Applications of extreme value statistics in physics

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TOPICAL REVIEW

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Abstract. We present a descriptive review of physical problems dealing with extreme values in several fields of physics. We consider different physical situations involving random variables that are correlated or not, and study the statistics of extremal variables, which is relevant for situations where height fluctuations, catastrophic events such as material failure, or power outrage occur. We describe the general theory and relate the cumulative limit distributions that can be accessible in experiments to microscopic models. In many cases however, the random variables are correlated, in interface problems for example, and the characteristics of the interaction are revealed in the asymptotic behavior of the limit distribution.

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1. Introduction

Problems involving extreme values of a large number of random variables are important in many fields of physics of fracture, engineering and statistics of disordered systems [1, 2, 3, 4, 5, 6, 7], galaxy clusters [8], geology, environment [9, 10], meteorology [11] and economy for financial markets and stock prices [12, 13], social science with data analysis in random networks [14], and statistics of athletic records [15]. It is an important field of research in the sense it probes occurrences of catastrophic events, and how often such events can occur. A simple example is given by a chain made of individual blocks tied together by a series of random forces. We can ask what is the minimal force to apply at the two ends to break the chain into two parts. This is equivalent to find the minimal value among a set of random variables. In other cases, it is the maximum value(s) of a set of random variables which is relevant. Applications to physics of disorder are numerous [16, 4, 17, 18]. In statistical mechanics, the state of a spin system in interaction at low temperature is determined by its lowest energy, or ground state. For spin glasses, the landscape of low energy states is generally a random set of values, and the minimum energies are separated by large barriers, infinite in the thermodynamical limit [16]. The important question that arises from all these problems is how to determine the class of extreme value distributions given the distribution of the individual random variables. Theses variables can be dependent through some direct or indirect processes, and therefore their joint probability can be or not factorized. In the simplest case, when the individual probabilities are identical and independent, an asymptotic answer can be given. In other cases, the main question arises on how to obtain approximate or exact results concerning the limit distributions, and if we can make a general classification of these functions. When correlations are present, it is pertinent to look at the strength of the interactions and check if the limit distributions, in the limit of large number of random variables, change to a new class of functions. For uncorrelated variables, it is known since a long time that only three classes of distributions exist. However new results emerge when correlations between the random variables are present [4, 19, 17, 20], in the sense that new classes of limit distributions appear, and one of the challenge in this field is to try to find new classification schemes.

In this paper, we first introduce the general theoretical framework to identify the limit distributions for identically and independently distributed (iid) random variables. This can be analyzed using renormalization group theory which is an elegant way to access to the limit distributions via a flow of parameters. Application to the maximal height distribution in interface problems is then presented, for which the maximal height can be defined relatively to an origin value or to the averaged height, leading to different classes of functions. In the consecutive section, extreme values have an important application to fracture problems, which is modeled by studying how cracks propagate in a given medium. The hypothesis of the weakest link is then discussed. The next part deals with correlated random variables, which are present in different models of height interfaces, and whose largest surface fluctuations can be measured experimentally.

$p_x(x)$	$x = a_N z + b_N$	G_γ	type
1 for $0 \leq x \leq 1$	$\frac{\sqrt{N}}{\sqrt{N+2}(N+1)} z + \frac{N}{N+1}$	$\mathrm{e}^{-(1-z)}$	Weibull, $\gamma = -1$
$\kappa \mathrm{e}^{-\kappa x}$, $x \geq 0$	$\kappa^{-1}[z + \log(N)]$	$\exp(-\mathrm{e}^{-z})$	Gumbel, $\gamma = 0$
$\frac{1}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2\sigma^2}$	$\sigma[\frac{1}{\sqrt{2\log(N)}} z + \sqrt{2\log(N)} - \frac{\log(4\pi N)}{2\sqrt{2\log(N)}}]$	$\exp(-\mathrm{e}^{-z})$	Gumbel, $\gamma = 0$
$\frac{\alpha}{1+\alpha} x^{-\alpha}$, $\alpha \geq 1$	$[N/(1+\alpha)]^{1/\alpha}(1 + z/\alpha)$	$\mathrm{e}^{-(1+z/\alpha)^{-\alpha}}$	Fréchet, $\gamma = 1/\alpha$

Table 1. Limit distributions for different simple examples of iid variables. In the last case, we take $p_x(x) = \frac{\alpha}{1+\alpha}$ when $0 \leq x \leq 1$ and $p_x(x) = \frac{\alpha}{1+\alpha} x^{-\alpha}$ when $x \geq 1$. In each example, we give the value of the γ parameter, see Eq. (2).

General ideas for obtaining the limit distribution in interacting problems are finally presented, based on scaling and saddle point method.

2. Generalities

In this section we present the main mathematical results concerning the extreme value statistics (EVS) for a set of iid random variables $\{x_1, \cdots, x_N\}$ when N is large, with individual probability (or parent distribution) $p_x(x)$ and cumulative distribution $F(x) = \int_{-\infty}^{x} p_x(u)\mathrm{d}u$. The main quantity associated to the EVS is the cumulative distribution of the maximum (and in the same manner for the minimum) value $\max(\{x_i\})$

$$\mathrm{Prob}(x \geq \max(\{x_i\})) = F(x)^N. \tag{1}$$

In particular, we want the distribution to be independent of N in the asymptotic limit. This can be achieved by noticing that F is monotonically increasing and that $F(x) \leq 1$. The main contribution to the previous expression is for values of x for which $F(x) \simeq 1$. One may then try to find two sequences of real numbers $a_N > 0$ and b_N such that the rescaling $x = a_N z + b_N$ makes the limit series $F(a_N z + b_N)^N$ converge to a finite function. It is known [21, 22, 23, 24, 25] that the resulting distribution belongs to one of the three classes of functions, which can be written in a compact form as

$$\lim_{N\to\infty} F(a_N z + b_N)^N = \left\{ \begin{array}{c} G_\gamma(z) = \exp\left(-(1+\gamma z)^{-1/\gamma}\right),\ \gamma \neq 0,\ (1+\gamma x) \geq 0 \\ \gamma < 0 \text{ Weibull}, \gamma > 0 \text{ Fréchet} \\ G_0(z) = \exp(-\exp(-z)),\ \gamma = 0 \text{ Fisher-Tippett-Gumbel} \end{array} \right. \tag{2}$$

Depending on the value of the general parameter γ, we can identify the three standard functions resulting from the EVS: the Weibull ($\gamma < 0$), Fréchet ($\gamma > 0$), and Fisher-Tippett-Gumbel ($\gamma \to 0$) forms respectively, with $G_0(z) = \exp(-\exp(-z))$ in the latter case. For most of physical problems, parameters a_N and b_N can be replaced by the standard deviation and average value of the maximum $\langle\max(\{x_i\})\rangle$ respectively. A simple example is given by a set of random variables with a uniform distribution between 0 and 1: $p_x(x) = 1$ for $x \in [0, 1]$. Then $F(x)^N = x^N$. The distribution of the maximum is $\mathrm{d}F(x)^N/\mathrm{d}x = Nx^{N-1}$, from which we can compute the average value $\langle\max(\{x_i\})\rangle = \int_0^1 \mathrm{d}u N u^N = N/(N+1) = b_N$ and $\langle\max(\{x_i\})^2\rangle = \int_0^1 \mathrm{d}u N u^{N+1} =$

$N/(N+2)$. Then the variance $a_N^2 = N/(N+1)^2(N+2)$ behaves like N^{-2} and the expectation value $b_N = \langle \max(\{x_i\}) \rangle \simeq 1 - N^{-1}$. Replacing these two parameters into the scaling function $F(a_n z + b_n)^N = (1 + (z-1)/N)^N$ and taking the limit $N \to \infty$, one obtains the asymptotic and finite distribution $G_{-1}(z) = \exp(-(1-z))$, which is exactly associated with a Weibull distribution of parameter $\gamma = -1$ given above Eq. (2). We have summarized in Table 1 different limit distributions for simple cases. In general, to find the coefficients a_N and b_N, it is easier to consider the asymptotic expansion of $F(x)$ for x large, since $F(x)^N$ is dominated by this limit $F(x) \to 1$. In the second example of Table 1, one can look for solutions $\exp(-\kappa x) = \exp(-z)/N$ with z independent of N, such that $(1 - N^{-1}\exp(-z))^N \to \exp(-\exp(-z))$. For the Gaussian case, the procedure is the same although it requires more algebra.

2.1. Renormalization group applied to EVS

The basic idea to apply the renormalization group (RG) to EVS [26, 27, 28] is to view the limit distributions as fixed points of the renormalization flow in a coarse-graining process, based on a decimation method and applied to a system of N iid variables $\{x_i\}_{i=1,\cdots,N}$. This set is first divided into p blocks of N/p variables. In each of these blocks $i = 1, \cdots, p$, we select the maximum value $y_i = \max(\{x_{p(i-1)+j}, j = 1, \cdots, p\})$. The new set $\{y_i\}$ is composed of iid random variables, with individual distribution given by $F(x)^p$, since they are the maximum value of a set of p variables. The continuous version, for p non integer and close to unity, can be obtained by incorporating the variation of parameters a_N and b_N defined in the previous section with the scaling $x = a_N z + b_N$. In that case, the transformation is given by the mapping

$$F(z) \to F(a_p z + b_p)^p = F(z, p) \tag{3}$$

at each iteration of the renormalization process with scaling factor p. Since F is bounded between 0 and unity, it is convenient to rewrite it as a generic form

$$F(z, s) = \exp(-\exp(-f(z, s))),\ s = \log(p), \tag{4}$$

with $f(z, s)$ a real function that can take any value. In order to work with differential equations, we will choose p close to unity and parametrize the flow with the parameter $s = \log(p)$. Initial conditions are given by $F(z, 0) = F(z)$ or $f(z, 0) = f(z)$, the initial distribution for the iid variables. The functions $a(s)$ and $b(s)$ are unknown and determined uniquely by additional conditions. One possibility is to fix the values of $F(z, s)$ and its derivative at the origin $z = 0$, for example $F(0, s) = \partial_z F(0, s) = \exp(-1)$ at each step of the iterations s, which gives two additional conditions $f(0, s) = 0$ and $\partial_z f(0, s) = 1$ that are sufficient to evaluate the functions $a(s)$ and $b(s)$ [28]. Then, the RG equation Eq. (3) is identical to the functional equation

$$f(z, s) = f(a(s)z + b(s)) - s. \tag{5}$$

Using the additional conditions at $z = 0$, one obtains

$$a(s)f'(s) = 1,\ b(s) = f^{-1}(s). \tag{6}$$

We can deduce from these two equalities a relation between $a(s)$ and $b(s)$: $a(s) = b'(s)$. By differentiating Eq. (5) with respect to s, the differential equation of the flow is equal to

$$\partial_s f(z,s) = (a'(s)z + b'(s))f'(a(s)z + b(s)) - 1.$$

In particular, for $z = 0$, the condition $\partial_s f(0,s) = a(s)f'(b(s)) - 1$ holds. The previous equation can be replaced by a partial differential equation, after eliminating the f' function on the right hand side by considering the partial derivative with respect to z, and thus eliminating the dependency on the initial probability function

$$\partial_s f(z,s) = \left(1 + \frac{a'(s)}{a(s)} z\right) \partial_z f(z,s) - 1. \tag{7}$$

The ratio $\gamma(s) = a'(s)/a(s)$ has a physical meaning. If we differentiate the previous flow equation with respect with z,

$$\partial^2_{zs} f(z,s) = \gamma(s)\partial_z f(z,s) + (1 + \gamma(s)z)\, \partial^2_{zz} f(z,s),$$

and take $z = 0$, one obtains $\partial^2_{zs} f(0,s) = \gamma(s)\partial_z f(0,s) + \partial^2_{zz} f(0,s)$. However, $\partial_z f(0,s) = 1$ due to the condition imposed above and therefore is independent of s, so that $\partial^2_{zs} f(0,s) = \partial^2_{sz} f(0,s) = 0$. Finally, we find that the coefficient $\gamma(s)$ is directly interpreted as the curvature of function $-f$ at the origin

$$\gamma(s) = -\partial^2_{zz} f(0,s). \tag{8}$$

The flow equation Eq. (7) is a closed equation, since the coefficient $a(s)$, and subsequently $\gamma(s)$, depends on the initial distribution $f(z,0) = f(z)$ through a differential equation: $f(b(s)) = s$ and $a(s) = b'(s)$. A practical way to compute $\gamma(s)$ is to write directly its dependency with the original cumulative distribution $F(x)$

$$\gamma(s) = \frac{\partial}{\partial s} \log\left(\frac{\partial}{\partial s} f^{-1}(s)\right),\ f^{-1}(s) = F^{-1}(\exp(-\mathrm{e}^{-s})). \tag{9}$$

The fixed point or stationary solutions of Eq. (7) are given, in the limit of large s, by the function $f(z,s) = g(z)$ independent of s. In that case, it is necessary that $\gamma(s)$ tends to a constant γ in the same limit. One therefore has to solve the differential equation

$$(1 + \gamma z)g'(z) = 1, \tag{10}$$

with initial condition $g(0) = 0$. The unique solution is then given by $g(z) = \frac{1}{\gamma}\log(1+\gamma z)$. This leads to the general set of limit distributions Eq. (2) which depends only on the parameter $\gamma = \lim_{s\to\infty}\gamma(s)$. We can take the second example of Table 1, $F(z) = 1 - \exp(-\kappa z) = \exp(-\mathrm{e}^{-f(z)})$, and compute the factor $\gamma(s)$ using Eq. (9) which is independent of κ

$$\gamma(s) = \frac{1 - \exp(-\mathrm{e}^{-s}) + \exp(-s + \mathrm{e}^{-s})}{\exp(-\mathrm{e}^{-s}) - 1}. \tag{11}$$

In the large s limit, one has the expansion $\gamma(s) \simeq \frac{1}{2}\mathrm{e}^{-s} \to 0$. This is consistent with the asymptotic value $\gamma = 0$, and therefore the limit distribution is the Gumbel distribution. Another simple example is given by the uniform distribution $p_x(x) = 1$ in the interval

$x \in [0,1]$. In that case $f(z) = -\log(-\log(z))$, $b(s) = \exp(-\mathrm{e}^{-s})$ and one obtains after some algebra

$$\gamma(s) = \frac{1 - 3\mathrm{e}^{-s} + \mathrm{e}^{-2s}}{1 - \mathrm{e}^{-s}}. \tag{12}$$

The asymptotic value is $\gamma = -1$, and the stationary distribution is an exponential as found above by guessing the correct scaling parameters. We have seen in this section the RG method for the statistics of extreme values for iid variables. It gives a set of equations for parameter γ that converges to one of the three general distributions Eq. (2). We can mention the important fact that the RG method can also be applied to correlated cases in constrained Brownian motions for example [29]. The authors developed in particular an analysis based on the method that computes the extrema distribution of Bessel processes. In the following, we study the cases where the maximum is taken relatively to an initial point or from a average value. This can be the case experimentally where a set of statistical values, such as interface heights, is recorded relatively to a given reference which can be a random variable itself. It is indeed important to know if the class of universal distributions Eq. (2) is stable or not, which is the subject of the next section.

2.2. Relative maximum height distribution

The influence of sensitivity of the maximum value distribution on the initial value has been discussed in several works, for problems involving a measure with respect to a reference point. For example, the maximum height of an interface can be measured with respect to an initial level [30], or to the average value [31, 32, 33], which is itself a fluctuating variable. This happens experimentally when a time series $h(t)$ is recorded during an interval T, from which one can extract time correlation functions. In the discrete version, one considers the set of N iid variables $\{h_i\}_{i=1,\cdots,N}$, which are measured at every time step τ, with the substitution $t = i\tau$ and $T = N\tau$. Therefore in the continuous limit one has $h_i \to h(t)$. Interfaces can also be described by a similar set of random variables that describe the height as function of the spatial position $h_{\mathbf{r}}$, such as h_i is the discretized version, which can depend on time as well $h_i \to h_i(t)$. For finite system one writes $L = Na$ where a is the elementary step and L the total length. Statistical distribution of the heights is of interest in several models, such as the Edwards-Wilkinson (EW) [34, 35] and Kardar-Parisi-Zhang (KPZ) [36] models for example. The maximum value $h_{max} = \max(\{h_i\})$ can be compared relatively to the initial or average value for example, and one can consider respectively $\mathrm{Prob}(h \geq \max(\{h_i - h_1\}))$ or $\mathrm{Prob}(h \geq \max(\{h_i - \overline{h}_N\}))$, where $\overline{h}_N = \sum_i h_i/N$ is the spatial average height for a given configuration of N variables (for time series one has $\overline{h}_T = \int_0^T \mathrm{d}\tau h(\tau)/T$ in the continuum limit during a time window T). If fluctuations of $\overline{h}_N$ are large, in the particular case where $\overline{h}_N$ does not converge as $N \to \infty$ for example, they have a direct influence on the limit distribution of the maximum. The problem appears when the roughness (or width) w_N is commonly studied for determining the critical exponents of an interface ‡, and therefore its universality class, namely

$$w_N^2 = \langle \overline{(h_i - \overline{h}_N)^2} \rangle_N. \tag{13}$$

Here the brackets $\langle \cdots \rangle_N$ denote the average over all the possible configurations for N variables, and we assume translational invariance so that the influence of boundary effects is negligible. The maximum value $\max(\{h_i\}) - \overline{h}_N$ can also be considered and its fluctuations as well

$$\Delta_N^2 = \langle \overline{(\max(\{h_i\}) - \overline{h}_N)^2} \rangle_N \neq w_N^2. \tag{14}$$

2.2.1. Maximum height relative to an initial value The first case is relevant in hydrology [37, 38], for determining the fluctuations of water levels for which extrema are associated to flooding, relatively to a reference point, study of annual or seasonal peak flows, wind or fluid velocity [39, 40, 41]. This can be translated into a problem of height statistics, for which we can generally write the following equation

$$\begin{aligned}\mathrm{Prob}(h \geq \max(\{h_i - h_1\})) &= \theta(h) \int_{-\infty}^{+\infty} \mathrm{d}h_1 p_h(h_1) \int_{-\infty}^{h+h_1} \mathrm{d}h_2 p_h(h_2) \cdots \int_{-\infty}^{h+h_1} \mathrm{d}h_N p_h(h_N) \\ &= \theta(h) \int_{-\infty}^{+\infty} \mathrm{d}h_1 p_h(h_1) F(h+h_1)^{N-1}, \end{aligned} \tag{15}$$

for a given distribution p_h, where $\theta(h) = 1$ if $h > 0$ and 0 otherwise. Introducing the limit cumulative distribution function G such that $F^N(a_N z + b_N) \to G(z)$, with a_N and b_N the scaling parameters, one obtains

$$\mathrm{Prob}(h \geq \max(\{h_i - h_1\})) \simeq \theta(h) \int_{-\infty}^{+\infty} \mathrm{d}h_1 p_h(h_1) G(a_N^{-1}(h + h_1 - b_N)). \tag{16}$$

We can isolate three cases, depending on the limit of a_N when N is large. The results are [30]

$$\mathrm{Prob}(h \geq \max(\{h_i - h_1\})) \simeq \left\{ \begin{array}{cl} \int_0^\infty \mathrm{d}u p_h(u + b_N - h), & a_N \to 0, \\ \int_{-\infty}^\infty \mathrm{d}z a_N p_h(a_N z + b_N - h) G(z), & a_N \to \text{constant}, \\ G(a_N^{-1}(h - b_N)), & a_N \to \infty. \end{array} \right. \tag{17}$$

The probability density can then be deduced by differentiation with respect to h. In the intermediate case, the probability depends on a convolution between the limit function G and parent distribution p_h. As in the previous section, we can write $F(z) = \exp(-\exp(-f(z)))$ and $G(z) = \exp(-\exp(-g(z)))$, where functions f and g are related by the scaling limit, equivalent to Eq. (5)

$$g(z) = \lim_{N\to\infty} f(a_N z + b_N) - \log(N). \tag{18}$$

This equation comes indeed from a direct identification between variables s and $\log(N)$ in the renormalization group flow when N is increasing. Also, additional conditions

‡ For interface problems, the roughness w_N^2 depends on time $w_N^2 \to w_N(t)^2$ if the heights $h_i(t)$ are measured at a given time t.

$p_h(x)$	a_N	b_N	G	a_N
$x^\delta \mathrm{e}^{-(x/\xi)^\alpha}$	$\xi\alpha^{-1}(\log(N))^{1/(\alpha-1)}$	$\xi(\log(N))^{1/\alpha}$	Gumbel	$a_N \to 0$, constant, ∞
$(x_c-x)^{-1+\alpha}$	$\alpha^{-1}N^{-1/\alpha}$	$x_c - N^{-1/\alpha}$	Weibull	$a_N \to 0$
$x^{-1-\alpha}$	$\alpha^{-1}N^{1/\alpha}$	$N^{1/\alpha}$	Fréchet	$a_N \to \infty$

Table 2. Limit distributions for the maximum measured relatively to an origin point, and for several examples of iid distributions. The asymptotic behavior of p_h is given in the first column. In the second example, p_h vanishes when $x > x_c$, while $\alpha > 0$ otherwise.

previously fixed at $z = 0$, such that $F(b_N)^N = \partial_z F(a_N + b_N)^N|_{z=0} = \mathrm{e}^{-1}$, are equivalent to the equations

$$f(b_N) = \log(N),\ f'(b_N) = a_N^{-1}. \tag{19}$$

We easily deduce from these two equalities the relation between a_N and b_N of the previous section: $a_N = \partial b_N/\partial\log(N)$. Now we can discuss the limit behavior of parameters a_N and b_N for the three conditions Eq. (17). As an application, let us consider the general parent distribution $p_h(x) = x^\beta \exp(-(x/\xi)^\alpha)$, with α and β real. Since function $f(x)$ is diverging asymptotically as $F(x) \to 1$, it is easy to obtain the asymptotic of f, using

$$p_h(x) \simeq f'(x)\exp(-f(x)). \tag{20}$$

Then in this example, the dominant behavior is given by $f(x) \simeq (x/\xi)^\alpha$ plus some additional corrections. Solving $f(b_N) = \log(N)$ when N is large leads to $b_N = \xi(\log(N))^{1/\alpha}$ and $a_N = \xi\alpha^{-1}(\log N)^{1/\delta-1}$. There are therefore three possibilities for determining the behavior of $\mathrm{Prob}(h \geq \max(\{h_i - h_1\}))$ depending on the value of α: if $\alpha > 1$ then $a_N \to 0$, or if $\alpha = 1$ then $a_N \to$ constant, and if $\alpha < 1$ then $a_N \to \infty$. Two other examples are given in Table 2 by the asymptotic value of p_h, where G belongs to the Weibull and Fréchet class of limit distributions.

2.2.2. Maximum height relative to an average value Unlike the previous case, one can consider to study the maximal height relatively to the average value. This can be found in studies of interface roughness in general [42]. Usually the width as defined in Eq. (13) gives the amplitudes of fluctuations of the relative height $h_i(t) - \overline{h}_N$, which is a fluctuating variable. The problem is simplified by considering the relative heights $u_i = h_i - \overline{h}_N$ satisfying the constraint $\sum_i u_i = 0$. This can be useful in studying correlated relative heights for interface problems [31, 32]. For example, one can introduce a Boltzmann weight corresponding to a Gaussian interaction between adjacent sites on the surface, such as in the EW model, and which softens the roughness, see section below. In the case of iid variables however, the cumulative distribution is written as follow

$$\mathrm{Prob}(h \geq \max(\{h_i - \overline{h}_N\}) \geq 0) = \prod_{i=1}^{N} \int_{-\infty}^{+\infty} \mathrm{d}h_i p_h(h_i)\theta(h - h_i + \overline{h}_N).$$

In this expression, the different integrals can not be factorized since the theta functions mixes the different variables h_i together because of the term $\overline{h}_N$. It is however useful to introduce the characteristic function, or Fourier transform, defined by $\tilde{p}_h(k) = \int \mathrm{d}h p_h(h)\exp(-ikh)$, such that

$$\mathrm{Prob}(h \geq \max(\{h_i - \overline{h}_N\})) =$$
$$\prod_{i=1}^{N}\int_{-\infty}^{+\infty}\frac{\mathrm{d}k_i}{2\pi}\tilde{p}_h(k_i)\prod_{i=1}^{N}\int_{-\infty}^{+\infty}\mathrm{d}h_i\exp\left(ik_ih_i\right)\prod_{i=1}^{N}\theta(h-h_i+\overline{h}_N). \tag{21}$$

Then the change of variable $h_i \to u_i$ leads to a factorization of the right part of the previous integral

$$\prod_{i=1}^{N}\int_{-\infty}^{+\infty}\mathrm{d}h_i\exp\left(ik_ih_i\right)\prod_{i=1}^{N}\theta(h-h_i+\overline{h}_N)$$
$$=\int\frac{\mathrm{d}\lambda}{2\pi}\int\mathrm{d}g\prod_{i=1}^{N}\int_{-\infty}^{+\infty}\mathrm{d}h_i\exp\left(ik_ih_i\right)\exp\left[i\lambda(g-\overline{h}_N)\right]\prod_{i=1}^{N}\theta(h-h_i+g)$$
$$=\int\frac{\mathrm{d}\lambda}{2\pi}\int\mathrm{d}g\prod_{i=1}^{N}\int_{-\infty}^{+\infty}\mathrm{d}u_i\exp\left[igk_i+iu_i\left(k_i-\frac{\lambda}{N}\right)\right]\theta(h-u_i). \tag{22}$$

We can then perform the integration over variables u_i, and inverse Fourier transform the product of integrals over the k_is. One finally obtains an integral expression for the cumulative distribution

$$\mathrm{Prob}(h \geq \max(\{h_i - \overline{h}_N\})) = \int\frac{\mathrm{d}\lambda}{2\pi}\int\mathrm{d}g\exp(ig\lambda)\left[\int_{-\infty}^{g+h}\mathrm{d}u\exp\left(-iu\frac{\lambda}{N}\right)p_h(u)\right]^N \tag{23}$$

A precise evaluation of the integral has to be made in each case. Indeed if the average value or variance have large fluctuations, the limit distribution may not follow the general theorem Eq. (2) since the above formula combines the difference of two random variables with similar behavior. If instead the fluctuations of the average value are not important, we may expect a smooth behavior of the limit distribution. For example, we can consider two typical cases where the probability distribution decays exponentially or with an algebraic power and for which an exact result can be obtained. In the first case, we consider a set of iid random variables distributed according to $p_h(u) = \alpha\exp(\alpha u)\theta(-u)$ and for which a finite average value exists. One obtains

$$\mathrm{Prob}(h \geq \max(\{h_i - \overline{h}_N\})) = \int\frac{\mathrm{d}\lambda}{2\pi}\int\mathrm{d}g\exp\left[ig\lambda - N\log\left(1-\frac{i\lambda}{N\alpha}\right)\right.$$
$$\left.+(N\alpha - i\lambda)\min(0,g+h)\right]. \tag{24}$$

We then expand the logarithm term inside the exponential argument up to second order in λ/N and perform Gaussian integration. One obtains finally a scaling expression in $z = (h-\alpha^{-1})\alpha\sqrt{N}$ such that the cumulative distribution has the limit form

$$\mathrm{Prob}(h \geq \max(\{h_i - \overline{h}_N\})) \simeq \frac{1}{2}+\frac{1}{2}\mathrm{erf}(z/\sqrt{2})+\frac{1}{\sqrt{2\pi N}}\exp(-z^2/2), \tag{25}$$

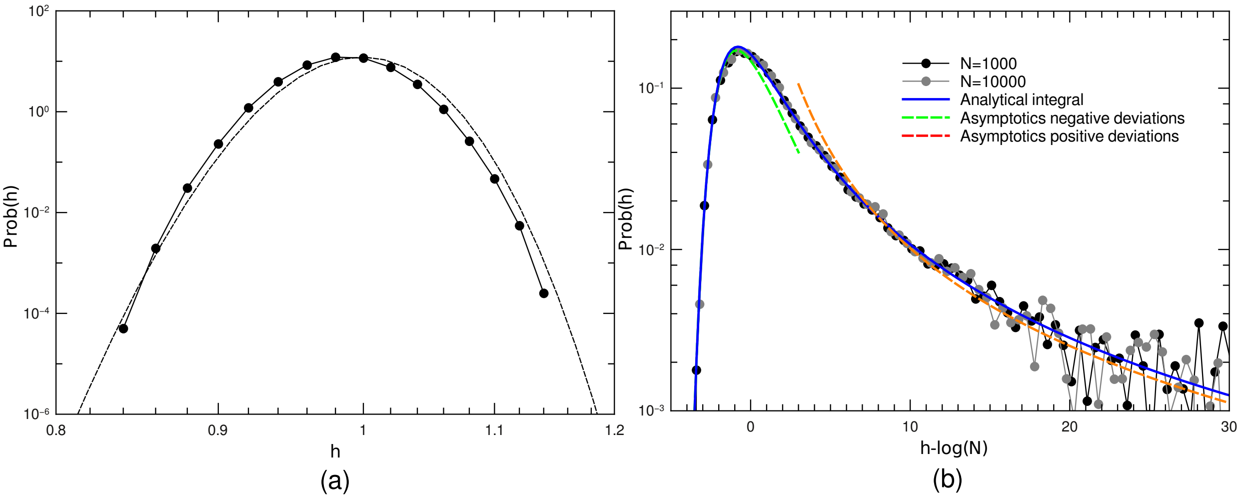

Figure 1. (a) Distribution of $h = \max(\{h_i - \overline{h}_N\})$ for an exponential law $p_h(u) = \alpha \exp(u)\theta(-\alpha u)$, $\alpha = 1$ and for a size $N = 10^3$. Dashed line is the exact result Eq. (25). (b) Distribution for a power law $p_h(u) = (1-u)^{-2}\theta(-u)$ and for two different sizes $N = 10^3$ and $N = 10^4$. Blue line is the integral Eq. (28). Green and red dashed lines are the asymptotics results (see text).

which finally leads to a Gaussian distribution after deriving with respect to z, as shown on Fig. 1(a). This is to be compare to the limit distribution for the maximum value only $\mathrm{Prob}(h \geq \max(\{h_i\}))$, which is given by the exponential law Eq. (2) with $\gamma = -1$ and scaling parameter $z = \alpha N h$ instead. More interesting cases are given by a distribution with a power law tail for which the mean value of N random variables has large fluctuations. One solvable situation is given by the individual distribution $p_h(u) = (1-u)^{-2}\theta(-u)$, for which we would like to define the limit of Eq. (23) when N is large. The corresponding expression for the cumulative distribution is given by

$$\mathrm{Prob}(h \geq \max(\{h_i - \overline{h}_N\})) = \int \frac{\mathrm{d}\lambda}{2\pi} \int \mathrm{d}g \exp(i\lambda(g-1)) \left[\int_{1-\min(g+h,0)}^{\infty} \frac{\mathrm{d}u}{u^2} \mathrm{e}^{iu\lambda/N} \right]^N . (26)$$

In this expression we can expand the integral inside the brackets

$$\int_a^{\infty} \frac{\mathrm{d}u}{u^2} \exp(iu\lambda/N) \simeq a^{-1} - \frac{i\lambda}{N} \left[\gamma - 1 - \log(N) + \log(-i\lambda a) \right],$$

where $a = 1 - \min(g+h, 0)$. Replacing this expansion inside Eq. (26), one obtains

$$\mathrm{Prob}(h \geq \max(\{h_i - \overline{h}_N\})) = \int \frac{\mathrm{d}\lambda}{2\pi} \int \mathrm{d}g \frac{\exp(i\lambda(g-1))}{a^N} \times \\ \exp(-i\lambda a[\gamma - 1 - \log(N) + \log(-i\lambda a)]). \qquad (27)$$

In the large N limit, it is easy to see that a^N goes to infinity when $g + h < 0$. Then the integral is non vanishing only for the values of $g \geq -h$ or $a = 1$

$$\mathrm{Prob}(h \geq \max(\{h_i - \overline{h}_N\})) = \frac{1}{2} + i \int \frac{\mathrm{d}\lambda}{2\pi} P\left(\frac{1}{\lambda}\right) \exp(-i\lambda[h + \gamma - \log(N) + \log(-i\lambda)]).$$

Here P is the principal value. It is easier to express the distribution itself by deriving with respect to h and to consider the semi-infinite positive interval of integration to obtain a real expression

$$\text{Prob}(h = \max(\{h_i - \overline{h}_N\})) = \int_0^\infty \frac{\mathrm{d}x}{\pi} \cos(x[h + \gamma - \log(N) + \log(x)]) \exp(-\pi x/2). \quad (28)$$

The resulting integral is then function of the scaling parameter $z = h + \gamma - \log(N)$ for which the distribution is independent of N and has a finite limit. In Fig. 1(b) we have performed numerical simulations for two different sizes and the results collapse to the previous distribution. Interestingly the asymptotic behavior is of very different nature for negative of positive deviations, $z \ll -1$ and $z \gg 1$ respectively. One obtains in particular the following results for each case

$$\text{Prob}(h = \max(\{h_i - \overline{h}_N\})) \simeq \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}(1+z) - \exp(-1-z)\right],\ z \ll -1 \quad (29)$$

and

$$\text{Prob}(h = \max(\{h_i - \overline{h}_N\})) \simeq \frac{1}{z^2} + \frac{\gamma - 3/2}{z^3},\ z \gg 1. \quad (30)$$

The distribution falls with a double exponential for negative deviations, and has a tails in $1/z^2$ for positive ones, similar to the original distribution p_h for the individual variables. We can generalize the previous case to the class of distributions $p_h(u) = \alpha(1-u)^{-1-\alpha}\theta(-u)$ where $\alpha > 1$, and which do not have a finite variance when $\alpha \leq 2$ §. In particular we have the expansion

$$\int_a^\infty \frac{\alpha \mathrm{d}u}{u^{1+\alpha}} \exp(iu\lambda/N) \simeq a^{-\alpha} + \frac{i\lambda}{N}\frac{\alpha a^{1-\alpha}}{\alpha-1} - \left(\frac{i\lambda}{N}\right)^\alpha \frac{\pi\alpha}{\sin(\pi\alpha)\Gamma(1+\alpha)} - \frac{\lambda^2}{2N^2}\frac{\alpha a^{2-\alpha}}{\alpha-2}. \quad (31)$$

As before the dominant contributions comes from $a = 1$, otherwise $a^{-\alpha N}$ vanishes as N goes to infinity. Also for $1 < \alpha < 2$ the term proportional to $N^{-\alpha}$ is dominant compare to N^{-2} and the third term can be ignored. In this case one has

$$\text{Prob}(h = \max(\{h_i - \overline{h}_N\})) = \int \frac{\mathrm{d}\lambda}{2\pi} \exp\left[-i\lambda\Big(h - \frac{1}{\alpha-1}\Big) + c_\alpha(-i\lambda)^\alpha N^{1-\alpha}\right],$$

with $c_\alpha = -\pi\alpha/\sin(\pi\alpha)\Gamma(1+\alpha) > 0$. A natural scaling is given by the change of variable $z = c_\alpha^{-1/\alpha} N^{(\alpha-1)/\alpha}\Big(h - \frac{1}{\alpha-1}\Big)$. This leads to the following scaling expression for the distribution for $1 < \alpha < 2$

$$\text{Prob}(h) = c_\alpha^{-1/\alpha} N^{(\alpha-1)/\alpha} \int_0^\infty \frac{\mathrm{d}x}{\pi} \cos[xz + \sin(\pi\alpha/2)x^\alpha] \exp[\cos(\pi\alpha/2)x^\alpha]. \quad (32)$$

This integral is directly related to the Laplace representation of Kohlrausch stretched exponentials in term of density function [43] $\rho(z,\alpha)$, such that $\exp(-u^\alpha) = \int_0^\infty \mathrm{d}z\rho(z,\alpha)\exp(-zu)$, in particular it is similar to the Pollard form for exponents in the range $0 < \alpha \leq 1$. It appears to be a stable Lévy stable distribution [44] and has exact expressions for few rational values of α. For $z \gg 1$, the distribution falls off like the power law $1/z^{1+\alpha}$, which is consistent with the behavior at $\alpha = 1$. This density

§ The case $0 < \alpha \leq 1$ for which the average value $\int \mathrm{d}u u p_h(u)$ is infinite can be studied as well following the same method

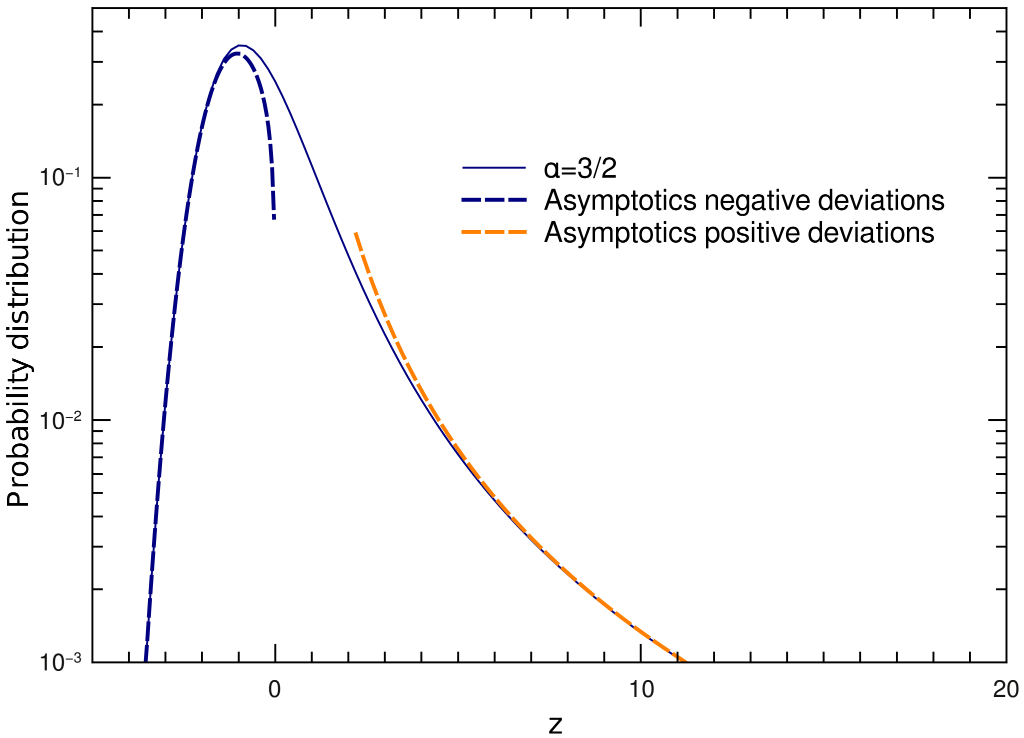

Figure 2. Rescaled distribution of $h = \max(\{h_i - \overline{h}_N\})$ for a power law $p_h(u) = \alpha(1-u)^{-\alpha-1}\theta(-u)$ when $\alpha = 3/2$ using the integral Eq. (32) as function of $z = c_\alpha^{-1/\alpha} N^{(\alpha-1)/\alpha}[h - (\alpha-1)^{-1}]$ (see text for constant c_α), in the limit where N is infinite. Blue dashed line is the asymptotic result Eq. (35) for negative deviations, and the orange dashed line is the series expansion Eq. (34) for positive values.

function for the stretched exponential, or Laplace inverse transform, has an asymptotic expansion when $0 < \alpha \leq 1$ [43] which is given by

$$\rho(z,\alpha) = -\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\sin(\pi\alpha k)\frac{\Gamma(1+\alpha k)}{z^{1+\alpha k}}. \tag{33}$$

For $\alpha > 1$ this expansion is obviously no more valid. However the first term gives a correct asymptotic behavior for $1 < \alpha < 2$, as seen on Fig. 2

$$\mathrm{Prob}(h = \max(\{h_i - \overline{h}_N\})) \simeq -c_\alpha^{-1/\alpha} N^{(\alpha-1)/\alpha}\frac{\sin(\pi\alpha)\Gamma(1+\alpha)}{\pi z^{1+\alpha}}. \tag{34}$$

The asymptotic analysis for $z \ll -1$ leads to a different behavior of the distribution at the unique saddle point, and the distribution behaves like

$$\mathrm{Prob}(h = \max(\{h_i - \overline{h}_N\})) \simeq \frac{c_\alpha^{-1/\alpha} N^{(\alpha-1)/\alpha}(-z)^{\frac{\alpha-2}{2(\alpha-1)}}}{\sqrt{2\pi\alpha^{1/(\alpha-1)}(\alpha-1)}} \exp\left(-\frac{\alpha-1}{\alpha^{\alpha/(\alpha-1)}}(-z)^{\alpha/(\alpha-1)}\right) \tag{35}$$

In Fig. 2 is plotted the resulting integral for the value $\alpha = 3/2$, as well as the asymptotic results Eq. (35) and Eq. (34). When α is close to 2, the term $N^{-\alpha}$ in Eq. (31) appears to be comparable to the last term $N^{-\alpha} = N^{-2}$ and one obtains, after expanding around $\alpha = 2$ and some algebra, a Gaussian distribution for the rescaled variable

$z = \sqrt{N/2\log(N)}(h-1)$

$$\mathrm{Prob}(h = \max(\{h_i - \overline{h}_N\})) = \sqrt{\frac{N}{4\pi\log(N)}}\exp\left(-\frac{1}{4}\frac{N(h-1)^2}{\log(N)}\right). \quad (36)$$

Finally, for $\alpha > 2$, the last term in Eq. (31) is more dominant than the term in $N^{-\alpha}$, and the distribution is Gaussian with a scaling variable $z = \sqrt{(\alpha-2)N/\alpha}\Big(h - \frac{1}{\alpha-1}\Big)$

$$\mathrm{Prob}(h = \max(\{h_i - \overline{h}_N\})) = \sqrt{\frac{2\pi(\alpha-2)N}{\alpha}}\exp\left[-\frac{(\alpha-2)N}{2\alpha}\left(h - \frac{1}{\alpha-1}\right)^2\right]. \quad (37)$$

This result is not surprising since for large values of $\alpha > 2$, the mean and variance are finite, and we recover the previous result given by Eq. (25) for the exponential law for which these quantities are also finite.

2.3. Fracture and breakdown problems

The importance of EVS in problems involving fractures is enlightened by the weakest link hypothesis [1, 2, 45]. Given a system made of subvolumes attached with different coupling values, the failure occurs when a strain applied to the whole system breaks the weakest link [46, 47, 48]. The distribution of local couplings can take different form for small values, decreasing as a power law or as an exponential. In the former case, it can be shown that the strength failure distribution has a Weibull form, and Gumbel like in the latter case [49], which can be checked numerically. This has application to numerous problems, in particular electrical breakdown in random networks modeled by the random fuse model [49, 50, 51, 52, 53, 54, 55]. Let us define the probability $F_N(\sigma)$ that the system composed of N segments does not break under strain σ. In this model the distribution of couplings are iid random variables. Therefore, we can apply as in section 2.1 a coarse-graining recursion relation for p blocks

$$F_N(\sigma) = F_{N/p}(a_p z + b_p)^p, \quad (38)$$

with the appropriate scaling $\sigma = a_p z + b_p$. The total probability is indeed the product of probabilities for each block in the case of independent variables. By replacing directly p by N, the renormalization group leads to $F_N(\sigma) \to F_1(a_N z + b_N)^N$ which is the usual form Eq. (3) with the cumulative function for individual stress F_1 used here and equivalent to F in the previous sections. For a d-dimensional system, there are $N = (L/L_0)^d$ elements, with L_0 being the lowest characteristic size, the dimension of elementary blocks for example. Then the limit distribution $G(z)$ is defined by $\lim_{L\to\infty} F_{L_0}(a_L z + b_L)^{(L/L_0)^d}$. This leads to the three fundamental solutions Eq. (2). Out of the three solutions, only the Gumbel and Weibull forms are relevant empirically for fracture problems. Note however that uncritical use of the EVS distributions to describe for experimental strength data has been questioned by the observation that the Normal distribution is a better fitting function for some materials [56].

There are several models of damage propagation that leads to either catastrophic rupture (brittle fracture) from a single crack propagation, or ductile fracture, where small cracks form (microvoids) and merge to provoke the rupture of the material after a phase of deformation [57, 58, 59, 60]. We can mention for example the electrical breakdown problem where a crack is a rupture of local fuse beyond a current threshold that goes through it, therefore increasing the stress on surrounding fuses which successively break as well, leading to an avalanche effect. Also problems in elastic media have been studied [57, 61, 62] in a similar way as the fuse model. Here the fuses are replaced by springs possessing a stretching limit, which can be random or not. Under a given macroscopic strain, the shear modulus (ratio between the shear stress and the displacement) is vanishing in the neighborhood of the failure region. A power law can be observed between this modulus and the distance $p - p_c$ to the percolation threshold p_c of the network, where $p > p_c$ is the probability of finding originally a non zero spring constant between two sites.

In the random fuse model [47, 48, 50, 52, 63], a regular d-dimensional lattice of linear size L is composed of individual fuses, placed with probability $0 < p < 1$ on each link, with p chosen above the percolation threshold p_c. A voltage V_0, equivalent to a global strain σ, is applied between the top and bottom of the lattice, see Fig. 3(a). Each fuse breaks when the local current going through it exceeds a threshold value (fixed usually to be unity, or can be chosen at random). One detects the first fuse that breaks in the system at value V_1 by using Green's formalism [64] to solve the linear set of Kirchhoff equations that determine the distribution of the local currents when increasing the voltage. This fuse is removed and the current distribution is then recalculated until the second fuse breaks. The procedure is repeated until the lattice is totally disconnected at value V_b, when the fraction of the remaining fuses is close to the percolation threshold, and the current can not flow anymore. For a regular lattice, without any disorder, the minimal critical voltage to apply in order to break a link is equal to L volt (one volt for each fuse). Since the current is the same in each row, all fuses break altogether at once at this value, then $V_1 = V_b$. The global strain can be related to the voltage or the current by defining the reduced variable $\sigma = V_0/L$. When p is less than unity, for a given configuration of fuses, $V_1 \neq V_b$ in general. For one defect, i.e. one fuse missing on a vertical line, see Fig. 3(b), it has been demonstrated, using a Green function [51, 52] in two dimensions, that a lower external voltage equal to $\frac{\pi}{4}L$ volt is sufficient to break the fuse on the vertical link next to the missing fuse. Therefore $V_1/L = \frac{\pi}{4}$, leading to a first order transition at the critical voltage, in the thermodynamic limit. Also, once the first fuse breaks, the others on the horizontal direction break as well, since they carry even larger load, and total rupture follows: $V_b/L = V_1/L = \frac{\pi}{4}$. The stress enhancement at the tip of a defect (the surrounding fuses) of size n (n consecutive vertical links or fuses missing on the horizontal direction) has been estimated in d dimension to be $I' = I(1 + k_d n^{1/2(d-1)})$ [52, 65, 46, 66], where k_d is a constant depending on the shape of the defect, for example the ellipsoid structure in Fig. 3(b), where n can be approximated by the ratio between the two semi-axis. This means that the breakdown of the network is dominated by the largest value of the defect size, when n is large. Indeed, the current at the tip of the horizontal defect is growing with the defect linear size and the voltage

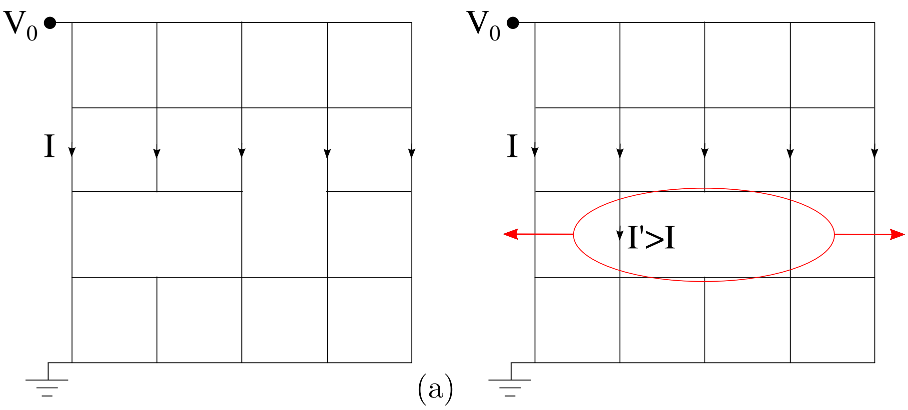

Figure 3. (a) Random fuse model. A lattice grid is subjected to a potential V_0. Each link is a resistance of 1 ohm and is distributed randomly with probability p. Every resistance breaks when the current exceeds one ampère with increasing voltage. (b) Model with one defect. The current I' through the side link is enhanced $I' > I$, and all fuses on the same line break altogether at the same time when the voltage reaches $V_1 = L\frac{\pi}{4}$, lowering the critical voltage at breakdown. The ellipsoid shape in red is a continuum version of the discrete bond model.

V_1 is reduced accordingly. We can connect the distribution of these defects with the distribution of the threshold strain V_1/L [49]. We first consider the probability $F_N(n)$ that no defect of size larger that n exists in the whole network formed by N elements of hypercubes $N = (L/L_0)^d$, with L_0 the characteristic length mentioned above. We can estimate this probability using the same arguments as above. We are indeed making the assumption that the distribution $F_N(n)$ has the same form inside the N hypercubes or in the total network of volume L^d. This means the scaling relation is similar to Eq. (38)

$$F_N(a_N n + b_N) = F_1(n)^N. \tag{39}$$

If the distribution of defects is exponential, which has been demonstrated to be the case for p below the percolation threshold [67], then it can be shown that the limit distribution has a Gumbel form

$$F_N(n) = \exp\left[-cL^d \exp(-kn)\right]. \tag{40}$$

Otherwise, for an algebraic distribution of defects, it has the Fréchet form (see Table 1)

$$F_N(n) = \exp\left[-cL^d n^{-\alpha}\right]. \tag{41}$$

with α a positive exponent. To check the assumption and validity of this scaling argument, one can first compute in two dimensions [52] the probability $P(n)$ that the *horizontal* rows contain no cluster of more than n defects [52]. The computation can be performed with simplified boundary conditions where the left end of each row is attached to the right end of the preceding row, in a spiral fashion. Therefore, the problem becomes one-dimensional and clusters of occupied bonds are separated by clusters of unoccupied ones. It is easy to see that there are the same number c of clusters of k_i unoccupied or

l_i occupied bonds with $i = 1, \cdots, c$. This can be formulated as

$$P(n) = \frac{1}{2i\pi} \oint \mathrm{d}z z^{-L^2-1} \sum_{c=1}^{\infty} \prod_{i=1}^{c} \sum_{k_i=1}^{n} \sum_{l_i=1}^{\infty} (1-p)^{k_i} p^{l_i} z^{\sum_i k_i + l_i}. \tag{42}$$

Here we used the Kronecker integral representation around the complex unit circle $\delta_{k,0} = \frac{1}{2i\pi} \oint \mathrm{d}z z^{k-1}$ to impose the constraint $\sum_i k_i + l_i = L^2$. The three sums can be performed and one obtains

$$P(n) = \frac{1}{2i\pi} \oint \mathrm{d}z z^{-L^2-1} \frac{(1-pz)[1-(1-p)z]}{1 - z + (1-p)^{n+1} z^{n+2}} \tag{43}$$

In the limit of large L, the main contribution of the integral comes from the vicinity of the pole $z = 1+\epsilon$, which is close to unity, with $\epsilon \simeq p(1-p)^{n+1}$ when n is relatively large and p not too close to the percolation threshold where clusters of defects can interact with each other. In that case the integral is dominated by the term $z^{-L^2} \simeq (1+\epsilon)^{-L^2} \simeq \exp(-L^2\epsilon)$ and therefore

$$P(n) \simeq \exp\left[-p(1-p)L^2 \exp(-n[-\log(1-p)])\right]. \tag{44}$$

This distribution has a Gumbel shape and is coherent with the scaling hypothesis Eq. (40) seen above, with $c = p(1-p)$ and $k = -\log(1-p)$. From there we can deduce the general failure or fracture distribution by considering the relation between the load enhancement until the breakdown threshold $I' = 1$, as a function of defect size n in the random fuse model. As we have shown before, the larger is the defect, the lower the voltage that is needed to break the fuses of the edges, since a cascade of failures occurs suddenly. The total failure of the system is then dominated by the largest defect at V_1, when I' exceeds unity. Since $V_1/L \simeq I$, one obtains $n \simeq L/V_1$, for the random fuse model in two dimensions, up to some constant, and therefore we can construct the probability of failure upon an external stress V_1 by inverting the previous relation

$$F_N(V_1/L) \simeq 1 - \exp\left[-p(1-p)L^2 \exp(-L[-\log(1-p)]/V_1)\right]. \tag{45}$$

The validity of this expression has been checked extensively in numerical studies of the random fuse model [47], where the distribution of defects is indeed exponential for small cracks, but the exponential coefficient or slope displays a system size dependent crossover in the tail of the defect distribution, when small cracks merge to form larger defects by local correlations, which may lead to a different class of universality. This affects the general behavior of the survival distribution near the percolation threshold. Otherwise, it can be checked that the following scaling coefficients in Eq. (39) are valid in the domains of values p close to unity in two dimensions: $a_N = \frac{1}{2}[\log(N)^{3/2}]$ and $b_N = 1/\sqrt{\log(N)}$ [47] (we remind that N is the number of sites). The Weibull form is present instead when the distribution of defects is algebraic, leading instead to

$$F_N(V_1/L) \simeq 1 - \exp\left[-cL^2(V_1/L)^{\alpha}\right], \tag{46}$$

with α a positive constant. Practically, the two distributions Eq. (45) and Eq. (46) can be differentiated if one considers numerically the quantity $\log(-\log[1-F(V_1/L)]/L^2)$ as function of L/V_1 or $\log(L/V_1)$, respectively, which should display an asymptotic linear behavior for V_1 small [49] when the appropriate form is chosen.

3. EVS for correlated variables

Until now we have seen that for iid variables the three stable distributions Eq. (2) are commonly found in problems where the scaling Eq. (3) and Eq. (38) relations can be applied, assuming simply that there are no correlations between random variables that affect the general behavior of the limit distribution. In presence of correlations, this scaling relation does not hold anymore and the calculation of the limit distribution for the extreme value of correlated random variables becomes way more involved, to say the least. We expect new classes of distributions, depending on the range of correlations for example.

3.1. An exactly solvable case: Tracy-Widom distributions

In this section we will briefly present a problem of extreme value of correlated random variables, appearing in the context of random matrices.

Random matrices are matrices whose elements are random variables, supposed here to be independent. They have been used in physics to describe spectral properties of large nuclei: the actual Hamiltonian describing such nuclei is so complex that it can be effectively described as random [68]. The particular case of Gaussian random matrices is interesting since it allows to derive analytically generic properties of these matrices. In this case the set of Gaussian random matrices H is associated with a probability distribution $P(H)dH \propto \exp\left[-\mathrm{Tr}\left(H^2\right)/(2\sigma^2)\right]dH$, where dH is the Haar measure on the ensemble of random matrices under consideration [69]. Other ensembles of matrices have also been considered but will not be discussed here (see for instance [70]).

Understanding the ground state energy distribution amounts to compute the distribution of the smallest eigenvalue of random Hamiltonians, i.e. of random Hermitian matrices. We therefore see that it is natural in this context to address the question of the distribution of the extreme eigenvalue of random matrices. The distribution of matrix elements leads to a probability distribution of the matrices eigenvalues, and one can then wonder if these distributions are organized in some kind of universality classes depending on some generic symmetry properties of the matrices and independent of the details of the element distribution.

A first step to address this question is to see whether eigenvalues $\{\lambda_i\}$ of random matrices are independent random variables, in other words if their join probability distribution factorizes. The derivation of this joint probability distribution can be found in the seminal book by Mehta [69], and proceeds as follow. The idea is to diagonalize the random matrix H by the mean of a matrix U belonging to a group, depending on the symmetry properties of H, so that $H = UDU^{-1}$. Then the statistical distribution of H "propagates" to D *via* this change of variables. Eigenvalues are then expressed in terms of the uncorrelated elements of H, but the previous product structure induces correlations between the eigenvalues so that their joint probability distribution does not factorize.

One can then show that the joint probability distribution of the eigenvalues

$P(\lambda_1, \cdots, \lambda_N)$ can be written as

$$P(\lambda_1, \cdots, \lambda_N) = C \exp\left(-\frac{1}{2\sigma^2}\sum_{k=1}^{N}\lambda_k^2\right)\prod_{i<j}|\lambda_i - \lambda_j|^\beta, \quad (47)$$

with C a normalization constant [69, 71]. The value of the exponent β depends on the ensemble under consideration: $\beta = 1$ for Gaussian orthogonal matrices (GOE), $\beta = 2$ for unitary random matrices and $\beta = 4$ for symplectic matrices).

The important point here is to note that expression Eq. (47) can not be factorized, so that the random eigenvalues λ_j are *not* independent. Thus one can not rely on the factorization property to derive the corresponding extreme value distributions. Quite remarkably, it has been possible to find the distribution of the largest eigenvalues of Gaussian random matrices, leading to the celebrated Tracy-Widom distributions.

Let M be $N \times N$ random matrix belonging either to the Gaussian Orthogonal Ensemble (GOE, $\beta = 1$), the Gaussian Unitary Ensemble (GUE, $\beta = 2$)) or the Gaussian Symplectic Ensemble (GSE, $\beta = 4$), and consider the largest eigenvalue of M. For fixed N and β, let $F_{N,\beta}(\lambda) = P_\beta(\lambda > \max(\{\lambda_i\}))$ be the probability that λ is greater than the largest eigenvalue. Then the distribution of the rescaled variable s, defined by $\lambda = \sigma N^{-1/6}s + 2\sigma\sqrt{N}$, converges towards $F_\beta(s)$ in the limit $N \to \infty$. Tracy and Widom showed [14, 72, 73] that the expression of this limit is given by

$$\lim_{N\to\infty} F_{N,\beta}(\sigma N^{-1/6}s + 2\sigma\sqrt{N}) = F_\beta(s), \quad (48)$$

with

$$F_2(s) = \exp\left(-\int_s^\infty (x-s)q(x)^2 \mathrm{d}x\right), \quad (49)$$

$$F_1(s) = \exp\left(-\frac{1}{2}\int_s^\infty q(x)\mathrm{d}x\right)F_2(s)^{1/2}, \quad (50)$$

$$F_4(s/\sqrt{2}) = \cosh\left(\frac{1}{2}\int_s^\infty q(x)\mathrm{d}x\right)F_2(s)^{1/2}, \quad (51)$$

where $q(x)$ is the unique solution of Painlevé II equation, $q''(x) = xq(x) + 2q(x)^3$, satisfying the boundary condition $q(x) \underset{x\to\infty}{\sim} \mathrm{Ai}(x)$ where $\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{t^3}{3} + xt\right)\mathrm{d}t$ is the Airy function, solution of $q''(x) = xq(x)$. This result is quite an achievement given the complexity of this problem, so much so that some have considered it as "one of the most exciting recent results in mathematical physics" [70]. Derivation of this distribution is quite involved, even in the elegant and "simple" derivation by Nadal *et al* [74] for $\beta = 2$, and goes beyond the scope of the present review. Examples of Tracy-Widom distributions are presented on Fig. 4. The connection between Tracy-Widom distributions and one-dimensional directed polymers and KPZ equation has been established both in mathematics [75, 76, 77, 78] and in physics [79, 80, 81], showing that the probability distribution function of the heights $h(x,t)$ converges at for large t towards the Tracy-Widom distribution with $\beta = 2$.

Such a display of heavy mathematics for two particular cases of correlation could appear discouraging at first, but it turns out that various problems in physics do belong to

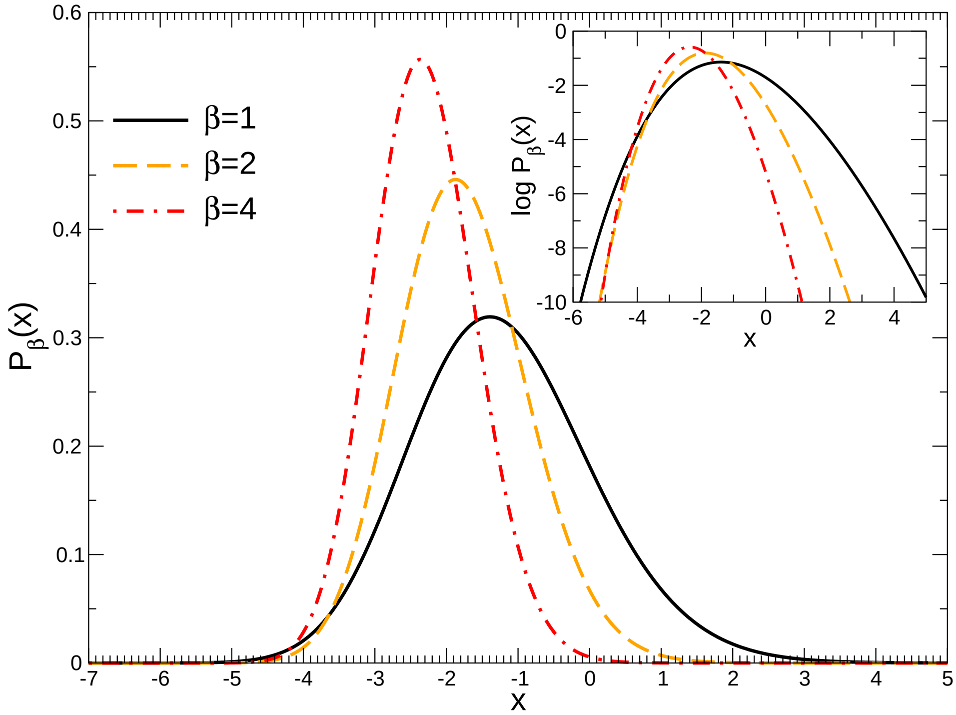

Figure 4. Examples of Tracy-Widom probability density functions $P_\beta(x)$ for $\beta = 1$ (plain black line), $\beta = 2$ (dashed orange line) and $\beta = 4$ (dotted dashed red line). Inset: same PDFs on a lin-log plot.

the universality classes of these two examples. This is the reason why Tracy-Widom distributions have been obtained in a variety of physics models [82], related either to the original random matrix problem [83, 84], or to KPZ equation [85], and also recently to the exact free-energy distribution of directed polymers on the semi-infinite plane and which converges at large time to the Tracy-Widom distribution $\beta = 4$ of the Gaussian symplectic ensemble (GSE) [86]. Universality of the Tracy-Widom distribution has also been discussed in the context of third-order phase transition associated to large deviations of the largest eigenvalue [87]. For the KPZ class of models, the asymptotic height distribution at large times is indeed directly linked to the distribution of the largest eigenvalue of special random matrix ensembles, unitary or orthogonal (GUE or GOE) [69]. The Tracy-Widom distribution depends especially on the geometry of the interface, either circular or linear respectively ‖. When the physical problem of interest does not belong to these two classes for which an exact solution is known, an exact expression for the probability distribution is not available in general, but one can use various approximation techniques to estimates the tails of the distribution for instance, as we shall see later in the examples of logarithmic 4.1 and power law interactions 4.2.

‖ The class of universality depends mainly on how the system (or number of sites) grows with time, this was in particular studied numerically for the KPZ equation in two dimensions [88].

3.2. Correlated heights

Physical systems involving correlated variables can be found in numerous classes of interface structures in one dimension [31, 32, 89, 90, 91, 92] or in two-dimensions [93, 90, 94, 95, 96], and which can be investigated by studying the statistics of maximal height Eq. (14) instead of usual roughness Eq. (13). Maximal value distributions can also be related experimentally to the height distribution in correlated interface problems, for example in the KPZ universality class, such as the interface between two turbulent states in quasi-two-dimensional liquid-crystals [97, 98]. In this case, the heights $h_i(t)$ can be indeed written as as function of a random variable χ_i satisfying one of the Tracy-Widom distribution [99]

$$h_i(t) = vt + (D^2\lambda t/8\nu)\chi_i, \tag{52}$$

where v is the uniform growth velocity of the interface, ν is the coefficient of diffusion of the KPZ equation, λ the coefficient of the non-linear term and D the strength of the Gaussian disorder in the Langevin equation. For general two-dimensional dynamical interfaces, where no exact analytical result is available for correlated systems, it is possible to obtain the signature of the KPZ universality class using skewness and kurtosis values, as well as universal correlation functions [96]. We can cite in particular two-dimensional experimental surfaces made by etching, molecular beam epitaxy or chemical deposition. We may at first assume that exponentially decreasing correlations between variables will not affect the theorem leading to the three limit distributions presented above, and discussion has been focused on long range algebraic correlations.

Let us consider the EW model in two dimensions [93] and study the distribution of the maximal fluctuations. It has to be noticed that the height distribution for the KPZ equation in the *steady* state converges to the same Gaussian limit as the EW model [100, 89], thanks to a modified discretization scheme that leads to same Fokker-Planck equation. The width distribution was previously studied using field theory [101] by computing the characteristic function and considering the poles of the Green function in a sequence of approximations. In general the joint probability between heights can be seen as a Boltzmann weight for all possible configurations and correlations are incorporated into an action with a stiffness between height differences, so that the cumulative distribution for the maximal *fluctuation* (the largest width and not the maximal height) in the continuum limit is given by

$$F(h) = \frac{\int \mathcal{D}h_{\mathbf{r}} \exp(-S[h_{\mathbf{r}}]) \prod_{\mathbf{r}} \theta(h - |h_{\mathbf{r}}|)}{\int \mathcal{D}h_{\mathbf{r}} \exp(-S[h_{\mathbf{r}}])}, \tag{53}$$

with the spatial average $\int \mathrm{d}\mathbf{r}\, h_{\mathbf{r}}$ set to zero. For a Gaussian interface the correlations are quadratic and $S[h_{\mathbf{r}}] = \frac{K}{2}\int \mathrm{d}\mathbf{r}(\nabla h_{\mathbf{r}})^2$ where K is the surface tension. This quantity selects all surface configurations whose width does not exceed h. This means that $F(h) \to 1$ when h is large. The model is defined on a finite two-dimensional lattice of size $L \times L$, or, in the discrete version, one a regular lattice containing $N = (L/a)^2$ sites where a is the elementary lattice step. The notation $\prod_{\mathbf{r}} \theta(h - |h_{\mathbf{r}}|)$ has only a meaning in the discrete case but can be extended formally in the continuum case. One way of dealing with the constraint on the maximal width is to introduce an effective mass $\mu(h)$ (or inverse correlation length) in an effective Hamiltonian that mimics the cut-off at h. Basically we replace $S[h_{\mathbf{r}}]$ and the maximal constraint on the fluctuations by $S_{\text{eff}}[h_{\mathbf{r}}] = \frac{1}{2}\int \mathrm{d}\mathbf{r}[K(\nabla h_{\mathbf{r}})^2 + \mu(h)^2 h_{\mathbf{r}}^2]$ with $\mu(h)$ vanishing when h is large. This effective mass can be determined self-consistently by deriving Eq. (53) with respect to h

$$\frac{\partial}{\partial h}\log F(h) = \frac{\int \mathcal{D}h_{\mathbf{r}} \exp(-S[h_{\mathbf{r}}]) \sum_{\mathbf{r}} \delta(h - |h_{\mathbf{r}}|) \prod_{\mathbf{r}'\neq\mathbf{r}} \theta(h - |h_{\mathbf{r}'}|)}{\int \mathcal{D}h_{\mathbf{r}} \exp(-S[h_{\mathbf{r}}])} \tag{54}$$

This integral can be performed if we decompose the delta function in two parts $\delta(h - |h_{\mathbf{r}}|) = \delta(h - h_{\mathbf{r}}) + \delta(h + h_{\mathbf{r}})$ and use a Fourier transform $h_{\mathbf{q}} = (2\pi)^{-1}\int \mathrm{d}\mathbf{r} \exp(-i\mathbf{q}.\mathbf{r})h_{\mathbf{r}}$ and $h_{\mathbf{r}} = (2\pi)^{-1}\int \mathrm{d}\mathbf{q} \exp(i\mathbf{q}.\mathbf{r})h_{\mathbf{q}}$. Since the action is even in the height variables, the two delta functions gives the same contribution and are evaluated with the effective action which is diagonalized in this basis $S_{\text{eff}}[h_{\mathbf{q}}] = \frac{1}{2}\int \mathrm{d}\mathbf{q}[K\mathbf{q}^2 + \mu(h)^2]h_{\mathbf{q}}h_{-\mathbf{q}}$, with $h_{-\mathbf{q}} = h_{\mathbf{q}}^*$. We can notice that a problem of correlated random variables can often be mapped onto a problem with independent variables in the Fourier space if it is quadratic, but the new variables are non-uniformly distributed. This is the case here where low $\mathbf{q}$-modes are dominant in the action, see for example the order parameter distribution in two-dimension XY-model [102] dominated by spin-waves on the critical line. Using an integral representation of the delta function, one obtains

$$\frac{F'(h)}{F(h)} = 2\sum_{\mathbf{r}}\int \frac{\mathrm{d}\lambda}{2\pi}\mathrm{e}^{i\lambda h}\frac{\int \mathcal{D}h_{\mathbf{q}} \exp\left(-S_{\text{eff}}[h_{\mathbf{q}}] - i\lambda(2\pi)^{-1}\int \mathrm{d}\mathbf{q}\mathrm{e}^{i\mathbf{q}.\mathbf{r}}h_{\mathbf{q}}\right)}{\int \mathcal{D}h_{\mathbf{q}} \exp(-S_{\text{eff}}[h_{\mathbf{q}}])}. \tag{55}$$

The integral can be computed by considering the discrete version, using $\int \mathrm{d}\mathbf{q} \to (2\pi/L)^2\sum_{\mathbf{q}}$, and $h_{\mathbf{q}} = x_{\mathbf{q}} + iy_{\mathbf{q}}$ leading to

$$\frac{F'(h)}{F(h)} = 2\sum_{\mathbf{r}}\int \frac{\mathrm{d}\lambda}{2\pi}\mathrm{e}^{i\lambda h}\frac{\prod'_{\mathbf{q}\neq0}\int \mathrm{d}x_{\mathbf{q}}\mathrm{d}y_{\mathbf{q}}\mathrm{e}^{-\frac{4\pi^2}{L^2}[K\mathbf{q}^2+\mu(h)^2](x_{\mathbf{q}}^2+y_{\mathbf{q}}^2) - i\lambda\frac{4\pi}{L^2}(x_{\mathbf{q}}\cos(\mathbf{q}.\mathbf{r}) - y_{\mathbf{q}}\sin(\mathbf{q}.\mathbf{r}))}}{\prod'_{\mathbf{q}\neq0}\int \mathrm{d}x_{\mathbf{q}}\mathrm{d}y_{\mathbf{q}}\mathrm{e}^{-\frac{2\pi^2}{L^2}[K\mathbf{q}^2+\mu(h)^2](x_{\mathbf{q}}^2+y_{\mathbf{q}}^2)}},$$

where the prime sign on the product takes into account only half of the modes $0 < |\mathbf{q}| \leq \pi/L$, and the zero mode is excluded because the spatial average $\int \mathrm{d}\mathbf{r}\, h_{\mathbf{r}}$ was set to zero. Defining the width $w_N^2(h) = L^{-2}\sum_{\mathbf{q}\neq0}(K\mathbf{q}^2 + \mu(h)^2)^{-1}$, one obtains, after integrating the successive Gaussian integrals over $x_{\mathbf{q}}$, $y_{\mathbf{q}}$ and λ, and noticing that the dependence on the position is removed

$$\frac{F'(h)}{F(h)} = \frac{2N}{\sqrt{2\pi}w_N(h)}\exp\left(-\frac{h^2}{2w_N(h)^2}\right). \tag{56}$$

This simple form is deduced from the Gaussian approximation of the effective action. Another relation can be deduced from the effective Hamiltonian itself,

$$\begin{aligned}\frac{F'(h)}{F(h)} &= \frac{\partial}{\partial h}\log\frac{\prod'_{\mathbf{q}\neq0}\int \mathrm{d}x_{\mathbf{q}}\mathrm{d}y_{\mathbf{q}}\mathrm{e}^{-\frac{4\pi^2}{L^2}[K\mathbf{q}^2+\mu(h)^2](x_{\mathbf{q}}^2+y_{\mathbf{q}}^2)}}{\prod'_{\mathbf{q}\neq0}\int \mathrm{d}x_{\mathbf{q}}\mathrm{d}y_{\mathbf{q}}\mathrm{e}^{-\frac{4\pi^2}{L^2}K\mathbf{q}^2(x_{\mathbf{q}}^2+y_{\mathbf{q}}^2)}} \\ &= -\frac{1}{2}\frac{\partial}{\partial h}\mu(h)^2\sum_{\mathbf{q}\neq0}(K\mathbf{q}^2 + \mu(h)^2)^{-1}.\end{aligned} \tag{57}$$

The two equations Eq. (56) and Eq. (57) give a set of self-consistent equations that are solved for $\mu(h)$. The advantage of this method is that it can be generalized in any dimension. The solution has been checked numerically up to $N = 128 \times 128$. Asymptotically, one can estimate the behavior of $F(h)$ for large h since $\mu(h) \to 0$. In particular one finds

$$\begin{aligned} w_N(h)^2 &\simeq (2\pi K)^{-1} \log[\mu_0/\mu(h)],\ \mu_0/\mu(h) \ll N^{1/2}, \\ &\simeq (2\pi K)^{-1} \log(N^{1/2}),\ \mu_0/\mu(h) \gg N^{1/2}, \end{aligned} \tag{58}$$

where $\mu_0 = \sqrt{K}\pi/a$ is a characteristic mass. Solving Eq. (56) and Eq. (57) with these values gives the solutions in the two regimes $\mu(h) \simeq \mu_0 \exp(-\sqrt{\pi K/2}h)$ for $1 \ll h \ll \sqrt{2/\pi K} \log N^{1/2}$ and $\mu(h) \simeq \mu_0 \exp(-\pi K h^2/2\log(N^{1/2}))$ for $h \gg \sqrt{2/\pi K}\log(N^{1/2})$. Then the distribution of the maximal fluctuations is given by [93]

$$F'(h) = \begin{cases} N\exp\left(-cN\mathrm{e}^{-\sqrt{2\pi K}h} - \sqrt{2\pi K}h\right) & 1 \ll h \ll \sqrt{2/\pi K}\log(N^{1/2}) \\ N\exp\left(-cN\mathrm{e}^{-\pi K h^2/2\log(N)} - \frac{\pi K}{2\log(N)}h^2\right) & h \gg \sqrt{2/\pi K}\log(N^{1/2}), \end{cases}$$

where c is some constant. In the limit of large N the distribution is consistent with a Gumbel form with the standard scaling $h = a_N z + b_N$. One can identify in particular the average height $\langle h_{\mathbf{r}}\rangle_N = \log(N)/\sqrt{2\pi K} = b_N$ and standard deviation $a_N = c'(2\pi K)^{-1/2}$ as parameters of the scaling Eq. (2), with $c' \simeq 0.69$ numerically. The roughness diverges logarithmically with N, $w_N(h) \simeq \sqrt{\log(N)}$, while in one dimension the roughness $w_N(h) \simeq \sqrt{N}$. However the parameter c is not unity, unlike the Gumbel distribution where $c = 1$, and is found numerically to be $c \simeq 1.58$. This value is close to the asymptotic exact value $c = \pi/2$ present in the general distribution of the classical XY-model order parameter, describing the fluctuations of the modulus of the total spin in the critical phase at low temperature [103, 102, 104]. Although the form of the distribution is similar, in the latter case the fluctuations are associated with a sum of correlated variables, not extrema. However, in the Fourier space the variables decouple with non identical weights, leading to the assumption that only few modes with $|\mathbf{q}|$ small are dominant in the sum. This is in contrast with the distribution for sums of iid variables [105]. The value of c is independent of N or K and is universal and characteristic of correlated systems. Indeed $c = 1$ corresponds to the maximum value distribution of iid variables, and its deviation may correspond to a renormalization of correlations. Interestingly, we may relate c and c' using the result in the next section for the generalized Gumbel form in the correlated case, see Eq. (91) below, and from which it appears that the exact asymptotic relation $c = 1/c'$ is coherent with the numerical value .

In the one dimensional EW model, it is possible to obtain exact results using path integral formalism and quantum operator method [31, 32, 33, 89] starting from Eq. (53). The cumulative distribution $F(h)$ in Eq. (53) can be reformulated on a one dimensional and periodic chain of size $L = Na$ as [32]

$$F(h) = C(L)\int_{-\infty}^{h}\mathrm{d}u\int_{h_0=u}^{h_L=u}\mathcal{D}h_x \exp\left(-\frac{1}{2}\int_0^L \mathrm{d}x(\partial_x h_x)^2\right)\delta\left(\int_0^L h_x \mathrm{d}x\right)\prod_x \theta\left(h - h_x\right),$$

where $h_0 = h_L$. The difference with Eq. (53) is that the zero average value is taken explicitly in a delta function, and one considers the maximal height. $C(L)$ is a normalization constant equal to $C(L) = \sqrt{2\pi}L^{3/2}$. A change of variable $y_x = h - h_x$ simplifies the expression and isolate the dependence on h in the delta function

$$F(h) = C(L)\int_0^\infty \mathrm{d}u \int_{y_0=u}^{y_L=u} \mathcal{D}y_x \exp\left(-\frac{1}{2}\int_0^L \mathrm{d}x(\partial_x y_x)^2\right) \times$$
$$\delta\left(\int_0^L y_x \mathrm{d}x - hL\right)\prod_x \theta\left(y_x\right). \tag{59}$$

In this representation, all y_x are positive and their average is simply given by h. The Laplace transform with respect to $hL > 0$ leads to an effective action given by $S[y_x] = \int_0^L \mathrm{d}x[\frac{1}{2}(\partial_x y_x)^2 + \lambda y_x]$, where λ is the parameter of the transformation. The Hamiltonian associated to this action is $\hat{H} = -\frac{1}{2}\partial_{yy} + V(y)$, with $V(y) = \lambda y$ for $y > 0$ and $V = \infty$ otherwise, which is imposed by the product of θ functions in Eq. (59). The path integration formalism above can be also interpreted as a quantum trace or partition function

$$\int_0^\infty \mathrm{d}(hL)F(h)\mathrm{e}^{-\lambda hL} = C(L)\int_0^\infty \mathrm{d}u\langle u|\exp(-\hat{H}L)|u\rangle, \tag{60}$$

using a basis of state vectors $|u\rangle$. The eigenfunctions $\phi_E(y)$ of the operator $\hat{H}$, $\hat{H}|\phi_E\rangle = E|\phi_E\rangle$, are given by Airy functions $\phi_E(y) = \mathrm{Ai}[(2\lambda)^{1/3}(y - E/\lambda)]$ [32], which should vanish at $y = 0$. This imposes a discrete set of energy levels $E_k = \alpha_k\lambda^{2/3}2^{-1/3}$, where $-\alpha_k$ with $k \geq 1$ are the zeroes of the Airy function on the negative real axis, for example $\alpha_1 \simeq 2.3381$ and $\alpha_2 \simeq 4.0879$. Inverting the Laplace transform back to the cumulative distribution gives

$$F(h) = \sqrt{2\pi}L^{3/2}\int_{a-i\infty}^{a+i\infty} \frac{\mathrm{d}\lambda}{2i\pi}\mathrm{e}^{\lambda hL}\sum_{k\geq 1}\mathrm{e}^{-\alpha_k\lambda^{2/3}2^{-1/3}L}, \tag{61}$$

where a is some constant that is located on the right of the singularities of the integrand. By a rescaling $\lambda \to \lambda L^{-3/2}$, the cumulative distribution is a function of the ratio $z = h/\sqrt{L}$ only, and can be written in a dimensionless form as

$$F(h = z\sqrt{L}) = \sqrt{2\pi}\int_{a-i\infty}^{a+i\infty} \frac{\mathrm{d}\lambda}{2i\pi}\mathrm{e}^{\lambda z}\sum_{k\geq 1}\mathrm{e}^{-\alpha_k\lambda^{2/3}2^{-1/3}}. \tag{62}$$

The probability distribution function $F'(h) = \sqrt{L}f(z)$ depends on the scaling function $f(z)$. Since the Laplace transform of $f(z)$ depends on the zeros of the Airy function, it is commonly called the Airy distribution. It describes the fluctuations of the area under a Brownian excursion on the unit time interval, with boundary conditions fixed by setting the particle at the zero ordinate at both ends [106], and with the additional condition that the particle stays in the positive domain with an infinite wall below the horizontal axis. Moreover the Laplace inversion in this case can be done exactly [89] by computing the moments M_n of order n of the function $f(z)$. These are equal to

$$M_n = \sqrt{\pi}2^{(4-n)/2}\frac{\Gamma(1+n)}{\Gamma([3n-1]/2)}K_n,$$

$$K_n = \frac{3n-4}{4} K_{n-1} + \sum_{i=1}^{n-1} K_i K_{n-i}, \tag{63}$$

where the K_n satisfy non-linear equations with initial condition $K_0 = -1/2$. Formally, the Laplace transform was also inverted by Takács [107, 106]

$$f(z) = \frac{2\sqrt{6}}{z^{10/3}} \sum_{k=1}^{\infty} \mathrm{e}^{-b_k z^{-2}} b_k^{2/3} U(-5/6, 4/3, b_k z^{-2}),\ b_k = \frac{2\alpha_k^3}{27}, \tag{64}$$

where $U(a,b,z)$ is the confluent hypergeometric function. The asymptotic of the solution Eq. (62) can be found by analyzing the saddle point. When z is small we may apply the saddle point method to the argument $\varphi_k(\lambda) = \lambda z - \alpha_k \lambda^{2/3} 2^{-1/3}$ for each k ¶. The solution of $\partial_\lambda \varphi_k = 0$ is unique $\lambda = (2\alpha_k 2^{-1/3} 3^{-1})^3 z^{-3}$. Replacing this value in φ_k, we can then perform a Gaussian integration around the saddle point and keep the first dominant exponential term $k = 1$, which leads to

$$F(h = z\sqrt{L}) \simeq z^{-2} \exp\left(-\frac{2\alpha_1^3}{27 z^2}\right). \tag{65}$$

This form resembles a Fréchet function, up to corrective terms. In the limit when z goes to infinity, it can be shown that the probability distribution function vanishes like a Gaussian $F'(z) \simeq \exp(-6z^2)$. The corrective terms in the asymptotic form for small values of z Eq. (65) is an indication that correlations modify the general theorem for the limit distribution Eq. (2). The Airy function has many applications beyond physics, in particular in computer science, see Majumdar for a review [108] on this topics. Extension to problems involving correlations that decay like a power-law with exponent α different than 2 as for the EW problem has been treated as well [33, 26] and also for evaluating maximum relative heights of elastic membranes in disordered media, using fractional derivatives [109]. The case $\alpha = 0$ corresponds for example to white noise. For $0 \leq \alpha < 1$ the limit distribution appears to belong to the Gumbel form. The case $\alpha \geq 1$ is characterized by a diverging variance. In the limit when $\alpha \to \infty$ [33], for short correlations, the limit distribution is given by a Gaussian decay

$$F(z) = 1 - \exp\left(-\frac{\pi z^2}{4}\right). \tag{66}$$

This result has to be compare to the asymptotic results Eq. (119) and Eq. (120) in the section below in the case of power-law interactions between adjacent particles in a one-dimensional system and in the same limit of the interaction exponent. Asymptotically one also obtains a Gaussian decay, as well as additional logarithmic corrections.

4. Examples of asymptotic analysis for processes with logarithmic and power-law interactions

In this section we consider the application of asymptotic analysis applied to EVS in systems with interacting potentials. Logarithmic potentials were introduced in the

¶ It is more correct to rescale first $\lambda \to \lambda/z$ and say that the arguments of the exponential terms of the series have large variations when $z \to 0$, but the result is identical.

Random Energy Model (REM) for example by Carpentier and Le Doussal [110, 111], for which the partition function Z_β is the sum of N Boltzmann weights with random and logarithmically correlated energies V_i, and function of the inverse temperature β

$$Z_\beta = \sum_{i=1}^{N} \exp\left(-\beta V_i\right). \tag{67}$$

The original model with uncorrelated energies was introduced by Derrida [112], where V_i were chosen as iid Gaussian variables with variance proportional to $\log(N)$. The distribution of the free energy, $F = -\beta^{-1}\log(Z_\beta)$, reduces, in the limit of low temperature, to the distribution of the energy minima $V_{min} = \min(\{V_i\})$ or ground state energy for a set of Gaussian iid variables. The correlated case were introduced on a circle geometry (*circular-log* model [20]) with energies V_i satisfying logarithmic correlations through the covariance matrix $\langle V_i V_j\rangle_d = -2\log(|z_i - z_j|)$, where the brackets are the average over disorder, and $z_j = \exp(2i\pi j/N)$ the position of the sites on the unit circle. From the evaluation of the moments $\langle Z_\beta^n\rangle_d$, the distribution of the partition function can be reconstructed. In particular, in the glass phase, below the critical temperature β_c, the probability density for the rescaled variable $f = \beta_c[F + 2\log(N)]$ takes the form of a Fourier integral

$$\mathrm{Prob}(f) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\lambda}{2\pi} \mathrm{e}^{-i\lambda f} \frac{\Gamma^2(1+i\lambda)}{\Gamma(1+i\frac{\beta_c}{\beta}\lambda)} \tag{68}$$

In the zero temperature limit, or $\beta \to \infty$, the Fourier transform can be computed and one obtains the probability distribution for the rescaled free energies in terms of a modified Bessel function

$$\mathrm{Prob}(f) = 2\mathrm{e}^{f} K_0(2\mathrm{e}^{f/2}). \tag{69}$$

In the limit $f \to -\infty$, the distribution has the asymptotic behavior $\mathrm{Prob}(f) \simeq -f\exp(f/2)$, which deviates from the Gumbel asymptotic form, while in the limit $f \to \infty$ one has $\mathrm{Prob}(f) \simeq \sqrt{\pi}\exp(\frac{3}{4}f - 2\mathrm{e}^{f/2})$. This function arises also for particles interacting with a logarithmic potential, see in the next section the function P_1, Eq. (85). A continuous version of the circular-log model generated by a Gaussian distributed logarithmic random correlated potential was proposed as an extension [113], and which has applications in number theory [114], see section 4.1.3.

It is also interesting to consider special cases of distributions for a set of random variables distributed along a chain and interacting with a local potential, in such a way to simulate a gas of particles in interaction. For example one can imagine a gas of particles in a gravitational field confined in a cylinder. The position of the upper particle can be seen as the sum of the distances between variables, which links the EVS to the study of sum of random variables non-identically distributed. It can be shown that the interactions between particles lead to volume fluctuations that are non-Gaussian and that can be described by extreme value distributions [115]. In particular the parameter of these distributions is physically related to the pressure acting on the piston.

In this section we study a class of interactions that act only between adjacent particles, located in a cylinder for example. We consider in particular logarithmic interactions and how they affect the general limit distribution of the system, then we will study power law interactions, and extract the asymptotic behaviors.

4.1. Logarithmic interactions

Let us consider N random and positive iid variables x_i, distributed according to a Poisson law $p_x(x) = \kappa\exp(-\kappa x)$. As demonstrated before in section 2, the limit distribution of the maximum value $\max(\{x_i\})$ is known to be a Gumbel form. After re-indexing variables x_i such that $x_1 \le x_2 \le \cdots \le x_N$, we can introduce distance variables $u_i = x_i - x_{i-1} \ge 0$, with boundary condition $u_1 = x_1$ or $x_0 = 0$. In this case, the maximum of the set $\{x_i\}$ is simply x_N, which is also equal to the sum of the distances $s_N = u_1 + \cdots + u_N = x_N = \max(\{x_i\})$ [116], and therefore

$$\mathrm{Prob}(x = x_N) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\lambda}{2\pi} \int_0^{\infty} \prod_{i=1}^{N} \mathrm{d}u_i P_u(u_1, u_2, \cdots, u_N) \exp\left(i\lambda x - i\lambda \sum_i u_i\right), \tag{70}$$

where P_u is the joint distribution for the set of variables $\{u_i\}$. It can be expressed using p_x

$$P_u(u_1, u_2, \cdots, u_N) \propto \int_0^{\infty} \prod_{i=1}^{N} \mathrm{d}x_i p_x(x_i) \delta(u_i - x_i + x_{i-1}) \propto \mathrm{e}^{-\kappa \sum_{k=1}^{N}(N-k+1)u_k}. \tag{71}$$

We then define $\kappa_k = \kappa(N - k + 1)$ as the coefficients of proportionality ensuring the normalization of the joint probability, and one obtains

$$P_u(u_1, u_2, \cdots, u_N) = \prod_{k=1}^{N} \kappa_k \exp\left(-\kappa_k u_k\right). \tag{72}$$

The distribution for the maximum can be expressed as function of parameters κ_k

$$\begin{aligned}\mathrm{Prob}(x = x_N) &= \int_{-\infty}^{\infty} \frac{\mathrm{d}\lambda}{2\pi} \exp\left(i\lambda x\right) \prod_{k=1}^{N} \int_0^{\infty} \kappa_k \mathrm{d}u_k \exp\left(-(\kappa_k + i\lambda)u_k\right) \\ &= \int_{-\infty}^{\infty} \frac{\mathrm{d}\lambda}{2\pi} \exp\left(i\lambda x - \sum_{k=1}^{N} \log\left(1 + i\frac{\lambda}{\kappa_k}\right)\right).\end{aligned} \tag{73}$$

The sum in the integrand is diverging as N goes to infinity. One way to regularize this term is to use the formula

$$\sum_{k=1}^{\infty} \left[\log\left(1 + \frac{x}{k}\right) - \frac{x}{k}\right] = -\log\Gamma(1 + x) - \gamma x, \tag{74}$$

so that, by adding and subtracting $i\lambda/\kappa_k$ in the integrand to isolate the divergence, one obtains

$$\mathrm{Prob}(x = x_N) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\lambda}{2\pi} \exp\left[i\lambda\left(x - \sum_{k=1}^{N} \frac{1}{\kappa_k} + \frac{\gamma}{\kappa}\right) - i\lambda\frac{\gamma}{\kappa} + \sum_{k=1}^{N} \left\{\frac{i\lambda}{\kappa_k} - \log\left(1 + i\frac{\lambda}{\kappa_k}\right)\right\}\right].$$

Defining the rescaled parameter $z=\kappa(x-\sum_{k=1}^{N}\kappa_k^{-1}+\gamma/\kappa)\simeq\kappa x-\log(N)-1/2N$, the distribution, in the limit of large N, can be expressed as

$$\mathrm{Prob}(x=x_N)=\int_{-\infty}^{\infty}\frac{\kappa\mathrm{d}\lambda}{2\pi}\exp\left(i\lambda z\right)\Gamma\left(1+i\lambda\right). \tag{75}$$

The normalization follows from the fact that $\Gamma(1)=1$ and that z is proportional to κx. This integral can be computed by noticing that $\Gamma(1+i\lambda)=\int_0^\infty\mathrm{d}t\,\exp(i\lambda\log(t)-t)$. Putting this identity in Eq. (75), and integrating over λ, one finally recovers the derivative of the Gumbel distribution for the density of maxima

$$\mathrm{Prob}(x=x_N)=\kappa\int_0^{\infty}\mathrm{d}t\,\delta(\log(t)+z)\mathrm{e}^{-t}=\kappa\exp(-z-\exp(-z))=\kappa G_0'(z). \tag{76}$$

Now we consider the same set of particles, randomly located at sites x_k, and introduce correlation effects between adjacent particles only. We consider here a two-body correlation function V or local potential such that the joint probability P_x can be written now as

$$P_x(x_1,\cdots,x_N)\propto\exp\left[-\kappa\sum_i x_i-\frac{1}{2}\sum_{i,j}V(x_i,x_j)\right], \tag{77}$$

with the normalization factor as the coefficient of proportionality. Let us choose the potential such as $V(x_i,x_j)$ is non zero only for adjacent particles x_i and $x_{i\pm1}$. In this case, it is easy to see, after relabeling the x_ks, that the potential can be written only as function of the distances $\frac{1}{2}\sum_{i,j}V(x_i,x_j)=\sum_i V(u_i)$, which has the property to factorize the joint distribution

$$P_u(u_1,u_2,\cdots,u_N)\propto\prod_{k=1}^{N}\exp\Big[-\kappa_k u_k-V(u_k)\Big], \tag{78}$$

Let us consider the logarithmic case $V(u)=-\beta\log(u)$, where $\beta>-1$ in order to have a finite normalization factor. After setting for simplicity $\kappa=1$ and reordering the indices of the u_k such that $\kappa_k=k$, one obtains

$$P_u(u_1,u_2,\cdots,u_N)=\prod_{k=1}^{N}k^{1+\beta}u_k^{\beta}\exp(-ku_k). \tag{79}$$

The Fourier transform of the probability density distribution function $\mathrm{Prob}(x=x_N)$ is given by

$$\begin{aligned}\mathrm{Prob}(x=x_N)&=\int_{-\infty}^{\infty}\frac{\mathrm{d}\lambda}{2\pi}\exp\left(i\lambda x\right)\prod_{k=1}^{N}\int_0^{\infty}k^{1+\beta}\mathrm{d}u_k u_k^{\beta}\exp\left(-(k+i\lambda)u_k\right)\\&=\int_{-\infty}^{\infty}\frac{\mathrm{d}\lambda}{2\pi}\exp\left(i\lambda x-(1+\beta)\sum_{k=1}^{N}\log\left(1+i\frac{\lambda}{k}\right)\right).\end{aligned} \tag{80}$$

As before, we can rescale the variable x in order to regularize the integral when N is large:

$$\mathrm{Prob}(x=x_N)=\int_{-\infty}^{\infty}\frac{\mathrm{d}\lambda}{2\pi}\exp\left(i\lambda z\right)\Gamma^{1+\beta}(1+i\lambda), \tag{81}$$

where the rescaled variable is $z=x-(1+\beta)\log(N)$.

4.1.1. Case of β integer When β is an integer, $\beta = n$, we define $P_n(z) = \mathrm{Prob}(x = x_N)$ and use the previous integral representation of the Γ function in order to express the distribution function as a multiple integral

$$P_n(z) = \int_0^\infty \mathrm{d}t_1 \cdots \mathrm{d}t_{n+1} \exp\left(-t_1 - \cdots - t_{n+1}\right) \delta(\log(t_1 \cdots t_{n+1}) + z). \tag{82}$$

A recursion relation can be found from this expression, by integrating over the last variable t_{n+1}

$$P_n(z) == \int_0^\infty \frac{\mathrm{d}t_1 \mathrm{e}^{-t_1}}{t_1} \cdots \frac{\mathrm{d}t_n \mathrm{e}^{-t_n}}{t_n} \exp\Big(-t_1^{-1} \cdots t_n^{-1} w\Big) w, \tag{83}$$

where $w = \exp(-z)$. One obtains, more specifically the recursion relation $P_n(z) = \int_0^\infty \mathrm{d}t_n \mathrm{e}^{-t_n} P_{n-1}(t_n^{-1} w)$. This recursion relation can also be written, after a change of variable $t_n \to 1/t$, as

$$P_n(z) = \int_0^\infty \frac{\mathrm{d}t}{t^2} \mathrm{e}^{-1/t} P_{n-1}(tw),\ P_0(z) = G_0'(-\log(w)) = w \exp(-w). \tag{84}$$

As examples, the first two distributions are expressed using special functions

$$P_1(z) = 2wK_0(2\sqrt{w}),\ P_2(z) = G^{3\,0}_{0\,3}\left(w \,\middle|\, \begin{matrix} - & - & - \\ 1 & 1 & 1 \end{matrix}\right) = \int_\gamma \frac{\mathrm{d}s}{2i\pi} \Gamma^3(1-s) w^s, \tag{85}$$

where P_2, and in general P_n, can be expressed using Meijer G-functions [117] and defined by

$$G^{m,n}_{p,q}\left(w \,\middle|\, \begin{matrix} a_1 & \cdots & a_p \\ b_1 & \cdots & b_p \end{matrix}\right) = \int_\gamma \frac{\mathrm{d}s}{2i\pi} \frac{\prod_{i=1}^m \Gamma(b_i + s) \prod_{i=1}^n \Gamma(1 - a_i - s)}{\prod_{i=n+1}^p \Gamma(a_i + s) \prod_{i=m+1}^q \Gamma(1 - b_i - s)} w^{-s}. \tag{86}$$

Here the contour γ lies between the poles of $\Gamma(1 - a_i - s)$ and the poles of $\Gamma(b_i + s)$. If $n = p = 0$ or $m = q = 0$ the coefficients a_i or b_j are missing respectively, as for the expression of P_2. The distribution P_1 is found in the previous work on logarithmic correlations on a unit circle for the REM model [20, 113], below the spins glass transition and in the limit of zero temperature, see Eq. (69). It also appears in a different context related to the distribution of number of distinct and common sites visited by independent walkers [118], as the result of the convolution of two Gumbel distributions.

4.1.2. Asymptotic limits in the general case Starting with Eq. (81) in the general case and for any real value of $\beta > 0$, we would like to study the tails of the density distribution, which is equivalent to study the saddle points of the argument function in the integral over λ

$$\varphi(\lambda) = i\lambda z + (1+\beta) \log \Gamma(1 + i\lambda), \tag{87}$$

in the asymptotic region where $|z| \gg 1$. This is tantamount to find in the complex plane the unique or multiple solutions of the saddle point equation

$$\frac{\partial \varphi(\lambda)}{\partial \lambda} = iz + i(1+\beta) \frac{\Gamma'(1 + i\lambda)}{\Gamma(1 + i\lambda)} = 0. \tag{88}$$

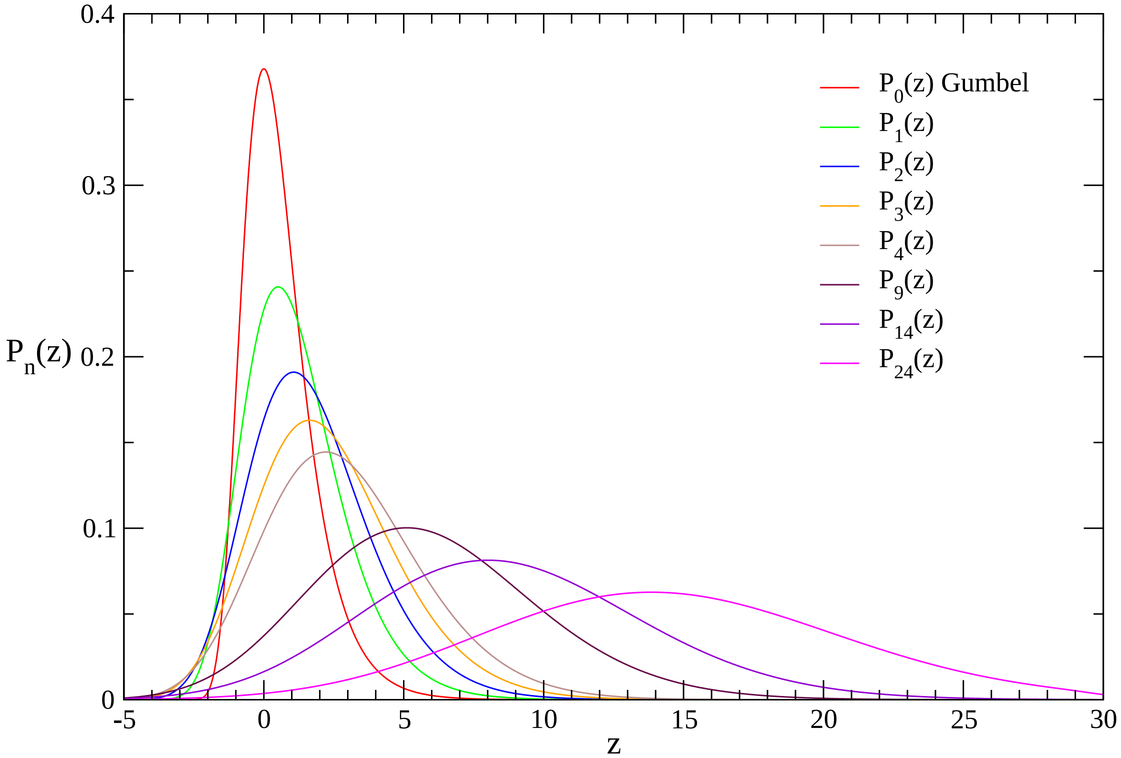

Figure 5. (Color online) Plots of the limit density distributions $P_n(z)$ for several values of n.

Let us first consider the case $z \ll -1$. We can find an obvious solution on the negative imaginary axis $\lambda = -iu$, by deforming the integration path according to $\mathcal{C}^-$ in Fig. 6, with $u \gg 1$, so that

$$(1+\beta)\frac{\Gamma'(1+u)}{\Gamma(1+u)} \simeq (1+\beta)\left[\log(u) + \frac{1}{2u}\right] = -z. \tag{89}$$

It is straightforward to see that $u = u^* = \exp(-z/(1+\beta)) \gg 1$ is a solution. We can also verify that the second derivative of φ is negative

$$\frac{\partial^2\varphi(\lambda)}{\partial\lambda^2} \simeq -\frac{1+\beta}{u^*} = -(1+\beta)\exp\left(\frac{z}{1+\beta}\right) < 0. \tag{90}$$

Integrating around the saddle point value, one obtains [+]

$$\begin{aligned}\mathrm{Prob}(z \ll -1) &\simeq \exp\left[u^* z + (1+\beta)\log\Gamma(1+u^*) - \frac{1}{2}\log(-\varphi''(-iu^*))\right] \\ &\simeq \exp\left[-(1+\beta)\exp\left(-\frac{z}{1+\beta}\right) - \frac{2+\beta}{2(\beta+1)}z\right].\end{aligned} \tag{91}$$

It turns out that the Gumbel form for this limit (double exponentially falloff) is preserved up to some constant factors depending on the exponent β. This result is to be compare to the constant $c = (1+\beta)$ found in section 3.2 and which is consistent with the value of $c' = (1+\beta)^{-1}$ for a correlation strength β. In this other limit $z \gg 1$,

[+] We can use the asymptotic limit formula $\log\Gamma(1+u) \simeq u\log(u) - u + \frac{1}{2}\log(u) + \frac{1}{2}\log(2\pi)$ for $u \gg 1$.

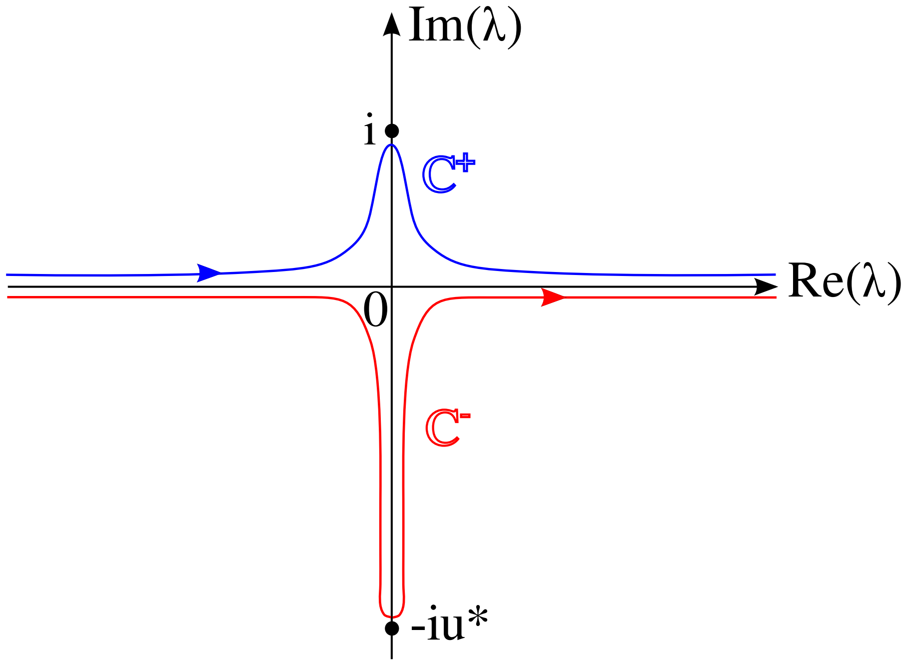

Figure 6. (Color online) modified path of integration for the saddle point analysis. $\mathcal{C}^+$ is the path chosen when $z \gg 1$ and $\mathcal{C}^-$ when $z \ll -1$. The saddle point solution u^* for $z \ll -1$ is equal to $u^* = \exp(-z/(1+\beta)) \gg 1$ (see text).

we can use the series expansion Eq. (74) for function $\log\Gamma(1+i\lambda)$ to obtain the extrema of the integrand by deforming the integral path on the imaginary axis without crossing the singularities. The series Eq. (74) diverges when u is close to -1, $u \geq -1$, which corresponds actually to the first term of the expansion $\log\Gamma(1+u) \simeq -\log(1+u)$, when $u \to -1^+$. The integration path is therefore modified in order to approach this singularity, according to $\mathcal{C}^+$ in Fig. 6. The other singularities are located at $\lambda = 2i, 3i, \cdots$ but the path can not be deformed without crossing the first singularity at $\lambda = i$. This gives $u^* = -1 + (1+\beta)/z + \gamma(1+\beta)^2/z^2 + \cdots$ as saddle point solution. The second derivative of φ is equal to $\varphi'' \simeq -z^2/(1+\beta) + 2\gamma z$, which is negative, and one obtains, after a Gaussian integration

$$\mathrm{Prob}(z \gg 1) \simeq z^\beta \exp\left[-z - \frac{\gamma\beta(1+\beta)}{z}\right]. \tag{92}$$

The exponential decay of the Gumbel density is therefore modified and enlarged by a power law term with exponent β. We can compare this result with the exact expansion of the modified Bessel function of the second kind K_0 that appears in the expression of P_1 in Eq. (85), with $\beta = 1$. Indeed, we have $P_1(w = \mathrm{e}^{-z}) \simeq \sqrt{\pi} w^{3/4} \exp(-2\sqrt{w})$, when $w \gg 1$, and $P_1(w) \simeq w(-\log w - 2\gamma)$ when $w \ll 1$, in agreement with Eq. (91) and Eq. (92) respectively.

4.1.3. Link with number theory The link between logarithmic interaction models and number theory problem appears when one considers the Riemann zeta function defined by

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right). \tag{93}$$

It was postulated that the zeros of this function lies on the line $s = \frac{1}{2} + it$ where t is real (Riemann hypothesis). Selberg [119] proved that $\log(|\zeta(\frac{1}{2} + it)|)$ behaves like Gaussian random variables with zero mean and variance equal to $\frac{1}{2}\log(\log(t))$. In particular, these variables are strongly correlated if we define the potential [114, 120]

$$V_t(x) = -2\log\{|\zeta(\frac{1}{2} + i(t+x))|\}, \tag{94}$$

whose correlations, after averaging on a given interval, are

$$\langle V_t(x_1)V_t(x_2)\rangle = \begin{cases} -2\log|x_1 - x_2|, & \frac{1}{\log(t)} \ll |x_1 - x_2| \ll 1 \\ 2\log(\log(t)), & |x_1 - x_2| \ll \frac{1}{\log(t)} \end{cases} \tag{95}$$

These properties have connections with random matrix theory of unitary matrices [114]. Indeed if we consider the ensemble of $N \times N$ unitary matrix and choose one element U_N such that its eigenvalues are $\{\mathrm{e}^{i\phi_1}, \cdots, \mathrm{e}^{i\phi_N}\}$ on the unit circle, then we can define the characteristic polynomial

$$p_N(\theta) = \det\left(1 - U_N \mathrm{e}^{-i\theta}\right) = \prod_{i=1}^{N} \left(1 - \mathrm{e}^{i(\phi_i - \theta)}\right). \tag{96}$$

We can then introduce a random potential $V_N(\theta) = -2\log(|p_N(\theta)|)$ such that in the limit of large N, the distribution of $\log(|p_N(\theta)|)$ tends to a Gaussian or normal law with zero mean and variance equal to $2\log(N)$ [121]. In particular, the correlation functions in this limit are equal to

$$\langle V_N(\theta_1)V_N(\theta_2)\rangle = -2\log\left\{2|\sin\left(\frac{1}{2}[\theta_1 - \theta_2]\right)|\right\}, \tag{97}$$

which presents the same logarithmic behavior as Eq. (95). Especially one can establish the correspondence $N \simeq \log(t)$ in the large N limit between the two theories. The distribution of the maximum values of $\log(|p_N(\theta)|)$ was studied [114] in the same context as the circular-log model presented before [20], and the distribution is exactly equal to the distribution found in Eq. (69), making an additional link between random matrix theory and random energy landscape.

4.2. Power law interaction

In this section, we consider instead the power law repulsive potential $V(u) = au^{-\beta}$, with $a > 0$ for the integrals to be convergent and exponent $\beta > 0$. The limit case when $\beta = 0$ corresponds to the logarithmic interaction, as seen previously. The joint probability is proportional to the product

$$P_u(u_1, u_2, \cdots, u_N) \propto \prod_{k=1}^{N} \exp(-ku_k - au_k^{-\beta}), \tag{98}$$

up to a general normalization factor. The distribution of the maximum $x_N = \max(\{x_i\})$ can be written as

$$\mathrm{Prob}(x = x_N) \propto \int_{-\infty}^{\infty} \frac{\mathrm{d}\lambda}{2\pi} \exp{(i\lambda x)} \prod_{k=1}^{N} \int_0^{\infty} \mathrm{d}u_k \exp\left(-(k+i\lambda)u_k - a u_k^{-\beta}\right)$$
$$\propto \int_{-\infty}^{\infty} \frac{\mathrm{d}\lambda}{2\pi} \exp\left(i\lambda x - \sum_{k=1}^{N} \log{(k+i\lambda)} + \sum_{k=1}^{N} \log R_\beta[a(k+i\lambda)^\beta] \right), \qquad (99)$$

with the function R_β defined by the integral $R_\beta(x) = \int_0^\infty \mathrm{d}u \exp(-u - xu^{-\beta})$. For example, we have $R_1(x) = 2\sqrt{x}K_1(2\sqrt{x}) \simeq \sqrt{\pi}x^{1/4}\exp(-2\sqrt{x})$, when $x \gg 1$. Let $\varphi(u) = -u - xu^{-\beta}$ be the argument in the integral representation of R_β. When x is large, we can use the saddle point approximation for evaluating $R_\beta(x)$. The equation $\varphi'(u) = 0$ gives the unique solution $u = u^* = (\beta x)^{1/(1+\beta)}$. Then, since $\varphi''(u) = -(1+\beta)/(x\beta)^{1/(1+\beta)} < 0$, one obtains, after the Gaussian integration around the saddle point

$$R_\beta(x) \simeq \sqrt{\frac{2\pi\beta^{1/(1+\beta)}}{1+\beta}} x^{1/(2(1+\beta))} \exp\left(-x^{1/(1+\beta)}\left[\beta^{1/(1+\beta)} + \beta^{-\beta/(1+\beta)}\right]\right). \qquad (100)$$

This expression can be used when the potential interaction a is moderately large. Replacing R_β in Eq. (99) by its asymptotic value (strong potential), one obtains, up to some constant, the argument function

$$\varphi(\lambda) = i\lambda x - \frac{2+\beta}{2(1+\beta)} \sum_{k=1}^{N} \log\left(1 + \frac{i\lambda}{k}\right) - C_\beta a^{1/(1+\beta)} \sum_{k=1}^{N} (k+i\lambda)^{\beta/(1+\beta)}, \qquad (101)$$

with constant C_β defined by

$$C_\beta = \beta^{1/(1+\beta)} + \beta^{-\beta/(1+\beta)}.$$

When N is large, we can regularize the argument by choosing an adequate rescaled variable z such that

$$z = x - \left(1 - \frac{\eta}{2}\right)\left[\gamma + \sum_k k^{-1}\right] - C_\beta a^{1-\eta} \sum_k k^{\eta-1}, \qquad (102)$$

where $\eta = \frac{\beta}{1+\beta} < 1$. Then, up to some constant, one obtains the regularized part of φ, expressed with convergent sums only

$$\varphi(\lambda) = i\lambda z - \left(1 - \frac{\eta}{2}\right) \sum_{k=1}^{N} \left[\log\left(1 + \frac{i\lambda}{k}\right) - i\frac{\lambda}{k}\right] - \left(1 - \frac{\eta}{2}\right) i\lambda\gamma$$
$$- C_\beta a^{1-\eta} \sum_{k=1}^{N} \left[(k+i\lambda)^\eta - k^\eta - i\lambda\eta k^{\eta-1}\right]. \qquad (103)$$

In the limit of large N, the function above is therefore well defined, and one has

$$\varphi(\lambda) = i\lambda z + \left(1 - \frac{\eta}{2}\right) \log\Gamma(1+i\lambda) - C_\beta a^{1-\eta} \sum_{k=1}^{\infty} \left[(k+i\lambda)^\eta - k^\eta - i\lambda\eta k^{\eta-1}\right]. \qquad (104)$$

We can notice that $\varphi(0) = 0$, therefore the distribution Prob(z) for the rescaled variable z is correctly normalized. Also, it has to be reminded that the limit $a = 0$ can not be taken directly since the exponent η in the factor of the logarithm in the above expression does not vanish as expected in the limit of small values of a. The expression Eq. (104) is indeed asymptotically valid only for large or moderate values of a.

4.2.1. Saddle point solutions Now, we are seeking the asymptotic behavior of the distribution Prob(z), and the argument function φ has to be extremized with respect to the variable λ:

$$\frac{1}{i}\frac{\partial}{\partial\lambda}\varphi(\lambda) = z + \left(1 - \frac{\eta}{2}\right)\frac{\Gamma'(1+i\lambda)}{\Gamma(1+i\lambda)} - C_\beta a^{1-\eta}\eta\sum_{k=1}^{\infty}\left[(k+i\lambda)^{\eta-1} - k^{\eta-1}\right] = 0. \tag{105}$$

When $z \ll -1$, we may try, as before, a solution of the form $\lambda = -iu$, with u real. The series in Eq. (105) can be rewritten as an integral. Indeed, if we define the function

$$S_\eta(u) = \sum_{k=1}^{\infty}\left[(k+u)^{\eta-1} - k^{\eta-1}\right] \tag{106}$$

we can use the identity $k^{-\eta} = \Gamma^{-1}(\eta)\int_0^\infty \mathrm{d}t\, t^{\eta-1}\mathrm{e}^{-kt}$, so that

$$S_\eta(u) = \frac{1}{\Gamma(1-\eta)}\int_0^\infty \mathrm{d}t\, t^{-\eta}\frac{\mathrm{e}^{-ut}-1}{\mathrm{e}^t - 1},\ 0 < \eta < 1. \tag{107}$$

This function is negative when $u > 0$ and positive when $u < 0$, with $S_\eta(0) = 0$. It diverges as $S_\eta(u) \simeq (1+u)^{\eta-1}$ when u is approaching -1 from above. When u is large, it diverges with u as a power law

$$S_\eta(u \gg 1) \simeq -\frac{u^\eta}{\Gamma(1-\eta)}\int_0^\infty \frac{\mathrm{d}t}{t^{1+\eta}}\left(1 - \mathrm{e}^{-t}\right) = -\frac{u^\eta}{\eta}. \tag{108}$$

The integral is indeed equal to $\Gamma(1-\eta)/\eta$ using one integration by parts. The saddle point equation takes two forms, depending on whether u is close to -1^+ or large $u \gg 1$:

$$z \simeq \left(1 - \frac{\eta}{2}\right)(1+u)^{-1} + \eta C_\beta a^{1-\eta}(1+u)^{\eta-1},\ u \to -1^+, \tag{109}$$

$$z \simeq -\left(1 - \frac{\eta}{2}\right)\left(\log u + \frac{1}{2u}\right) - C_\beta a^{1-\eta}u^\eta,\ u \gg 1. \tag{110}$$

By inspecting the asymptotic solutions in the two limits, we conclude that Eq. (109) corresponds to the solution for $z \gg 1$ and Eq. (110) to $z \ll -1$. In the first case, we obtain, similarly to the logarithm potential, $u^*(z \gg 1) = -1 + (1 - \eta/2)/z$, whereas in the second case we obtain

$$u^*(z \ll -1) \simeq \mathcal{K}_\beta^{1/\eta}|z|^{1/\eta} - \mathcal{K}_\beta^{1/\eta}\left(1 - \frac{\eta}{2}\right)\frac{1}{\eta^2}|z|^{(1-\eta)/\eta}\log(\mathcal{K}_\beta|z|), \tag{111}$$

where $\mathcal{K}_\beta = (C_\beta a^{1-\eta})^{-1}$ can be considered as an effective inverse length for the potential range, which is consistent with a simple dimensional analysis. This solution differs from the previous case since the first logarithmic term is less divergent, as long as $\beta > 0$. The class of the saddle point solutions is different from the logarithm interaction in the latter case. Expanding Eq. (104) around solution Eq. (109), up to second order, and integrating over the Gaussian measure, one obtains the following dominant asymptotic behavior

$$\mathrm{Prob}(z \gg 1) \simeq z^{-\frac{1}{2}\frac{\beta}{(1+\beta)}} \exp\left(-z\right). \tag{112}$$

In the other limit $z \ll -1$, we need to study in detail the behavior of the argument function φ Eq. (104) when u is large, around the saddle point Eq. (110), and in particular the sum defined by

$$W_\eta(u) = \sum_{k=1}^{\infty} \left[(k+u)^\eta - k^\eta - u\eta k^{\eta-1}\right], \tag{113}$$

which is a primitive of S_η: $\partial_u W_\eta(u) = \eta S_\eta(u)$, with $W_\eta(0) = 0$. It is convenient to use the integral representation Eq. (107) of S_η, which leads to the solution

$$W_\eta(u) = \frac{\eta u^\eta}{\Gamma(1-\eta)} \int_0^\infty \frac{\mathrm{d}t}{t^{1+\eta}} \frac{1-t-\mathrm{e}^{-t}}{\mathrm{e}^{t/u}-1}. \tag{114}$$

In the limit of large u, the denominator tends to zero and the integrand is diverging. We may correct the divergence using the dominant behavior of the expansion $(\mathrm{e}^{t/u}-1)^{-1} \simeq u/t$. Then we can rewrite the integral as

$$\begin{aligned}
W_\eta(u) &= \frac{\eta u^\eta}{\Gamma(1-\eta)} \left[u \int_0^\infty \frac{\mathrm{d}t}{t^{2+\eta}} (1-t-\mathrm{e}^{-t}) - u^{1-\eta} \int_0^\infty \frac{\mathrm{d}t}{t^\eta} \left(1 - \frac{1-\mathrm{e}^{-ut}}{ut}\right)\left(\frac{1}{\mathrm{e}^t-1} - \frac{1}{t}\right)\right] \\
&= \frac{\eta u^\eta}{\Gamma(1-\eta)} \left[u \int_0^\infty \frac{\mathrm{d}t}{t^{2+\eta}} (1-t-\mathrm{e}^{-t}) - u^{1-\eta} \int_0^\infty \frac{\mathrm{d}t}{t^\eta} \left(\frac{1}{\mathrm{e}^t-1} - \frac{1}{t}\right)\right. \\
&\left. + u^{-\eta} \int_0^\infty \frac{\mathrm{d}t}{t^{1+\eta}} \left(1-\mathrm{e}^{-ut}\right)\left(\frac{1}{\mathrm{e}^t-1} - \frac{1}{t}\right)\right].
\end{aligned} \tag{115}$$

In the last integral, it is not possible to replace directly $1-\mathrm{e}^{-ut}$ by unity in the large u limit because of the divergence coming from the term $t^{-1-\eta}$ near $t=0$. Instead we can use the series expansion $(\mathrm{e}^t-1)^{-1} \simeq t^{-1} - 1/2 + t/12$ when t is small to remove the divergence, by adding and subtracting $1/2$ to $(\mathrm{e}^t-1)^{-1} - t^{-1}$. We then perform an integration by parts in the first integral, to finally obtain the asymptotic expansion

$$\begin{aligned}
W_\eta(u) &\simeq \frac{\eta}{\Gamma(1-\eta)} \left[-\frac{u^{1+\eta}}{1+\eta} \int_0^\infty \frac{\mathrm{d}t}{t^{1+\eta}} (1-\mathrm{e}^{-t}) + u \int_0^\infty \frac{\mathrm{d}t}{t^\eta} \left(\frac{1}{t} - \frac{1}{\mathrm{e}^t-1}\right)\right. \\
&\left. - \frac{u^\eta}{2} \int_0^\infty \frac{\mathrm{d}t}{t^{1+\eta}} (1-\mathrm{e}^{-t}) + \int_0^\infty \frac{\mathrm{d}t}{t^{1+\eta}} \left(\frac{1}{\mathrm{e}^t-1} - \frac{1}{t} + \frac{1}{2}\right)\right].
\end{aligned} \tag{116}$$

After this result is replaced in Eq. (104), this leads to the following asymptotic form for the probability density, up to corrective terms in the argument of the order of $\mathcal{O}(|z|^{1/\beta})$

$$\begin{aligned}
\mathrm{Prob}(z \ll -1) &\simeq \exp\left[-\mathcal{K}_\beta^{1+1/\beta} \frac{\beta}{1+2\beta} |z|^{2+1/\beta} + \mathcal{K}_\beta^{1+1/\beta} \frac{2+\beta}{2\beta} |z|^{1+1/\beta} \log(\mathcal{K}_\beta |z|)\right. \\
&- (\mathcal{K}_\beta |z|)^{1+1/\beta} \left\{ \frac{2+\beta}{2(1+\beta)} + \frac{\beta I(\beta)}{\mathcal{K}_\beta (1+\beta)\Gamma((1+\beta)^{-1})} \right\} \\
&\left. + \frac{(2+\beta) I(\beta)}{2\beta\Gamma((1+\beta)^{-1})} \left(\mathcal{K}_\beta |z|\right)^{1/\beta} \log(\mathcal{K}_\beta |z|) + \mathcal{O}(|z|^{1/\beta})\right],
\end{aligned} \tag{117}$$

where the integral $I(\beta)$ and its asymptotic behavior are given by

$$I(\beta) = \int_0^\infty \frac{\mathrm{d}t}{t^{\beta/(1+\beta)}} \left(\frac{1}{t} - \frac{1}{\mathrm{e}^t - 1} \right) \simeq \beta \int_0^\infty \mathrm{d}t \left(\frac{1}{t^2} - \frac{1}{4\sinh^2(t/2)} \right) = \frac{1}{2}\beta,\ \beta \gg 1. \tag{118}$$

4.2.2. Limit of large β In the limit when β is large, which corresponds to the hard core limit, the previous asymptotic regimes have a limit form. In the first case Eq. (112) becomes easily

$$\mathrm{Prob}(z \gg 1) \simeq \exp\left(-z - \frac{1}{2}\log z \right). \tag{119}$$

In the other case Eq. (117), we can use the behavior of I in this limit, given in Eq. (118), as well as the asymptotics limit $\Gamma((1+\beta)^{-1}) \simeq \beta$ and $\mathcal{K}_\beta \simeq 1$, independent of parameter a. In this case it is straightforward to show that Eq. (117) becomes

$$\mathrm{Prob}(z \ll -1) \simeq \exp\left(-\frac{1}{2}|z|^2 + \frac{1}{2}|z|\log|z| - |z| + \frac{1}{4}\log|z| \right). \tag{120}$$

The different results found in this section can be summed up as follow. For logarithmic interactions, the general limit distribution, whose asymptotic behavior is given by Eq. (91) and Eq. (92), is close to a modified Gumbel form involving the strength parameter β in factor of the exponential argument, plus logarithmic corrections, in the regime of negative deviations. In the power-law case, the general asymptotic behavior given by Eq. (112) and Eq. (117) is close to a stretched exponential or Weibull form in the same regime, with exponent $2 + 1/\beta > 2$. For positive deviations, both cases lead to an exponential decay with logarithmic corrections. Therefore correlations modify the general limit distributions Eq. (2) by introducing corrective terms as well as general exponents that can be measured precisely.

5. Conclusion

In this review, we studied the physical applications of the extreme value statistics to several cases described by models of maximal height distribution for various types of physical problems, such as interface, fracture, random fuse network, and gas of particles in interaction. The general theorem of the three limit distributions can be applied in case of iid variables, but correlations lead in general to corrections that appear in the tail behavior where universal exponents related to the interactions are accessible. The fracture problem can be modeled by simple scaling arguments leading to either slow crack propagation in ductile material or catastrophic fracture in brittle material. Both can be studied by analyzing how the stress load is redistributed locally when a crack is formed. The case of interface with correlated heights is sensitive to the strength parameter of the correlations at long distance. All asymptotic corrections due to correlations can in general be extracted using a saddle point analysis with a proper rescaling. Recently there have been a development of applications of EVS in relation to random matrix theory and to the distribution of the largest eigenvalue for certain classes of matrices, in relation with number theory [114]. In particular it can be shown that the fluctuations of the maximal height on a path for certain random surfaces [122], or the radial distribution of a curved surface belonging to the KPZ universality class [123], can be mapped onto the largest value distribution of the Gaussian Unitary Ensemble, and given by the Tracy-Widom distribution [72], which can be further verified by analyzing the precise value of the skewness and kurtosis.

References

[1] Weibull W 1951 *J. App. Mech.* **18** 293

[2] Hunt R A and McCartney L N 1979 *Int. J. Fracture* **15** 365–375

[3] Castillo E 1988 *Extreme value theory in engineering* (New York: Academic Press, Inc.)

[4] Bouchaud J P and Mézard M 1997 *J. Phys. A: Math. Gen.* **30** 7997

[5] Coles S 2001 *An introduction to statistical modeling of extreme values* (London: Springer Series in Statistics)

[6] Majumdar S N and Krapivsky P L 2003 *Physica A*

[7] Alava M J, Nukala P K V V and Zapperi S 2006 *Adv. Phys.* **55** 349–476

[8] Chongchitnan S and Silk J 2012 *Phys. Rev. D* **85** 063508

[9] Alvarado E, Sandberg D V and Pickford S G 1998 *Northwest Science (Special Issue)* **72** 66–75

[10] Beirlant J, Goegebeur Y, Segers J and Teugels J 2005 *Statistics of extremes: Theory and applications* (John Wiley & Sons Ltd.)

[11] Abarbane H, Koonin S, Levine H, MacDonald G and Rothaus O 1992 Statistics of extreme events with application to climate Tech. Rep. JSR-90-30S JASON

[12] Embrechts P, Klüppelberg C and Mikosch T 1997 *Modelling extremal events for insurance and finance* (Berlin: Springer Verlag)

[13] Novak S 2011 *Extreme value methods with applications to finance* (London: Chapman & Hall/CRC Monographs on Statistics & Applied Probability)

[14] Tracy C A and Widom H 2008 The distributions of random matrix theory and their applications *Stanford institute for theoretical economics summer 2008 workshop: Complex data in economics and finance: Spatial models, social networks and factor models* (Standford)

[15] Gembris D, Taylor J G and Suter D 2007 *J. of Appl. Stat.* **34** 529–545

[16] Rammal R 1985 *J. Phys. France* **46** 1837–1842

[17] Carpena P, Bernaola-Galván P and Ivanov P 2004 *Phys. Rev. Lett.* **93** 176804

[18] Moloney N R and Davidsen J 2009 *Phys. Rev. E* **79** 041131

[19] Dean D and Majumdar S 2001 *Phys. Rev. E* **64** 046121

[20] Fyodorov Y V and Bouchaud J P 2008 *J. Phys. A: Math. and Theor.* **41** 372001

[21] Fisher R A and Tippett L H C 1928 *P. Cam. Philol. Soc.* **24** 180–190

[22] Gumbel E 1935 *Annales de l'I.H.P.* **5** 115–158

[23] Gnedenko B V 1943 *Ann. Math.* **44** 423–453

[24] Galambos J 1978 *The asymptotic theory of extreme order statistics* (New-York: John Wiley & Sons)

[25] de Haan L and Ferreira A 2006 *Extreme value theory: An introduction* (New-York: Spinger)

[26] Györgyi G, Moloney N, Ozogány K and Rácz Z 2008 *Phys. Rev. Lett.* **100** 210601

[27] Györgyi G, Moloney N, Ozogány K, Rácz Z and Droz M 2010 *Phys. Rev. E* **81** 041135

[28] Bertin E and Györgyi G 2010 *J. Stat. Mech.: Th. E.* **2010** P08022

[29] Schehr G and Le Doussal P 2010 *J. Stat. Mech.: Th. E.* P01009

[30] Burkhardt T, Györgyi G, Moloney N and Rácz Z 2007 *Phys. Rev. E* **76** 041119

[31] Raychaudhuri S, Cranston M, Przybyla C and Shapir Y 2001 *Phys. Rev. Lett.* **87**(13) 136101

[32] Majumdar S N and Comtet A 2004 *Phys. Rev. Lett.* **92**(22) 225501

[33] Györgyi G, Moloney N, Ozogány K and Rácz Z 2007 *Phys. Rev. E* **75** 021123

[34] Edwards S F and Wilkinson D R 1982 *P. Roy. Soc. Lond. A Mat.* **381** 17–31

[35] Antal T, Droz M, Györgyi G and Rácz Z 2001 *Phys. Rev. Lett.*

[36] Kardar M, Parisi G and Zhang Y C 1986 *Phys. Rev. Lett.* **56**(9) 889–892

[37] Anderson P L and Meerschaert M M 1998 *Water Resour. Res.* **34** 2271–2280

[38] Katz R W, Parlange M B and Naveau P 2002 *Adv. in Water Resour.* **25** 1287–1304

[39] Noullez A and Pinton J F 2002 *Eur. Phys. J B* **28**

[40] Felici M, Lucarini V, Speranza A and Vitolo R 2007 *J. Atmos. Sci.* **64** 2137–2158

[41] Planet R, Santucci S and Ortín J 2009 *Phys. Rev. Lett.* **102** 094502

[42] Barabasi A L and Stanley H E 1995 *Fractal concepts in surfaces growth* (Cambridge, UK: Cambridge University Press)

[43] Anderssen R S, Husain S A and Loy R J 2004 *Anziam Journal* **45** C800–C816

[44] Bendler J T 1984 *J. Stat. Phys.* **36** 625

[45] Weibull W 1939 *A statistical theory of the strength of materials* (Stockholm: Generalstabens litografiska anstalts förlag)

[46] Phoenix S L and Beyerlein I J 2000 *Phys. Rev. E* **62**(2) 1622–1645

[47] Manzato C, Shekhawat A, Nukala P K V V, Alava M J, Sethna J P and Zapperi S 2012 *Phys. Rev. Lett.* **108** 065504

[48] Bertalan Z, Shekhawat A, Sethna J P and Zapperi S 2014 *Phys. Rev. Appl.* **2** 034008

[49] Duxbury P M and Leath P L 1987 *J. Phys. A: Math. Gen.* **20** L411

[50] de Arcangelis L, Redner S and Hermann H 1985 *J. Phys. Lett. Paris* **46** L585

[51] Duxbury P M, Beale P D and Leath P L 1986 *Phys. Rev. Lett.* **57** 1052

[52] Duxbury P M, Leath P L and Beale P D 1987 *Phys. Rev. B* **36**(1) 367–380

[53] Gilabert A, Vanneste C, Sornette D and Guyon E 1987 *J. Phys. France* **48** 763–770

[54] Kahng B, Batrouni G, Redner S, de Arcangelis L and HJ H 1988 *Phys. Rev. B* **37** 7625–7637

[55] Barthelemy M, da Silveira R and Orland H 2002 *Europhys. Lett.)* **57** 831

[56] Lu C, Danzer R and Fischer F D 2002 *Phys. Rev. E* **65**(6) 067102

[57] Sahimi M and Goddard J D 1986 *Phys. Rev. B* **33**(11) 7848–7851

[58] Halicioglu T and Cooper D M 1984 *Mater. Sci. and Eng.* **62** 121

[59] Halicioglu T and Cooper D M 1986 *Mater. Sci. and Eng.* **79** 157–163

[60] Reurings F and Alava M J 2005 *Eur. Phys. J B* **47** 85–91

[61] Ashurst W T and Hoover W G 1976 *Phys. Rev. B* **14**(4) 1465–1473

[62] Paskin A, Som D K and Diennes G J 1983 *Acta Metall.* **31** 1841

[63] de Arcangelis L and Hermann H 1989 *Phys. Rev. B* **39** 2678

[64] Bhagavatula R, Chen K, Jayaprakash C and Xu H J 1994 *Phys. Rev. E* **49**(6) 5001–5006

[65] Li Y S and Duxbury P M 1987 *Phys. Rev. B* **36**(10) 5411–5419

[66] Krenlampi P P 2007 *Eur. Phys. J B* **59** 385–390

[67] Kunz H and Souillard B 1978 *Phys. Rev. Lett.* **40**(3) 133–135

[68] Dyson F J 1962 *J. Math. Phys.* **3** 140–156

[69] Mehta M L 1991 *Random matrices* (Boston, New York, San Diego: Academic Press)

[70] Biroli G, Bouchaud J P and Potters M 2007 *Europhys. Lett.* **78** 10001

[71] Forrester P J 2010 *Log-gases and random matrices (LMS-34)* (Princeton University Press)

[72] Tracy C A and Widom H 1994 *Comm. Math. Phys.* **159** 151–174

[73] Tracy C A and Widom H 1996 *Comm. Math. Phys.* **177** 727–754

[74] Nadal C and Majumdar S N 2011 *J. Stat. Mech.: Th. E.* **2011** P04001

[75] Amir G, Corwin I and Quastel J 2011 *Comm. Pur. Appl. Math.* **64** 466–537

[76] Sasamoto T and Spohn H 2010 *J. Stat. Phys.* **140** 209–231

[77] Sasamoto T and Spohn H 2010 *Nucl. Phys. B* **834** 523–542

[78] Sasamoto T and Spohn H 2010 *Phys. Rev. Lett.* **104** 230602

[79] Calabrese P, Le Doussal P and Rosso A 2010 *Europhys. Lett.* **90** 20002

[80] Dotsenko V 2010 *Europhys. Lett.* **90** 20003

[81] Dotsenko V 2010 *J. Stat. Mech.: Th. E.* **2010** P07010

[82] Majumdar S N 2007 *Les Houches* **85** 179–216
[83] Biroli G, Bouchaud J P and Potters M 2007 *J. Stat. Mech.: Th. E.* **2007** P07019
[84] Nishigaki S and Sugino F 2014 *J. High Energy Phys.* **2014** 104
[85] Majumdar S N and Nechaev S 2004 *Phys. Rev. E* **69**(1) 011103
[86] Gueudré T and Doussal P L 2012 *Europhys. Lett.)* **100** 26006
[87] Majumdar S N and Schehr G 2014 *J. Stat. Mech.: Th. E.* **2014** P01012
[88] Carrasco I S S, Takeuchi K A, Ferreira S C and Oliveira T J 2014 *New J. Phys.* **16** 123057
[89] Majumdar S N and Comtet A 2005 *J. Stat. Phys.* **119** 777–826
[90] Schehr G and Majumdar S N 2006 *Phys. Rev. E* **73**(5) 056103
[91] Rambeau J and Schehr G 2009 *J. Stat. Mech.: Th. E.* **2009** P09004
[92] Rambeau J and Schehr G 2010 *Europhys. Lett.* **91** 60006
[93] Lee D S 2005 *Phys. Rev. Lett.* **95**(15) 150601
[94] Oliveira T J and Aarão Reis F D A 2008 *Phys. Rev. E* **77**(4) 041605
[95] Almeida R A L, Ferreira S O, Oliveira T J and Reis F D A A a 2014 *Phys. Rev. B* **89**(4) 045309
[96] Halpin-Healy T and Palasantzas G 2014 *Europhys. Lett.* **105** 50001
[97] Takeuchi K and Sano M 2010 *Phys. Rev. Lett.* **104** 230601
[98] Takeuchi K A and Sano M 2012 *J. Stat. Phys.* **147** 853890
[99] Prähofer M and Spohn H 2000 *Phys. Rev. Lett.* **84**(21) 4882–4885
[100] Lam C H and Shin F G 1998 *Phys. Rev. E* **58**(5) 5592–5595
[101] Rácz Z and Plischke M 1994 *Phys. Rev. E* **50**(5) 3530–3537
[102] Bramwell S T, Christensen K, Fortin J Y, Holdsworth P C W, Jensen H J, Lise S, López J M, Nicodemi M, Pinton J F and Sellitto M 2000 *Phys. Rev. Lett.* **84**(17) 3744–3747
[103] Bramwell S T, Holdsworth P C W and Pinton J F 1998 *Nature* **396** 552
[104] Bramwell S T, Fortin J Y, Holdsworth P C W, Peysson S, Pinton J F, Portelli B and Sellitto M 2001 *Phys. Rev. E* **63**(4) 041106
[105] Bertin E and Clusel M 2006 *J. Phys. A: Math. Gen.* **39** 7607–7619
[106] Takács L 1995 *J. App. Prob.* **32** 375–395
[107] Takács L 1991 *Adv. Appl. Prob.* **23** 557–585
[108] Majumdar S N 2005 *Curr. Sci. India* **88** 2076–2092
[109] Rambeau J, Bustingorry S, Kolton A B and Schehr G 2011 *Phys. Rev. E* **84**(4) 041131
[110] Carpentier D and Le Doussal P 2001 *Phys. Rev. E* **63**(2) 026110
[111] Carpentier D and Le Doussal P 2006 *Phys. Rev. E* **73**(1) 019910
[112] Derrida B 1981 *Phys. Rev. B* **24**(5) 2613–2626
[113] Fyodorov Y V, Doussal P L and Rosso A 2009 *J. Stat. Mech.: Th. E.* **2009** P10005
[114] Fyodorov Y V, Hiary G A and Keating J P 2012 *Phys. Rev. Lett.* **108**(17) 170601
[115] Bertin E, Clusel M and Holdsworth P 2008 *J. Stat. Mech.: Th. E.* P07019
[116] Bertin E 2005 *Phys. Rev. Lett.* **95** 170601
[117] Prudnikov A P, Brychkov Yu A and Marichev O I 1990 *Integrals and series: More special functions, vol. 3* (New York: Gordon and Breach Science Publishers) translated from the Russian by G. G. Gould. Table erratum Math. Comp. v. 65 (1996), no. 215, p. 1384.
[118] Kundu A, Majumdar S N and Schehr G 2013 *Phys. Rev. Lett.* **110**(22) 220602
[119] Laurincikas A 1996 *Limit theorems for the Riemann zeta-function. Series: Mathematics and its applications* vol 352 (Kluwer Academic Publishers Group, Dordrecht)
[120] Fyodorov Y V and Keating J P 2013 *Philos. Tr. R. Soc. Lond. A* **372**
[121] Hughes C P, Keating J P and O'Connell N 2001 *Comm. Math. Phys.* **220** 429–451
[122] Johansson K 2000 *Comm. Math. Phys.* **209** 437–476
[123] Alves S G, Oliveira T J and Ferreira S C 2011 *Europhys. Lett.* **96** 48003