,n}^U \\ &= \sum_{\substack{\mathbf{p} \text{ partition} \\ \text{of } \{1, \dots, |\nu|\} \\ \text{in pairs}}} \int_{\gamma_1, \dots, \gamma_{|\nu|/2}} \left\langle \left(\prod_{l=1}^{m(\nu)} \operatorname{Re} \left(\operatorname{Tr} \left(\Pi_{\nu,l,\mathbf{p}}(M, U, \tilde{A})(\gamma_1, \dots, \gamma_{|\nu|/2}) \right) \right) \right) \right\rangle_{\beta=4,n} \\ & \quad \times \left\langle F(\operatorname{Tr}(\Phi^2)/2 + L(\gamma_1) + \dots + L(\gamma_{|\nu|/2})) \right\rangle_{\beta=4,n}^{\mu_{\mathbf{p}}^{x_1, x_2, \dots, x_{|\nu|}}}(d\gamma_1, \dots, d\gamma_{|\nu|/2}). \end{aligned}$$

Proof. This follows from Lemma 2.3. Indeed, the action of $U(x, y)$ on \mathbb{R}^n ($\beta = 1$), \mathbb{C}^n ($\beta = 2$), respectively \mathbb{H}^n ($\beta = 4$), induces an action on $E_{\beta,n}$ by conjugation:

$$M \mapsto U(x, y)MU(x, y)^*.$$

$\mathcal{U}(x, y)$ will denote the corresponding linear orthogonal operator on $E_{\beta,n}$. The holonomy of the connection $(\mathcal{U}(x, y))_{\{x,y\} \in E}$ along a path γ is related to that of $(U(x, y))_{\{x,y\} \in E}$ along γ by

$$\operatorname{hol}^{\mathcal{U}}(\gamma)(M) = \operatorname{hol}^U(\gamma)M\operatorname{hol}^U(\gamma)^*, \quad M \in E_{\beta,n}.$$

So one applies Lemma 2.3 to the connection $(\mathcal{U}(x, y))_{\{x,y\} \in E}$. \square

To finish the proof of Theorem 3.3, one needs to check, for every fixed partition in pairs \mathbf{p} , the following expressions of expectations with respect to the law of $(M^{(i)})_{1 \leq i \leq |\nu|/2}$:

$$\begin{aligned} & \left\langle \left(\prod_{l=1}^{m(\nu)} \operatorname{Tr} \left(\Pi_{\nu,l,\mathbf{p}}(M, U, A)(\gamma_1, \dots, \gamma_{|\nu|/2}) \right) \right) \right\rangle_{\beta=1,n} \\ &= \sum_{\rho \in \mathcal{R}_{\nu,\mathbf{p}}} w_{\nu,\beta=1}(\rho) \prod_{\mathfrak{t} \in \mathcal{T}_{\nu}(\rho)} \operatorname{Tr}(\Pi_{\mathfrak{t}}(U, A)(\gamma_1, \dots, \gamma_{|\nu|/2})), \end{aligned}$$

$$\begin{aligned} & \left\langle \left(\prod_{l=1}^{m(\nu)} \operatorname{Tr} \left(\Pi_{\nu,l,\mathbf{p}}(M, U, A)(\gamma_1, \dots, \gamma_{|\nu|/2}) \right) \right) \right\rangle_{\beta=2,n} \\ &= \sum_{\rho \in \mathcal{R}_{\nu,\mathbf{p}}} w_{\nu,\beta=2}(\rho) \prod_{\overline{\mathfrak{t}} \in \overline{\mathcal{T}}_{\nu}(\rho)} \operatorname{Tr}(\Pi_{\overline{\mathfrak{t}}}(U, A)(\gamma_1, \dots, \gamma_{|\nu|/2})), \end{aligned}$$

$$\begin{aligned} & \left\langle \left(\prod_{l=1}^{m(\nu)} \operatorname{Re} \left(\operatorname{Tr} \left(\Pi_{\nu,l,\mathbf{p}}(M, U, \tilde{A})(\gamma_1, \dots, \gamma_{|\nu|/2}) \right) \right) \right) \right\rangle_{\beta=4,n} \\ &= \sum_{\rho \in \mathcal{R}_{\nu,\mathbf{p}}} w_{\nu,\beta=4}(\rho) \prod_{\mathfrak{t} \in \mathcal{T}_{\nu}(\rho)} \operatorname{Re}(\operatorname{Tr}(\tilde{\Pi}_{\mathfrak{t}}(U, \tilde{A})(\gamma_1, \dots, \gamma_{|\nu|/2}))). \end{aligned}$$

The identities above follow already from Lemma 4.4. One has to apply the latter to matrices $\mathcal{A}(k, k')$, respectively $\tilde{\mathcal{A}}(k, k')$, instead of $A(k, k')$, respectively $\tilde{A}(k, k')$, where $\mathcal{A}(k, k')$ and $\tilde{\mathcal{A}}(k, k')$ are as follows:

$$\begin{aligned} \mathcal{A}(a_i, a_j) &= \operatorname{hol}^U(\gamma_i)^* A(a_i, a_j) \operatorname{hol}^U(\gamma_j), & \tilde{\mathcal{A}}(a_i, a_j) &= \operatorname{hol}^U(\gamma_i)^* \tilde{A}(a_i, a_j) \operatorname{hol}^U(\gamma_j), \\ \mathcal{A}(a_i, b_j) &= \operatorname{hol}^U(\gamma_i)^* A(a_i, b_j), & \tilde{\mathcal{A}}(a_i, b_j) &= \operatorname{hol}^U(\gamma_i)^* \tilde{A}(a_i, b_j), \\ \mathcal{A}(b_i, a_j) &= A(b_i, a_j) \operatorname{hol}^U(\gamma_j), & \tilde{\mathcal{A}}(b_i, a_j) &= \tilde{A}(b_i, a_j) \operatorname{hol}^U(\gamma_j), \\ \mathcal{A}(b_i, b_j) &= A(b_i, b_j), & \tilde{\mathcal{A}}(b_i, b_j) &= \tilde{A}(b_i, b_j). \end{aligned}$$

Then, for every $l \in \{1, \dots, m(\nu)\}$,

$$\begin{aligned}\mathrm{Tr}(\Pi_{\nu,l,\mathbf{p}}(M, U, A)(\gamma_1, \dots, \gamma_{|\nu|/2})) &= \mathrm{Tr}(\Pi_{\nu,l,\mathbf{p}}(M, \mathcal{A})), \\ \mathrm{Re}(\mathrm{Tr}(\Pi_{\nu,l,\mathbf{p}}(M, U, \tilde{A})(\gamma_1, \dots, \gamma_{|\nu|/2}))) &= \mathrm{Re}(\mathrm{Tr}(\Pi_{\nu,l,\mathbf{p}}(M, \tilde{\mathcal{A}}))).\end{aligned}$$

Further, if $\rho \in \mathcal{R}_{\nu,\mathbf{p}}$ and $\mathfrak{t} \in \mathcal{T}_\nu(\rho)$,

$$\begin{aligned}\mathrm{Tr}(\Pi_{\mathfrak{t}}(U, A)(\gamma_1, \dots, \gamma_{|\nu|/2})) &= \mathrm{Tr}(\Pi_{\mathfrak{t}}(\mathcal{A})), \\ \mathrm{Re}(\mathrm{Tr}(\tilde{\Pi}_{\mathfrak{t}}(U, \tilde{A})(\gamma_1, \dots, \gamma_{|\nu|/2}))) &= \mathrm{Re}(\mathrm{Tr}(\tilde{\Pi}_{\mathfrak{t}}(\tilde{\mathcal{A}}))).\end{aligned}$$

If ρ is the only ribbon pairing in $\mathcal{R}_{\nu,\mathbf{p}}$ with only straight edges (the setting for $\beta = 2$), if $\vec{\mathfrak{t}} \in \vec{\mathcal{T}}_\nu(\rho)$ and \mathfrak{t} is the corresponding unoriented trail, then

$$\mathrm{Tr}(\Pi_{\vec{\mathfrak{t}}}(U, A)(\gamma_1, \dots, \gamma_{|\nu|/2})) = \mathrm{Tr}(\Pi_{\mathfrak{t}}(\mathcal{A})).$$

ACKNOWLEDGEMENTS

This work was supported by the French National Research Agency (ANR) grant within the project MALIN (ANR-16-CE93-0003).

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