Unique ergodicity for infinite area Translation Surfaces

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UNIQUE ERGODICITY FOR INFINITE AREA TRANSLATION SURFACES

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ABSTRACT. We consider infinite staircase translation surfaces with varying step sizes. We show that for typical step sizes, up to scaling, the translation flow has a unique invariant, non-atomic, ergodic Radon measure in almost every direction.

1. INTRODUCTION

One of the most fundamental results in the theory of (compact) translation surfaces is the theorem of Kerchoff, Masur and Smillie which states that the geodesic flow on any compact translation surface is uniquely ergodic in almost every direction [KeMaSm]. The importance of this result is such that there are several articles explaining the proofs [Ar, GoLa, Mo] as well as [MaTa]. Since the result holds for all translation surfaces, it holds for translation surfaces arising from billiards in rational polygons.

In the past decade there has been intensive study of translation surface and polygonal billiards with infinite area. In particular, research has concentrated on trying to understand if the ergodicity conclusion of the Kerchoff, Masur, Smillie theorem holds. There are examples of infinite billiard tables/translation surfaces which are not ergodic in almost every direction with respect to the natural invariant area measure [FrUl, FrHu], and others which are ergodic in almost every direction with respect to the natural invariant area measure [HoHuWe, HuWe, MSTr2, RaTr] (see also [FrUl1] for some partial results in this direction); some of these examples are shown in Figure 1. There is a special class of (compact) translation surfaces called Veech surfaces, on a (compact) Veech surface each minimal direction is in fact uniquely ergodic [Ve1][Ve2]; and thus one might think that they are natural candidates for unique ergodicity in infinite area as well. This is not the case, it turns out that for the only known Veech example of an infinite area translation surface, the translation flow has many ergodic invariant Radon measures in almost every direction [HoHuWe].
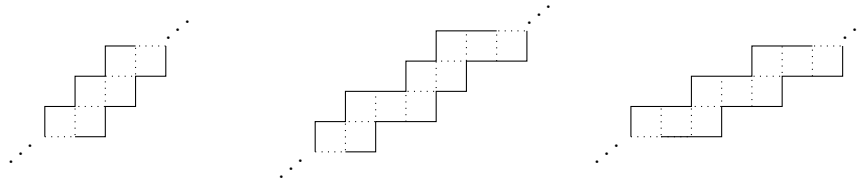
FIGURE 1. Periodic translation surfaces formed by iden-
tifying opposite sides. The first two are ergodic in almost
every direction [HoHuWe, RaTr], while for the third the
set of ergodic directions has measure 0 [FrUl].

Hooper has classified the invariant measures of the translation flow in
many direction for certain other examples [Ho].

Our main result is that there are many infinite area translation sur-
faces for which the area measure is the unique (up to scaling) invariant
ergodic Radon measure for the translation flow in almost every direc-
tion. We present our result in the specific setting of infinite staircases
(Theorem 2). In fact our proof shows that the generic staircase satis-
fies this unique ergodicity result. The strategy of the proof is based on
the approximation techniques first developed for polygonal billiards in
[KaZe] to prove topological transitivity, and then in [KeMaSm, Vo] to
prove ergodicity of generic billiards. Here we refine these techniques to
prove a kind of unique ergodicity rather than just ergodicity. We re-
mark that unique ergodicity has been observed for non-compact trans-
lation surfaces of finite area by Hooper [Ho], however in this case we
have classical finite measure unique ergodicity.

2. DEFINITIONS AND MAIN RESULTS

2.1. **Ergodic theory.** Let T be a measurable map on a measurable
space $(\Omega, \mathcal{B})$, and suppose μ is a σ-finite measure on $(\Omega, \mathcal{B})$ s.t. $\mu(\Omega) =
\infty$. We say that μ is *invariant*, if $\mu(T^{-1}(E)) = \mu(E)$ for all $E \in \mathcal{B}$.
We say that μ is *ergodic*, if for every set $E \in \mathcal{B}$ such that $T^{-1}(E) = E$
either $\mu(E) = 0$ or $\mu(\Omega \setminus E) = 0$.

Suppose Ω_0 is a locally compact second countable metric space with
Borel σ-algebra $\mathcal{B}_0$. Let $C_c(\Omega_0) := \{f : \Omega_0 \to \mathbb{R} : f$ continuous with
compact support$\}$. A regular Borel measure μ on Ω_0 is called a *Radon
measure*, if $\mu(C) < \infty$ for every compact set $C \subset \Omega_0$,

We will need to deal with Borel maps T which are only defined on a
subset $\Omega \subset \Omega_0$ with $\Omega \in \mathcal{B}_0$.

Let $\mathcal{B} := \{E \cap \Omega : E \in \mathcal{B}_0\}$. A measure μ on $(\Omega, \mathcal{B})$ is called *locally
finite*, if $\mu_0(E) := \mu(E \cap \Omega)$ is a Radon measure on $(\Omega_0, \mathcal{B}_0)$.

Note that the null measure will never be considered.

2.2. **Staircases.** A *staircase translation surface* or simply a *staircase* is a translation surface obtained by gluing an enumerated, ordered, collection of same size rectangles, say 2 by 1, all of whose sides are parallel to the coordinate axes in the following way. We place a rectangle in the plane with the center of the rectangle at the origin, label it as the 0th rectangle. The bottom of the first rectangle will intersect the top of the 0th rectangle with length of intersection $w_0 \in (0,1)$. Choosing a sequence $w := \{w_i \in (0,1) : i \in \mathbb{Z}\}$, we continue this procedure inductively to produce a staircase like collection of rectangles. Then we identify opposite sides of the staircase to form a translation surface which we will call S_w (see Figure 2). Note that corners of the rectangles give rise to conical points. Every such corner, considered as a point on the staircase surface, is a conical point with angle 6π.

The set of all staircases is then coded by the set $(0,1)^{\mathbb{Z}}$. We consider the product topology on this space, its closure is $[0,1]^{\mathbb{Z}}$, a Baire space. In the proof we will also consider parameters $w \in [0,1]^{\mathbb{Z}}$. In the case $w_n = 0$ for some n, after removing the singular points S_w is not connected. These non-connected staircases will play a very important role in the proof.

Fix a direction θ and consider the translation flow $\psi_t^{\theta} = \psi_t^{\omega,\theta}$ on the surface S_w. It will be convenient to use the section of this flow defined by intersecting S_{ω} with the collections of lines $y = n$ where $n \in \mathbb{Z}$, we identify this section with the set $X := \mathbb{Z} \times [0,2)$. Note that in S_{ω} the points 0 and 2 are identified, thus each set $\{n\} \times [0,2) \subset X$ is a circle. After the identification, the set X does not depend on the parameter ω, but sometimes we need to emphasize the nature of this sets as phase spaces for dynamical systems, we will then write X^{ω}. Let $X^N = X^{\omega,N} \subset X^{\omega}$ denote the set $X^N := \{-N+1,\ldots,N\} \times [0,2)$. Likewise, the sets do not depend on the direction θ chosen, when needed for clarity we write X^{θ} or $X^{\omega,\theta}$.

Let $T^{\theta} = T^{\omega,\theta}$ be the first return map of the flow ψ_t^{θ} to the section X. Note that T^{θ} preserves the *length measure* μ, and that this measure is an infinite measure.

Proposition 1. *For any ω such that $\liminf_{i \to \pm\infty} w_i = 0$, $(T^{\omega,\theta}, X, \mu)$ is a conservative system for all θ.*

This proposition is closely related to results in [Tr, MS, MSTr1, MSTr2]. We still provide a proof for completeness (§4.1). We remark that the set of ω satisfying this condition is a G_{δ} set.

Theorem 2. *There is a dense G_{δ} subset G of $[0,1]^{\mathbb{Z}}$ and a dense G_{δ} set of full measure of directions Θ, such that for each $w \in G$, we have $w \in (0,1)^{\mathbb{Z}}$ and, up to scaling, μ is the unique T^{θ}-invariant, non-atomic, ergodic Radon measure, for every $\theta \in \Theta$.*

It follows that, up to scaling, the area measure is the unique ψ_t^{θ}-invariant ergodic Radon measure, for every $\theta \in \Theta$.
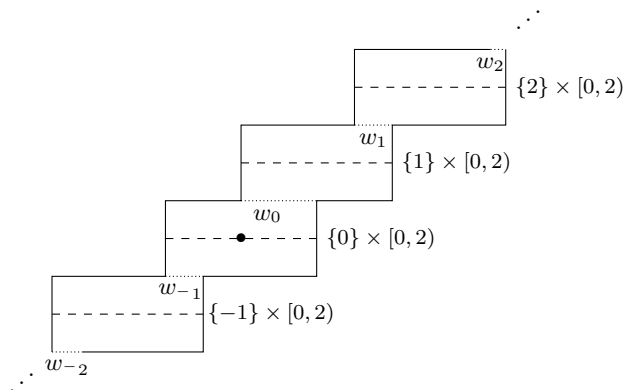
FIGURE 2. The staircase, opposite sides are identified. The section X consists of the dashed lines in the middle of the rectangles.

2.3. **Comparison to other definitions of unique ergodicity.** In his survey article [Sa] Sarig defines two notions of unique ergodicity in the infinite measure setting which are stronger than the one we prove.

A point $x \in \Omega$ is called *generic* for μ if for all continuous f, g with compact support such that $g \geq 0$ and $\int g \, d\mu > 0$ we have

$$\frac{\sum_{k=0}^{\ell} f\left(T^k(x)\right)}{\sum_{k=0}^{\ell} g\left(T^k(x)\right)} \to \frac{\int_{\Omega} f(y) \, d\mu(y)}{\int_{\Omega} g(y) \, d\mu(y)}.$$

Sarig calls a map T *uniquely ergodic* if (1) up to scaling, T admits a unique Radon invariant measure not supported on a single orbit; and (2) every point is generic for this measure.

This strong version of unique ergodicity does not hold in either of our settings. In fact, in [MSTr1] we showed that the existence of non-recurrent points, this result holds in all the settings we consider here, thus (2) can not hold.

Sarig calls a map T *uniquely ergodic in the broad sense* if (1) up to scaling, T admits a unique Radon ergodic invariant measure not supported on a single orbit; and (2) every non-exceptional non-periodic point is generic for this measure (see [Sa]).

Here a point x is called (forward) *exceptional* for a map T if the measure $\sum_{n>0} \delta_{T^n(x)}$ is locally finite, where δ_y denote the point mass at y.

We do not know if the proof of Theorem 2 can be strengthened to show unique ergodicity in the broad sense.

3. PROOF OF THE STAIRCASE RESULT

Proof of Theorem 2. Throughout the proof we will occasionally confound the parameter ω with the staircase S_ω.

The strategy of the proof is as follows: we choose a dense set $\{S_{\omega^i}\}$ of staircases which satisfy the goal dynamical property of unique ergodicity in almost every direction on compact sets which exhaust the staircase. The parameter of these staircases will satisfy $\omega \in [0,1)^{\mathbb{Z}} \setminus (0,1)^{\mathbb{Z}}$. Then we will show that staircases which are sufficiently well approximated by this dense set will satisfy the goal dynamical property on the whole phase space.

A staircase S_ω is N-*ringed* if $w_N = w_{-N} = 0$ and $w_j \notin \{0,1\}$ for all $|j| < N$ (the terminology ringed comes from the corresponding topologically more complicated definition for wind-tree configurations which we used in [MSTr1, MSTr2, MSTr3]). We will consider the dynamics on the set X^N, i.e., inside the ring, the set $X \setminus X^N$ plays no role in our proof.

Let $\{\omega^i\}$ be a dense set of parameters such that each S_{ω^i} is an N_i-ringed configuration, with N_i increasing with i.

By [KeMaSm] the translation flow is uniquely ergodic (in the classical finite measure sense) in almost every direction inside the ring, and thus the map $T^{\omega^i,\theta}|_{X^N}$ is also uniquely ergodic in almost every direction. To transfer this finite area unique ergodicity to sufficiently well approximable generic staircases, we suppose that δ_i are strictly positive numbers and consider the dense G_δ set

$$\mathcal{G} := \bigcap_{m=1}^{\infty} \bigcup_{i=m}^{\infty} \mathcal{U}_{\delta_i}(\omega^i)$$

where $\mathcal{U}_\delta(\omega)$ denotes the set of all parameters ω' such that $|\omega'_j - \omega_j| < \delta$ for $1 \leq j \leq 1/\delta$. Note that each such $\mathcal{U}_\delta(\omega)$ is open in the product topology. We will show that the δ_i can be chosen in such a way that $\mathcal{G} \subset (0,1)^{\mathbb{Z}}$ (i.e., the staircases in $\mathcal{G}$ are connected), and all the staircases in $\mathcal{G}$ have a unique (up to scaling) Radon T^θ-invariant ergodic measure for all $\theta \in \Theta$, where Θ is a G_δ set of full measure that will be constructed in the proof.

The first requirement on δ_i is that $1/\delta_i > N_i$, thus all configurations in $\mathcal{U}_{\delta_i}(\omega^i)$ will be close to the part of ω^i inside the ring.

For each $N \in \mathbb{N}$ let

$$\{h_j^{+N} : X^N \to \mathbb{R}\}_{j \geq 1}$$

be a countable collection of continuous, strictly positive functions which are dense with respect to the sup norm in the set of all continuous non-negative functions. We choose an enumeration $\{h_n^+\}_{n \geq 1}$ of the set $\mathcal{H}_{N_0} := \cup_{n \geq 1, N \geq N_0} \{h_n^{+N}\}$. We can think of each h_n^+ as a function with compact support defined on X. In our proofs we will identify the space X with $X \times \{\theta\}$ and thus can think of h_n^+ as defined on $X \times \{\theta\}$.

By construction, each of the collections $\mathcal{H}_{N_0}$ is dense with respect to the sup norm in the set of all continuous compactly supported non-negative functions on X. Hence this collection is also dense in the set of all continuous non-negative functions on X.

We consider a second collection of continuous, positive functions. For each $N \in \mathbb{N}$ let

$$\{h_n^N : X^N \to \mathbb{R}\}_{n \geq 1}$$

be a countable collection of continuous, positive functions which are dense with respect to the sup norm in the set of all continuous non-negative functions, satisfying the following additional assumption. Let $\mathcal{Y} := \{Y_i\}$ be an enumeration of compact intervals in X which have rational endpoints. We assume that the collection $\{h_n^N\}$ satisfies that for each $Y_i \in \mathcal{Y}$ such that $Y_i \subset X^N$ and $\varepsilon > 0$ we can find a h_j^N taking values in $[0,1]$ and intervals $Y_i^1 \in \mathcal{Y}$ and $Y_i^2 \in \mathcal{Y}$ satisfying $Y_i^1 \subset Y_i \subset Y_i^2 \subset X^N$ such that the function h_j^N is identically equal to one on Y_i^1 and identically equal to zero on $(Y_i^2)^c$ and the total length of $Y_i^2 \setminus Y_i^1$ is at most ε. We choose an enumeration $\{h_j\}_{j \geq 1}$ of the set $\cup_{j,N}\{h_j^N\}$. Again we think of each h_j as a function with compact support defined on X. By construction, this collection is dense with respect to the sup norm in the set of all continuous non-negative functions on X.

Fix a surface S_ω, a direction $\theta \in \mathbb{S}^1$, a point $z \in X$, and $(j,n) \in \mathbb{N}^2$. Consider the Hopf average

$$H_{j,n,\ell}^{\omega,\theta}(z) := \frac{\sum_{k=0}^{\ell} h_j\big((T^{\omega,\theta})^k(z)\big)}{\sum_{k=0}^{\ell} h_n^+\big((T^{\omega,\theta})^k(z)\big)}.$$

Our strategy is to study unique ergodicity via Hopf averages. We start with the ringed configuration ω^i. We consider the Hopf averages of a finite subcollection of the $\{h_j, h_n^+\}$ and the times ℓ_i where these averages almost converge. We will show that for a small perturbation S_ω of S_{ω^i}, for many directions, for each point z either the forward or the backward orbit segment of length ℓ_i stays close enough to a good orbit on S_{ω^i} to control the Hopf averages on S_ω. Since for a dense G_δ this behavior happens on infinitely many scales, we can conclude that the forward or the backward Hopf average converges to the ratio of the integrals with respect to Lebesgue measure. Then we apply the following criterion for unique ergodicity.

Remember that μ is the length measure on X.

Lemma 3. *Fix $\omega \in (0,1)^{\mathbb{Z}}, \theta \in \mathbb{S}^1$. Suppose that there exists an increasing sequence ℓ_k tending to infinity such that for each $z \in X$ either*

$$\lim_{k \to \infty} H_{j,n,\ell_k}^{\omega,\theta}(z) = \frac{\int_X h_j(y)\, d\mu(y)}{\int_X h_n^+(y)\, d\mu(y)} \ \forall(j,n) \in \mathbb{N}^2 \ or$$

$$\lim_{k \to \infty} H_{j,n,-\ell_k}^{\omega,\theta}(z) = \frac{\int_X h_j(y)\, d\mu(y)}{\int_X h_n^+(y)\, d\mu(y)} \ \forall(j,n) \in \mathbb{N}^2,$$

then, up to scaling, there exists at most one conservative, $T^{\omega,\theta}$-invariant, non-atomic, ergodic, Radon measure; if it exists it is the measure μ.

Proof. Suppose that m is a conservative, $T^{\omega,\theta}$-invariant, ergodic Radon measure. Thus there is an N_0 such that $m(X^N) > 0$ for all $N \geq N_0$. All the functions h_n^+ will be chosen from $\mathcal{H}_{N_0}$. Fix n, by assumption, h_n^+ is strictly positive on some X^N, so we have $\int_X h_n^+(y)\, d\mu(y) > 0$ and $\int_X h_n^+(y)\, dm(y) > 0$. By the Hopf ergodic theorem, for every $j \in \mathbb{N}$ and m-a.e. $z \in X$ we have

$$\lim_{\ell \to \infty} H_{j,n,\ell}^{\omega,\theta}(z) = \lim_{\ell \to \infty} H_{j,n,-\ell}^{\omega^i,\theta}(z) = \frac{\int_X h_j(y)\, dm(y)}{\int_X h_n^+(y)\, dm(y)}.$$

But by the assumptions of this lemma, this number must coincide with $\frac{\int_X h_j(y)\, d\mu(y)}{\int_X h_n^+(y)\, d\mu(y)}$. Thus for each $j \in \mathbb{N}$ we have

(1) $$\int_X h_j(y)\, dm(y) = C \int_X h_j(y)\, d\mu(y)$$

with

$$C = \frac{\int_X h_n^+(y)\, dm(y)}{\int_X h_n^+(y)\, d\mu(y)}.$$

Consider any compact interval $Y := \{k\} \times \widehat{Y} \subset X$. We apply the triangle inequality.

(2) $$|m(Y) - C\mu(Y)| <$$

$$\left|m(Y) - \int_X h_j(y)\, dm(y)\right| +$$

$$\left|\int_X h_j(y)\, dm(y) - C \int_X h_j(y)\, d\mu(y)\right| +$$

$$\left|C \int_X h_j(y)\, d\mu(y) - C\mu(Y)\right|$$

$$=: I + II + III.$$

Equation (1) tells us that $II = 0$.

By the assumptions on the collection $\{h_j\}$ for each $n > 0$ there is a h_{j_n} taking values in $[0,1]$ with intervals $Y_n^1 \subset Y \subset Y_n^2$ such that $h_{j_n} = 1$ on Y_n^1, $h_{j_n} = 0$ on $(Y_n^2)^c$ and $\mu(Y_n^2 \setminus Y_n^1) < \min(\varepsilon/C, 1/n)$. Thus for each j_n we have $III < \varepsilon$. We can assume without loss of generality that $Y_n^1 \subset Y_{n+1}^1$ and $Y_n^2 \supset Y_{n+1}^2$ for all $n \geq 1$.

But $\cup_{n\geq 1}Y_n^1 = \cap_{n\geq 1}Y_n^2 = Y$, so $\cap_{n\geq 1}(Y_n^2 \setminus Y_n^1) = \partial Y$. Since ∂Y consists of 2 points only and m is non-atomic, we have $m(\partial Y) = 0$, thus $\lim_{n\to\infty} m(Y_n^2 \setminus Y_n^1) = 0$. Thus for n large enough, we have and $m(Y_n^2 \setminus Y_n^1) < \varepsilon$ and thus $I < \varepsilon$. Combining the above facts yields $|m(Y) - C\mu(Y)| < 2\varepsilon$, since $\varepsilon > 0$ and Y are arbitrary we conclude that $m = C\mu$. □

Notice that in the above lemma, the only rigidity we proved is with respect to conservative ergodic measures. But, the only $T^{\omega,\theta}$-invariant ergodic measure on X which are not conservative, are measures supported on a single, bi-infinite non-periodic orbit of $T^{\omega,\theta}$. Such a measure is not a Radon measure if the bi-infinite orbit is dense because every neighborhood is visited an infinite number of times.

Proposition 4. *If the assumptions of Lemma 3 are verified, then up to scaling, the length measure μ is the unique $T^{\omega,\theta}$-invariant, ergodic Radon measure.*

Proof. Fix a point z and an open set U with compact support. Choose a function h_j in our family whose support is a subset of U. Fix an h_n^+. Then by Lemma 3 for k large enough either $H_{j,n,\ell_k}^{\omega,\theta}(z)$ or $H_{j,n,-\ell_k}^{\omega,\theta}(z)$ is strictly positive; in particular the bi-infinite orbit of z has visited the set U. □

Thus for our proof we need to construct a set of directions Θ and show that for every $\omega \in \mathcal{G}$ the convergence suppositions of the Lemma hold for $\theta \in \Theta$, and all $(j,n) \in \mathbb{N}^2$.

Recall that the surfaces S_{ω^i} are N_i ringed. Let $J_i \in \mathbb{N}$ be the maximal $J \in \mathbb{N}$ such that for all $1 \leq j \leq J$ the support of h_j is contained in X^{N_i}, i.e., inside the the ring of S_{ω^i}. Let $\mathcal{J}_i := \{1,\ldots,J_i\}^2$. Notice that $J_{i+1} \geq J_i$ and that $\lim_{i\to\infty} J_i = \infty$.

For each i let A_i be the set of directions for which the map $T^{\omega^i,\theta}$ is uniquely ergodic when restricted to X^{N_i}. By [KeMaSm] the set A_i is of full measure. We additionally assume that A_i does not contain a saddle connection direction, this removes an at most countable set from the set of uniquely ergodic directions. Let $\gamma_i > 0$ be a sequence tending to 0 and fix i. We apply the Corollary 8 of the appendix on the uniform convergence of Hopf averages to any ω^i, $\theta \in A_i$, $(j,n) \in \mathcal{J}_i$ to conclude that there is an $\widehat{\ell}$ which depends on i,θ,j,n such that

$$(3) \qquad \left| H_{j,n,\ell}^{\omega^i,\theta}(z) - \frac{\int_X h_j(y)\,d\mu(y)}{\int_X h_n^+(y)\,d\mu(y)} \right| < \gamma_i$$

for all $\ell \geq \widehat{\ell}$ and all $z \in X^{N_i}$ (except those z whose orbit hits a singular point before time ℓ). Choose $\widehat{\ell}$ sufficiently large so that (3) holds for all $(j,n) \in \mathcal{J}_i$; note that $\widehat{\ell}$ depends only on $\theta \in A_i$ and i.

Next we uniformize this estimate to a large set of directions. We choose $\ell_i \geq N_i$ and sets $B_i \subset A_i \subset \mathbb{S}^1$ so that $\lambda(B_i^c) < \gamma_i$ and $\widehat{\ell}(i, \theta) \leq \ell_i$ for all $\theta \in B_i$. Consider the finite set of saddle connections of length at most $2\ell_i$, and a small open neighborhood of this set. By choosing the open set very small and modifying B_i we can additionally assume that it does not intersect this neighborhood.

Now we would like to extend these estimates to the neighborhood $\mathcal{U}_{\delta_i}(\omega^i)$ for a sufficiently small strictly positive δ_i. We will impose various conditions on δ_i in an incremental way. The first requirement is

    A0 $\delta_i < \min\{\omega_j^i : -n_i < j < n_i\}$.

This requirement ensures that for all $\omega \in \mathcal{U}_{\delta_i}(\omega^i)$ we have $\omega_j > 0$ for all $-n_i < j < n_i$ which will ensure that the tables in the dense G_δ set $\mathcal{G}$ are connected.

We will explicitly state the next three requirements, **A1 - A3** for forward orbits, the analogous requirements are also assumed for backward orbits, but will not be stated explicitly. Our goal is now to define a small neighborhood C_i of B_i, and then a large open subset D_i of C_i, and for each $\theta \in D_i$ to define a map $\zeta^+ = \zeta^+(\omega^i, \omega, \theta)$ defined on a large subset of X^{N_i} (for ω) onto X^{N_i} (for ω^i) which is close to the identity, and a map ζ^- satisfying similar conditions. The $+$ maps will control the future behavior of orbits, the $-$ map will control the past. In one of our first articles on the Ehrenfest model where we studied ergodicity [MSTr2] it was sufficient to use the identity map for the maps $\zeta^\pm$. In our article on infinite ergodic index [MSTr3] we we have defined a related map, but the form of the map considered in that article is not adapted to the study of unique ergodicity.
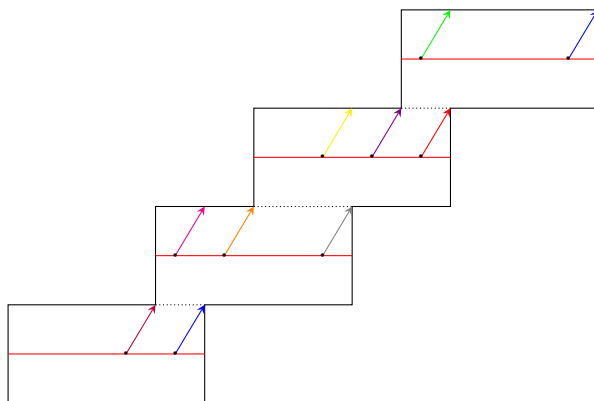

FIGURE 3. The set $\Sigma^{\omega^i, \theta, N_i}$.

To define ζ^+ for a direction $\theta \in B_i$ which has no saddle connection shorter than $2\ell_i$, we develop the argumentation when $\theta \in \mathbb{S}^1$ is in the interior of the first quadrant, and leave to the reader to derive the corresponding formulas for θ in the other quadrants. Consider the
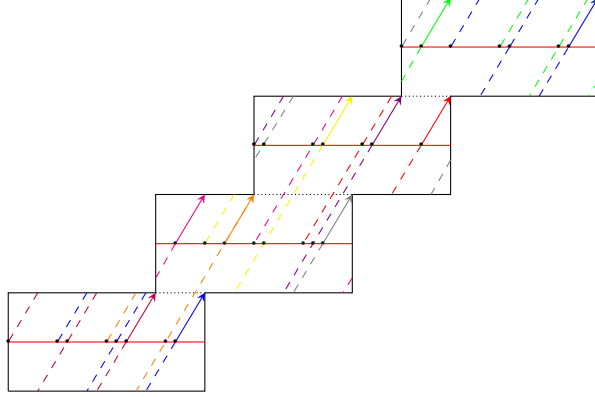
FIGURE 4. Partition of X^{N_i} into sets of continuity of $(T^{\omega^i,\theta})^{\ell_i}$.

finite collection of points

$$\Sigma^{\omega^i,\theta,N_i} := \left\{ (k, j - 1/2\tan(\theta)) : j \in \{\omega^i_{k-1}, 2 - \omega^i_k, 2\} \right\} \cap X^{N_i}$$

(see Figure 3). $\Sigma^{\omega^i,\theta,N_i}$ consists of the points inside the ring for which the trajectory hits a singularity before hitting the next cross-section.

All the points in $\Sigma^{\omega^i,\theta,N_i}$ are distinct except for the pair of points $\{N_i\} \times \{2 - 1/2\tan(\theta)\}$ and $\{N_i\} \times \{2 - \omega^i_{N_i} - 1/2\tan(\theta)\}$ which are the same point since $\omega^i_{N_i} = 0$, as well as the pair of points $\{-N_i + 1\} \times \{2 - 1/2\tan(\theta)\}$ and $\{-N_i + 1\} \times \{2 - \omega^i_{-N_i} - 1/2\tan(\theta)\}$ which are the same point since $\omega^i_{-N_i} = 0$. We call such a point z a *blocking point*. Here, and at the discussion below, it's worth to recall that for each integer k, $\{k\} \times [0, 2)$ is a circle and we will always consider the coordinates of the circle modulo 2.

Consider the collection of points

$$\widehat{\Sigma}^{\omega^i,\theta,N_i} := \{(T^{\omega^i,\theta})^{-j} z : j \in \{0, 1, \ldots, \ell_i\} \text{ and } z \in \Sigma^{\omega^i,\theta,N_i}\}.$$

Any point in this set corresponds to a unique $j \in \{0, \ldots, \ell_i\}$ and a unique $z \in \Sigma^{\omega^i,\theta,N_i}$. This set partitions X^{N_i} into a finite number of open intervals, $\{I_j^{\omega^i,\theta}\}$, such that on each of these intervals the forward map $(T^{\omega^i,\theta})^{\ell_i}$ is continuous (see Figure 4). This partition can be defined more generally than for $\theta \in B_i$, it makes sense as soon as as there are no saddle connections of length at most $2\ell_i$. Thus for $\theta \in B_i$ and a small enough neighborhood $U(\theta)$, each interval $I_j^{\omega^i,\theta'}$ varies continuously with respect to $\theta' \in U(\theta)$. Moreover we assume that each B_i avoids a small fixed neighborhood of the (finite) collection of saddle connections of length at most $2\ell_i$ in X^{N_i}.

Let $\eta_i(\theta) > 0$ be the minimum of the lengths of the intervals $I_j^{\omega^i,\theta'}$. Let

$$C_i := \bigcup_{\theta \in B_i} U(\theta).$$

By assumption the set C_i avoids a small neighborhood of the collection of saddle connections of length at most $2\ell_i$ in X^{N_i}.

We naturally extend the definition of the function $\eta_i(\cdot)$, originally defined on B_i, to the larger set C_i, it is a continuous function of θ. We additionally suppose that $U(\theta)$ is sufficiently small so that $\eta(\theta') > \eta_i(\theta)/2$ for all $\theta' \in U(\theta)$. In particular all the intervals $I_j^{\omega',\theta'}$ maintain positive length. By construction C_i is an open set containing B_i, thus $\lambda(C_i^c) < \gamma_i$. Furthermore, starting with (3), by continuity in θ we can suppose that for all $\theta \in C_i$ we have

$$(4) \qquad \left| H_{j,n,\ell}^{\omega^i,\theta}(z) - \frac{\int_X h_j(y)\,d\mu(y)}{\int_X h_n^+(y)\,d\mu(y)} \right| < 2\gamma_i.$$

for all $\ell \geq \widehat{\ell}$ and all $z \in X^{N_i}$ (except those z whose orbit hits a singular point before time ℓ).

We can do the same for backward orbits, we will choose ℓ_i to work for forward and backward orbits. We will denote by C_i the open set of directions satisfying both the forward and backward conditions.

For $\eta_i > 0$ consider the open set

$$D_i := \{\theta \in C_i : \eta_i(\theta) > \eta_i\}.$$

Since the function $\eta_i(\theta)$ is strictly positive for every $\theta \in C_i$, we can choose a small number $\eta_i > 0$ such that the normalized Lebesgue measure of the set D_i is at least $1 - \gamma_i$. So for all $\theta \in D_i$ the lengths of all the intervals $\{I_j^{\omega^i,\theta}\}$ are at least $\eta_i > 0$.
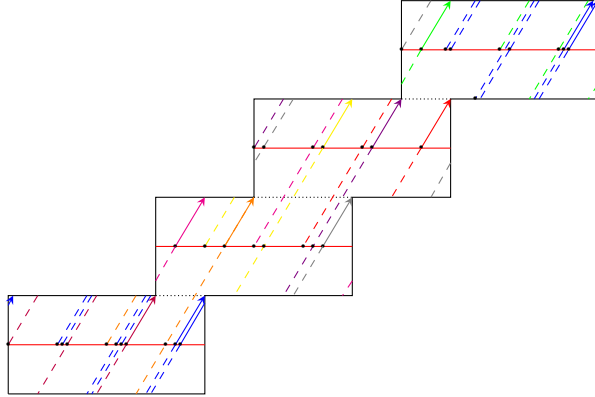
FIGURE 5. The partition changes continuously, the blocking points bifurcate.

Now we consider a configuration ω, not necessarily ringed, but close to a ringed configuration. Assume now that $\theta \in D_i$ and note that the point $(T^{\omega,\theta})^{-j}(z)$ varies continuously with respect to ω close to ω^i, for any z which is not a blocking point for ω^i.

If w is very close to w_i such that $\omega_{\pm N_i} \neq 0$, then each of the two blocking points, $\{N_i\} \times \{2 - 1/2\tan(\theta)\}$ and $\{-N_i + 1\} \times \{2 - 1/2\tan(\theta)\}$,

bifurcates into a pair of points for S_ω. The bifurcated points create two new intervals $\tilde{I}_+^{\omega,\theta} = \{N_i\} \times (2 - \omega_{N_i} - 1/2\tan(\theta), 2 - 1/2\tan(\theta))$ and $\tilde{I}_-^{\omega,\theta} = \{-N_i + 1\} \times (2 - 1/2\tan(\theta), 2 + \omega_{-N_i} - 1/2\tan(\theta))$ in the partition induced by the set $\Sigma^{w,\theta}$ (see Figure 5).

Then, if we consider the partition $\{I_j^{w,\theta}\}$ defined in a similar way as the partition $\{I_j^{\omega^i,\theta}\}$, we have at most $2(\ell_i+1)$ new intervals $\{(T^{\omega,\theta})^{-j}\tilde{I}_\pm^{\omega,\theta} : j \in \{0, \ldots, \ell_i\}\}$. We call this collection of new intervals $\mathcal{I}^{w,\theta,N_i}$.

Let $V^{\omega,\theta,N_i} := \cup_{I \in \mathcal{I}^{w,\theta,N_i}} I$ be the union of all the intervals in the collection $\{I_j^{w,\theta}\}$. Since $\{I_j^{\omega^i,\theta}\}$ is in bijection with a subset of $\{I_j^{\omega,\theta}\}$ (except for at most $2(\ell_i + 1)$ new intervals) we can define an injective, piecewise affine transformation $\zeta^+ = \zeta^{+,\omega^i,\omega,n,\theta}$ from V^{ω^i,θ,N_i} to V^{ω,θ,N_i} that sends every $I_j^{\omega^i,\theta}$ affinely to the corresponding $I_j^{\omega,\theta}$. Note that, for ω sufficiently close to ω^i, this piecewise affine transformation is close to the identity.

The domain of ζ^+ is V^{ω^i,θ,N_i}. For any interval $I_j^{w^i,\theta}$ the ζ^+ map could be extended by continuity from the inside to the boundary of this interval. Let z be an extremity of this interval and let z' by the image of z obtained by the extension by continuity of ζ^+ to z. The point z' is a boundary point of the interval $\zeta^+(I_j^{w^i,\theta})$ which is in the collection $\mathcal{I}^{\omega,\theta,N_i}$. It is important to underline that the definition of the map ζ^+ guarantees that the time $\ell \leq \ell_i$ at which $T^{\omega,\theta}$ forward orbit of z' hits a singular point coincides with the time at which the $T^{\omega^i,\theta}$ forward orbit of z hits a singular point.

The map ζ^+ is not defined on the set $X^{N_i} \setminus V^{\omega^i,\theta,N_i}$. In Figure 5 this set consists of all the boundary points described above which are portrayed as dashed lines as well as the points in between the close blue orbit segments which have bifurcated from a blocking orbit segment. Except for the boundary points described above, the set $X^{N_i} \setminus V^{\omega^i,\theta,N_i}$ has no useful information for our proof.

Fix $\theta \in \overline{D_i}$ and suppose that ω is close enough to ω^i so that the bijection ζ^+ is defined. Then we can define $\eta_i(\omega, \theta)$ to be the infimum length of all the intervals $I_j^{\omega,\theta}$. For ω sufficiently close to ω^i, for each $\theta \in C_i$, the function $\eta_i(\omega, \theta)$ varies continuously with ω since the set $\overline{C_i}$ does not contain any saddle connection directions of length at most $2\ell_i$. Let $\eta_i(\omega) := \inf_{\theta \in \overline{D_i}} \eta_i(\omega, \theta) \geq 0$. We have that $\eta_i(\omega^i) = \eta_i(\omega^i, \theta_0)$ for some $\theta_0 \in \overline{D_i}$ and thus $\eta_i(\omega)$ is locally a continuous function of ω.

By the definition of D_i we have $\eta_i(\omega^i) \geq \eta_i$, we require that

A1 δ_i is so small that $\eta_i(\omega)$ varies continuously with $\omega \in \mathcal{U}_{\delta_i}(\omega^i)$ and $\eta_i(\omega) > \eta_i/2$ for each $\omega \in \mathcal{U}_{\delta_i}(\omega^i)$.
We make the analogous assumption for the interval collections arising from backward orbits and the analogous bijection ζ^-

We require a further closeness condition on those orbits which we can control.

A2 We assume that $\delta_i > 0$ is so small that for all $\omega \in \mathcal{U}_{\delta_i}(\omega^i)$, for all $z \in X^{N_i}$ for which ζ^+ is defined, for all $\theta \in D_i$, we have

$$\left| H^{\omega,\theta}_{j,n,\ell_i}(z) - H^{\omega^i,\theta}_{j,n,\ell_i}(\zeta^+(z)) \right| < \gamma_i.$$

Again we make the analogous assumption for ζ^-.

In the proof we will only consider directions in $\overline{D_i}$, since we are not able to control the Hopf averages in the complement. As already mentioned, we also are not able to control the Hopf averages of the forward orbits of points in $X^{N_i} \setminus V^{\omega,\theta,N_i}$. Our main goal is to control the backwards Hopf averages of such orbits.

A key point of the proof is that if $\theta \in D_i$, then for each $z \in X^{N_i}$ at least one of $\zeta^+(z)$ or $\zeta^-(z)$ is defined. To do this consider any $\theta \in D_i$ and the finite set $\widehat{\Sigma}^+ := \widehat{\Sigma}^{\omega^i,\theta,N_i}$ and the analogous set $\widehat{\Sigma}^-$. These two sets do not intersect, for if they did there would be a saddle connection of length at most $2\ell_i$. Let $\iota_i(\theta) > 0$ be the minimum of the distances between pairs of points in these two collections and $\iota_i := \inf_{\theta \in \overline{D_i}} \iota_i(\theta)$. Since C_i avoids a neighborhood of the directions of the saddle connections of length at most $2\ell_i$ we have $\iota_i > 0$.

We define $\iota_i(\omega)$ in an analogous way.

A3 We assume that $\delta_i > 0$ is so small that for each $\omega \in U_{\delta_i}(\omega^i)$ $\iota_i(\omega) > \iota_i/2$.
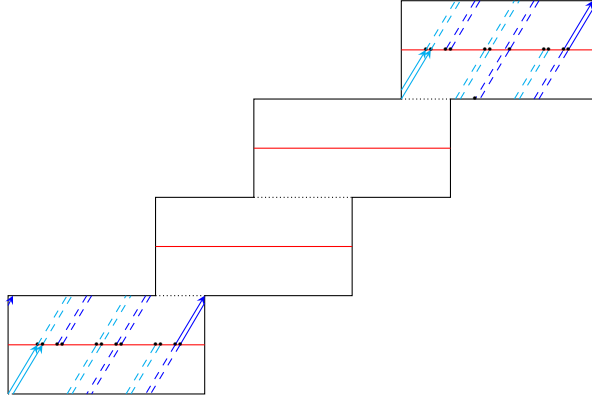

FIGURE 6. Uncontrolled forward orbits do not intersect uncontrolled backward orbits.

Fix i, by the triangle inequality, for any $z \in X^{N_i}$ such that $\theta \in D_i$ and $\zeta^+(z)$ is defined, we have

$$\left| H^{\omega,\theta}_{j,n,\ell_i}(z) - \frac{\int_X h_j(y)\,d\mu}{\int_X h_n^+(y)\,d\mu} \right| \leq \left| H^{\omega,\theta}_{j,n,\ell_i}(z) - H^{\omega^i,\theta}_{j,n,\ell_i}(\zeta^+(z)) \right|$$
$$+ \left| H^{\omega^i,\theta}_{j,n,\ell_i}(\zeta^+(z)) - \frac{\int_X h_j(y)\,d\mu}{\int_X h_n^+(y)\,d\mu} \right|.$$

By **A2**, the first term is smaller than γ_i. Now we consider the second term.

By assumption $\theta \in D_i$ and thus if the orbit of $\zeta^+(z)$ does not hits a singular point before time ℓ, then (4) implies

$$\left| H_{j,n,\ell_i}^{\omega^i,\theta}(\zeta^+(z)) - \frac{\int_X h_j(y)\,d\mu}{\int_X h_n^+(y)\,d\mu} \right| < 2\gamma_i$$

for all $1 \leq j \leq i$, and $1 \leq n \leq N_i$. Combining with **A2** we conclude

$$\left| H_{j,n,\ell_i}^{\omega,\theta}(z) - \frac{\int_X h_j(y)\,d\mu}{\int_X h_n^+(y)\,d\mu} \right| < 3\gamma_i$$

for all z where ζ^+ is defined, with $\theta \in D_i$, for all $(j,n) \in \mathcal{J}_i$, for all $\omega \in \mathcal{U}_{\delta_i}(\omega^i)$ as long as the $T^{\omega^i,\theta}$ forward orbit of $\zeta^+(z)$ of length ℓ_i does not hit a singular point. As already pointed out we can thus consider points $z \in X^{N_i}$ such that ζ^+ is defined an the the $T^{\omega,\theta}$ forward orbit of z of length ℓ_i does not hit a singular point.

We make the analogous estimate on the past

$$\left| H_{j,n,-\ell_i}^{\omega,\theta}(z) - \frac{\int_X h_j(y)\,d\mu}{\int_X h_n^+(y)\,d\mu} \right| < 3\gamma_i$$

for all z where ζ^- is defined, with $\theta \in D_i$, for all $(j,n) \in \mathcal{J}_i$, for all $g \in \mathcal{U}_{\delta_i}(\omega^i)$ as long as the backwards orbit of $\zeta^-(z)$ of length ℓ_i is defined.

Using **A3**, since $\lambda(D_i) > 1 - 2\gamma_i$, the direction set $\Theta = \cap_{M=1}^{\infty} \cup_{i=M}^{\infty} D_i$ has full measure. Fix $\omega \in \mathcal{G}$ and $\theta \in \Theta$, then there is an infinite sequence i_k such that $\omega \in \mathcal{U}_{\delta_{i_k}}(\omega^{i_k})$ and $\theta \in \Theta_{i_k}$ for all k. Since $J_{i_k} \to \infty$ we can conclude that for all $z \in X$ with $\theta \in \Theta$, for each $j \geq 1$, for each $n \geq 1$ either

$$\lim_{k\to\infty} H_{j,n,\ell_{i_k}}^{\omega,\theta}(z) = \frac{\int_X h_j(y)\,d\mu}{\int_X h_n^+(y)\,d\mu}.$$

or

$$\lim_{k\to\infty} H_{j,n,-\ell_{i_k}}^{\omega,\theta}(z) = \frac{\int_X h_j(y)\,d\mu}{\int_X h_n^+(y)\,d\mu}.$$

Proposition 4 finishes the proof of the theorem. $\square$

4. Appendix

4.1. **Conservativity of $(T^{\omega,\theta}, X, \mu)$.** Let $T = T^{\omega,\theta}$. An *(almost invariant) box sequence* is an increasing sequence (Y_n) of open subsets of X such that:

   (1) $X = \cup Y_n$
   (2) each Y_n is of finite measure
   (3) $\lim_{n\to\infty} \mu(T(Y_n) \setminus Y_n) = 0$.

Lemma 5 (Boxes lemma). *If there is an almost invariant box sequence then (T, X, μ) is conservative.*

Proof. Let (Y_n) be a box sequence for T. Let U be a non-empty open set. We want to show that U is *non-wandering*, i.e., that there is an i such that $T^i(U) \cap U \neq \emptyset$.

Let $V_n = U \cap Y_n$. By the definition of a box sequence, we have that $U = \cup V_n$. Thus, for some n_0, V_{n_0} is non-empty. Let $V = V_{n_0}$ and $n \geq n_0$ large enough so $\mu(T(Y_n) \setminus Y_n) < \mu(V)$. Then we have $V \subset Y_n$.

For each $i \geq 1$, let A_i^n be the set of points in V who escape from Y_n at time i, namely

$$A_i^n = \{x \in V : T^i(x) \notin Y_n \text{ and } \forall 0 \leq k < i, T^k(x) \in Y_n\}.$$

Let us suppose the sets $T^i(A_i^n)$ are pairwise disjoint. Note that $\cup_i T^i(A_i^n) \subset T(Y_n) \setminus Y_n$, then we have

$$\mu\left(\bigcup_i A_i^n\right) \leq \Sigma_i \mu\left(A_i^n\right) = \Sigma_i \mu\left(T^i(A_i^n)\right)$$

$$= \mu\left(\bigcup_i T^i(A_i^n)\right) \leq \mu(T(Y_n) \setminus Y_n)) < \mu(V).$$

Thus the set $F = V \setminus \cup_i A_i^n$, which is the set of points in V whose (positive) iterates always stay in Y_n, is of positive measure. Given that Y_n is of finite measure, the Poincaré recurrence Theorem applied to $\cup_{i \geq 0} T^i(F)$ implies that this set F is non-wandering. Thus V and U are also non-wandering.

On the other hand if the sets $T^i(A_i^n)$ are not pairwise disjoint, then there exists $i \neq j$ such that $T^i(A_i^n) \cap T^j(A_j^n) \neq \emptyset$. Note that A_i^n, A_j^n, $T^i(A_i^n)$ and $T^j(A_j^n)$ have all non-empty interiors. Thus it follows that $T^{i-j}(A_i^n) \cap A_j^n \neq \emptyset$. But A_i^n and A_j^n are both subsets of V. Thus V (and also U) is non-wandering. $\qquad\square$

Proof. (Proposition 1) Consider a strictly increasing sequence i_n and a strictly decreasing sequence j_n such that $w_{i_k} \to 0$ and $w_{j_k} \to 0$. Then the sequence of sets $X_n := \{j_k + 1, \ldots, i_k\} \times [0, 2)$ is a sequence of boxes for all θ. $\qquad\square$

4.2. **Cantor representation.** Consider the mapping $T : X \times \mathbb{S}^1 \to X \times \mathbb{S}^1$, induced by the first return map to the cross section X of the translation flow in the direction θ on a compacted connected translation surface. For each θ, the map $T|_{X \times \{\theta\}}$ is an interval exchange transformation (up to the behavior on the endpoints of the intervals). The following construction can be found in [GaKrTr]. Label the sets of continuity of T by a finite enumerated alphabet $\mathcal{A}$, these are connected regions, with piecewise smooth boundary (one can even choose coordinates so that they are polygonal regions [Ka]). Let X^∞ be the set of (x, θ) whose forward orbit does not hit a singularity (nor an endpoint

of the cross section if the cross section does not start and end at a singular point), and for $(x,\theta) \in X^\infty$ let $i(x,\theta) \in \mathcal{A}^{\mathbb{N}}$ be the coding map defined by $i(x,\theta)_j = k \in \mathcal{A}$ if and only if $T^j(x,\theta)$ is in the kth element of $\mathcal{A}$. Let $K := \overline{i(X^\infty)} \subset \mathcal{A}^{\mathbb{N}}$. Let $\widehat{T}$ denote the shift map on K, then by definition $\widehat{T} \circ i = i \circ T$ on X^∞. Each point $(x,\theta) \in X \times \mathbb{S}^1 \setminus X^\infty$ has exactly two images in K. The main result of [GaKrTr] is that the inverse $i^{-1} : K \to X \times \mathbb{S}^1$ satisfies that the set $i^{-1}(s)$ is in fact a single point for any $s \in i(X^\infty)$ except when s is a periodic point. If s is a periodic point then $i^{-1}(s)$ is a periodic cylinder in X.

Finally let $K^\theta := \overline{i(X^\infty \cap (X \times \{\theta\}))}$, this is the Cantor representation of T^θ. Suppose that there are no saddle connections in the direction θ, then there are no periodic points in this direction, and thus $i^{-1} : K^\theta \to \mathbb{R}$ is a well defined continuous map and so we can define the lift $\widehat{h} : K^\theta \to \mathbb{R}$ of $h \in C(X \times \{\theta\})$ by $\widehat{h}(z) = h(i^{-1}(z)$ and conclude that $\widehat{h} \in C(K^\theta)$.

4.3. **Uniform convergence of Hopf averages.** In Section 3 the first return maps we consider, $T^{\omega^i,\theta}|_{X^{N_i}}$ are IETs, thus they are not continuous. Therefore we need to discuss in detail the notion of unique ergodicity and its relation to uniform convergence of Hopf averages, as well as how the convergence varies as we vary the angle on the translation surface. To do this we will use the Cantor representation of the previous subsection, however we will use this representation for a single direction. Thus we will use the notation Y for $X \times \{\theta\}$ for a certain direction θ, $Y^\infty := X^\infty \cap Y$ and we will use K^θ and $i : Y \to K^\theta$ as above.

Lemma 6. *An IET (T, Y) without saddle connection is uniquely ergodic if and only if it's Cantor representation $(\widehat{T}, K^\theta)$ is uniquely ergodic*

Proof. The only if direction is clear.

Consider any pair of $\widehat{T}$ invariant measures $\widehat{\nu}_1, \widehat{\nu}_2$. Using the coding map i we pull back these two measures to a pair of invariant measures ν_1 and ν_2 on Y. Since T is uniquely ergodic these measures satisfy $\nu_1 = \nu_2$. Thus for any measurable set $\widehat{C} \subset K^\theta$, setting $C := i^{-1}(\widehat{C})$ we have

$$\nu_1(C) = \nu_2(C).$$

If $\widehat{\nu}_1 \neq \widehat{\nu}_2$ then there is a measurable set $\widehat{B} \subset K^\theta$ such that $\widehat{\nu}_1(\widehat{B}) \neq \widehat{\nu}_2(\widehat{B})$. Let $B := i^{-1}(\widehat{B})$ and consider $B^\infty := B \cap Y^\infty$ and $B_1 := B \setminus B^\infty$. By construction the set B_1 is at most countable since it is contained in the orbits of the singular points of T. But if either $\nu_1(B_1) > 0$ or $\nu_2(B_1) > 0$ then we have an atomic measure, which can not happen since we have no saddle connection; thus $\nu_1(B_1) = \nu_2(B_1) = 0$ and $\nu_1(B) = \nu_1(B^\infty)$ and the same for ν_2. But $i|_{B^\infty}$ is a bijection to its

image, thus by definition of the pull back measure we conclude that $\widehat{\nu}_1(B) = \widehat{\nu}(B)$. $\qquad\square$

We need the following result which we did not find in the literature, it is well known if $g \equiv 1$, i.e., for Birkhoff averages.

Proposition 7. *If $\widehat{T} : K \to K$ is uniquely ergodic (in the finite measure sense) then for every $f, g \in C(K)$ (with $\int_K g \, d\mu \neq 0$)*

$$\frac{\sum_{k=0}^{\ell} f\big(\widehat{T}^k(z)\big)}{\sum_{k=0}^{\ell} g\big(\widehat{T}^k(z)\big)}$$

converges uniformly to a constant.

The proof is essentially identical to the proof in the Birkhoff case (see for example Theorem 6.19 in [Wa]).

Proof. From the Hopf ergodic theorem the constant must be equal to $\int_K f \, d\mu / \int_K g \, d\mu$ where μ is the unique invariant probability measure. From the uniform converges of Birkhoff sums we can find an $L_0 > 0$ such that for all $\ell \geq L_0$ and all $z \in K$ we have $\sum_{k=0}^{\ell} g\big(\widehat{T}^k(z)\big) > 0$ and thus the Hopf sums are well defined. Suppose that the convergence is not uniform for some f, g. Then there exists $\varepsilon > 0$ such that for all $L \geq L_0$ there exists $\ell > L$ and there exists $z_\ell \in K$ with

$$\left| \frac{\sum_{k=0}^{\ell} f\big(\widehat{T}^k(z_\ell)\big)}{\sum_{k=0}^{\ell} g\big(\widehat{T}^k(z_\ell)\big)} - \frac{\int_K f \, d\mu}{\int_K g \, d\mu} \right| \geq \varepsilon.$$

Set $\mu_\ell := \frac{1}{\ell} \sum_{i=0}^{\ell-1} \delta_{\widehat{T}^i z_\ell}$, then

$$\frac{\sum_{k=0}^{\ell} f\big(\widehat{T}^k(z_\ell)\big)}{\sum_{k=0}^{\ell} g\big(\widehat{T}^k(z_\ell)\big)} = \frac{\int_K f \, d\mu_\ell}{\int_K g \, d\mu_\ell}$$

and thus

$$\left| \frac{\int_K f \, d\mu_\ell}{\int_K g \, d\mu_\ell} - \frac{\int_K f \, d\mu}{\int_K g \, d\mu} \right| \geq \varepsilon.$$

Choose a convergent subsequence $\mu_{\ell_k} \to \mu_\infty$, then μ_∞ is an invariant probability measure and

$$\left| \frac{\int_K f \, d\mu_\infty}{\int_K g \, d\mu_\infty} - \frac{\int_K f \, d\mu}{\int_K g \, d\mu} \right| \geq \varepsilon.$$

Thus $\mu_\infty \neq \mu$. $\qquad\square$

Corollary 8. *Suppose $T : X \to X$ is a uniquely ergodic IET. Then for every $f, g \in C(X)$ (with $\int_X g \, d\mu \neq 0$)*

$$\frac{\sum_{k=0}^{\ell} f\big(T^k(z)\big)}{\sum_{k=0}^{\ell} g\big(T^k(z)\big)}$$

converges uniformly to a constant.

Proof. We consider the Cantor representation $(\widehat{T}, K)$ of (T, X). By Lemma 6 it is uniquely ergodic. Thus we can lift f, g to $\widehat{f}, \widehat{g}$ and apply Proposition 7 to $\widehat{f}$ and $\widehat{g}$. Projecting back to X yields the uniform convergence of the Hopf averages of f and g. $\qquad\square$

References

[Ar] P. Arnoux, *Ergodicité générique des billards polygonaux (d'après Kerckhoff, Masur, Smillie)*. Séminaire Bourbaki, Vol. 1987/88. Astérisque No. 161–162 (1988), Exp. No. 696, 5, 203–221 (1989).

[FrHu] K. Frączek and P. Hubert *Recurrence and non-ergodicity in generalized wind-tree models* Math. Nachrichen 291 (2018) 1686–711.

[FrUl] K. Frączek and C. Ulcigrai *Non-ergodic $\mathbb{Z}$-periodic billiards and infinite translation surfaces* Invent. Math. 197 (2014) 241–298.

[FrUl1] K. Frączek and C. Ulcigrai *Ergodic directions for billiards in a strip with periodically located obstacles* Communications in Mathematical Physics 327 (2014) 643–663.

[GaKrTr] G. Galperin, T. Krüger and S. Troubetzkoy *Local instability of orbits in polygonal and polyhedral billiards* Comm. Math. Phys. 169 (1995), no. 3, 463–473.

[GoLa] S. Gouëzel and E. Lanneau *Un théorème de Kerckhoff, Masur et Smillie: unique ergodicité sur les surfaces plates* (French) École de Théorie Ergodique, 113–145, Sémin. Congr., 20, Soc. Math. France, Paris, 2010.

[HoHuWe] P. Hooper, P. Hubert and B. Weiss *Dynamics on the infinite staircase* Discrete Cont. Dyn. Sys. 33 (2013) 4341–4347.

[Ho] P. Hooper, *The invariant measures of some infinite interval exchange maps* Geom. Topol. 19 (2015)1895–2038.

[HuWe] P. Hubert, and B. Weiss *Ergodicity for infinite periodic translation surfaces* Composito Math. 149 (2013) 1364–1380.

[Ka] A. Katok *The growth rate for the number of singular and periodic orbits for a polygonal billiard* Comm. Math. Phys. 111 (1987), no. 1, 151–160.

[KaZe] A.B. Katok and A.N. Zemlyakov *Topological transitivity of billiards in polygons* Mat. Zametki 18 (1975), no. 2, 291–300.

[KeMaSm] S. Kerckhoff, H. Masur, and J. Smillie, *Ergodicity of billiard flows and quadratic differentials*, Annals of Math. (2) 124 (1986), no. 2, 293–311.

[MS] A. M. Málaga Sabogal *Étude d'une famille de transformations préservant la mesure de $\mathbb{Z} \times T$* PhD dissertation, Paris-Sud 11 University, 2014

[MSTr1] A. Málaga Sabogal and S. Troubetzkoy *Minimality of the Ehrenfest wind-tree model* Journal Modern Dynamics 10 (2016), 209–228.

[MSTr2] A. Málaga Sabogal and S. Troubetzkoy *Ergodicity of the Ehrenfest wind-tree model* Comptes Rendus Mathematique 354 (2016) 1032–1036.

[MSTr3] A. Málaga Sabogal and S. Troubetzkoy *Infinite ergodic index of the Ehrenfest wind-tree model* Comm. Math. Phys. 358 (2018) 995–1006.

[MaTa] H. Masur and S. Tabachnikov, *Rational billiards and flat structures*, Handbook of dynamical systems, Vol. 1A, 1015–1089, North-Holland, Amsterdam, 2002.

[Mo] T. Monteil *Introduction to the theorem of Kerkhoff, Masur and Smillie* Workshop Arbeitsgemeinschaft: Mathematical Billards, Apr 2010, Franckfurt, Germany. pp.955-1015, 2010, Oberwolfach Report

[RaTr] D. Ralston and S. Troubetzkoy *Ergodic infinite group extensions of geodesic flows on translation surfaces* Journal Modern Dynamics 6 (2012) 477–497.

[Sa] O. Sarig *Unique ergodicity for infinite measures* Proceedings of the International Congress of Mathematicians Hyderabad, India, 2010 1777-1803;

[Tr] S. Troubetzkoy *Recurrence in generic staircases* Discrete and Continuous Dynamical Systems - Series A 32, 3 (2012) 1047-1053

[Ve1] W. Veech *Moduli spaces of quadratic differentials* J. D'Analyse Math., 55, (1990) 117–171.

[Ve2] W. Veech *Teichmuller curves in moduli space. Eisenstein series and an application to triangular billiards.* Invent. Math., 97, (1989) 553–583.

[Vo] Ya.B. Vorobets *Ergodicity of billiards in polygons: explicit examples* Russian Math. Surveys, 51 (1996) 756-757.

[Wa] P. Walters *An Introduction to Ergodic Theory* Springer-Verlag (1982).

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