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A localized version of the basic triangle theorem

G.H.E. Duchamp

University Paris 13, Sorbonne Paris City, 93430 Villetaneuse, France,

N.P. Gargava

École Polytechnique Fédérale de Lausanne, CH-1015, Lausanne, Switzerland

V. Hoang Ngoc Minh

University of Lille, 1 Place Dliot, 59024 Lille, France,

P. Simonnet

University of Corsica, 20250 Corte, France.

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Abstract

In this short note, we give a localized version of the basic triangle theorem, first published in 2011 (see [6]) in order to prove the independence of hyperlogarithms over various function fields. This version provides direct access to rings of scalars and avoids the recourse to fraction fields as that of meromorphic functions for instance.

1 Original theorem

Non commutative differential equations with left multiplier can be expressed in the context of general differential algebras. Notations about alphabets and (noncommutative) series are standard and can be found in [1].

Theorem 1.1 (Th 1 in [6]) Let (\mathcal{A}, d) be a k -commutative associative differential algebra with unit ($\ker(d) = k$, a field) and \mathcal{C} be a differential subfield of \mathcal{A} (i.e. $d(\mathcal{C}) \subset \mathcal{C}$ and $k \subset \mathcal{C}$). Let X be some alphabet (i.e. some set) and we define $\mathbf{d} : \mathcal{A}\langle\langle X \rangle\rangle \rightarrow \mathcal{A}\langle\langle X \rangle\rangle$ to be the map given by $\langle \mathbf{d}(S) \mid w \rangle = d(\langle S \mid w \rangle)$. We suppose that $S \in \mathcal{A}\langle\langle X \rangle\rangle$ is a solution of the differential equation

$$\mathbf{d}(S) = MS ; \langle S \mid 1_{X^*} \rangle = 1_{\mathcal{A}} \quad (1)$$

where the multiplier M is a homogeneous series (a polynomial in the case of finite X) of degree 1, i.e.

$$M = \sum_{x \in X} u_x x \in \mathcal{C}\langle\langle X \rangle\rangle . \quad (2)$$

The following conditions are equivalent :

- i) The family $(\langle S \mid w \rangle)_{w \in X^*}$ of coefficients of S is free over \mathcal{C} .
- ii) The family of coefficients $(\langle S \mid y \rangle)_{y \in X \cup \{1_{X^*}\}}$ is free over \mathcal{C} .
- iii) The family $(u_x)_{x \in X}$ is such that, for $f \in \mathcal{C}$ and $\alpha \in k^{(X)}$ (i.e. $\text{supp}(\alpha)$ is finite)

$$d(f) = \sum_{x \in X} \alpha_x u_x \implies (\forall x \in X)(\alpha_x = 0) . \quad (3)$$

- iv) The family $(u_x)_{x \in X}$ is free over k and

$$d(\mathcal{C}) \cap \text{span}_k((u_x)_{x \in X}) = \{0\} . \quad (4)$$

Proof. For convenience of the reader, we enclose here the demonstration given in Th 1 [6].

(i) \implies (ii) Obvious.

(ii) \implies (iii)

Suppose that the family $(\langle S \mid y \rangle)_{y \in X \cup \{1_{X^*}\}}$ (coefficients taken at letters and the empty word) of coefficients of S were free over \mathcal{C} and let us consider the relation as in (3)

$$d(f) = \sum_{x \in X} \alpha_x u_x . \quad (5)$$

We form the polynomial $P = -f1_{X^*} + \sum_{x \in X} \alpha_x x$. One has $\mathbf{d}(P) = -d(f)1_{X^*}$ and

$$d(\langle S \mid P \rangle) = \langle \mathbf{d}(S) \mid P \rangle + \langle S \mid \mathbf{d}(P) \rangle = \langle MS \mid P \rangle - d(f)\langle S \mid 1_{X^*} \rangle = \left(\sum_{x \in X} \alpha_x u_x \right) - d(f) = 0 \quad (6)$$

whence $\langle S \mid P \rangle$ must be a constant, say $\lambda \in k$. For $Q = P - \lambda.1_{X^*}$, we have

$$\text{supp}(Q) \subset X \cup \{1_{X^*}\} \text{ and } \langle S \mid Q \rangle = \langle S \mid P \rangle - \lambda \langle S \mid 1_{X^*} \rangle = \langle S \mid P \rangle - \lambda = 0 .$$

This, in view of (ii), implies that $Q = 0$ and, as $Q = -(f + \lambda)1_{X^*} + \sum_{x \in X} \alpha_x x$, one has, in particular, $\text{supp}(\alpha) = \emptyset$ (and, as a byproduct, $f = -\lambda$ which is indeed the only possibility for the L.H.S. of (3) to occur).

(iii) \iff (iv)

Obvious, (iv) being a geometric reformulation of (iii).

(iii) \implies (i)

Let \mathcal{K} be the kernel of $P \mapsto \langle S | P \rangle$ (linear $\mathcal{C}\langle X \rangle \rightarrow \mathcal{A}$) i.e.

$$\mathcal{K} = \{P \in \mathcal{C}\langle X \rangle \mid \langle S | P \rangle = 0\} . \quad (7)$$

If $\mathcal{K} = \{0\}$, we are done. Otherwise, let us adopt the following strategy.

First, we order X by some well-ordering $<$ ([2] III.2.1) and X^* by the graded lexicographic ordering \prec defined as follows

$$u \prec v \iff |u| < |v| \text{ or } (u = pxs_1, v = pys_2 \text{ and } x < y) . \quad (8)$$

It is easy to check that X^* is also a well-ordered by \prec . For each nonzero polynomial P , we denote by $\text{lead}(P)$ its leading monomial; i.e. the greatest element of its support $\text{supp}(P)$ (for \prec).

Now, as $\mathcal{R} = \mathcal{K} \setminus \{0\}$ is not empty, let w_0 be the minimal element of $\text{lead}(\mathcal{R})$ and choose a $P \in \mathcal{R}$ such that $\text{lead}(P) = w_0$. We write

$$P = fw_0 + \sum_{u \prec w_0} \langle P | u \rangle u ; f \in \mathcal{C} \setminus \{0\} . \quad (9)$$

The polynomial $Q = \frac{1}{f}P$ is also in \mathcal{R} with the same leading monomial, but the leading coefficient is now 1; and so Q is given by

$$Q = w_0 + \sum_{u \prec w_0} \langle Q | u \rangle u . \quad (10)$$

Differentiating $\langle S | Q \rangle = 0$, one gets

$$\begin{aligned} 0 &= \langle \mathbf{d}(S) | Q \rangle + \langle S | \mathbf{d}(Q) \rangle = \langle MS | Q \rangle + \langle S | \mathbf{d}(Q) \rangle = \\ &= \langle S | M^\dagger Q \rangle + \langle S | \mathbf{d}(Q) \rangle = \langle S | M^\dagger Q + \mathbf{d}(Q) \rangle \end{aligned} \quad (11)$$

with

$$M^\dagger Q + \mathbf{d}(Q) = \sum_{x \in X} u_x (x^\dagger Q) + \sum_{u \prec w_0} d(\langle Q | u \rangle) u \in \mathcal{C}\langle X \rangle . \quad (12)$$

It is impossible that $M^\dagger Q + \mathbf{d}(Q) \in \mathcal{R}$ because it would be of leading monomial strictly less than w_0 , hence $M^\dagger Q + \mathbf{d}(Q) = 0$. This is equivalent to the recursion

$$d(\langle Q | u \rangle) = - \sum_{x \in X} u_x \langle Q | xu \rangle ; \text{ for } x \in X, v \in X^* . \quad (13)$$

From this last relation, we deduce that $\langle Q | w \rangle \in k$ for every w of length $\deg(Q)$ and, because $\langle S | 1_{X^*} \rangle = 1_{\mathcal{A}}$, one must have $\deg(Q) > 0$. Then, we write $w_0 = x_0 v$ and compute the coefficient at v

$$d(\langle Q | v \rangle) = - \sum_{x \in X} u_x \langle Q | xv \rangle = \sum_{x \in X} \alpha_x u_x \quad (14)$$

with coefficients $\alpha_x = -\langle Q | xv \rangle \in k$ as $|xv| = \deg(Q)$ for all $x \in X$. Condition (3) implies that all coefficients $\langle Q | xv \rangle$ are zero; in particular, as $\langle Q | x_0 v \rangle = 1$, we get a contradiction. This proves that $\mathcal{K} = \{0\}$.

□

2 Localization

We will now establish the following extension of Theorem 1 in [6]. Let (\mathcal{A}, d) be a k -commutative associative differential algebra with unit ($\ker(d) = k$, a field). We consider a solution of the differential equation

$$\mathbf{d}(S) = MS ; \langle S | 1_{X^*} \rangle = 1_{\mathcal{A}} \quad (15)$$

where the multiplier M is a homogeneous series (a polynomial in the case of finite X) of degree 1, i.e.

$$M = \sum_{x \in X} u_x x \in \mathcal{A} \langle\langle X \rangle\rangle . \quad (16)$$

Proposition 2.1 (Thm1 in [6], Localized form) *Let (\mathcal{A}, d) be a commutative associative differential ring ($\ker(d) = k$ being a field) and \mathcal{C} be a differential subring (i.e. $d(\mathcal{C}) \subset \mathcal{C}$) of \mathcal{A} which is an integral domain containing the field of constants. We suppose that, for all $x \in X$, $u_x \in \mathcal{C}$ and that $S \in \mathcal{A} \langle\langle X \rangle\rangle$ is a solution of the differential equation (15) and that $(u_x)_{x \in X} \in \mathcal{C}^X$.*

The following conditions are equivalent :

- i) *The family $(\langle S | w \rangle)_{w \in X^*}$ of coefficients of S is free over \mathcal{C} .*
- ii) *The family of coefficients $(\langle S | y \rangle)_{y \in X \cup \{1_{X^*}\}}$ is free over \mathcal{C} .*
- iii') *For all $f_1, f_2 \in \mathcal{C}$, $f_2 \neq 0$ and $\alpha \in k^{(X)}$, we have the property*

$$W(f_1, f_2) = f_2^2 \left(\sum_{x \in X} \alpha_x u_x \right) \implies (\forall x \in X) (\alpha_x = 0) . \quad (17)$$

where $W(f_1, f_2)$, the wronskian, stands for $d(f_1)f_2 - f_1d(f_2)$.

Proof. (i. \implies ii.) being trivial, remains to prove (ii. \implies iii'.) and (iii'. \implies i.). To this end, we localize the situation w.r.t. the multiplicative subset $\mathcal{C}^\times := \mathcal{C} \setminus \{0\}$ as can be seen in the following commutative cube

$$\begin{array}{ccccc}
\mathcal{C} & \xrightarrow{\varphi_{\mathcal{C}}} & Fr(\mathcal{C}) & & \\
\downarrow d & \searrow j & \downarrow d_{frac} & \searrow j_{frac} & \\
& \mathcal{A} & \xrightarrow{\varphi_{\mathcal{A}}} & \mathcal{A}[(\mathcal{C}^\times)^{-1}] & \\
& \downarrow d & \downarrow & \downarrow d_{frac} & \\
\mathcal{C} & \xrightarrow{\varphi_{\mathcal{C}}} & Fr(\mathcal{C}) & & \\
& \searrow j & \downarrow d_{frac} & \searrow j_{frac} & \\
& \mathcal{A} & \xrightarrow{\varphi_{\mathcal{A}}} & \mathcal{A}[(\mathcal{C}^\times)^{-1}] &
\end{array} \tag{18}$$

We give here a detailed demonstration of the commutation which provides, in passing, the labelling of the arrows.

Left face. — Comes from the fact that $d(\mathcal{C}) \subset \mathcal{C}$, j being the canonical embedding.

Upper and lower faces. — We first construct the localization

$\varphi_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{A}[(\mathcal{C}^\times)^{-1}]$ w.r.t. the multiplicative subset $\mathcal{C}^\times \subset \mathcal{A} \setminus \{0\}$ (recall that \mathcal{C} has no zero divisor). Now, from standard theorems (see [5], ch2 §2 remark 3 after Def. 2, for instance), we have

$$\ker(\varphi_{\mathcal{A}}) = \{u \in \mathcal{A} \mid (\exists v \in \mathcal{C}^\times)(uv = 0)\} \tag{19}$$

For every intermediate ring $\mathcal{C} \subset \mathcal{B} \subset \mathcal{A}$, we remark that the composition

$$\mathcal{B} \hookrightarrow \mathcal{A} \xrightarrow{\varphi_{\mathcal{A}}} \mathcal{A}[(\mathcal{C}^\times)^{-1}]$$

realises the ring of fractions $\mathcal{B}[(\mathcal{C}^\times)^{-1}]$ which can be identified with the subalgebra generated by $\varphi_{\mathcal{A}}(\mathcal{B})$ and the set of inverses $\varphi_{\mathcal{A}}(\mathcal{C}^\times)^{-1}$. Applying this to \mathcal{C} , and remarking that $\mathcal{C}[(\mathcal{C}^\times)^{-1}] \simeq Fr(\mathcal{C})$, we get the embedding j_{frac} and the commutation of upper and lower faces.

Front and rear faces. — From standard constructions (see e.g. the book [11]), there exists a unique $d_{frac} \in \mathfrak{Det}(\mathcal{A}[(\mathcal{C}^\times)^{-1}])$ such that these faces commute.

Right face. — Commutation comes from the fact that $d_{frac}j_{frac}$ and $j_{frac}d_{frac}$ co-

incide on $\varphi_{\mathcal{C}}(\mathcal{C})$ hence on $\varphi_{\mathcal{C}}(\mathcal{C}^\times)$ and on their inverses. Therefore on all $Fr(\mathcal{C})$. From the constructions it follows that the arrows (derivations, morphisms) are arrows of k -algebras.

Now, we set

$$(i) \quad \bar{S} = \sum_{w \in X^*} \varphi_{\mathcal{A}}(\langle S | w \rangle) w \in \mathcal{A}[\mathcal{C}^\times]^{-1} \langle\langle X \rangle\rangle$$

$$(ii) \quad \bar{M} = \sum_{x \in X} \varphi_{\mathcal{C}}(u_x) x \in \mathcal{A}[\mathcal{C}^\times]^{-1} \langle\langle X \rangle\rangle$$

it is clear, from the commutations, that $(\mathcal{A}[\mathcal{C}^\times]^{-1} \langle\langle X \rangle\rangle, \mathbf{d}_{frac})$ where \mathbf{d}_{frac} is the extension of d_{frac} to the series, is a differential algebra and that

$$\mathbf{d}_{frac}(\bar{S}) = \bar{M}\bar{S}; \quad \langle \bar{S} | 1 \rangle = 1 \quad (20)$$

we are now in the position to resume the proofs of (ii. \implies iii'.) and (iii'. \implies i.).
ii. \implies iii'.) Supposing (ii), we remark that the family of coefficients

$$(\langle \bar{S} | y \rangle)_{y \in X \cup \{1_{X^*}\}}$$

is free over \mathcal{C}^1 . Indeed, let us suppose a relation

$$\sum_{y \in X \cup \{1_{X^*}\}} g_y \langle \bar{S} | y \rangle = 0 \text{ with } (g_y)_{y \in X \cup \{1_{X^*}\}} \in \mathcal{C}^{(X \cup \{1_{X^*}\})} \quad (21)$$

this relation is equivalent to

$$\varphi_{\mathcal{A}}\left(\sum_{y \in X \cup \{1_{X^*}\}} g_y \langle S | y \rangle\right) = 0 \quad (22)$$

which, in view of (19), amounts to the existence of $v \in \mathcal{C}^\times$ such that

$$0 = v\left(\sum_{y \in X \cup \{1_{X^*}\}} g_y \langle S | y \rangle\right) = \sum_{y \in X \cup \{1_{X^*}\}} v g_y \langle S | y \rangle \quad (23)$$

which implies $(\forall y \in X \cup \{1_{X^*}\})(v g_y = 0)$ but, \mathcal{C} being without zero divisor, one gets

$$(\forall y \in X \cup \{1_{X^*}\})(g_y = 0) \quad (24)$$

which proves the claim. This implies in particular, by chasing denominators, that the family of coefficients

$$(\langle \bar{S} | y \rangle)_{y \in X \cup \{1_{X^*}\}}$$

¹ As $\varphi_{\mathcal{C}}$ is injective on \mathcal{C} we identify $\varphi_{\mathcal{C}}(\mathcal{C})$ and \mathcal{C} , this can be unfolded on request, of course.

is free over $Fr(\mathcal{C})$. This also implies² that $\varphi_{\mathcal{A}}$ is injective on

$$span_{\mathcal{C}}(\langle S | y \rangle)_{y \in X \cup \{1_{X^*}\}} \quad (25)$$

To finish the proof that (ii. \implies iii'), let us choose $f_1, f_2 \in \mathcal{C}$ with $f_2 \neq 0$ and set some relation which reads

$$W(f_1, f_2) = f_2^2 \left(\sum_{x \in X} \alpha_x u_x \right) \quad (26)$$

with $\alpha \in k^{(X)}$, then

$$\left(\sum_{x \in X} \alpha_x u_x \right) = \frac{W(f_1, f_2)}{f_2^2} = d_{frac} \left(\frac{f_1}{f_2} \right) \quad (27)$$

but, in view of Th1 in [6] applied to the differential field $Fr(\mathcal{C})$, we get $\alpha \equiv 0$.
(iii' \implies i.) The series \bar{S} satisfies

$$\mathbf{d}(\bar{S}) = \bar{M}\bar{S} ; \langle \bar{S} | 1_{X^*} \rangle = 1_{\mathcal{A}[\mathcal{C}^\times]^{-1}} = 1_{Fr(\mathcal{C})} \quad (28)$$

and remarking that

- (i) all f in the differential field $Fr(\mathcal{C})$ can be expressed as $f = \frac{f_1}{f_2}$
- (ii) condition (iii') for $(S, \mathcal{A}, \mathcal{C}, d, X)$ implies condition (iii) for $(\bar{S}, \mathcal{A}[\mathcal{C}^\times]^{-1}, Fr(\mathcal{C}), d_{frac}, X)$ ³ which, in turn, implies the $Fr(\mathcal{C})$ -freeness of $(\langle \bar{S} | w \rangle)_{w \in X^*}$ hence its \mathcal{C} -freeness and, by inverse image⁴ the \mathcal{C} -freeness of $(\langle S | w \rangle)_{w \in X^*}$.

□

Remark 2.2 It seems reasonable to think that the whole commutation of the cube could be understood by natural transformations within an appropriate category. If yes, this will be inserted in a forthcoming version.

² And indeed is equivalent under the assumption of (ii).

³ Once again we identify, with no loss, $k \subset \mathcal{C}$, the latter being identified with its image through $\varphi_{\mathcal{A}[\mathcal{C}^\times]^{-1}}$.

⁴ If the image (through a A -linear arrow) of a family is A -free then the family itself is A -free.

3 Closed subgroup property and algebraic independence.

4 Appendix.

4.1 Closed subgroup (Cartan) theorem in Banach algebras

This section is meant to be withdrawn afterwards⁵ and integrated within the introduction.

Let \mathcal{B} be a Banach algebra (with unit e) and G a closed subgroup of the open set \mathcal{B}^\times . By a path “drawn on T ” ($T \subset \mathcal{B}$) is understood any function $\varphi : I \rightarrow T$ where I is a open real interval.

The first step is to establish what would be seen as the Lie algebra of G .

Let $L = L(G)$ be the set of tangent vectors of G at the origin i.e.

$$L(G) = \{\gamma'(0) \mid \gamma : I \rightarrow G \text{ is differentiable at } 0 (= 0_{\mathbb{R}}) \text{ and } \gamma(0) = e\} \quad (29)$$

Proposition 4.1 *The set $L(G)$ has the following properties*

- 1) If $u \in L(G)$, then, the one-parameter group $t \mapsto e^{t \cdot u}$ is drawn on G
- 2) $L(G)$ is a closed Lie subalgebra of \mathcal{B}
- 3) Let $g \in \mathcal{B}$ s.t. $\|g - e\| < 1$, then

$$\log(g) \in L(G) \implies g \in G$$

Proof. 1) Let γ be such a tangent path (differentiable at 0 and s.t. $\gamma(0) = e$), then one can write

$$\gamma(t) = e + t \cdot \gamma'(0) + t \cdot \varepsilon_1(t) \text{ with } \lim_{t \rightarrow 0} \varepsilon_1(t) = 0_{\mathcal{B}} \quad (30)$$

and then, $t \in [-1, 1]$ being fixed,

$$\gamma\left(\frac{t}{n}\right) = e + \frac{t}{n} \cdot \gamma'(0) + \frac{t}{n} \cdot \varepsilon_2(n) \text{ with } \lim_{n \rightarrow \infty} \varepsilon_2(n) = 0_{\mathcal{B}} \quad (31)$$

now, for $\|h\| < 1$, one has (in \mathcal{B}) $\log(e + h) = h + \|h\| \cdot \varepsilon_3(h)$ so, with $h = \frac{t}{n} \cdot \gamma'(0) + \frac{t}{n} \cdot \varepsilon_2(n)$, we get

$$n \log\left(\gamma\left(\frac{t}{n}\right)\right) = t \cdot (\gamma'(0) + \varepsilon_2(n)) + \|t \cdot (\gamma'(0) + \varepsilon_2(n))\| \varepsilon_3(h) \quad (32)$$

⁵ Following the advise of Gauss “no self-respecting architect leaves the scaffolding in place after completing the building”.

hence $\lim_{n \rightarrow \infty} n \log(\gamma(\frac{t}{n})) = t \cdot \gamma'(0)$ and then

$$\lim_{n \rightarrow \infty} \gamma(\frac{t}{n})^n = \lim_{n \rightarrow \infty} e^{n \cdot \log(\gamma(\frac{t}{n}))} = e^{t \cdot \gamma'(0)} \quad (33)$$

as γ is drawn on G , each $\gamma(\frac{t}{n})^n$ belongs to G and it is the same for the limit (G is closed), then $e^{t \cdot \gamma'(0)} \in G$ for all $t \in [-1, 1]$. For general $t \in \mathbb{R}$, one just has to use the archimedean property that, for some $N \in \mathbb{N}_{\geq 1}$, $\frac{t}{N} \in [-1, 1]$ and remark that

$$e^{t \cdot \gamma'(0)} = (e^{\frac{t}{N} \cdot \gamma'(0)})^N.$$

2) To prove that $L(G)$ is a Lie subalgebra of \mathcal{B} it suffices to provide suitable paths. Let $u, v \in L(G)$, we have

Path	Tangent vector (at zero)
$e^{t \cdot u} e^{t \cdot v}$	$u + v$
$e^{\alpha t \cdot u}$	αu
Let $\gamma_1(t) = e^{r(t) \cdot u} e^{r(t) \cdot v} e^{-r(t) \cdot u} e^{-r(t) \cdot v}$ with $r(t) = \sqrt{2 \cdot t}$ for $t \geq 0$ $\gamma(t) = \gamma_1(t)$ for $t \geq 0$ and $= (\gamma_1(-t))^{-1}$ for $t \leq 0$	$[u, v]$

Remains to show that $L(G)$ is closed. To see this, let us consider a sequence $(u_n)_{n \in \mathbb{N}}$ in $L(G)$ which converges to u . For every fixed $t \in \mathbb{R}$, $\lim_{n \rightarrow \infty} e^{t \cdot u_n} = e^{t \cdot u}$ because $\exp: \mathcal{B} \rightarrow \mathcal{B}$ is continuous, then $t \rightarrow e^{t \cdot u}$ is drawn on G and $u \in L(G)$.

3) Set $u = \log(g)$. Now, as $u \in L(G)$, the one-parameter group $t \rightarrow e^{t \cdot u}$ is drawn on G and $g = (e^{t \cdot u})|_{t=1}$ (in the neighbourhood $\|g - e\| < 1$, we have $\exp(\log(g)) = g$). \square

Now, we have an analogue of Cartan's theorem in the context of Banach algebras

Theorem 4.2 *Let $G \subset \mathcal{B}$ be a closed subgroup of \mathcal{B}^\times and $L(G)$ as above. Let $I \subset \mathbb{R}$ be a non-void open interval and $M: I \rightarrow L(G)$ to be a continuous path drawn on $L(G)$. Let $t_0 \in I$, $g_0 \in G$. Then*

1) *The system*

$$\Sigma(t_0, M, g_0) \quad \begin{cases} \frac{d}{dt}(S(t)) = M(t).S(t) & \text{(NCDE)} \\ S(t_0) = g_0 & \text{(Init. Cond.)} \end{cases} \quad (34)$$

admits a unique solution $S : I \rightarrow \mathcal{B}$.

2) *This solution S is a path drawn on G .*

Proof. 1-2) I sketch the proof below

1) Let J be an open real interval, $t_0 \in J$ and $m \in C^0(J, L(G))$. In order to paste them together, we call “local solution” (of $\Sigma(t_0, m, g)$), a C^1 map $J \rightarrow G$ fulfilling the following system

$$\Sigma(t_0, m, g) \quad \begin{cases} \frac{d}{dt}(T(t)) = m(t).T(t) & \text{(NCDE)} \\ T(t_0) = g & \text{(Init. Cond.)} \end{cases} \quad (35)$$

2) We first prove that there exists a local solution to any system $\Sigma(0, m, e)$ for $t_0 = 0$, $g = e$. As m is continuous, there is an open interval J , containing zero, in which $|t| < 1$ and is such that $\|m_1(t)\| \leq B < 1$. In these conditions, Picard’s process

$$\begin{cases} T_0 = e \\ T_{n+1}(t) = \int_0^t m(s).T_n(s)ds \end{cases} \quad (36)$$

converges absolutely (in $C^0(J, \mathcal{B})$) to a function $T \in C^0(J, \mathcal{B})$ such that

$$T(t) = e + \int_0^t m(s).T(s)ds \quad (37)$$

this proves that, in fact, $T \in C^1(J, \mathcal{B})$. Remains to prove that T is drawn on G .

3) If t and B are sufficiently small **est1**, we have $\|T(t) - e\| < 1$ and can compute $\Omega(t) = \log(T)$, which is C^1 and, by Magnus expansion (see below **4.2**), satisfies

$$\Omega'(t) = \frac{ad_\Omega}{e^{ad_\Omega} - Id_{\mathcal{B}}}[\Omega] \quad (38)$$

where, the symbol $\frac{ad_\Omega}{e^{ad_\Omega} - Id_{\mathcal{B}}}$ denotes the substitution of ad_Ω in the series

$\sum_{n \geq 0} B_n \frac{z^n}{n!}$ (B_n being the Bernouilli numbers) **est2**. But we know that $\Omega(t)$ is

the limit of the following process

$$\begin{cases} \Omega_1(t) = \int_0^t m(s) ds \\ \Omega_n(t) = \sum_{j=1}^{n-1} \frac{B_j}{j!} \sum_{\substack{k_1+\dots+k_j=n-1 \\ k_i \geq 1}} \int_0^t ad_{\Omega_{k_1}} \cdots ad_{\Omega_{k_j}} [m(s)] ds \end{cases} \quad (39)$$

each Ω_n is drawn on $L(G)$ as shows the preceding recursion. Hence $T(t) = e^{\Omega(t)}$ is drawn on G .

- 4) We can now shift the situation in order to compute a local solution of any system $\Sigma(t_0, m, g)$ as follows (given J an open real interval, $t_0 \in J$ and $m \in C^0(J, L(G))$)
- Find a local solution R of $\Sigma(0, m_1, e)$ with $m_1(t) = m(t + t_0) - m(t_0)$. For it

$$R'(t) = (m(t + t_0) - m(t_0))R(t) ; T(0) = e \quad (40)$$

- Define $T(t) := R(t)e^{t \cdot m(t_0)}g$, one has

$$T'(t) = m(t)T(t) ; T(t_0) = g \quad (41)$$

- 5) We now return to our original system $\Sigma(t_0, M, g_0)$. By the previous item (4) we know that it admits at least a local solution (J, S) . We remark also that if we have two local solutions (J_1, S_1) , (J_2, S_2) they coincide on $J_1 \cap J_2$, thus the union of all graphs on local solutions of $\Sigma(t_0, M, g_0)$ is functional and is the maximal solution (J_{max}, S_{max}) of $\Sigma(t_0, M, g_0)$. Now, if we had $b_m = \sup(J_{max}) < \sup(I)$, we could consider the system $\Sigma(b_m, M, e)$ and a local solution T of it on some $]a, b[$ with $a < b_m < b$, now taking some intermediate point t_1 within $]a, b_m[$, we observe that $TS_{max}T^{-1}(t_1)S_{max}(t_1)$ and S_{max} coincide on $]a, b_m[$ and the union of their graphs would be a strict extension of S_{max} . A contradiction, then $b_m = \sup(I)$. A similar reasoning proves that $a_m = \inf(I)$ and then theorem 4.2 is proved. □

4.2 About Magnus expansion and Poincaré-Hausdorff formula

Formal derivation of Poincaré-Hausdorff and Magnus formulas

Let $(\mathbb{C}\{X\}, \partial)$ be the differential algebra freely generated by X (a formal variable)⁶. X be a formal variable and $(\mathbb{C}X, \partial)$ be We define a comultiplication Δ by asking that all $X^{[k]}$ be primitive note that Δ commutes with the derivation. Setting,

⁶ It is, in fact, the free algebra $\mathbb{C}\langle (X^{[k]})_{k \geq 0} \rangle$ (with $X^{[0]} = X$) endowed with $\partial(X^{[k]}) = X^{[k+1]}$, the construction is similar to what is to be found in [11], but in the noncommutative realm.

in $\widehat{\mathbb{C}\{X\}}$, $D = \partial(e^X)e^{-X}$, direct computation shows that D is primitive and hence a Lie series, which can therefore be written as a sum of Dynkin trees. On the other hand, the formula

$$D = \sum_{k \geq 1} \frac{1}{k!} \sum_{l=0}^{k-1} X^l (\partial X) X^{k-1-l} \cdot \sum_{n \geq 0} \frac{(-X)^n}{n!} \quad (42)$$

suggests that all bedegrees (in $X, \partial X$) are of the form $[n, 1]$ and thus, there exists an univariate series $\Phi(Y) = \sum_{n \geq 0} a_n Y^n$ such that $D = \Phi(ad_X)[\partial X]$. Using left and right multiplications by X (resp. noted g, d), we can rewrite (42) as

$$D = \left(\sum_{k \geq 1} \frac{1}{k!} \sum_{l=0}^{k-1} g^l d^{k-1-l} [\partial X] \right) e^{-X} \quad (43)$$

but, from the fact that g, d commute, the inner sum $\sum_{l=0}^{k-1} g^l d^{k-1-l}$ is ruled out by the the following identity (in $\mathbb{C}[Y, Z]$, but computed within $\mathbb{C}(Y, Z)$)

$$\sum_{l=0}^{k-1} Y^l Z^{k-1-l} = \frac{Y^k - Z^k}{Y - Z} = \frac{((Y - Z) + Z)^k - Z^k}{Y - Z} = \sum_{j=1}^k \binom{k}{j} (Y - Z)^j Z^{k-j} \quad (44)$$

Taking notice that $(g - d) = ad_X$ and plugging (44) into (42), one gets

$$\begin{aligned} D &= \left(\sum_{k \geq 1} \frac{1}{k!} \sum_{j=1}^k \binom{k}{j} (ad_X)^{j-1} d^{k-j} [\partial X] \right) e^{-X} = \\ &= \frac{1}{ad_X} \left(\sum_{k \geq 1} \sum_{j=1}^k \frac{1}{j!(k-j)!} (ad_X)^j d^{k-j} [\partial X] \right) e^{-X} = \frac{e^{ad_X} - 1}{ad_X} [X'] \end{aligned} \quad (45)$$

which is Poincaré-Hausdorff formula.

Application

Let G be a Lie group with Lie algebra L . Let $X(t)$ be a C^1 path drawn within L (setting as above i.e. $X(0) = 0$), then

$$(e^{X(t)})' e^{-X(t)} = \frac{e^{ad_X} - 1}{ad_X} [X'(t)] \quad (46)$$

In particular, if $S(t)$ is a solution of the system

$$\Sigma(0, M, e) \quad \begin{cases} \frac{d}{dt}(S(t)) = M(t).S(t) & \text{(NCDE)} \\ S(0) = e & \text{(Init. Cond.)} \end{cases}$$

then $\Omega = \log(S)$, at a neighbourhood of $t = 0$ ⁷ must satisfy $\Omega' = \frac{ad_\Omega}{e^{ad_\Omega} - 1}[M]$ (this identity⁸ lives in the completion of the non-commutative free differential algebra generated by the single X , constructed like in [11] but non commutative⁹). This guarantees the existence of a local solution drawn on G .

5 Appendix (between us, to be withdrawn for the paper and kept elsewhere)

Free generation

I begin here by what was taught to me by Christian Lair (below at Cartier's seminar)

<https://www.youtube.com/watch?v=zfn4Enr1bLw>

in 1992¹⁰. All further upgrades (in particular w.r.t. modern well-spread notations and concepts and possibly considering [12], which has the merit to bridge with computing sciences) are open to discussion. Let \mathcal{C}, \mathcal{D} be categories and $G: \mathcal{D} \rightarrow \mathcal{C}$ a (covariant) functor. For each $X \in \mathcal{C}$, we call "freely generated" by X , a pair (j, \hat{X}) , where $\hat{X} \in \mathcal{D}$ and $j \in \text{Hom}_{\mathcal{C}}(X, G(\hat{X}))$ with the following universal property: For all object $M \in \mathcal{D}$ and $f \in \text{Hom}_{\mathcal{C}}(X, G(M))$ it exists a unique arrow $\hat{f} \in$

⁷ Such that the norm of ad_Ω for the topology of bounded convergence be strictly less than 2π (the radius of convergence of $\phi(z) = \frac{z}{e^z - 1}$) for which, it is sufficient that $\|\Omega\| < \pi$.

⁸ The fraction

$$\frac{ad_X}{e^{ad_X} - 1}$$

means, of course, $\phi(ad_X)$ where

$$\phi(t) = \frac{t}{e^t - 1} = \sum_{n \geq 0} B_n \frac{t^n}{n!}$$

the family $(B_n)_{n \geq 0}$ being that of Bernoulli numbers.

⁹ In fact, this (highly noncommutative) differential algebra $(\mathbb{C}^{diff}\langle X \rangle, \partial)$ may be realized as the free algebra $\mathbb{C}\langle (X^{[k]})_{k \geq 0} \rangle$ (with $X^{[0]} = X$) endowed with the derivation defined by $\partial(X^{[k]}) = X^{[k+1]}$ (see e.g. [4], Ch I, §2.8 *Extension of derivations*).

¹⁰ This visit and Lair's enlightments motivated subsequently the paper [8].

$\text{Hom}_{\mathcal{D}}(\hat{X}, M)$ such that

$$f = G(\hat{f}) \circ j \quad (47)$$

this is illustrated by the diagram below (where the dashed arrows stand for “mixed” ones i.e. within some $\text{Hom}_{\mathcal{C}}(A, G(B))$ and the vertical bar is meant to be continuous **todo**).

$$\begin{array}{ccc} \mathcal{C} & & \mathcal{D} \\ & | & \\ X & \xrightarrow{f} & Y \\ & \searrow j & \uparrow \hat{f} \\ & & \hat{X} \end{array}$$

Remark 5.1 It may happen that the solution of this universal does not exist for some objects $X \in \mathcal{C}$, we will note Dom the class of objects in \mathcal{C} for which it exists. For example

- Let k be a field, \mathcal{C} be the category of sets and \mathcal{D} be that of finite dimensional vector spaces, then Dom is the subcategory of finite sets.
- Let k be a field, \mathcal{C} be the category of sets and \mathcal{D} be that finite dimensional k -AAU, then Dom consists only of the empty set.

It can be shown that, in all cases, Dom is a full subcategory of \mathcal{C} .

Lemma 5.2 The mapping $X \rightarrow [\hat{X}]$ defines a functor from Dom to $\tilde{\mathcal{D}}$, where $\tilde{\mathcal{D}}$ is the category of isomorphism classes of \mathcal{D} .

Proof. The functor is well-defined because the object \hat{X} is well-defined up to isomorphism (follows from the universal property). Now suppose $f : X \rightarrow X'$ is a morphism in Dom . Then, by the universal property, we induce a uniquely defined morphism $\hat{f} : [\hat{X}] \rightarrow [\hat{X}']$ given by the following diagram (in \mathcal{C}).

$$\begin{array}{ccc} X & \xrightarrow{j_X} & G(\hat{X}) \\ \downarrow f & & \downarrow G(\hat{f}) \\ X' & \xrightarrow{j_{X'}} & G(\hat{X}') \end{array}$$

□

We set the image of f as the morphism $[\hat{f}]$ in $\tilde{\mathcal{D}}$ (that is, the morphism \hat{f} up to composition with isomorphisms on left or right). By uniqueness of \hat{f} in \mathcal{D} , the composition law will be respected by this functor.

Below a quick list of the categories of use in combinatorics (k is a given field), morphisms are standard.

- 1) **St**, the category of sets
- 2) **Mon**, the category of monoids
- 3) **CMon**, the category of commutative monoids
- 4) **Gp**, the category of groups
- 5) **Vect_k**, the category of k -vector spaces
- 6) **Lie_k**, the category of k -Lie algebras
- 7) **AAU_k**, the category of k -Associative Algebras with Unit
- 8) **ACAU_k**, the category of k -Associative and Commutative Algebras with Unit
- 9) **Mg**, the category of Magmas i.e. sets with only a binary laws (without conditions)
- 10) **Alg_k**, the category of k -Algebras (without conditions)
- 11) **DiffAlg_k**, the category of k -Associative Differential Algebras with Unit.

All of these have a standard forgetful functor to **St**, called in a standard way $G_{1,n}$ (and, likewise, for $G_{i,j}$ when $i < j$). We describe below the corresponding free functors which are their left-adjoints and their factors which are the most useful.

- **1→2**) Here, a set X (arbitrary, called alphabet) being given, the pair $(j, \hat{X}) = (j_2, X^*)$ where X^* is the monoid of words (with concatenation and the empty word as unit) and j is the embedding $X \hookrightarrow X^*$ which send a letter to the corresponding one-letter-word hence, here, $F_{2,1}(X) = X^*$
- **1→3**) A set X (arbitrary, called alphabet) being given, the pair $(j, \hat{X}) = (j_3, \mathbb{N}^{(X)})$ where $\mathbb{N}^{(X)}$ is the set of finitely supported mappings $X \rightarrow \mathbb{N}$ endowed with addition and $j_3(x)$ is the function $y \rightarrow \delta_{x,y}$ (Kronecker delta). Note that the free commutative monoid admits a multiplicative version $\{X^\alpha\}_{\alpha \in \mathbb{N}^{(X)}}$ which is the monoid of monomials of any algebra multivariate polynomials $k[X]$. This monoid is also that of te multisets.
- **1→4**) An (arbitrary, but fixed) set X being given (called alphabet), take a disjoint copy \bar{X} of X and build an involution $X \cup \bar{X} \rightarrow X \cup \bar{X}$; $x \mapsto \bar{x}$ by the rule $\bar{\bar{x}} = x$, then the presented monoid¹¹

$$F_{4,1}(X) = \langle X \cup \bar{X} ; (x\bar{x} \sim 1_{X \cup \bar{X}}) \rangle_{\mathbf{Mon}} \quad (48)$$

with j_{41} being the composite $x \mapsto s \circ j_{21}$ where s is the quotient map

¹¹ See [12] Definition 4.1.1.19 and around.

$(X \cup \bar{X})^* \rightarrow F_{4,1}(X)$. This construction realizes the free group.

- **1→5)** The free functor is here given by $F(X) = k^{(X)}$ (the space of finitely supported functions $X \rightarrow k$) and $j_{51}(x) = (y \mapsto \delta_{xy})$ ([Exercise](#)).
- **1→6)** One of the most interesting construction. We begin by the construction of the free magma $F_{9,1}(X)$ i.e. the set $\mathcal{T} = \sqcup_{n \geq 1} \mathcal{T}_n$ (graded by the number of leaves) of binary trees with leaves in X defined by the grammar

$$\mathcal{T}_1 = X ; \mathcal{T}_n = \sqcup_{p+q=n} \text{append}(\mathcal{T}_p, \mathcal{T}_q) \tag{49}$$

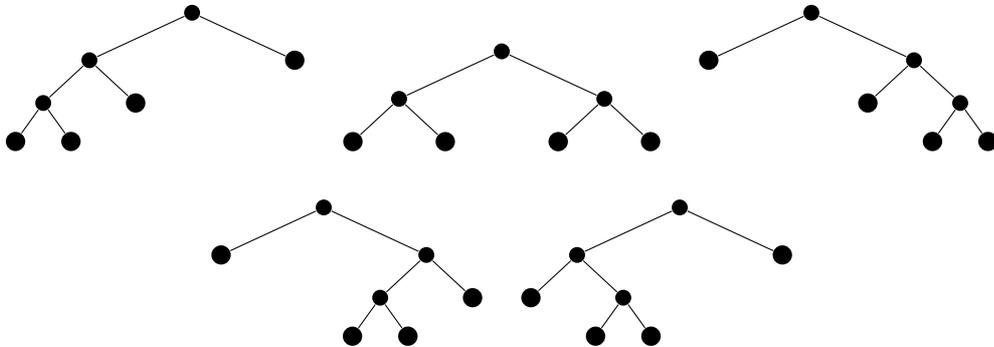
where, here, the operation $\text{append}(-, -)$ consists in appending two trees to the same root. Below another presentation of the grammar for \mathcal{T}

$$\mathcal{T} = X + \begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ \mathcal{T} \quad \mathcal{T} \end{array}$$

in other words a binary tree (with leaves in X) is either an element of X or a suspension $t = (t_1, t_2)$. Then, j_{91} is just the canonical embedding $X \hookrightarrow \mathcal{T}$. Now the (totally ¹²) free Algebra is the algebra $k[\mathcal{T}(X)]$ of the magma $\mathcal{T}(X)$, and the Free Lie Algebra is the quotient of $k[\mathcal{T}(X)]$ by the ideal generated by the elements (see [4])

$$[t_1, t_2] + [t_2, t_1] ; [t_1, [t_2, t_3]] - [[t_1, t_2], t_3] - [t_2, [t_1, t_3]] \tag{50}$$

Now, due to the transitivity of the Free Functors, the universal enveloping algebra of the Free Lie Algebra is the Free AAU. It will be the basis of the construction of the (noncommutative) Free Differential Algebra **todo tomorrow**.



¹² By abuse of language, certain authors call “ k -Free Algebra” what we will call $(X$ “Free k -AAU”

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