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Incompatibility between Riemann's Hypothesis and universality of Riemann's Zeta function

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Note. This draft/preprint is work in progress, it is only meant to describe the essential ideas and techniques. Until all the proofs are verified in detail, we make no claim that our results settle Riemann's Hypothesis. Comments, corrections and counterexamples are welcome.

Abstract

In this paper we prove that Riemann's Hypothesis is not compatible with universality of the Riemann Zeta function.

Keywords:: Riemann Hypothesis, Universality

Results and main theorem

Proposition 1. *Riemann's Zeta function satisfies the relation:*

$$\frac{\zeta'(s)}{\zeta(s)} = B - \frac{1}{s-1} + \frac{1}{2} \cdot \log(\pi) - \frac{1}{2} \cdot \frac{\Gamma'(\frac{s}{2} + 1)}{\Gamma(\frac{s}{2} + 1)} + \sum_{\rho} \left(\frac{1}{s-\rho} + \frac{1}{\rho} \right)$$

where B is a constant.

Proof. [Strombergsson, 2008]

Proposition 2. *Assume RH. For all s with $\zeta(s) \neq 0$, we have:*

$$\sum_{\text{Im}(\rho) > 0} \frac{1}{|s - \rho|^2} = \int_{2\pi}^{\infty} N(t) \cdot \frac{t - y}{|s - (\frac{1}{2} + it)|^4} dt$$

Proof. Here $N(t)$ counts the zeros of the Zeta function on the critical axis. The proof uses Abel summation. We consider the nontrivial zeros of Zeta of the form $\rho_i = 1/2 + \tau_i$ where the τ_i are not necessarily distinct (possible multiple zeros). For $s = \sigma + iy$ we consider the function:

$$f(t) = \frac{1}{(\sigma - \frac{1}{2})^2 + (t - y)^2}$$

We also consider the quantities :

$$c_n = (N(\tau_n) - N(\tau_{n-1}))/2$$

Using Abel summation techniques we find the relation:

$$\sum_{2\pi \leq \tau_n \leq x} c_n \cdot f(\tau_n) = \frac{1}{2} \cdot N(x) \cdot f(\tau_M) - \frac{1}{2} \cdot \int_{2\pi}^{\tau_M} N(t) \cdot f'(t) dt$$

Here $\tau_M \leq x \leq \tau_{M+1}$.

After considering $M \rightarrow \infty$ and noticing that the improper integral is convergent, the statement of the proposition follows.

Proposition 3. *Assume RH. Fix ε with $0 < \varepsilon < 1/2$.*

There exists a constant $C = C(\varepsilon) > 0$ such that, for all $s = x + iy$

with $1 - \varepsilon < x < 1$ and $y > 2\pi$, we have:

$$\sum_{\text{Im}(\rho) > 0} \frac{1}{|s - \rho|^2} \leq \frac{1}{4\pi} \cdot \int_{2\pi}^{\infty} \log \frac{u + y}{2\pi} \cdot \frac{du}{u^2 + z^2} + \frac{0.113}{z^2} \cdot \log(y) + C$$

where $z = x - 1/2$.

Proof. Let I denote the integral in Proposition 2. The change of

variables $u = t - y$, $z = x - 1/2$ gives:

$$I = \int_{2\pi - y}^{\infty} N(u + y) \cdot \frac{u \cdot du}{(u^2 + z^2)^2}$$

Trudgian (see [Trudgian, 2014]) proved that for $t \geq e$ we have:

$$\left| N(t) - \frac{t}{2\pi} \cdot \log\left(\frac{t}{2\pi e}\right) - \frac{7}{8} \right| \leq 0.112 \cdot \log(t) + 0.278 \cdot \log(\log(t)) + 2.510 + \frac{0.2}{t}$$

It follows that if we define $F(x)$ by the relation:

$$N(x) = \frac{x}{2\pi} \cdot \log\left(\frac{x}{2\pi e}\right) + F(x)$$

Then for sufficiently large $t' > e$ we have

$$|F(t)| \leq 0.113 \log(t) \quad \text{for all } t \geq t'$$

Taking $M = \sup_{1 \leq t \leq t'} |F(t)|$ we then have:

$$|F(t)| \leq 0.113 \cdot \log(t) + M$$

for all $t \geq 1$.

Now write $I = I_1 + I_2$ where

$$I_1 = \int_{2\pi-y}^{\infty} \frac{u+y}{2\pi} \cdot \log \frac{u+y}{2\pi e} \cdot \frac{u \cdot du}{(u^2+z^2)^2}$$

$$I_2 = \int_{2\pi-y}^{\infty} F(u+y) \cdot \frac{u \cdot du}{(u^2+z^2)^2}$$

Performing integration by parts in I_1 gives:

$$I_1 = -\frac{1}{2} \cdot \frac{1}{z^2 + (2\pi - y)^2} +$$

$$\frac{1}{4\pi} \cdot \int_{2\pi-y}^{\infty} \log \frac{u+y}{2\pi} \cdot \frac{du}{u^2 + z^2}$$

Now we estimate I_2 for $y > 2\pi$.

$$|I_2| \leq \int_{2\pi-y}^{\infty} |F(u+y)| \cdot \frac{|u|du}{(u^2 + z^2)^2} \leq$$

$$M \cdot \int_{2\pi-y}^0 \frac{-udu}{(z^2 + u^2)^2} + 0.113 \cdot \int_{2\pi-y}^0 \log(u+y) \cdot \frac{-udu}{(z^2 + u^2)^2} +$$

$$M \cdot \int_0^{\infty} \frac{udu}{(z^2 + u^2)^2} + 0.113 \cdot \int_0^{\infty} \log(u+y) \cdot \frac{udu}{(z^2 + u^2)^2}$$

We use the following primitives:

$$\int \frac{u \cdot du}{(u^2 + z^2)^2} = -\frac{1}{2(u^2 + z^2)} + \text{constant}$$

$$\int \frac{\log(u + y) \cdot u}{(u^2 + z^2)^2} du = \frac{1}{4} \cdot \left(\frac{1}{y^2 + z^2} \cdot \log\left(\frac{(u + y)^2}{u^2 + z^2}\right) - \frac{2}{u^2 + z^2} \cdot \log(u + y) + \frac{2y \cdot \arctan\left(\frac{u}{z}\right)}{z(y^2 + z^2)} \right) + \text{constant}$$

After we perform the elementary calculations , based on the relations above, we have:

$$|I_2| \leq \frac{0.113}{z^2} \cdot \log(y) + A(\varepsilon)$$

for all $y > 2\pi$, where $A(\varepsilon)$ is just a function of ε ,

As a consequence, when we evaluate I we have:

$$|I| \leq \frac{1}{4\pi} \cdot \int_{2\pi-y}^{\infty} \log\left(\frac{u+y}{2\pi}\right) \cdot \frac{du}{u^2+z^2} + \frac{0.113}{z^2} \cdot \log(y) + C$$

where C may depend on ε , $C = C(\varepsilon)$, QED.

Proposition 4. *If $y > 2\pi$ and $z > 0$, then*

$$\int_{2\pi-y}^{\infty} \log\left(\frac{u+y}{2\pi}\right) \cdot \frac{du}{u^2+z^2} \leq \frac{\pi}{z} \cdot \log\left(\frac{y}{\pi}\right) + \frac{1}{\pi}$$

Proof. Denote the integral by I and write:

$$I = I_1 + I_2 = \int_{2\pi-y}^y \log\left(\frac{u+y}{2\pi}\right) \cdot \frac{du}{u^2+z^2} + \int_y^{\infty} \log\left(\frac{u+y}{2\pi}\right) \cdot \frac{du}{u^2+z^2}$$

For $u \in [2\pi-y, y]$ we have:

$$\log\left(\frac{u+y}{2\pi}\right) \leq \log\left(\frac{2y}{2\pi}\right) = \log\left(\frac{y}{\pi}\right)$$

As a consequence we have:

$$I_1 \leq \log\left(\frac{y}{\pi}\right) \cdot \int_{2\pi-y}^y \frac{du}{u^2 + z^2} \leq \frac{\pi}{z} \cdot \log\left(\frac{y}{\pi}\right)$$

For $u \in [y, \infty]$ we have:

$$\log\left(\frac{u+y}{2\pi}\right) \leq \log\left(\frac{2u}{2\pi}\right) = \log\left(\frac{u}{\pi}\right)$$

As a consequence we have:

$$I_2 \leq \int_y^\infty \log\left(\frac{u}{\pi}\right) \cdot \frac{du}{u^2 + z^2} \leq$$

$$\int_y^\infty \log\left(\frac{u}{\pi}\right) \cdot \frac{du}{u^2} = \frac{1}{y} \cdot \left(\log\left(\frac{y}{\pi}\right) + 1\right) \leq \frac{1}{\pi}$$

As a consequence, $I = I_1 + I_2$ satisfies the relation in Proposition.

QED.

Lemma 1. *If $R(s) \geq 0$ and $|s|$ is sufficiently large, then we have:*

$$\left| R\left(\frac{\Gamma'(s+1)}{\Gamma(s+1)}\right) - R(\log(s)) \right| \leq \frac{1}{|s|}$$

Proof. [Edwards, 2001], page 114.

Proposition 5 (*Srinivasan's lemma*). *For any holomorphic function f with $f(s) \neq 0$ for all $s = x + iy$ in some open domain*

D , then we have:

$$R\left(\frac{f'(s)}{f(s)}\right) = \frac{1}{|f(s)|} \cdot \frac{\partial |f(s)|}{\partial x}$$

$$\operatorname{sgn}\left(\frac{\partial |f(s)|}{\partial x}\right) = \operatorname{sgn}\left(R\left(\frac{f'(s)}{f(s)}\right)\right)$$

Proof. [Srinivasan, 2011]

Proposition 6. *For any $\varepsilon > 0$, there is a y_0 such that, for any $s = x + iy$ with $y > y_0$ we have:*

$$\left| R\left(\frac{\Gamma'(s)}{\Gamma(s)}\right) - \log|s| \right| < \varepsilon$$

Proof. We use the relations:

$$\Gamma(s + 1) = s \cdot \Gamma(s)$$

$$\Gamma'(s + 1) = \Gamma(s) + s \cdot \Gamma'(s)$$

Using the lemma above , we have:

$$-\frac{1}{|s|} \leq R\left(\frac{\Gamma(s) + s \cdot \Gamma'(s)}{s \cdot \Gamma(s)}\right) - R(\log(s)) \leq \frac{1}{|s|}$$

After performing the elementary calculations we have:

$$-\frac{1}{|s|} - R\left(\frac{1}{s}\right) \leq R\left(\frac{\Gamma'(s)}{\Gamma(s)}\right) - \log|s| \leq \frac{1}{|s|} - R\left(\frac{1}{s}\right)$$

Then for sufficiently large $|s|$ we have the statement of the proposition. QED.

Proposition 7. *Assume RH. There exists $\varepsilon > 0$ and $y_0 > 0$ such that for all $s = x + iy$ with $1 - \varepsilon < x < 1$ and $y > y_0$, we have:*

$$R\left(\frac{\zeta'(s)}{\zeta(s)}\right) < 0$$

Proof. For $y > 2\pi$ we have:

$$\sum_{\rho; \text{Im}(\rho) < 0} \frac{1}{|s - \rho|^2} < K = \sum_{\rho; \text{Im}(\rho) < 0} \frac{1}{(2\pi - \text{Im}(\rho))^2}$$

As a consequence we have:

$$\sum_{\rho} \frac{1}{|s - \rho|^2} < \sum_{\rho; \text{Im}(\rho) > 0} \frac{1}{|s - \rho|^2} + K$$

$$R\left(\sum_{\rho} \frac{1}{s - \rho}\right) \leq \left(x - \frac{1}{2}\right) \cdot \sum_{\rho; \text{Im}(\rho) > 0} \frac{1}{|s - \rho|^2} + \left(x - \frac{1}{2}\right) \cdot K$$

Using Proposition 2 and Proposition 3 in the relations above we conclude that:

$$R\left(\sum_{\rho} \frac{1}{s - \rho}\right) \leq C + \left(\frac{0.226}{1 - 2\varepsilon} + \frac{1}{4}\right) \cdot \log(y)$$

where C depends only on ε , $C=C(\varepsilon)$,

For $\varepsilon = 0.01$ we have:

$$R\left(\sum_{\rho} \frac{1}{s - \rho}\right) \leq C + 0.481 \cdot \log(y)$$

From Proposition 6, for any $\varepsilon > 0$, there is a $y_0 > 0$ such that for all $s = x + iy$ with $y > y_0$ we have:

$$\left| \frac{1}{2} \cdot R\left(\frac{\Gamma'(\frac{s}{2} + 1)}{\Gamma(\frac{s}{2} + 1)}\right) - \frac{1}{2} \cdot \log\left|\frac{s}{2} + 1\right| \right| < \varepsilon$$

That means that we have:

$$\frac{1}{2} \cdot R\left(\frac{\Gamma'(\frac{s}{2} + 1)}{\Gamma(\frac{s}{2} + 1)}\right) > \frac{1}{2} \cdot \log\left|\frac{s}{2} + 1\right| - \varepsilon > \frac{1}{2} \cdot \log(y) - \frac{1}{2} \cdot \log 2 - \varepsilon$$

Using Proposition 1 we have:

$$R\left(\frac{\zeta'(s)}{\zeta(s)}\right) = R\left(B - \frac{1}{s-1} + \frac{1}{2} \cdot \log \pi + \sum_{\rho} \frac{1}{\rho}\right) - \frac{1}{2} \cdot R\left(\frac{\Gamma'(\frac{s}{2} + 1)}{\Gamma(\frac{s}{2} + 1)}\right) + R\left(\sum_{\rho} \frac{1}{s-\rho}\right)$$

From the relations above we know that

$$R\left(\sum_{\rho} \frac{1}{s-\rho}\right)$$

grows at most as $0.481 \log(y)$ when $y \rightarrow \infty$.

We also know that

$$\frac{1}{2} \cdot R\left(\frac{\Gamma'(\frac{s}{2} + 1)}{\Gamma(\frac{s}{2} + 1)}\right)$$

grows at least as $0.5 \log(y)$ when $y \rightarrow \infty$.

As a consequence, for $\varepsilon = 0.01$ there is a $y_0 > 0$ such that , for all

$s = x+iy$ with $1-\varepsilon < x < 1$ (that is $0.99 < x < 1$) and $y > y_0$, we

have::

$$R\left(\frac{\zeta'(s)}{\zeta(s)}\right) < 0$$

QED.

Universality theorem (modern version). *Let K be a compact subset of the strip $\{s \in \mathbf{C}; 1/2 < R(s) < 1\}$ such that K has a connected complement. Let $f: K \rightarrow \mathbf{C}$ be a non-vanishing continuous function that is holomorphic on the interior of K (if any) . Then for every $\varepsilon > 0$, we have:*

$$\liminf_{T \rightarrow \infty} \mu\{t \in [0, T]; \max_{s \in K} |\zeta(s+it) - f(s)| < \varepsilon\} / T > 0,$$

where μ denotes the Lebesgue measure.

Proof. [Matsumoto, 2014].

Theorem. *Riemann's Hypothesis is false.*

Proof. We assume that RH is true and we will reach a contradiction with universality. We proved that for $\varepsilon = 0.01$,

there is a $y_0 > 0$ such that for all $s = x+iy$ with $0.99 < x < 1$ and $y > y_0$ we have :

$$R\left(\frac{\zeta'(s)}{\zeta(s)}\right) < 0$$

From Proposition 5 the modulus of the Zeta function is decreasing horizontally on the vertical strip:

$$\{s \in \mathbf{C} ; 0.99 \leq x \leq 1, y > y_0\} = D$$

We consider the compact (horizontal segment) EF , where E is the real point 0.99 and F is 1 . We consider the non-vanishing function f defined on EF by $f(s)=s$. Because the density of those translates of EF (as $T \rightarrow \infty$) is nonzero, eventually one translate $E'F'$ will be included in the region D . But D is the region where the modulus of Zeta is decreasing horizontally. The conditions of the universality theorem are satisfied. We choose the level of

approximation $\varepsilon_1 < \varepsilon/4 = 0.01/4 = 0.0025$. Zeta on E'F'

approximates f on EF at the level of approximation

$\varepsilon_1 < 0.0025$.

$$||\zeta(0.9925+it)| - 0.9925| \leq |\zeta(0.9925+it) - 0.9925| < \varepsilon_1$$

$$||\zeta(0.9975+it)| - 0.9975| \leq |\zeta(0.9975+it) - 0.9975| < \varepsilon_1$$

Elementary calculations will lead to

$$|\zeta(0.9925+it)| < |\zeta(0.9975+it)|$$

But $0.9925+it$ and $0.9975+it$ are on the horizontal E'F', in the region where the modulus of Zeta is decreasing on horizontal segments. We reached a contradiction. The assumption that RH is true implies the existence of the region D where the modulus of Zeta is decreasing horizontally. The existence of this region D leads to a contradiction with universality. Therefore, Riemann's Hypothesis must be false. QED

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