On the 2-edge-coloured chromatic number of grids

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Abstract

The oriented (2-edge-coloured, respectively) chromatic number $\chi_o(G)$ ($\chi_2(G)$, respectively) of an undirected graph G is defined as the maximum oriented (2-edge-coloured, respectively) chromatic number of an orientation (signature, respectively) of G. Although the difference between $\chi_o(G)$ and $\chi_2(G)$ can be arbitrarily large, there are, however, contexts in which these two parameters are quite comparable.

We here compare the behaviour of these two parameters in the context of (square) grids. While a series of works has been dedicated to the oriented chromatic number of grids, we are not aware of any work dedicated to their 2-edge-coloured chromatic number. We investigate this throughout this paper. We show that the maximum 2-edge-coloured chromatic number of a grid lies between 8 and 11. We also focus on 2-row grids and 3-row grids, and exhibit bounds on their 2-edge-coloured chromatic number, some of which are tight. Although our results indicate that the oriented chromatic number and the 2-edge-coloured chromatic number of grids are close in general, they also show that these parameters may differ, even for easy instances.

Keywords: 2-edge-coloured chromatic number, oriented chromatic number, grids.

1. Introduction

Colouring problems are among the most important problems of graph theory, as they can model many real-life problems under a graph-theoretical formalism. In its most common sense, a *colouring* of an undirected graph G refers to a *proper vertex-colouring*, which is a colouring of $V(G)$ such that every two adjacent vertices of G get assigned distinct colours. Many variants of this definition have been introduced and studied in the literature, including variants dedicated to modified kinds of graphs, which are of interest in this paper.

Namely, our investigations are related to two kinds of modified graphs, called *oriented graphs* and 2-*edge-coloured graphs*. An *oriented graph* $\overrightarrow{G}$ is a directed graph obtained from an undirected simple graph G by orienting every edge uv either from u to v (resulting in an arc $\overrightarrow{uv}$) or conversely (resulting in an arc $\overrightarrow{vu}$). We sometimes also call $\overrightarrow{G}$ an *orientation* of G. Now, from G, we can also get a 2-*edge-coloured graph* (G, σ) by assigning a sign $\sigma(uv)$, being either $-$ (negative) or $+$ (positive), to every edge uv of G. We call (G, σ) a *signature* of G. In the literature, 2-edge-coloured graphs are sometimes also called *signified graphs*, from which we here borrow the terminology above.

One of the most judicious ways for extending the notion of proper vertex-colouring to oriented graphs and 2-edge-coloured graphs is through the notion of graph homomorphisms. That is, a proper k-vertex-colouring ϕ of an undirected graph G can be regarded as a *homomorphism* from G to K_k (the complete graph on k vertices), i.e., a mapping $\phi : V(G) \to V(K_k)$ preserving the edges (i.e., for every edge uv of G, we have that $\phi(u)\phi(v)$ is an edge of K_k). Quite similarly, we can define an *oriented homomorphism* as a vertex-mapping (from an oriented graph to another one) preserving not only the arcs but also the arc directions, and a 2-*edge-coloured homomorphism* as a vertex-mapping (from a 2-edge-coloured graph to another one) preserving not only the edges but also the edge signs. From this, an *oriented colouring* ϕ of an oriented graph can be defined as a

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vertex-colouring such that, for any two arcs $\overrightarrow{u_1v_1}$ and $\overrightarrow{u_2v_2}$, if $\phi(u_1) = \phi(v_2)$ then $\phi(v_1) \neq \phi(u_2)$. Analogously, a *2-edge-coloured colouring* ϕ of a 2-edge-coloured graph has the property that, for any two edges u_1v_1 and u_2v_2 with different signs, if $\phi(u_1) = \phi(v_1)$ then $\phi(u_2) \neq \phi(v_2)$.

Given a graph and a particular colouring variant, the main objective is usually to find a colouring of the graph that minimizes the number of colours. For an undirected graph G, the least number of colours in a proper vertex-colouring is called the *chromatic number* of G, commonly denoted by $\chi(G)$. From the homomorphism point of view, $\chi(G)$ can also be defined as the smallest k such that G admits a homomorphism to K_k. Concerning the aforementioned colouring variants for oriented graphs and 2-edge-coloured graphs, the associated chromatic parameters are called the *oriented chromatic number* and *2-edge-coloured chromatic number*, respectively, and are denoted by $\chi_o(\overrightarrow{G})$ and $\chi_2((G,\sigma))$, respectively (where $\overrightarrow{G}$ is an oriented graph, and (G,σ) is a 2-edge-coloured graph). The parameters χ_o and χ_2 can also be derived for undirected graphs: for an undirected graph G, $\chi_o(G)$ is defined as the maximum value of χ_o for an orientation of G, while $\chi_2(G)$ is defined as the maximum value of χ_2 for a signature of G. In other words, $\chi_o(G)$ and $\chi_2(G)$ indicate whether G is the underlying graph of oriented or 2-edge-coloured graphs needing many colours to be coloured. For more details on these two chromatic parameters, we refer the interested reader to the recent survey [7] by Sopena dedicated to the oriented chromatic number, and to the Ph.D. thesis [6] of Sen, which is dedicated, in particular, to both the oriented chromatic number and the 2-edge-coloured chromatic number.

Our investigations in this paper are motivated by the general relation between $\chi_o(G)$ and $\chi_2(G)$ for a given undirected graph G. Intuitively, one could expect these two parameters to be close somehow, as oriented graphs and 2-edge-coloured graphs are rather alike notions: in both an orientation and a signature of G, every edge has one of two possible "states" (being oriented in one way or the other, or being positive or negative). From a more local point of view, though, an oriented edge and a 2-edge-coloured edge are perceived differently by their two ends. In light of these two facts, it thus appears legitimate to wonder whether oriented graphs and 2-edge-coloured graphs have comparable behaviours (in general, or in particular cases). This aspect was notably investigated by Sen in his Ph.D. thesis [6].

In general, it has to be known that, for a given undirected graph G, the difference between $\chi_o(G)$ and $\chi_2(G)$ can be arbitrarily large, as noted by Bensmail, Duffy and Sen in [1]. A natural arising question is thus whether this behaviour is systematic or can be observed for a restricted number of graph classes only. Towards this question, we here focus on the class of (square) *grids*, where the grid $G(n,m)$ with n rows and m columns is defined as the undirected graph being the Cartesian product of the path with order n and the path with order m. While, to the best of our knowledge, no studies dedicated to the 2-edge-coloured chromatic number of grids were led, a series of works, namely [2, 4, 8], can be found in the literature on the oriented chromatic number of these graphs. In brief words, these works have (1) pointed out that the maximum oriented chromatic number of a grid lies between 8 and 11, and have (2) established the exact oriented chromatic number of grids with at most four rows. More details on these results will be given throughout this paper as they connect to our investigations.

We must also report that some upper bounds on the 2-edge-coloured chromatic number of grids can be derived from more general results. In particular, Nešetřil and Raspaud proved in [5] that every undirected graph G with acyclic chromatic number k has 2-edge-coloured chromatic number at most $k \cdot 2^{k-1}$; since grids were shown to have acyclic chromatic number at most 3 (see [3]), this implies that grids have 2-edge-coloured chromatic number at most 12.

We thus initiate the study of the 2-edge-coloured chromatic number of grids as such, our main objective being to investigate how close the oriented chromatic number and the 2-edge-coloured chromatic number of these graphs are. Before presenting our results, we first introduce, in Section 2, some definitions and terminology that are used throughout this paper. We then start, in Section 3, by providing a general constant upper bound on the 2-edge-coloured chromatic number of grids. Namely, we prove that $\chi_2(G(n,m)) \leq 11$ holds for every $n, m \geq 1$, which improves the upper bound of 12 mentioned above. We then get, in Sections 4 and 5, first lower bounds on the 2-edge-coloured chromatic number of grids by focusing on 2-edge-coloured grids with at most three rows. In particular, we point out that some 2-edge-coloured 3-row grids cannot be coloured with less than 7 colours. We also provide refined bounds on the 2-edge-coloured chromatic number of

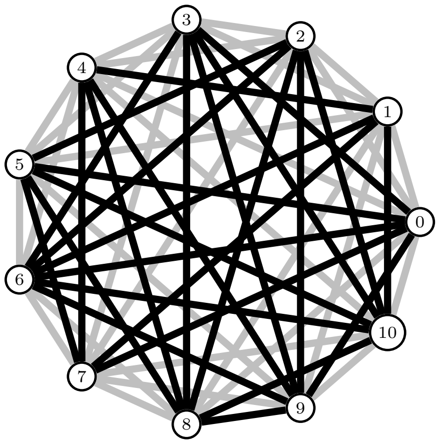

Figure 1: The 2-edge-coloured graph A_{11}. Black (gray, respectively) edges are positive (negative, respectively) edges.

2-row grids and 3-row grids, our bounds for 2-row grids being sharp. Generalizing the proofs of our lower bounds for 2-edge-coloured 3-row grids, we then show, in Section 6, that there exist 2-edge-coloured grids with 2-edge-coloured chromatic number at least 8. We finally conclude this paper by summarizing our results in Section 7, and by discussing how the oriented chromatic number and 2-edge-coloured chromatic number of grids compare.

2. Definitions and terminology

Throughout this paper, we use σ to refer to the implicit signature function of any 2-edge-coloured graph G. For every vertex v of G, we say that another vertex u is a $-$*-neighbour* (+*-neighbour*, respectively) of v if uv is a negative (positive, respectively) edge. The $-$*-degree* (+*-degree*, respectively) of v is its number of $-$-neighbours (+-neighbours, respectively)

Let A be a 2-edge-coloured graph. By an A*-colouring* of G, we refer to a homomorphism from G to A. We also say that G is *coloured by* A. To stick to the colouring point of view, the vertices of any colouring graph A are generally represented, in our proofs, by consecutive integers $0, \dots, |V(A)| - 1$. A downside of this notation is that, to refer to an edge $\alpha\beta$ of A, we sometimes have to write it under the form $\{\alpha, \beta\}$ to avoid any ambiguity. In that spirit, we denote k*-paths* (i.e., paths of length k) of A under the form $P = (\alpha_1, \dots, \alpha_{k+1})$, where $\alpha_1, \dots, \alpha_{k+1}$ are the consecutive vertices of P. Assuming the signs of the k edges of P are $s_1, \dots, s_k$, we sometimes say that P is an $s_1 \dots s_k$*-path*. Similarly as for paths, we denote by $(\alpha_1, \dots, \alpha_k, \alpha_1)$ any k*-cycle* (i.e., cycle of length k). Any 2-edge-coloured path or cycle is said *alternating* if no two of its consecutive edges have the same sign.

Some of our upper bounds in this paper are established from colourings by special 2-edge-coloured graphs which we call 2*-edge-coloured circulant graphs*. The definition is as follows (see Figure 2 (right) for an illustration). Let K_n be the complete graph with vertex set $\{0, \dots, n-1\}$, and $S \subseteq \{1, \dots, n-1\}$ be a set of integers. The 2*-edge-coloured circulant graph* $C(n, S)$ (generated by S) is the signature of K_n where the edge $\{i, (i+j) \pmod n\}$ is positive for every $j \in S$ and $i \in \{0, \dots, n-1\}$, while all other edges are negative.

3. A general upper bound

The only known upper bound on the oriented chromatic number of grids was exhibited by Fertin, Raspaud and Roychowdhury, who proved in [4] that $\chi_o(G(n, m)) \leq 11$ holds for every $n, m \geq 1$. In this section, we prove that, for every grid $G = G(n, m)$, we have $\chi_2(G) \leq 11$ as well. As mentioned in the introductory section, this improves a bound of 12 that can be derived from general results on the 2-edge-coloured chromatic number.

Vertex	$-$-neighbours	$+$-neighbours
0	$1, 2, 4, 8, 10$	$3, 5, 6, 7, 9$
1	$0, 2, 3, 5, 8, 9$	$4, 6, 7, 10$
2	$0, 1, 3, 4, 7$	$5, 6, 8, 9, 10$
3	$1, 2, 4, 5, 7$	$0, 6, 8, 9, 10$
4	$0, 2, 3, 5, 6, 10$	$1, 7, 8, 9$
5	$1, 3, 4, 6, 9$	$0, 2, 7, 8, 10$
6	$4, 5, 7, 8$	$0, 1, 2, 3, 9, 10$
7	$2, 3, 6, 8, 9, 10$	$0, 1, 4, 5$
8	$0, 1, 6, 7$	$2, 3, 4, 5, 9, 10$
9	$1, 5, 7, 10$	$0, 2, 3, 4, 6, 8$
10	$0, 4, 7, 9$	$1, 2, 3, 5, 6, 8$

Table 1: Adjacencies of A_{11}.

We more precisely prove that every 2-edge-coloured grid admits an A_{11}-colouring, where A_{11} is the signature of K_{11} depicted in Figure 1. To avoid any ambiguity, the $-$-neighbours and $+$-neighbours of every vertex of A_{11} are listed in Table 1. A_{11} has properties that will prove to be of interest to us, some of which are tedious to prove formally due to the lack of general symmetries of A_{11}. We point out some of these properties, that can easily be checked by hand using Table 1.

Observation 3.1. *A_{11} has the following properties:*

- *P1. Every vertex of A_{11} has $-$-degree (and $+$-degree) at least 4 and at most 6.*
- *P2. For every two vertices $u \neq v$ of A_{11}, there exist $++$-paths from u to v.*
- *P3. For every two vertices $u \neq v$ of A_{11}, there exist $--$-paths from u to v.*
- *P4. For every two vertices $u \neq v$ of A_{11}, there exist $+-$-paths from u to v.*
- *P5. For every two vertices $u \neq v$ of A_{11}, there exist $-+$-paths from u to v.*

To ease the checking of Properties P2 to P5, we provide, in Table 2, the exhaustive list of all $++$-paths, $--$-paths, $+-$-paths and $-+$-paths of A_{11}. Due to the large number of cases to consider, that table is postponed to the Appendix section, at the end of this paper.

We are now ready to prove our main result.

Theorem 3.2. *Every 2-edge-coloured grid is A_{11}-colourable. Therefore, for every $n, m \geq 1$, we have $\chi_2(G(n,m)) \leq 11$.*

Proof. Consider G any signature of $G(n,m)$. We construct an A_{11}-colouring ϕ of G in the following way. First, we assign a colour by ϕ to every vertex of the first row, from the first-column vertex to the last-column vertex. We then repeatedly do the following, row by row. Assuming all vertices of the $(i-1)$th row have been assigned a colour by ϕ, we then extend the partial A_{11}-colouring to the vertices of the ith row, from the first-column vertex to the last-column vertex. Once this has been performed for every row of G, we will end up with ϕ being an A_{11}-colouring of the whole grid G.

Let us consider the consecutive vertices $a_1, \dots, a_n$ of the first row of G, where a_1 (a_n, respectively) is the first-column (last-column, respectively) vertex. We start by setting e.g. $\phi(a_1) = 0$. We now claim that, assuming $\phi(a_{i-1})$ has been fixed (for some $i \geq 1$), we can correctly extend the partial A_{11}-colouring to a_i. When choosing $\phi(a_i)$, we just need to make sure that the sign of $\phi(a_{i-1})\phi(a_i)$ in A_{11} matches that of $a_{i-1}a_i$ in G. Since all vertices of A_{11} have $-$-degree and $+$-degree at least 4, recall Property P1 of Observation 3.1, we then have at least four colours that can correctly be assigned to $\phi(a_i)$. Repeating this argument for all successive vertices of the first row, we end up with a correct A_{11}-colouring of the first row of G.

Now assume all vertices $a_1, \dots, a_n$ of the $(i-1)$th row (for some $i \geq 1$) of G have been assigned a colour by ϕ, and consider the consecutive vertices $b_1, \dots, b_n$ of the ith row (where, for every j, a_j, b_j are the vertices of the jth column). Assume we want to colour the b_i's as going from b_1 to b_n. When considering a vertex b_i, we note that $\phi(b_i)$ must be chosen in such a way that the signs of $\phi(b_{i-1})\phi(b_i)$ and $\phi(b_i)\phi(a_i)$ in A_{11} match that of $b_{i-1}b_i$ and b_ia_i, respectively, in G. This implies that we need to make sure that, in A_{11}, there exist 2-edge-coloured 2-paths $\phi(b_{i-1})\phi(b_i)\phi(a_i)$ whose signs match that of $b_{i-1}b_ia_i$. According to Properties P1 to P5 of Observation 3.1, such paths always exist in A_{11}, provided $\phi(b_{i-1}) \neq \phi(a_i)$, or $\phi(b_{i-1}) = \phi(a_i)$ but $\phi(b_{i-1})\phi(b_i)$ and $\phi(b_i)\phi(a_i)$ have the same sign. In other words, we must avoid the situation where $\phi(b_{i-1}) = \phi(a_i)$ when the signs of $b_{i-1}b_i$ and b_ia_i are different. One problem is that, as noted in Table 2, there are configurations of colours and signs where only one colour can be correctly chosen as $\phi(b_i)$ (for instance, when $\phi(b_{i-1}) = 0$, $\phi(a_i) = 5$ and $b_{i-1}b_i$ and b_ia_i are both positive). This is an issue, as this might lead to b_{i+1} being not correctly colourable (typically when the unique possible colour for b_i is that of a_{i+1}, the edge b_ib_{i+1} is positive, and the edge $b_{i+1}a_{i+1}$ is negative).

Because of such configurations, we cannot just colour the b_i's one after another, as we may fall into a dead end. What we do instead, is computing and memorizing the possible colours for b_i by all possible correct partial A_{11}-colourings of the previous vertices $b_1, \dots, b_{i-1}$. More formally, for each vertex b_i, we consider the function $\psi(b_i)$ being the set of colours such that for each $\alpha \in \psi(b_i)$, there is an extension of ϕ to $b_1, \dots, b_i$ where $\phi(b_i) = \alpha$. What we prove below is that $|\psi(b_n)| > 0$, which implies that ϕ can correctly be extended to all b_i's, thus to the whole row.

We first consider $\psi(b_1)$. The possible colours for $\phi(b_1)$ are those such that the sign of $\phi(a_1)\phi(b_1)$ in A_{11} matches that of a_1b_1. This implies that $\psi(b_1)$ is highly dependent of $\phi(a_1)$. For instance, if $\phi(a_1) = 0$ and a_1b_1 is positive, then $\psi(b_1)$ is the set of all $+$-neighbours of vertex 0 in A_{11}. If $\phi(a_1) = 0$ and a_1b_1 is negative, then $\psi(b_1)$ is the set of all $-$-neighbours of vertex 0 in A_{11}. And so on. In other words, $\psi(b_1) \in \mathcal{L}_1$, where $\mathcal{L}_1$ is the union, over all vertices of A_{11}, of the $-$-neighbourhoods and $+$-neighbourhoods; thus, $\mathcal{L}_1$ can be extracted directly from Table 1.

Claim 3.3. *$\psi(b_1) = L$, where $L \in \mathcal{L}_1 := \{\{0,1,2,3,9,10\}, \{0,1,3,4,7\}, \{0,1,4,5\}, \{0,1,6,7\}, \{0,2,3,4,6,8\}, \{0,2,3,5,6,10\}, \{0,2,3,5,8,9\}, \{0,2,7,8,10\}, \{0,4,7,9\}, \{0,6,8,9,10\}, \{1,2,3,5,6,8\}, \{1,2,4,5,7\}, \{1,2,4,8,10\}, \{1,3,4,6,9\}, \{1,5,7,10\}, \{1,7,8,9\}, \{2,3,4,5,9,10\}, \{2,3,6,8,9,10\}, \{3,5,6,7,9\}, \{4,5,7,8\}, \{4,6,7,10\}, \{5,6,8,9,10\}\}$.*

One way for making sure that a bad configuration (as described earlier) does not occur, is to have all $\psi(b_i)$'s having sufficiently many elements (i.e., at least three). This is already the case for $\psi(b_1)$ by Claim 3.3, as $\psi(b_1) \in \mathcal{L}_1$.

Observation 3.4. *For every set $L \in \mathcal{L}_1$, we have $|L| \geq 3$. Consequently, $|\psi(b_1)| \geq 3$.*

We now consider $\psi(b_2)$. Note that $\psi(b_2)$ depends on $\psi(b_1)$ (which itself depends on $\phi(a_1)$), on the signs of b_1b_2 and b_2a_2, and on $\phi(a_2)$. Taking all these elements into consideration, and playing with Table 2, from a tedious checking it can be checked that the following holds true:

Claim 3.5. *$\psi(b_2) = L$, where either:*

- *$L \in \mathcal{L}_2 := \{\{0,1,2,3,9\}, \{0,1,2,3,10\}, \{0,1,2,9,10\}, \{0,1,3,9,10\}, \{0,1,4\}, \{0,1,5\}, \{0,1,6\}, \{0,1,7\}, \{0,2,3,4,6\}, \{0,2,3,4,8\}, \{0,2,3,5,6\}, \{0,2,3,5,8\}, \{0,2,3,5,9\}, \{0,2,3,5,10\}, \{0,2,3,6,8\}, \{0,2,3,6,10\}, \{0,2,3,8,9\}, \{0,2,3,9,10\}, \{0,2,4,6,8\}, \{0,2,5,6,10\}, \{0,2,5,8,9\}, \{0,2,7,8\}, \{0,2,7,10\}, \{0,2,8,10\}, \{0,3,4,6,8\}, \{0,3,5,6,10\}, \{0,3,5,8,9\}, \{0,4,5\}, \{0,4,7\}, \{0,4,9\}, \{0,6,7\}, \{0,6,8,9\}, \{0,6,8,10\}, \{0,6,9,10\}, \{0,7,8,10\}, \{0,7,9\}, \{0,8,9,10\}, \{1,2,3,5,6\}, \{1,2,3,5,8\}, \{1,2,3,6,8\}, \{1,2,3,9,10\}, \{1,2,4,7\}, \{1,2,4,8\}, \{1,2,4,10\}, \{1,2,5,6,8\}, \{1,2,8,10\}, \{1,3,4,6\}, \{1,3,4,7\}, \{1,3,4,9\}, \{1,3,5,6,8\}, \{1,3,6,9\}, \{1,4,5\}, \{1,4,6,9\}, \{1,4,8,10\}, \{1,5,7\}, \{1,5,10\}, \{1,6,7\}, \{1,7,8\}, \{1,7,9\}, \{1,7,10\}, \{1,8,9\}, \{2,3,4,5,9\}, \{2,3,4,5,10\}, \{2,3,4,6,8\}, \{2,3,4,9,10\}, \{2,3,5,6,8\}, \{2,3,5,6,10\}, \{2,3,5,8,9\}, \{2,3,5,9,10\}, \{2,3,6,8,9\}, \{2,3,6,8,10\}, \{2,3,6,9,10\}, \{2,3,8,9,10\}, \{2,4,5,9,10\}, \{2,4,8,10\}, \{2,7,8,10\}, \{3,4,5,9,10\}, \{3,4,6,9\}, \{3,5,6,7\}, \{3,5,6,9\}, \{3,5,7,9\}, \{3,6,7,9\}, \{4,5,7\}, \{4,5,8\}, \{4,6,7\}, \{4,6,10\}, \{4,7,8\}, \{4,7,9\}, \{4,7,10\}, \{5,6,7,9\}, \{5,6,8,9\}, \{5,6,8,10\}, \{5,6,9,10\}, \{5,7,8\}, \{5,7,10\}, \{5,8,9,10\}, \{6,7,10\}, \{6,8,9,10\}, \{7,8,9\}\}$, or*
- *L is a superset of some set $L' \in \mathcal{L}_1 \cup \mathcal{L}_2$.*

As an illustration, assume that $\psi(b_1) = \{0,1,4\}$ and that $\phi(a_2) = 0$. If b_1b_2 and b_2a_2 are both positive, then, looking at Table 1, we see that $0 \in \psi(b_1)$ implies $\{3,5,6,7,9\} \subseteq \psi(b_2)$, which makes $\psi(b_2)$ be a superset of $\{3,5,6,7,9\} \in \mathcal{L}_1$. If b_1b_2 is positive while b_2a_2 is negative, then $1 \in \psi(b_1)$ implies $\{4,10\} \in \psi(b_2)$ while $4 \in \psi(b_1)$ implies $\{1,8\} \in \psi(b_2)$; in total, we thus have $\psi(b_2) = \{1,4,8,10\} \in \mathcal{L}_2$.

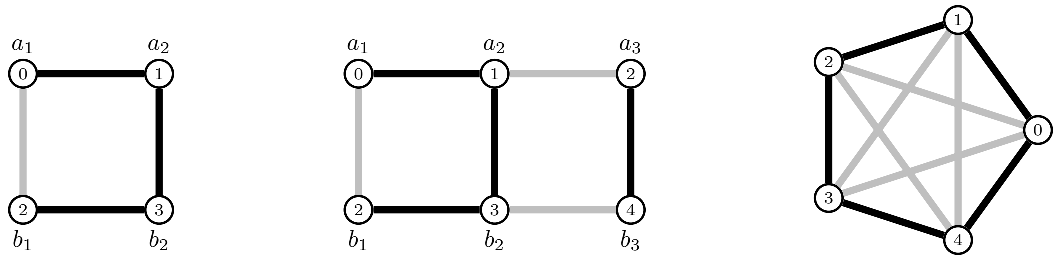

Figure 2: A 2-edge-coloured 4-colouring of a signature of $G(2,2)$ (left), a 2-edge-coloured 5-colouring of a signature of $G(2,3)$ (middle), and the 2-edge-coloured circulant graph $C(5,\{1\})$ (right). Black (gray, respectively) edges are positive (negative, respectively) edges.

To fully prove that Claim 3.5 holds, the same reasoning must be performed for every combination of $\psi(b_1), \phi(a_2), \sigma(b_1b_2), \sigma(b_2a_2)$, which is quite tedious due to the non-symmetric structure of A_{11}. For this reason, we provide in the online file `http://jbensmai.fr/code/signed-grids/A11-L2.txt` an exhaustive list of all cases.

Observation 3.6. *For every set $L \in \mathcal{L}_2$, we have $|L| \geq 3$. Consequently, $|\psi(b_2)| \geq 3$.*

The exact same process can then be performed for $\psi(b_3)$ (except that, here, $\psi(b_3)$ depends on $\psi(b_2), \phi(a_3), \sigma(b_2b_3), \sigma(b_3a_3)$). We here get:

Claim 3.7. *$\psi(b_3) = L$, where either:*

- $L \in \mathcal{L}_3 := \{\{0,1,2,10\}, \{0,1,3,9\}, \{0,1,9,10\}, \{0,2,4,8\}, \{0,2,5,8\}, \{0,2,5,10\}, \{0,2,6,10\}, \{0,2,8,9\}, \{0,3,4,6\}, \{0,3,5,6\}, \{0,3,5,9\}, \{0,4,6,8\}, \{0,5,6,10\}, \{0,5,8,9\}, \{0,7,8\}, \{0,7,10\}, \{1,2,5,8\}, \{1,2,6,8\}, \{1,3,5,6\}, \{1,3,9,10\}, \{1,4,6\}, \{1,4,8\}, \{1,4,9\}, \{1,4,10\}, \{1,5,6,8\}, \{2,4,5,10\}, \{2,4,9,10\}, \{2,6,8,10\}, \{2,8,9,10\}, \{3,4,5,9\}, \{3,4,6,8\}, \{3,5,6,10\}, \{3,5,8,9\}, \{3,6,8,9\}, \{3,6,9,10\}, \{4,5,9,10\}, \{5,6,7\}, \{5,7,9\}, \{6,7,9\}, \{7,8,10\}\}$*, or*
- *L is a superset of some set $L' \in \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3$.*

Again, we provide the external online file `http://jbensmai.fr/code/signed-grids/A11-L3.txt`, which contains a full analysis of all cases.

Observation 3.8. *For every set $L \in \mathcal{L}_3$, we have $|L| \geq 3$. Consequently, $|\psi(b_3)| \geq 3$.*

We are now done, because applying the same deduction process onto $\psi(b_4)$ gives that $\psi(b_4)$ (and thus each of $\psi(b_5), \dots, \psi(b_n)$) must be a superset of a set in $\mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3$. Again, the exhaustive process is described in details online at `http://jbensmai.fr/code/signed-grids/A11-L4.txt`.

Claim 3.9. *For every $i = 4, \dots, n$, we have $\psi(b_i) = L$, where L is a superset of some set $L' \in \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3$.*

In particular, $\psi(b_i)$ is thus defined for every b_i. Consequently, there is a way to extend ϕ to an A_{11}-colouring so that $\phi(b_n) \in \psi(b_n)$, thus to the whole row by the definition of ψ. Repeating this colouring process row by row, we end up with ϕ being an A_{11}-colouring of G. $\square$

4. 2-edge-coloured grids with two rows

The oriented chromatic number of 2-row grids was fully determined by Fertin, Raspaud and Roychowdhury in [4], who proved that $\chi_o(G(2,n)) = 6$ for every $n \geq 4$, while $G(2,2)$ and $G(2,3)$ have oriented chromatic number 4 and 5, respectively. We here completely determine the 2-edge-coloured chromatic number of 2-row grids by mainly showing that $\chi_2(G(2,n)) \leq 5$ for every $n \geq 3$. Hence, for this type of grids, the 2-edge-coloured chromatic number is always smaller than the oriented chromatic number.

We start off by noting that $G(2,2)$, which is the cycle of length 4, admits a signature for which each of the vertices must be coloured with a unique colour in any 2-edge-coloured colouring.

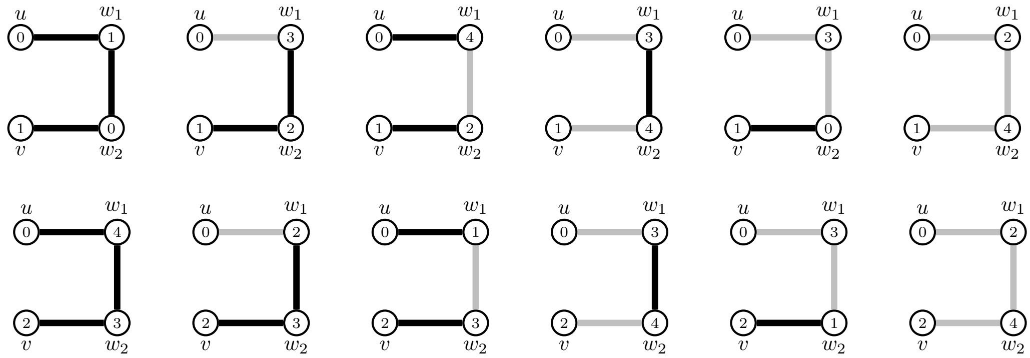

Figure 3: Examples of the 2-edge-coloured 3-paths of $C(5, \{1\})$ claimed in the proof of Observation 4.4, for $(u, v) = (0, 1)$ (top), and $(u, v) = (0, 2)$ (bottom). Black (gray, respectively) edges are positive (negative, respectively) edges.

Proposition 4.1. *We have* $\chi_2(G(2, 2)) = 4$.

Proof. Consider the signature of $G(2, 2)$ depicted in Figure 2 (left). In this 2-edge-coloured graph, every two non-adjacent vertices are joined by an alternating 2-path. Since, for every such alternating 2-path, the two end-vertices must receive distinct colours by any 2-edge-coloured colouring, we get that this signature of $G(2, 2)$ cannot be coloured with less than $|V(G(2, 2))|$ colours. □

Since $G(2, 2)$ is a subgraph of $G(2, n)$ for every $n \geq 2$, by Proposition 4.1 we get that $\chi_2(G(2, n)) \geq 4$ for every $n \geq 2$. In the following, we prove that, actually, $\chi_2(G(2, n)) \geq 5$ holds for every $n \geq 3$.

Proposition 4.2. *We have* $\chi_2(G(2, 3)) \geq 5$.

Proof. To be convinced of this statement, consider the signature of $G(2, 3)$ depicted in Figure 2 (middle), and assume, for contradiction, that it admits a 2-edge-coloured 4-colouring ϕ. We note that the vertices a_1, a_2, b_1, b_2 form exactly the signature of $G(2, 2)$ described in the proof of Proposition 4.1. As explained earlier, these four vertices must be assigned different colours by ϕ. Assume $\phi(a_1) = 0$, $\phi(a_2) = 1$, $\phi(b_1) = 2$ and $\phi(b_2) = 3$ without loss of generality. Now, because a_3 is adjacent to a_2, and a_3 is joined by alternating 2-paths to both a_1 and b_2, clearly we must have $\phi(a_3) = 2$. But now, b_3 cannot be assigned any of colours 1, 2 or 3 for the same reasons, while it cannot be assigned colour 0 since a_1b_1 and a_3b_3 have different signs and $\phi(a_3) = \phi(b_1) = 2$. So b_3 cannot be assigned a colour by ϕ, contradicting our initial hypothesis. □

Again, since $G(2, 3)$ is a subgraph of $G(2, n)$ for every $n \geq 3$, Proposition 4.2 implies that $\chi_2(G(2, n)) \geq 5$ holds for every $n \geq 3$. Actually, it turns out that five colours are sufficient to colour any signature of any 2-row grid.

Proposition 4.3. *For every* $n \geq 1$, *we have* $\chi_2(G(2, n)) \leq 5$.

Proof. We actually show that every signature of $G(2, n)$, where $n \geq 1$, can be coloured by the 2-edge-coloured circulant graph $C(5, \{1\})$ (see Figure 2 (right)). To that aim, let us first point out the following property of $C(5, \{1\})$.

Observation 4.4. *For every two distinct vertices* u, v *of* $C(5, \{1\})$, *and for every set* $\{s_1, s_2, s_3\}$ *of* $\{-, +\}^3$, *there exists a 3-path* uw_1w_2v *in* $C(5, \{1\})$ *such that* $\sigma(uw_1) = s_1$, $\sigma(w_1w_2) = s_2$, $\sigma(w_2v) = s_3$.

Proof. Due to the signature-preserving automorphisms of $C(5, \{1\})$, we may restrict our attention to the cases $(u, v) = (0, 1)$ and $(u, v) = (0, 2)$. Furthermore, only six of the sets among $\{-, +\}^3$ have to be considered. To see that the claim holds, refer to Figure 3, which gathers examples of the claimed twelve 3-paths of $C(5, \{1\})$. □

Back to the proof of Proposition 4.3, we now describe how to get a colouring ϕ by $C(5, \{1\})$ of any signature G of $G(2, n)$ with $n \geq 1$. Let us denote by $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ the consecutive vertices of the first and second rows of G, respectively, where a_i, b_i are the vertices of the ith column for every $i = 1, \ldots, n$. As a first step, we colour a_1 and b_1. For this purpose, we choose an edge $\{\alpha, \beta\}$ of $C(5, \{1\})$ having sign $\sigma(a_1b_1)$ and set $\phi(a_1) = \alpha$ and $\phi(b_1) = \beta$.

To complete the colouring by $C(5, \{1\})$, it now suffices to repeatedly apply the following procedure. Assuming vertices a_{i-1} and b_{i-1} have been coloured in the previous step, we extend ϕ to a_i and b_i. Let s_1, s_2, s_3 be the signs of $a_{i-1}a_i, a_ib_i, b_ib_{i-1}$, respectively. According to Observation 4.4 (applied to $u = \phi(a_{i-1})$, $v = \phi(b_{i-1})$ and s_1, s_2, s_3), there exists a 3-path $(\phi(a_{i-1}), \alpha, \beta, \phi(b_{i-1}))$ in $C(5, \{1\})$ whose edges have sign s_1, s_2, s_3, respectively. By hence setting $\phi(a_i) = \alpha$ and $\phi(b_i) = \beta$, we get an extension of ϕ to a_i and b_i. $\square$

From all the previous results, we end up with the following characterization of the 2-edge-coloured chromatic number of 2-row grids.

Theorem 4.5. *We have:*

- $\chi_2(G(2, 2)) = 4$,
- $\chi_2(G(2, n)) = 5$ *for every* $n \geq 3$.

5. 2-edge-coloured grids with three rows

The investigations on the oriented chromatic number of 3-row grids were initiated by Fertin, Raspaud and Roychowdhury who proved, in [4], that $\chi_o(G(3, 3)), \chi_o(G(3, 4)), \chi_o(G(3, 5)) = 6$, while $\chi_o(G(3, n)) \in \{6, 7\}$ for every $n \geq 6$. Later on, Szepietowski and Targan completely determined, in [8], the values of $\chi_o(G(3, n))$ for every $n \geq 6$ by proving that $\chi_o(G(3, 6)) = 6$ while $\chi_o(G(3, n)) = 7$ for every $n \geq 7$.

Before presenting our results on 2-edge-coloured 3-row grids, we first introduce some definitions and terminology that are used throughout this section.

Whenever dealing with a (2-edge-coloured) 3-row grid $G = G(3, n)$, we assume that its vertices are labelled by $a_1, \ldots, a_n$, $b_1, \ldots, b_n$ and $c_1, \ldots, c_n$, where the a_i's are the consecutive vertices of the first row, the b_i's are the consecutive vertices of the second row, and the c_i's are the consecutive vertices of the third row. This labelling is such that, for every $i = 1, \ldots, n$, the vertices of the ith column are a_i, b_i, c_i (see Figure 4 (left) for an illustration).

Let A be a 2-edge-coloured graph, and assume now that G is a 2-edge-coloured 3-row grid. In the sequel, we will mainly A-colour G by extending a partial A-colouring ϕ from column to column, starting from the first column. When doing so, for each column i we get a set of possible *triplets* of colours, which are 3-element sets $(\alpha, \beta, \gamma) \in \{0, 1, \ldots, |V(A)| - 1\}^3$ such that, when extending ϕ to the ith column, we can correctly set $\phi(a_i) = \alpha$, $\phi(b_i) = \beta$ and $\phi(c_i) = \gamma$. Note that every triplet (α, β, γ) verifies $\beta \neq \alpha, \gamma$.

When extending ϕ to the ith column of G, the possible colours for a_i, b_i, c_i, i.e., the possible triplets $(\alpha_i, \beta_i, \gamma_i)$ of colours that can be assigned to this column, are highly dependent of the triplet $(\alpha_{i-1}, \beta_{i-1}, \gamma_{i-1})$ of colours assigned to the $(i-1)$th column. Also, assuming $\phi(a_{i-1}) = \alpha_{i-1}$, $\phi(b_{i-1}) = \beta_{i-1}$, $\phi(c_{i-1}) = \gamma_{i-1}$, the possible triplets $(\alpha_i, \beta_i, \gamma_i)$ depend on the set of five edges $\{a_{i-1}a_i, b_{i-1}b_i, c_{i-1}c_i, a_ib_i, b_ic_i\}$ which form a 2-edge-coloured subgraph that we call a 2-*comb*. Formally, a 2-comb refers to a graph obtained from a path $uw_1w_2w_3v$ of length 4 by joining w_2 to a new pendant vertex w. Under that labelling, we say that the 2-comb *joins* u, w, v and call $w_1w_2w_3$ the *spine* of the 2-comb. We note that any 2-edge-coloured 3-row grid can be obtained, starting from a 2-edge-coloured 2-path $a_1b_1c_1$, by repeatedly joining $a_ib_ic_i$ (being the original path $a_1b_1c_1$, or the spine of the lastly-added 2-comb) via a new 2-edge-coloured 2-comb with spine $a_{i+1}b_{i+1}c_{i+1}$.

Back to our context, the possible triplets $(\alpha_i, \beta_i, \gamma_i)$ for the ith column of G are precisely those 3-element sets such that A has a 2-comb joining $\alpha_{i-1}, \beta_{i-1}, \gamma_{i-1}$, with spine $\alpha_i\beta_i\gamma_i$, and whose edge signs are precisely the signs, in G, of the 2-comb with spine $a_ib_ic_i$ joining $a_{i-1}b_{i-1}c_{i-1}$.

5.1. Lower bounds

We start off by investigating general lower bounds on the 2-edge-coloured chromatic number of 3-row grids. To begin, note that for some signatures of $G(3, 3)$ at least six colours are needed.

Proposition 5.1. *We have* $\chi_2(G(3, 3)) \geq 6$.

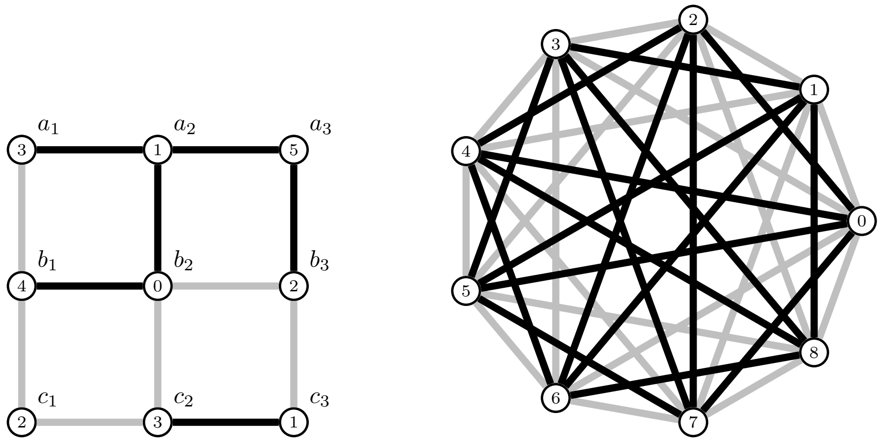

Figure 4: A 2-edge-coloured 6-colouring of a signature of $G(3,3)$ (left), and the 2-edge-coloured circulant graph $C(9,\{2,4\})$ (right). Black (gray, respectively) edges are positive (negative, respectively) edges.

Proof. Let G be the signature of $G(3,3)$ depicted in Figure 4 (left), and assume, for contradiction, that there is a signature A of K_5 such that G admits an A-colouring ϕ.

We note that every two vertices in $\{a_2, b_1, b_3, c_2\}$ are joined by an alternating 2-path. For this reason, all colours $\phi(a_1), \phi(a_2), \phi(b_1), \phi(b_3), \phi(c_2)$ must be different. As in Figure 4 (left), let us assume, without loss of generality, that $\phi(b_2) = 0$, $\phi(a_2) = 1$, $\phi(b_3) = 2$, $\phi(c_2) = 3$ and $\phi(b_1) = 4$. This reveals that, in A, edges $\{0,1\}$ and $\{0,4\}$ are positive, while $\{0,2\}$ and $\{0,3\}$ are negative.

Now consider c_3. Since b_2 and c_3 are joined by an alternating 2-path, we have either $\phi(c_3) = 1$ or $\phi(c_3) = 4$. At this point of the proof, we may assume that $\phi(c_3) = 1$. This reveals that, in A, edge $\{1,2\}$ is negative while $\{1,3\}$ is positive. Now consider c_1. Since c_1 is joined by an alternating 2-path to both b_2 and c_3, we must have $\phi(c_1) = 2$. Hence, edges $\{2,3\}$ and $\{2,4\}$ are negative in A. For similar reasons, vertex a_1 must receive colour 2 or 3 by ϕ. Actually, we cannot have $\phi(a_1) = 2$ since edge $\{1,2\}$ was shown to be negative in A. So, we have $\phi(a_1) = 3$.

We finally note that a_3 cannot be coloured with either of colours $0,1,2$ due to some edges or alternating 2-paths of G. Furthermore, we cannot have $\phi(a_3) = 3$ since edge $\{2,3\}$ is negative in A, or $\phi(a_3) = 4$ since edge $\{2,4\}$ is negative in A. Hence a_3 cannot be assigned a valid colour by ϕ, a contradiction. □

It turns out that some 2-edge-coloured 3-row grids need at least seven colours to be coloured. To verify this, it suffices to exhibit, for every signature A of K_6, a 2-edge-coloured 3-row grid G_A that cannot be A-coloured. Once we have such a grid G_A for every A, it then suffices to consider a large 2-edge-coloured 3-row grid G that contains all G_A's; there is then no signature of K_6 that can colour G, meaning that G has 2-edge-coloured chromatic number at least 7.

Let A be a fixed signature of K_6. Designing such a 2-edge-coloured 3-row grid G_A is tedious because we have to prove that there is no way to A-colour it. For that reasons, we made use of a computer, through the following approach. We start off from G_A being the 2-path $a_1b_1c_1$ signed in some way, and we consider $\mathcal{L}_1$ the set of triplets $(\alpha_1, \beta_1, \gamma_1)$ of colours that can be assigned to a_1, b_1, c_1 in an A-colouring. If this set $\mathcal{L}_1$ is empty, then A cannot colour G_A, and we are done. Otherwise, we make G_A one column larger by joining a_1, b_1, c_1 by a 2-comb with spine a_2, b_2, c_2. For a signature of the resulting five new edges (a_1a_2, b_1b_2, c_1c_2, a_2b_2, b_2c_2), we would like to find a bad signature, i.e., a signature such that, by all A-colourings of G_A, the set $\mathcal{L}_2$ of triplets $(\alpha_2, \beta_2, \gamma_2)$ of colours that can be assigned to a_2, b_2, c_2 is as small as possible. We note that, for a fixed signature of the 2-comb, computing $\mathcal{L}_2$ can be done easily from $\mathcal{L}_1$, by just consider every $(\alpha_1, \beta_1, \gamma_1) \in \mathcal{L}_1$, and checking, in A, what are the 2-combs with spine $\alpha_2\beta_2\gamma_2$ joining $\alpha_1, \beta_1, \gamma_1$ which have their signature matching that of the 2-comb in G. Then we can try out all possible signatures of the 2-comb in G, and find one that minimizes the size of $\mathcal{L}_2$. The same principle can be applied again and again iteratively, adding new 2-combs (with spine $a_ib_ic_i$ joining $a_{i-1}, b_{i-1}, c_{i-1}$) to G and computing the resulting sets $\mathcal{L}_3, \mathcal{L}_4, \ldots$ Hopefully, at some point a set $\mathcal{L}_i$ with $\mathcal{L}_i = \emptyset$ will be reached, meaning that a non-A-colourable 2-edge-coloured 3-row grid has been obtained.

It turns out that, for every fixed signature A of K_6, this strategy does result in a 2-edge-coloured 3-row grid G_A that cannot be A-coloured. We give a certificate of this in the online file `http://jbensmai.fr/code/signed-grids/G3n-lower-bound.txt`, which describes, for every A, the signature of a candidate as G_A, and the resulting sets $\mathcal{L}_i$. The number of non-equivalent signatures of K_6 is 78, as two signatures A_1, A_2 of K_6 are isomorphic as soon as the set of positive edges of A_1 induce a graph isomorphic to that induced by the set of positive edges of A_2, and two signatures A_1, A_2 of A are equivalent as soon as the set of positive edges of A_1 induce a graph isomorphic to that induced by the set of negative edges of A_2 (just invert all edge signs). Since the number of non-isomorphic graphs on 6 vertices is 156, this gives that only 78 non-equivalent signatures of K_6 exist. A remarkable fact is that, for every signature A of K_6, a claimed grid G_A we construct always has at most six columns. Thus, without trying to optimize further, an upper bound on the parameter n_0 in the next result is 78×6.

Theorem 5.2. *There exists a n_0 such that for every $n \geq n_0$, we have $\chi_2(G(3,n)) \geq 7$.*

5.2. Upper bounds

As in the previous section, we here systematically colour any 2-edge-coloured grid from column to column (as going from the first column to the last column), by essentially extending triplets of colours from 2-comb to 2-comb (i.e., colouring the first-column vertices first, then the second-column vertices, and so on), as they are attached to each other

Our upper bounds on the 2-edge-coloured chromatic number of 3-row grids rely on the existence of 2-edge-coloured circulant graphs with properties analogous to that described in the statement of Observation 4.4. More precisely, we are here interested in 2-edge-coloured circulant graphs that make the following proposition applicable.

Proposition 5.3. *Suppose we have a 2-edge-coloured graph A such that, for every three distinct vertices u, v, w of A, and for every set $\{s_1, s_2, s_3, s_4, s_5\}$ of $\{-, +\}^5$, there exists, in A, a 2-comb with spine $w_1w_2w_3$ joining u, w, v such that $\sigma(uw_1) = s_1$, $\sigma(ww_2) = s_2$, $\sigma(vw_3) = s_3$, $\sigma(w_1w_2) = s_4$, $\sigma(w_2w_3) = s_5$. Then every signature of $G(3,n)$ is A-colourable.*

Proof. We prove by induction on n, the number of columns, that every signature G of $G(3,n)$ can be A-coloured, provided A has the desired property. In case $n = 1$, we note that G is actually a 2-edge-coloured path on two edges. Since, by our assumptions, A has both positive edges and negative edges, and has positive edges adjacent to negative edges, it is easy to see that a_1, b_1, c_1 can be coloured.

Assume now that the claim is true for every n up to value $i-1$ and consider the case $n = i$. By the induction hypothesis, there exists an A-colouring ϕ of the $i-1$ first columns of G, which form a signature of $G(3, n-1)$. We now extend ϕ the ith column, i.e., to the vertices a_i, b_i, c_i. To that aim, consider the 2-edge-coloured 2-comb C of G joining $a_{i-1}, b_{i-1}, c_{i-1}$ with spine $a_ib_ic_i$. According to the initial assumption on A, no matter what the triplet $(\phi(a_{i-1}), \phi(b_{i-1}), \phi(c_{i-1}))$ is, and no matter what the signs of the edges of C are, we can find, in A, a 2-comb joining $\phi(a_{i-1}), \phi(b_{i-1}), \phi(c_{i-1})$, and with the same edge signs as C. Denote its spine by $\alpha_i\beta_i\gamma_i$. Then we can simply extend ϕ to a_i, b_i, c_i by setting $\phi(a_i) = \alpha_i$, $\phi(b_i) = \beta_i$, $\phi(c_i) = \gamma_i$. □

Hence, by showing that a 2-edge-coloured graph A with small order has the property described in Proposition 5.3, we immediately get that every 2-edge-coloured 3-row grid is A-colourable, thus that its 2-edge-coloured chromatic number is at most $|V(A)|$. Using a computer, we have determined that the smallest 2-edge-coloured circulant graphs having that property have order 10.

Proposition 5.4. *The smallest 2-edge-coloured circulant graphs $C(n,S)$ having the property described in Proposition 5.3 have $n = 10$. An example of a such graph is $C(10, \{2,4\})$.*

From Propositions 5.3 and 5.4, we thus directly get the following.

Theorem 5.5. *For every $n \geq 1$, we have $\chi_2(G(3,n)) \leq 10$.*

We now improve the upper bound in Theorem 5.5 down to 9, by showing that every 2-edge-coloured 3-row grid can be coloured by the 2-edge-coloured circulant graph $C(9, \{2,4\})$ (illustrated in Figure 4 (right)). The colouring strategy we use is again the column-to-column one that we have used earlier. We however have to be more careful here, because, as indicated by Proposition 5.4, there are situations where a colouring of the $(i-1)$th column cannot be extended to a colouring of the ith one, because $C(9,\{2,4\})$ does not admit all possible kinds of 2-edge-coloured 2-combs.

Following Proposition 5.4, we know that $C(9,\{2,4\})$ has *bad triplets*, namely triplets (α,β,γ) of vertices such that $C(9,\{2,4\})$ has no 2-comb, with a particular signature, joining α,β,γ. Hence, when colouring a new column of a 2-edge-coloured 3-row grid, we should avoid assigning a bad triplet as it might then be not possible to extend the partial colouring to the next column.

Using a computer program to enumerate all 3-element sets of colours (α,β,γ) and, for every signature, all 2-edge-coloured 2-combs joining α,β,γ in $C(9,\{2,4\})$, we came up with the following characterization of the bad triplets of $C(9,\{2,4\})$ (refer to `http://jbensmai.fr/code/signed-grids/C924-triplets.txt` for an exhaustive list of the possible ways to extend a $C(9,\{2,4\})$-colouring from a column to the next column):

Observation 5.6. *A triplet (α,β,γ) of $C(9,\{2,4\})$ is bad if and only if:*

- $(\beta,\gamma)\in\{(\alpha+2,\alpha+4),(\alpha-2,\alpha-4),(\alpha+3,\alpha+6),(\alpha-3,\alpha-6)\}$*, or*
- $(\beta,\alpha)\in\{(\gamma+2,\gamma+4),(\gamma-2,\gamma-4),(\gamma+3,\gamma+6),(\gamma-3,\gamma-6)\}$,

where the operations are understood modulo 9*. In other words,* (α,β,γ) *is bad if and only if* $(\alpha,\beta,\gamma)\in\{(0,3,6),\ (0,4,2),\ (0,5,7),\ (0,6,3),\ (1,4,7),\ (1,5,3),\ (1,6,8),\ (1,7,4),\ (2,5,8),\ (2,6,4),\ (2,7,0),\ (2,8,5),\ (3,0,6),\ (3,6,0),\ (3,7,5),\ (3,8,1),\ (4,0,2),\ (4,1,7),\ (4,7,1),\ (4,8,6),\ (5,0,7),\ (5,1,3),\ (5,2,8),\ (5,8,2),\ (6,0,3),\ (6,1,8),\ (6,2,4),\ (6,3,0),\ (7,1,4),\ (7,2,0),\ (7,3,5),\ (7,4,1),\ (8,2,5),\ (8,3,1),\ (8,4,6),\ (8,5,2)\}$.

When colouring a column, we should as well avoid assigning a non-bad triplet (α,β,γ) of colours such that, for a particular fixed signature, all 2-edge-coloured 2-combs with that signature, joining α,β,γ in $C(9,\{2,4\})$, have a bad spine, i.e., a spine $\alpha'\beta'\gamma'$ such that (α',β',γ') is bad. We call such a triplet *dangerous*. Once again, the dangerous triplets of $C(9,\{2,4\})$ can easily be generated using a computer, and, hence, characterized (again, refer to the full list above for an exhaustive checking of this result).

Observation 5.7. *A non-bad triplet (α,β,γ) of $C(9,\{2,4\})$ is dangerous if and only if:*

- $(\beta,\gamma)\in\{(\alpha+2,\alpha+5),(\alpha-2,\alpha-5),(\alpha+2,\alpha+6),(\alpha-2,\alpha-6),(\alpha+3,\alpha+5),(\alpha-3,\alpha-5),(\alpha+4,\alpha+6),(\alpha-4,\alpha-6)\}$*, or*
- $(\beta,\alpha)\in\{(\gamma+2,\gamma+5),(\gamma-2,\gamma-5),(\gamma+2,\gamma+6),(\gamma-2,\gamma-6),(\gamma+3,\gamma+5),(\gamma-3,\gamma-5),(\gamma+4,\gamma+6),(\gamma-4,\gamma-6)\}$,

where the operations are understood modulo 9*. In other words,* (α,β,γ) *is dangerous if and only if* $(\alpha,\beta,\gamma)\in\{(0,3,5),\ (0,3,7),\ (0,4,6),\ (0,4,7),\ (0,5,2),\ (0,5,3),\ (0,6,2),\ (0,6,4),\ (1,4,6),\ (1,4,8),\ (1,5,7),\ (1,5,8),\ (1,6,3),\ (1,6,4),\ (1,7,3),\ (1,7,5),\ (2,5,0),\ (2,5,7),\ (2,6,0),\ (2,6,8),\ (2,7,4),\ (2,7,5),\ (2,8,4),\ (2,8,6),\ (3,0,5),\ (3,0,7),\ (3,6,1),\ (3,6,8),\ (3,7,0),\ (3,7,1),\ (3,8,5),\ (3,8,6),\ (4,0,6),\ (4,0,7),\ (4,1,6),\ (4,1,8),\ (4,7,0),\ (4,7,2),\ (4,8,1),\ (4,8,2),\ (5,0,2),\ (5,0,3),\ (5,1,7),\ (5,1,8),\ (5,2,0),\ (5,2,7),\ (5,8,1),\ (5,8,3),\ (6,0,2),\ (6,0,4),\ (6,1,3),\ (6,1,4),\ (6,2,0),\ (6,2,8),\ (6,3,1),\ (6,3,8),\ (7,1,3),\ (7,1,5),\ (7,2,4),\ (7,2,5),\ (7,3,0),\ (7,3,1),\ (7,4,0),\ (7,4,2),\ (8,2,4),\ (8,2,6),\ (8,3,5),\ (8,3,6),\ (8,4,1),\ (8,4,2),\ (8,5,1),\ (8,5,3)\}$.

One should of course be cautious with non-bad and non-dangerous triplets (α,β,γ) of colours such that, for some signature, all 2-edge-coloured 2-combs with that signature, joining α,β,γ in $C(9,\{2,4\})$, have a bad or dangerous spine. However, it can be checked that every non-bad and non-dangerous triplet (α,β,γ) is *good*, in the sense that, in $C(9,\{2,4\})$, for every signature there is a 2-edge-coloured 2-comb with that signature, joining α,β,γ, and with a good spine, i.e., a spine $\alpha'\beta'\gamma'$ such that (α',β',γ') is good. For certificates, see the online file `http://jbensmai.fr/code/signed-grids/C924-good-triplets.txt`.

Observation 5.8. *Every non-bad and non-dangerous triplet is good.*

We are now ready to improve the bound in Theorem 5.5.

Theorem 5.9. *For every $n \geq 1$, we have $\chi_2(G(3,n)) \leq 9$.*

Proof. We actually prove, by induction on n, that every signature G of $G(3,n)$ can be coloured by $C(9,\{2,4\})$, implying the result. The colouring strategy we use is again the column-to-column strategy that we have been using so far, but restricted to good triplets of colours. More precisely, we show that the columns of G can be coloured one after another, in such a way that the triplets of colours, assigned by the colouring ϕ, are all good.

As a base case, assume $n = 1$. In case a_1b_1 and b_1c_1 are both positive, we can set e.g. $\phi(a_1) = 0$, $\phi(b_1) = 4$, $\phi(c_1) = 0$. If a_1b_1 and b_1c_1 are both negative, then we can here set e.g. $\phi(a_1) = 0$, $\phi(b_1) = 1$, $\phi(c_1) = 0$. Finally, if, say, a_1b_1 is positive while b_1c_1 is negative, then we can set e.g. $\phi(a_1) = 0$, $\phi(b_1) = 2$, $\phi(c_1) = 1$. In every case, we get that $(\phi(a_1), \phi(b_1), \phi(c_1))$ is a good triplet, according to Observation 5.8, which concludes this case.

Assume now that the claim is true for every n up to some value $i-1$, and consider the next step $n = i$. By the induction hypothesis, we can colour the $i-1$ first columns of G, as they form a signature of $G(3, n-1)$, in such a way that all triplets of colours are good. Let ϕ be such a colouring. We now extend ϕ to the ith column of G, namely to its vertices a_i, b_i, c_i, in a good way. To that aim, consider, in G, the 2-edge-coloured comb C joining $a_{i-1}, b_{i-1}, c_{i-1}$ with spine $a_ib_ic_i$. According to the definition of a good triplet, and because $(\phi(a_{i-1}), \phi(b_{i-1}), \phi(c_{i-1}))$ is good, there has to be, in $C(9,\{2,4\})$, a 2-edge-coloured comb with the same edge signs as C, joining $(\phi(a_{i-1}), \phi(b_{i-1}), \phi(c_{i-1}))$, and with a good spine $\alpha_i\beta_i\gamma_i$, i.e., $(\alpha_i, \beta_i, \gamma_i)$ is a good triplet. So we can extend ϕ to a_i, b_i, c_i by just setting $\phi(a_i) = \alpha_i, \phi(b_i) = \beta_i, \phi(c_i) = \gamma_i$. This proves the inductive step, and, hence, the claim. □

6. 2-edge-coloured grids with more rows

In this section, we extend, to grids with more rows, the principles described in Section 5 for verifying Theorem 5.2. From these, we deduce that there exist 2-edge-coloured 5-rows grids with 2-edge-coloured chromatic number at least 8.

Theorem 6.1. *There exists a n_0 such that for every $n \geq n_0$, we have $\chi_2(G(5,n)) \geq 8$.*

The existence of such a grid $G = G(5,n)$ with $\chi_2(G) \geq 8$ can be attested following the method described at the end of Section 5.1. Namely, we consider every signature A of K_7 (there are 522 such, recall the arguments given earlier), and our task is to construct a 2-edge-coloured grid G_A with at most five rows that cannot be A-coloured. If such a G_A can be constructed for every A, then a possible G will be any 2-edge-coloured 5-row grid containing all G_A's.

For each A, an example of a such G_A can be constructed as follows. For some $i \in \{2, \dots, 4\}$, we start from G_A being an i-path $a_1b_1c_1\dots$ signed in a particular way, and we then compute $\mathcal{L}_1$ the set of the possible tuples $(\alpha_1, \beta_1, \gamma_1, \dots)$ of colours that can be assigned to $a_1, b_1, c_1, \dots$ in an A-colouring of G_A. If $\mathcal{L}_1 = \emptyset$, then we are done. Otherwise, we add a new column $a_2b_2c_2\dots$ to G_A by adding the edges $a_1a_2, b_1b_2, c_1c_2, \dots$. We sign the resulting $2i-1$ new edges in such a way that the set $\mathcal{L}_2$ of the possible tuples $(\alpha_2, \beta_2, \gamma_2, \dots)$ of colours that can be assigned to $a_2, b_2, c_2, \dots$ in an A-colouring of G_A is as small as possible. We repeat this process until hopefully reaching an $\mathcal{L}_k$ that is empty, meaning that the 2-edge-coloured grid G_A constructed so far cannot be A-coloured.

In the online file `http://jbensmai.fr/code/signed-grids/lower-bound-8.txt`, we prove that such a G_A does exist for every signature A of K_7. More precisely, for each G_A we describe its signature, as well as the corresponding sets $\mathcal{L}_1, \mathcal{L}_2, \dots$ (which can be deduced successively). In most cases, we get that such G_A's with only three rows exist. In a few more cases, grids with four rows must be considered. For a very particular signature of K_7, we have to consider a grid with five rows.

7. Conclusion

In this article, we have investigated the 2-edge-coloured chromatic number of grids, our main goal being to compare how the oriented chromatic number and the 2-edge-coloured chromatic number behave in these graphs. We have provided several bounds for both general grids and 2-row or 3-row grids. In particular, we have shown that the maximum 2-edge-coloured chromatic number of a grid lies between 8 and 11. For 2-row grids, we managed to completely determine their 2-edge-coloured chromatic number, while, for 3-row grids, we have obtained partial results.

Concerning the relation between the oriented chromatic number and the 2-edge-coloured chromatic number, our results show that these two parameters are, as expected, quite close for grids. This is mainly established by the matching lower and upper bounds we know on the maximum value of these parameters for grids.

Some disparities, though, are worth mentioning. For 2-row grids, while the oriented chromatic number is 6 in general, the 2-edge-coloured chromatic number is 5 in general. We still do not know whether 3-row grids with 2-edge-coloured chromatic number 8 exist, but, if this were to hold, then that would be quite interesting as these grids have oriented chromatic number at most 7. In that spirit, it could as well be interesting considering 4-row grids, which have oriented chromatic number at most 7 according to [8].

Appendix: Exhaustive list of the 2-paths of A_{11}

Type of path	Candidates	Type of path	Candidates	Type of path	Candidates	Type of path	Candidates
0+?+ 1	6, 7	0−?− 1	2, 8	0+?− 1	3, 5, 9	0−?+ 1	4, 10
0+?+ 2	5, 6, 9	0−?− 2	1, 4	0+?− 2	3, 7	0−?+ 2	8, 10
0+?+ 3	6, 9	0−?− 3	1, 2, 4	0+?− 3	5, 7	0−?+ 3	8, 10
0+?+ 4	7, 9	0−?− 4	2, 10	0+?− 4	3, 5, 6	0−?+ 4	1, 8
0+?+ 5	7	0−?− 5	1, 4	0+?− 5	3, 6, 9	0−?+ 5	2, 8, 10
0+?+ 6	3, 9	0−?− 6	4, 8	0+?− 6	5, 7	0−?+ 6	1, 2, 10
0+?+ 7	5	0−?− 7	2, 8, 10	0+?− 7	3, 6, 9	0−?+ 7	1, 4
0+?+ 8	3, 5, 9	0−?− 8	1	0+?− 8	6, 7	0−?+ 8	2, 4, 10
0+?+ 9	3, 6	0−?− 9	1, 10	0+?− 9	5, 7	0−?+ 9	2, 4, 8
0+?+ 10	3, 5, 6	0−?− 10	4	0+?− 10	7, 9	0−?+ 10	1, 2, 8
1+?+ 2	6, 10	1−?− 2	0, 3	1+?− 2	4, 7	1−?+ 2	5, 8, 9
1+?+ 3	6, 10	1−?− 3	2, 5	1+?− 3	4, 7	1−?+ 3	0, 8, 9
1+?+ 4	7	1−?− 4	0, 2, 3, 5	1+?− 4	6, 10	1−?+ 4	8, 9
1+?+ 5	7, 10	1−?− 5	3, 9	1+?− 5	4, 6	1−?+ 5	0, 2, 8
1+?+ 6	10	1−?− 6	5, 8	1+?− 6	4, 7	1−?+ 6	0, 2, 3, 9
1+?+ 7	4	1−?− 7	2, 3, 8, 9	1+?− 7	6, 10	1−?+ 7	0, 5
1+?+ 8	4, 10	1−?− 8	0	1+?− 8	6, 7	1−?+ 8	2, 3, 5, 9
1+?+ 9	4, 6	1−?− 9	5	1+?− 9	7, 10	1−?+ 9	0, 2, 3, 8
1+?+ 10	6	1−?− 10	0, 9	1+?− 10	4, 7	1−?+ 10	2, 3, 5, 8
2+?+ 3	6, 8, 9, 10	2−?− 3	1, 4, 7	2+?− 3	5	2−?+ 3	0
2+?+ 4	8, 9	2−?− 4	0, 3	2+?− 4	5, 6, 10	2−?+ 4	1, 7
2+?+ 5	8, 10	2−?− 5	1, 3, 4	2+?− 5	6, 9	2−?+ 5	0, 7
2+?+ 6	9, 10	2−?− 6	4, 7	2+?− 6	5, 8	2−?+ 6	0, 1, 3
2+?+ 7	5	2−?− 7	3	2+?− 7	6, 8, 9, 10	2−?+ 7	0, 1, 4
2+?+ 8	5, 9, 10	2−?− 8	0, 1, 7	2+?− 8	6	2−?+ 8	3, 4
2+?+ 9	6, 8	2−?− 9	1, 7	2+?− 9	5, 10	2−?+ 9	0, 3, 4
2+?+ 10	5, 6, 8	2−?− 10	0, 4, 7	2+?− 10	9	2−?+ 10	1, 3
3+?+ 4	8, 9	3−?− 4	2, 5	3+?− 4	0, 6, 10	3−?+ 4	1, 7
3+?+ 5	0, 8, 10	3−?− 5	1, 4	3+?− 5	6, 9	3−?+ 5	2, 7
3+?+ 6	0, 9, 10	3−?− 6	4, 5, 7	3+?− 6	8	3−?+ 6	1, 2
3+?+ 7	0	3−?− 7	2	3+?− 7	6, 8, 9, 10	3−?+ 7	1, 4, 5
3+?+ 8	9, 10	3−?− 8	1, 7	3+?− 8	0, 6	3−?+ 8	2, 4, 5
3+?+ 9	0, 6, 8	3−?− 9	1, 5, 7	3+?− 9	10	3−?+ 9	2, 4
3+?+ 10	6, 8	3−?− 10	4, 7	3+?− 10	0, 9	3−?+ 10	1, 2, 5
4+?+ 5	7, 8	4−?− 5	3, 6	4+?− 5	1, 9	4−?+ 5	0, 2, 10
4+?+ 6	1, 9	4−?− 6	5	4+?− 6	7, 8	4−?+ 6	0, 2, 3, 10
4+?+ 7	1	4−?− 7	2, 3, 6, 10	4+?− 7	8, 9	4−?+ 7	0, 5
4+?+ 8	9	4−?− 8	0, 6	4+?− 8	1, 7	4−?+ 8	2, 3, 5, 10
4+?+ 9	8	4−?− 9	5, 10	4+?− 9	1, 7	4−?+ 9	0, 2, 3, 6
4+?+ 10	1, 8	4−?− 10	0	4+?− 10	7, 9	4−?+ 10	2, 3, 5, 6
5+?+ 6	0, 2, 10	5−?− 6	4	5+?− 6	7, 8	5−?+ 6	1, 3, 9
5+?+ 7	0	5−?− 7	3, 6, 9	5+?− 7	2, 8, 10	5−?+ 7	1, 4
5+?+ 8	2, 10	5−?− 8	1, 6	5+?− 8	0, 7	5−?+ 8	3, 4, 9
5+?+ 9	0, 2, 8	5−?− 9	1	5+?− 9	7, 10	5−?+ 9	3, 4, 6
5+?+ 10	2, 8	5−?− 10	4, 9	5+?− 10	0, 7	5−?+ 10	1, 3, 6
6+?+ 7	0, 1	6−?− 7	8	6+?− 7	2, 3, 9, 10	6−?+ 7	4, 5
6+?+ 8	2, 3, 9, 10	6−?− 8	7	6+?− 8	0, 1	6−?+ 8	4, 5
6+?+ 9	0, 2, 3	6−?− 9	5, 7	6+?− 9	1, 10	6−?+ 9	4, 8
6+?+ 10	1, 2, 3	6−?− 10	4, 7	6+?− 10	0, 9	6−?+ 10	5, 8
7+?+ 8	4, 5	7−?− 8	6	7+?− 8	0, 1	7−?+ 8	2, 3, 9, 10
7+?+ 9	0, 4	7−?− 9	10	7+?− 9	1, 5	7−?+ 9	2, 3, 6, 8
7+?+ 10	1, 5	7−?− 10	9	7+?− 10	0, 4	7−?+ 10	2, 3, 6, 8
8+?+ 9	2, 3, 4	8−?− 9	1, 7	8+?− 9	5, 10	8−?+ 9	0, 6
8+?+ 10	2, 3, 5	8−?− 10	0, 7	8+?− 10	4, 9	8−?+ 10	1, 6
9+?+ 10	2, 3, 6, 8	9−?− 10	7	9+?− 10	0, 4	9−?+ 10	1, 5

Table 2: Exhaustive list of the ++-paths, −−-paths, +−-paths and −+-paths of A_{11}.

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