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A localized version of the basic triangle theorem

Lie-theoretic aspects of the basic triangle theorem

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Abstract. In this paper, we examine some aspects of the BTT (basic triangle theorem), first published in 2011 (see [8]).

In a first part, we review the interplay between integration of Non Commutative Differential equations and paths drawn on Magnus groups and some of their closed subgroups.

In a second part, we provide a localized version of the BTT and apply it to prove the independence of hyperlogarithms over various function algebras. This version provides direct access to rings of scalars and avoids the recourse to fraction fields as that of meromorphic functions for instance.

1 Introduction

Iterated integrals (Lappo-Danielevskii), K.-T. Chen [6,7] (path spaces, loops spaces, algebraic topology), Brown, Kontsevich

In a second step, we will use an analogue of the well-known closed subgroup theorem (also called Cartan theorem for finite dimensional Lie groups) which, in the Banach Lie context can be stated as follows.

Let \mathcal{B} be a Banach algebra (with unit e) and G a closed subgroup of the open set \mathcal{B}^\times . By a path “drawn on T ” ($T \subset \mathcal{B}$) is understood any function $\varphi : I \rightarrow T$ where I is a open real interval.

The first step is to establish what would be seen as the Lie algebra of G .

Let $L = L(G)$ be the set of tangent vectors of G at the origin i.e.

$$L(G) = \{\gamma'(0) \mid \gamma : I \rightarrow G \text{ is differentiable at } 0 (= 0_{\mathbb{R}}) \text{ and } \gamma(0) = e\} \quad (1)$$

Proposition 1 (see [11]). *The set $L(G)$ has the following properties*

- 1) *If $u \in L(G)$, then, the one-parameter group $t \mapsto e^{t \cdot u}$ is drawn on G*
- 2) *$L(G)$ is a closed Lie subalgebra of \mathcal{B}*
- 3) *Let $g \in \mathcal{B}$ s.t. $\|g - e\| < 1$, then*

$$\log(g) \in L(G) \implies g \in G$$

2 BTT theorem

2.1 Background

Notations about alphabets and (noncommutative) series are standard and can be found in [1].

Set of variables, series, Dirac-Schützenberger duality, Magnus and Hausdorff groups. Series with constant and variable coefficients. Differential rings and algebras.

2.2 For the Magnus group

We can always consider a series with variable coefficients $S \in \mathcal{H}(\Omega)\langle\langle \mathcal{X} \rangle\rangle$ as a *function* i.e. with, for all $z \in \Omega$

$$S(z) := \sum_{w \in \mathcal{X}^*} \langle S \mid w \rangle(z) w \quad (2)$$

we get an embedding $\mathcal{H}(\Omega)\langle\langle \mathcal{X} \rangle\rangle \hookrightarrow \mathbb{C}\langle\langle \mathcal{X} \rangle\rangle^\Omega$. With this point of view in head, we can always consider series $S \in \mathcal{H}(\Omega)\langle\langle \mathcal{X} \rangle\rangle$ such that $\langle S \mid 1_{\mathcal{X}^*} \rangle = 1_{\mathcal{H}(\Omega)}$ as (holomorphic) paths *drawn on the Magnus group*. The Non commutative differential equations with left multiplier can be expressed in the context of general differential algebras.

Theorem 1 (See Th 1 in [8]). Let (\mathcal{A}, d) be a k -commutative associative differential algebra with unit ($\ker(d) = k$, a field) and \mathcal{C} be a differential subfield of \mathcal{A} (i.e. $d(\mathcal{C}) \subset \mathcal{C}$ and $k \subset \mathcal{C}$). Let X be some alphabet (i.e. some set) and we define $\mathbf{d} : \mathcal{A}\langle\langle X \rangle\rangle \rightarrow \mathcal{A}\langle\langle X \rangle\rangle$ to be the map given by $\langle\mathbf{d}(S) \mid w\rangle = d(\langle S \mid w\rangle)$. We suppose that $S \in \mathcal{A}\langle\langle X \rangle\rangle$ is a solution of the differential equation

$$\mathbf{d}(S) = MS ; \langle S \mid 1_{X^*} \rangle = 1_{\mathcal{A}} \quad (3)$$

where the multiplier M is a homogeneous series (a polynomial in the case of finite X) of degree 1, i.e.

$$M = \sum_{x \in X} u_x x \in \mathcal{C}\langle\langle X \rangle\rangle . \quad (4)$$

The following conditions are equivalent :

- i) The family $(\langle S \mid w \rangle)_{w \in X^*}$ of coefficients of S is free over \mathcal{C} .
- ii) The family of coefficients $(\langle S \mid y \rangle)_{y \in X \cup \{1_{X^*}\}}$ is free over \mathcal{C} .
- iii) The family $(u_x)_{x \in X}$ is such that, for $f \in \mathcal{C}$ and $\alpha \in k^{(X)}$ (i.e. $\text{supp}(\alpha)$ is finite)

$$d(f) = \sum_{x \in X} \alpha_x u_x \implies (\forall x \in X)(\alpha_x = 0) . \quad (5)$$

- iv) The family $(u_x)_{x \in X}$ is free over k and

$$d(\mathcal{C}) \cap \text{span}_k \left((u_x)_{x \in X} \right) = \{0\} . \quad (6)$$

Proof. For convenience of the reader, we enclose here the demonstration given in Th 1 [8].

(i) \implies (ii) Obvious.

(ii) \implies (iii)

Suppose that the family $(\langle S \mid y \rangle)_{y \in X \cup \{1_{X^*}\}}$ (coefficients taken at letters and the empty word) of coefficients of S were free over \mathcal{C} and let us consider the relation as in (5)

$$d(f) = \sum_{x \in X} \alpha_x u_x . \quad (7)$$

We form the polynomial $P = -f1_{X^*} + \sum_{x \in X} \alpha_x x$. One has $\mathbf{d}(P) = -d(f)1_{X^*}$ and

$$d(\langle S \mid P \rangle) = \langle \mathbf{d}(S) \mid P \rangle + \langle S \mid \mathbf{d}(P) \rangle = \langle MS \mid P \rangle - d(f)\langle S \mid 1_{X^*} \rangle = \left(\sum_{x \in X} \alpha_x u_x \right) - d(f) = 0 \quad (8)$$

whence $\langle S \mid P \rangle$ must be a constant, say $\lambda \in k$. For $Q = P - \lambda.1_{X^*}$, we have

$$\text{supp}(Q) \subset X \cup \{1_{X^*}\} \text{ and } \langle S \mid Q \rangle = \langle S \mid P \rangle - \lambda \langle S \mid 1_{X^*} \rangle = \langle S \mid P \rangle - \lambda = 0 .$$

This, in view of (ii), implies that $Q = 0$ and, as $Q = -(f+\lambda)1_{X^*} + \sum_{x \in X} \alpha_x x$, one has, in particular, $\text{supp}(\alpha) = \emptyset$ (and, as a byproduct, $f = -\lambda$ which is indeed the only possibility for the L.H.S. of (5) to occur).

(iii) \iff (iv)

Obvious, (iv) being a geometric reformulation of (iii).

(iii) \implies (i)

Let \mathcal{K} be the kernel of $P \mapsto \langle S \mid P \rangle$ (linear $\mathcal{C}\langle X \rangle \rightarrow \mathcal{A}$) i.e.

$$\mathcal{K} = \{P \in \mathcal{C}\langle X \rangle \mid \langle S \mid P \rangle = 0\} . \quad (9)$$

If $\mathcal{K} = \{0\}$, we are done. Otherwise, let us adopt the following strategy.

First, we order X by some well-ordering $<$ ([2] III.2.1) and X^* by the graded lexicographic ordering \prec defined as follows

$$u \prec v \iff |u| < |v| \text{ or } (u = pxs_1, v = pys_2 \text{ and } x < y). \quad (10)$$

It is easy to check that X^* is also a well-ordered by \prec . For each nonzero polynomial P , we denote by $\text{lead}(P)$ its leading monomial; i.e. the greatest element of its support $\text{supp}(P)$ (for \prec).

Now, as $\mathcal{R} = \mathcal{K} \setminus \{0\}$ is not empty, let w_0 be the minimal element of $\text{lead}(\mathcal{R})$ and choose a $P \in \mathcal{R}$ such that $\text{lead}(P) = w_0$. We write

$$P = fw_0 + \sum_{u \prec w_0} \langle P \mid u \rangle u ; f \in \mathcal{C} \setminus \{0\} . \quad (11)$$

The polynomial $Q = \frac{1}{f}P$ is also in \mathcal{R} with the same leading monomial, but the leading coefficient is now 1; and so Q is given by

$$Q = w_0 + \sum_{u \prec w_0} \langle Q | u \rangle u . \quad (12)$$

Differentiating $\langle S | Q \rangle = 0$, one gets

$$0 = \langle \mathbf{d}(S) | Q \rangle + \langle S | \mathbf{d}(Q) \rangle = \langle MS | Q \rangle + \langle S | \mathbf{d}(Q) \rangle = \langle S | M^\dagger Q \rangle + \langle S | \mathbf{d}(Q) \rangle = \langle S | M^\dagger Q + \mathbf{d}(Q) \rangle \quad (13)$$

with

$$M^\dagger Q + \mathbf{d}(Q) = \sum_{x \in X} u_x (x^\dagger Q) + \sum_{u \prec w_0} d(\langle Q | u \rangle) u \in \mathcal{C}\langle X \rangle . \quad (14)$$

It is impossible that $M^\dagger Q + \mathbf{d}(Q) \in \mathcal{R}$ because it would be of leading monomial strictly less than w_0 , hence $M^\dagger Q + \mathbf{d}(Q) = 0$. This is equivalent to the recursion

$$d(\langle Q | u \rangle) = - \sum_{x \in X} u_x \langle Q | xu \rangle ; \text{ for } x \in X , v \in X^* . \quad (15)$$

From this last relation, we deduce that $\langle Q | w \rangle \in k$ for every w of length $\deg(Q)$ and, because $\langle S | 1_{X^*} \rangle = 1_{\mathcal{A}}$, one must have $\deg(Q) > 0$. Then, we write $w_0 = x_0 v$ and compute the coefficient at v

$$d(\langle Q | v \rangle) = - \sum_{x \in X} u_x \langle Q | xv \rangle = \sum_{x \in X} \alpha_x u_x \quad (16)$$

with coefficients $\alpha_x = -\langle Q | xv \rangle \in k$ as $|xv| = \deg(Q)$ for all $x \in X$. Condition (5) implies that all coefficients $\langle Q | xv \rangle$ are zero; in particular, as $\langle Q | x_0 v \rangle = 1$, we get a contradiction. This proves that $\mathcal{K} = \{0\}$.

3 Localization

We will now establish the following extension of Theorem 1 in [8]. Let (\mathcal{A}, d) be a k -commutative associative differential algebra with unit ($\ker(d) = k$, a field). We consider a solution of the differential equation

$$\mathbf{d}(S) = MS ; \langle S | 1_{X^*} \rangle = 1_{\mathcal{A}} \quad (17)$$

where the multiplier M is a homogeneous series (a polynomial in the case of finite X) of degree 1, i.e.

$$M = \sum_{x \in X} u_x x \in \mathcal{A}\langle X \rangle . \quad (18)$$

Proposition 2 (Thm1 in [8], Localized form). *Let (\mathcal{A}, d) be a commutative associative differential ring ($\ker(d) = k$ being a field) and \mathcal{C} be a differential subring (i.e. $d(\mathcal{C}) \subset \mathcal{C}$) of \mathcal{A} which is an integral domain containing the field of constants.*

We suppose that, for all $x \in X$, $u_x \in \mathcal{C}$ and that $S \in \mathcal{A}\langle X \rangle$ is a solution of the differential equation (17) and that $(u_x)_{x \in X} \in \mathcal{C}^X$.

The following conditions are equivalent :

- i) The family $(\langle S | w \rangle)_{w \in X^*}$ of coefficients of S is free over \mathcal{C} .*
- ii) The family of coefficients $(\langle S | y \rangle)_{y \in X \cup \{1_{X^*}\}}$ is free over \mathcal{C} .*
- iii') For all $f_1, f_2 \in \mathcal{C}$, $f_2 \neq 0$ and $\alpha \in k^{(X)}$, we have the property*

$$W(f_1, f_2) = f_2^2 \left(\sum_{x \in X} \alpha_x u_x \right) \implies (\forall x \in X) (\alpha_x = 0) . \quad (19)$$

where $W(f_1, f_2)$, the wronskian, stands for $d(f_1)f_2 - f_1d(f_2)$.

Proof. (i. \implies ii.) being trivial, remains to prove (ii. \implies iii'.) and (iii'. \implies i.). To this end, we localize the situation w.r.t. the multiplicative subset $\mathcal{C}^\times := \mathcal{C} \setminus \{0\}$ as can be seen in the following commutative cube

$$\begin{array}{ccccc}
\mathcal{C} & \xrightarrow{\varphi_{\mathcal{C}}} & Fr(\mathcal{C}) & & \\
\downarrow d & \searrow j & \downarrow d_{frac} & \searrow j_{frac} & \\
& \mathcal{A} & \xrightarrow{\varphi_{\mathcal{A}}} & \mathcal{A}[(\mathcal{C}^{\times})^{-1}] & \\
& \downarrow d & \downarrow & \downarrow d_{frac} & \\
\mathcal{C} & \xrightarrow{\varphi_{\mathcal{C}}} & Fr(\mathcal{C}) & & \\
\downarrow d & \searrow j & \downarrow d_{frac} & \searrow j_{frac} & \\
& \mathcal{A} & \xrightarrow{\varphi_{\mathcal{A}}} & \mathcal{A}[(\mathcal{C}^{\times})^{-1}] &
\end{array} \tag{20}$$

We give here a detailed demonstration of the commutation which provides, in passing, the labelling of the arrows.

Left face. – Comes from the fact that $d(\mathcal{C}) \subset \mathcal{C}$, j being the canonical embedding.

Upper and lower faces. – We first construct the localization

$\varphi_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{A}[(\mathcal{C}^{\times})^{-1}]$ w.r.t. the multiplicative subset $\mathcal{C}^{\times} \subset \mathcal{A} \setminus \{0\}$ (recall that \mathcal{C} has no zero divisor). Now, from standard theorems (see [5], ch2 par. 2 remark 3 after Def. 2, for instance), we have

$$\ker(\varphi_{\mathcal{A}}) = \{u \in \mathcal{A} | (\exists v \in \mathcal{C}^{\times})(uv = 0)\} \tag{21}$$

For every intermediate ring $\mathcal{C} \subset \mathcal{B} \subset \mathcal{A}$, we remark that the composition

$$\mathcal{B} \hookrightarrow \mathcal{A} \xrightarrow{\varphi_{\mathcal{A}}} \mathcal{A}[(\mathcal{C}^{\times})^{-1}]$$

realises the ring of fractions $\mathcal{B}[(\mathcal{C}^{\times})^{-1}]$ which can be identified with the subalgebra generated by $\varphi_{\mathcal{A}}(\mathcal{B})$ and the set of inverses $\varphi_{\mathcal{A}}(\mathcal{C}^{\times})^{-1}$. Applying this to \mathcal{C} , and remarking that $\mathcal{C}[(\mathcal{C}^{\times})^{-1}] \simeq Fr(\mathcal{C})$, we get the embedding j_{frac} and the commutation of upper and lower faces.

Front and rear faces. – From standard constructions (see e.g. the book [15]), there exists a unique $d_{frac} \in \mathfrak{Der}(\mathcal{A}[(\mathcal{C}^{\times})^{-1}])$ such that these faces commute.

Right face. – Commutation comes from the fact that $d_{frac}j_{frac}$ and $j_{frac}d_{frac}$ coincide on $\varphi_{\mathcal{C}}(\mathcal{C})$ hence on $\varphi_{\mathcal{C}}(\mathcal{C}^{\times})$ and on their inverses. Therefore on all $Fr(\mathcal{C})$.

From the constructions it follows that the arrows (derivations, morphisms) are arrows of k -algebras.

Now, we set

1. $\bar{S} = \sum_{w \in X^*} \varphi_{\mathcal{A}}(\langle S | w \rangle) w \in \mathcal{A}[(\mathcal{C}^{\times})^{-1}] \langle\langle X \rangle\rangle$
2. $\bar{M} = \sum_{x \in X} \varphi_{\mathcal{C}}(u_x) x \in \mathcal{A}[(\mathcal{C}^{\times})^{-1}] \langle\langle X \rangle\rangle$

it is clear, from the commutations, that $(\mathcal{A}[(\mathcal{C}^{\times})^{-1}] \langle\langle X \rangle\rangle, \mathbf{d}_{frac})$ where \mathbf{d}_{frac} is the extension of d_{frac} to the series, is a differential algebra and that

$$\mathbf{d}_{frac}(\bar{S}) = \bar{M}\bar{S} ; \langle\bar{S} | 1\rangle = 1 \tag{22}$$

we are now in the position to resume the proofs of (ii. \implies iii'.) and (iii'. \implies i.).

ii. \implies iii'.) Supposing (ii), we remark that the family of coefficients

$$(\langle\bar{S} | y\rangle)_{y \in X \cup \{1_{X^*}\}}$$

is free over \mathcal{C}^1 . Indeed, let us suppose a relation

$$\sum_{y \in X \cup \{1_{X^*}\}} g_y \langle\bar{S} | y\rangle = 0 \text{ with } (g_y)_{y \in X \cup \{1_{X^*}\}} \in \mathcal{C}^{(X \cup \{1_{X^*}\})} \tag{23}$$

this relation is equivalent to

$$\varphi_{\mathcal{A}}\left(\sum_{y \in X \cup \{1_{X^*}\}} g_y \langle S | y \rangle\right) = 0 \tag{24}$$

which, in view of (21), amounts to the existence of $v \in \mathcal{C}^{\times}$ such that

$$0 = v\left(\sum_{y \in X \cup \{1_{X^*}\}} g_y \langle S | y \rangle\right) = \sum_{y \in X \cup \{1_{X^*}\}} v g_y \langle S | y \rangle \tag{25}$$

¹ As $\varphi_{\mathcal{C}}$ is injective on \mathcal{C} we identify $\varphi_{\mathcal{C}}(\mathcal{C})$ and \mathcal{C} , this can be unfolded on request, of course.

which implies $(\forall y \in X \cup \{1_{X^*}\})(vg_y = 0)$ but, \mathcal{C} being without zero divisor, one gets

$$(\forall y \in X \cup \{1_{X^*}\})(g_y = 0) \quad (26)$$

which proves the claim. This implies in particular, by chasing denominators, that the family of coefficients

$$(\langle \bar{S} \mid y \rangle)_{y \in X \cup \{1_{X^*}\}}$$

is free over $Fr(\mathcal{C})$. This also implies² that $\varphi_{\mathcal{A}}$ is injective on

$$span_{\mathcal{C}}(\langle S \mid y \rangle)_{y \in X \cup \{1_{X^*}\}} \quad (27)$$

To finish the proof that (ii. \implies iii'), let us choose $f_1, f_2 \in \mathcal{C}$ with $f_2 \neq 0$ and set some relation which reads

$$W(f_1, f_2) = f_2^2 \left(\sum_{x \in X} \alpha_x u_x \right) \quad (28)$$

with $\alpha \in k^{(X)}$, then

$$\left(\sum_{x \in X} \alpha_x u_x \right) = \frac{W(f_1, f_2)}{f_2^2} = d_{frac} \left(\frac{f_1}{f_2} \right) \quad (29)$$

but, in view of Th1 in [8] applied to the differential field $Fr(\mathcal{C})$, we get $\alpha \equiv 0$.
(iii' \implies i.) The series \bar{S} satisfies

$$\mathbf{d}(\bar{S}) = \bar{M}\bar{S} ; \langle \bar{S} \mid 1_{X^*} \rangle = 1_{\mathcal{A}[\mathcal{C}^\times]^{-1}} = 1_{Fr(\mathcal{C})} \quad (30)$$

and remarking that

1. all f in the differential field $Fr(\mathcal{C})$ can be expressed as $f = \frac{f_1}{f_2}$
2. condition (iii') for $(S, \mathcal{A}, \mathcal{C}, d, X)$ implies condition (iii) for $(\bar{S}, \mathcal{A}[\mathcal{C}^\times]^{-1}, Fr(\mathcal{C}), d_{frac}, X)^3$ which, in turn, implies the $Fr(\mathcal{C})$ -freeness of $(\langle \bar{S} \mid w \rangle)_{w \in X^*}$ hence its \mathcal{C} -freeness and, by inverse image⁴ the \mathcal{C} -freeness of $(\langle S \mid w \rangle)_{w \in X^*}$.

Remark 1. i) It seems reasonable to think that the whole commutation of the cube could be understood by natural transformations within an appropriate category. If yes, this will be inserted in a forthcoming version.

ii) In fact, in the localized form and with \mathcal{C} not a differential field, (iii) is strictly weaker than (iii'), as shows the following family of counterexamples

1. $\Omega = \mathbb{C} \setminus (]-\infty, 0])$
2. $X = \{x_0\}, u_0 = z^\beta, \beta \notin \mathbb{Q}$
3. $\mathcal{C}_0 = \mathbb{C}\{\{z^\beta\}\} = \mathbb{C}.1_\Omega \oplus span_{\mathbb{C}}\{z^{(k+1)\beta-l}\}_{k,l \geq 0}$
4. $S = 1_\Omega + \left(\sum_{n \geq 1} \frac{z^{n(\beta+1)}}{(\beta+1)^n n!} \right)$

Application 1 As a result of the theory of domains (see [12]), the conc-characters $(\alpha x_0)^*, (\beta x_1)^*$ are in the domain of Li_\bullet (see [12] for details), then due to the fact that $\mathcal{H}(\Omega)$ is nuclear, their shuffle $(\alpha x_0)^* \sqcup (\beta x_1)^* = (\alpha x_0 + \beta x_1)^*$ is also in $\text{Dom}(\text{Li}_\bullet)$. Let us compute

$$\text{Li}(\alpha x_0)^* \sqcup (\beta x_1)^* = \text{Li}(\alpha x_0)^* \text{Li}(\beta x_1)^* = z^\alpha (1-z)^\beta \quad (31)$$

Now, for a family of functions $\mathcal{F} = (f_i)_{i \in I}$, let us note $\mathbb{C}\{f_i\}_{i \in I}$ the algebra generated by \mathcal{F} within $\mathcal{H}(\Omega)$ and then set $\mathcal{C}_{\mathbb{C}} := \mathbb{C}\{z^\alpha (1-z)^\beta\}_{\alpha, \beta \in \mathbb{C}}$. We, at once, remark that, as $\mathcal{M} = \{z^\alpha (1-z)^\beta\}_{\alpha, \beta \in \mathbb{C}}$ is a monoid,

$$\mathcal{C}_{\mathbb{C}} = span_{\mathbb{C}}(z^\alpha (1-z)^\beta)_{\alpha, \beta \in \mathbb{C}} = \mathbb{C}[z^\alpha (1-z)^{-\beta}]_{\alpha, \beta \in \mathbb{C}}$$

as well.

In this application, we give a detailed proof that the family $(\text{Li}_w)_{w \in X^*}$ is $\mathcal{C}_{\mathbb{C}}$ -linearly independent. Let us suppose $P_i \in \mathcal{C}_{\mathbb{C}}, i = 1 \dots 3^5$ such that

$$P_1(z) + P_2(z) \log(z) + P_3(z) \left(\log\left(\frac{1}{1-z}\right) \right) = 0_\Omega$$

² And indeed is equivalent under the assumption of (ii).

³ Once again we identify, with no loss, $k \subset \mathcal{C}$, the latter being identified with its image through $\varphi_{\mathcal{A}[\mathcal{C}^\times]^{-1}}$.

⁴ If the image (through a A -linear arrow) of a family is A -free then the family itself is A -free.

⁵ i.e. elements of the algebra of the monoid $\mathcal{M} = \{z^\alpha (1-z)^\beta\}_{\alpha, \beta \in \mathbb{C}} \{z^\alpha (1-z)^\beta\}_{\alpha, \beta \in \mathbb{C}}$

We first prove that $P_2 = \sum_{i \in F} c_i z^{\alpha_i} (1-z)^{\beta_i}$ is zero using the deck transformation D_0 of index one around zero.

One has $D_0^n(\sum_{i \in F} c_i z^{\alpha_i} (1-z)^{\beta_i}) = \sum_{i \in F} c_i z^{\alpha_i} (1-z)^{\beta_i} e^{2i\pi \cdot n \alpha_i}$, the same calculation holds for all P_i which proves that all $D_0^n(P_i)$ are bounded. But one has $D_0^n(\log(z)) = \log(z) + 2i\pi \cdot n$ and then

$$\begin{aligned} D_0^n(P_1(z) + P_2(z) \log(z) + P_3(z) (\log(\frac{1}{1-z}))) &= \\ D_0^n(P_1(z)) + D_0^n(P_2(z)) (\log(z) + 2i\pi \cdot n) + D_0^n(P_3(z)) \log(\frac{1}{1-z}) &= 0 \end{aligned}$$

It suffices to build a sequence of integers $n_j \rightarrow +\infty$ such that $\lim_{j \rightarrow \infty} D_0^{n_j}(P_2(z)) = P_2(z)$ which is a consequence of the following lemma.

Lemma 1. Let us consider a homomorphism $\varphi : \mathbb{N} \rightarrow G$ where G is a compact (Hausdorff) group, then it exists $u_j \rightarrow +\infty$ such that

$$\lim_{j \rightarrow \infty} \varphi(u_j) = e$$

Proof. First of all, due to the compactness of G , the sequence $\varphi(n)$ admits a subsequence $\varphi(n_k)$ convergent to some $\ell \in G$. Now one can refine the sequence as n_{k_j} such that

$$0 < n_{k_1} - n_{k_0} < \dots < n_{k_{j+1}} - n_{k_j} < n_{k_{j+2}} - n_{k_{j+1}} < \dots$$

With $u_j = n_{k_{j+1}} - n_{k_j}$ one has $\lim_{j \rightarrow \infty} \varphi(u_j) = e$.

End of the proof One applies the lemma to the morphism

$$n \mapsto (e^{2i\pi \cdot n \alpha_i})_{i \in F} \in \mathbb{U}^F$$

4 Closed subgroup property and algebraic independance.

5 Appendix.

5.1 Closed subgroup (Cartan) theorem in Banach algebras

This section is meant to be withdrawn afterwards⁶ and integrated within the introduction.

Let \mathcal{B} be a Banach algebra (with unit e) and G a closed subgroup of the open set \mathcal{B}^\times . By a path “drawn on T ” ($T \subset \mathcal{B}$) is understood any function $\varphi : I \rightarrow T$ where I is a open real interval.

The first step is to establish what would be seen as the Lie algebra of G .

Let $L = L(G)$ be the set of tangent vectors of G at the origin i.e.

$$L(G) = \{\gamma'(0) \mid \gamma : I \rightarrow G \text{ is differentiable at } 0 (= 0_{\mathbb{R}}) \text{ and } \gamma(0) = e\} \quad (32)$$

Proposition 3. The set $L(G)$ has the following properties

- 1) If $u \in L(G)$, then, the one-parameter group $t \mapsto e^{t \cdot u}$ is drawn on G
- 2) $L(G)$ is a closed Lie subalgebra of \mathcal{B}
- 3) Let $g \in \mathcal{B}$ s.t. $\|g - e\| < 1$, then

$$\log(g) \in L(G) \implies g \in G$$

Proof. 1) Let γ be such a tangent path (differentiable at 0 and s.t. $\gamma(0) = e$), then one can write

$$\gamma(t) = e + t \cdot \gamma'(0) + t \cdot \epsilon_1(t) \text{ with } \lim_{t \rightarrow 0} \epsilon_1(t) = 0_{\mathcal{B}} \quad (33)$$

and then, $t \in [-1, 1]$ being fixed,

$$\gamma\left(\frac{t}{n}\right) = e + \frac{t}{n} \cdot \gamma'(0) + \frac{t}{n} \cdot \epsilon_2(n) \text{ with } \lim_{n \rightarrow \infty} \epsilon_2(n) = 0_{\mathcal{B}} \quad (34)$$

now, for $\|h\| < 1$, one has (in \mathcal{B}) $\log(e+h) = h + \|h\| \cdot \epsilon_3(h)$ so, with

$h = \frac{t}{n} \cdot \gamma'(0) + \frac{t}{n} \cdot \epsilon_2(n)$, we get

$$n \log\left(\gamma\left(\frac{t}{n}\right)\right) = t \cdot (\gamma'(0) + \epsilon_2(n)) + \|t \cdot (\gamma'(0) + \epsilon_2(n))\| \epsilon_3(h) \quad (35)$$

⁶ Following the advise of Gauss “no self-respecting architect leaves the scaffolding in place after completing the building”.

hence $\lim_{n \rightarrow \infty} n \log(\gamma(\frac{t}{n})) = t \cdot \gamma'(0)$ and then

$$\lim_{n \rightarrow \infty} \gamma(\frac{t}{n})^n = \lim_{n \rightarrow \infty} e^{n \cdot \log(\gamma(\frac{t}{n}))} = e^{t \cdot \gamma'(0)} \quad (36)$$

as γ is drawn on G , each $\gamma(\frac{t}{n})^n$ belongs to G and it is the same for the limit (G is closed), then $e^{t \cdot \gamma'(0)} \in G$ for all $t \in [-1, 1]$. For general $t \in \mathbb{R}$, one just has to use the archimedean property that,

for some $N \in \mathbb{N}_{\geq 1}$, $\frac{t}{N} \in [-1, 1]$ and remark that $e^{t \cdot \gamma'(0)} = (e^{\frac{t}{N} \cdot \gamma'(0)})^N$.

2) To prove that $L(G)$ is a Lie subalgebra of \mathcal{B} it suffices to provide suitable paths. Let $u, v \in L(G)$, we have

Path	Tangent vector (at zero)
$e^{t \cdot u} e^{t \cdot v}$	$u + v$
$e^{\alpha t \cdot u}$	αu
Let $\gamma_1(t) = e^{r(t) \cdot u} e^{r(t) \cdot v} e^{-r(t) \cdot u} e^{-r(t) \cdot v}$ with $r(t) = \sqrt{2} \cdot t$ for $t \geq 0$ $\gamma(t) = \gamma_1(t)$ for $t \geq 0$ and $= (\gamma_1(-t))^{-1}$ for $t \leq 0$	$[u, v]$

Remains to show that $L(G)$ is closed. To see this, let us consider a sequence $(u_n)_{n \in \mathbb{N}}$ in $L(G)$ which converges to u . For every fixed $t \in \mathbb{R}$, $\lim_{n \rightarrow \infty} e^{t \cdot u_n} = e^{t \cdot u}$ because $\exp : \mathcal{B} \rightarrow \mathcal{B}$ is continuous, then $t \rightarrow e^{t \cdot u}$ is drawn on G and $u \in L(G)$.

3) Set $u = \log(g)$. Now, as $u \in L(G)$, the one-parameter group $t \rightarrow e^{t \cdot u}$ is drawn on G and $g = (e^{t \cdot u})|_{t=1}$ (in the neighbourhood $\|g - e\| < 1$, we have $\exp(\log(g)) = g$).

Now, we have an analogue of Cartan's theorem in the context of Banach algebras

Theorem 2. *Let $G \subset \mathcal{B}$ be a closed subgroup of \mathcal{B}^\times and $L(G)$ as above. Let $I \subset \mathbb{R}$ be a non-void open interval and $M : I \rightarrow L(G)$ to be a continuous path drawn on $L(G)$. Let $t_0 \in I$, $g_0 \in G$. Then*

1) *The system*

$$\Sigma(t_0, M, g_0) \quad \begin{cases} \frac{d}{dt}(S(t)) = M(t) \cdot S(t) & \text{(NCDE)} \\ S(t_0) = g_0 & \text{(Init. Cond.)} \end{cases} \quad (37)$$

admits a unique solution $S : I \rightarrow \mathcal{B}$.

2) *This solution S is a path drawn on G .*

Proof. 1-2) I sketch the proof below

1) Let J be an open real interval, $t_0 \in J$ and $m \in C^0(J, L(G))$. In order to paste them together, we call "local solution" (of $\Sigma(t_0, m, g)$), a C^1 map $J \rightarrow G$ fulfilling the following system

$$\Sigma(t_0, m, g) \quad \begin{cases} \frac{d}{dt}(T(t)) = m(t) \cdot T(t) & \text{(NCDE)} \\ T(t_0) = g & \text{(Init. Cond.)} \end{cases} \quad (38)$$

2) We first prove that there exists a local solution to any system $\Sigma(0, m, e)$ for $t_0 = 0$, $g = e$. As m is continuous, there is an open interval J , containing zero, in which $|t| < 1$ and is such that $\|m_1(t)\| \leq B < 1$. In these conditions, Picard's process

$$\begin{cases} T_0 = e \\ T_{n+1}(t) = \int_0^t m(s) \cdot T_n(s) ds \end{cases} \quad (39)$$

converges absolutely (in $C^0(J, \mathcal{B})$) to a function $T \in C^0(J, \mathcal{B})$ such that

$$T(t) = e + \int_0^t m(s) \cdot T(s) ds \quad (40)$$

this proves that, in fact, $T \in C^1(J, \mathcal{B})$. Remains to prove that T is drawn on G .

- 3) If t and B are sufficiently small [est1](#), we have $\|T(t) - e\| < 1$ and can compute $\Omega(t) = \log(T)$, which is C^1 and, by Magnus expansion (see below 5.2), satisfies

$$\Omega'(t) = \frac{ad_\Omega}{e^{ad_\Omega} - Id_B}[\Omega] \quad (41)$$

where, the symbol $\frac{ad_\Omega}{e^{ad_\Omega} - Id_B}$ denotes the substitution of ad_Ω in the series $\sum_{n \geq 0} B_n \frac{z^n}{n!}$ (B_n being the Bernoulli numbers) [est2](#). But we know that $\Omega(t)$ is the limit of the following process

$$\begin{cases} \Omega_1(t) = \int_0^t m(s) ds \\ \Omega_n(t) = \sum_{j=1}^{n-1} \frac{B_j}{j!} \sum_{\substack{k_1 + \dots + k_j = n-1 \\ k_i \geq 1}} \int_0^t ad_{\Omega_{k_1}} \cdots ad_{\Omega_{k_j}}[m(s)] ds \end{cases} \quad (42)$$

each Ω_n is drawn on $L(G)$ as shows the preceding recursion. Hence $T(t) = e^{\Omega(t)}$ is drawn on G .

- 4) We can now shift the situation in order to compute a local solution of any system $\Sigma(t_0, m, g)$ as follows (given J an open real interval, $t_0 \in J$ and $m \in C^0(J, L(G))$)
- Find a local solution R of $\Sigma(0, m_1, e)$ with $m_1(t) = m(t + t_0) - m(t_0)$. For it

$$R'(t) = (m(t + t_0) - m(t_0))R(t) ; T(0) = e \quad (43)$$

- Define $T(t) := R(t)e^{t-m(t_0)}g$, one has

$$T'(t) = m(t)T(t) ; T(t_0) = g \quad (44)$$

- 5) We now return to our original system $\Sigma(t_0, M, g_0)$. By the previous item (4) we know that it admits at least a local solution (J, S) . We remark also that if we have two local solutions $(J_1, S_1), (J_2, S_2)$ they coincide on $J_1 \cap J_2$, thus the union of all graphs on local solutions of $\Sigma(t_0, M, g_0)$ is functional and is the maximal solution (J_{max}, S_{max}) of $\Sigma(t_0, M, g_0)$. Now, if we had $b_m = \sup(J_{max}) < \sup(I)$, we could consider the system $\Sigma(b_m, M, e)$ and a local solution T of it on some $]a, b[$ with $a < b_m < b$, now taking some intermediate point t_1 within $]a, b_m[$, we observe that $T S_{max} T^{-1}(t_1) S_{max}(t_1)$ and S_{max} coincide on $]a, b_m[$ and the union of their graphs would be a strict extension of S_{max} . A contradiction, then $b_m = \sup(I)$. A similar reasoning proves that $a_m = \inf(I)$ and then theorem 2 is proved.

5.2 About Magnus expansion and Poincaré-Hausdorff formula

Formal derivation of Poincaré-Hausdorff and Magnus formulas Let $(\mathbb{C}\{X\}, \partial)$ be the differential algebra freely generated by X (a formal variable)⁷. X be a formal variable and $(\mathbb{C}X, \partial)$ be We define a comultiplication Δ by asking that all $X^{[k]}$ be primitive note that Δ commutes with the derivation. Setting, in $\widehat{\mathbb{C}\{X\}}$, $D = \partial(e^X)e^{-X}$, direct computation shows that D is primitive and hence a Lie series, which can therefore be written as a sum of Dynkin trees.

On the other hand, the formula

$$D = \sum_{k \geq 1} \frac{1}{k!} \sum_{l=0}^{k-1} X^l (\partial X) X^{k-1-l} \cdot \sum_{n \geq 0} \frac{(-X)^n}{n!} \quad (45)$$

suggests that all bedegrees (in $X, \partial X$) are of the form $[n, 1]$ and thus, there exists an univariate series $\Phi(Y) = \sum_{n \geq 0} a_n Y^n$ such that $D = \Phi(ad_X)[\partial X]$. Using left and right multiplications by X (resp. noted g, d), we can rewrite (45) as

$$D = \left(\sum_{k \geq 1} \frac{1}{k!} \sum_{l=0}^{k-1} g^l d^{k-1-l} [\partial X] \right) e^{-X} \quad (46)$$

but, from the fact that g, d commute, the inner sum $\sum_{l=0}^{k-1} g^l d^{k-1-l}$ is ruled out by the the following identity (in $\mathbb{C}[Y, Z]$, but computed within $\mathbb{C}(Y, Z)$)

$$\sum_{l=0}^{k-1} Y^l Z^{k-1-l} = \frac{Y^k - Z^k}{Y - Z} = \frac{((Y - Z) + Z)^k - Z^k}{Y - Z} = \sum_{j=1}^k \binom{k}{j} (Y - Z)^j Z^{k-j} \quad (47)$$

⁷ It is, in fact, the free algebra $\mathbb{C}\langle (X^{[k]})_{k \geq 0} \rangle$ (with $X^{[0]} = X$) endowed with $\partial(X^{[k]}) = X^{[k+1]}$, the construction is similar to what is to be found in [15], but in the noncommutative realm.

Taking notice that $(g - d) = ad_X$ and plugging (47) into (45), one gets

$$D = \left(\sum_{k \geq 1} \frac{1}{k!} \sum_{j=1}^k \binom{k}{j} (ad_X)^{j-1} d^{k-j} [\partial X] \right) e^{-X} = \frac{1}{ad_X} \left(\sum_{k \geq 1} \sum_{j=1}^k \frac{1}{j!(r-j)!} (ad_X)^j d^{k-j} [\partial X] \right) e^{-X} = \frac{e^{ad_X} - 1}{ad_X} [X'] \quad (48)$$

which is Poincaré-Hausdorff formula.

Application Let G be a Lie group with Lie algebra L . Let $X(t)$ be a C^1 path drawn within L (setting as above i.e. $X(0) = 0$), then

$$(e^{X(t)})' e^{-X(t)} = \frac{e^{ad_X} - 1}{ad_X} [X'(t)] \quad (49)$$

In particular, if $S(t)$ is a solution of the system

$$\Sigma(0, M, e) \quad \begin{cases} \frac{d}{dt}(S(t)) = M(t) \cdot S(t) & \text{(NCDE)} \\ S(0) = e & \text{(Init. Cond.)} \end{cases}$$

then $\Omega = \log(S)$, at a neighbourhood of $t = 0^8$ must satisfy $\Omega' = \frac{ad_\Omega}{e^{ad_\Omega} - 1} [M]$ (this identity⁹ lives in the completion of the non-commutative free differential algebra generated by the single X , constructed like in [15] but non commutative¹⁰). This guarantees the existence of a local solution drawn on G .

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⁸ Such that the norm of ad_Ω for the topology of bounded convergence be strictly less than 2π (the radius of convergence of $\phi(z) = \frac{z}{e^z - 1}$) for which, it is sufficient that $\|\Omega\| < \pi$.

⁹ The fraction

$$\frac{ad_X}{e^{ad_X} - 1}$$

means, of course, $\phi(ad_X)$ where

$$\phi(t) = \frac{t}{e^t - 1} = \sum_{n \geq 0} B_n \frac{t^n}{n!}$$

the family $(B_n)_{n \geq 0}$ being that of Bernoulli numbers.

¹⁰ In fact, this (highly noncommutative) differential algebra $(\mathbb{C}^{diff}\langle X \rangle, \partial)$ may be realized as the free algebra $\mathbb{C}\langle (X^{[k]})_{k \geq 0} \rangle$ (with $X^{[0]} = X$) endowed with the derivation defined by $\partial(X^{[k]}) = X^{[k+1]}$ (see e.g. [4], Ch I, par. 2.8 *Extension of derivations*).

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