

The Liénard equation has at least n periodic solutions when $F(x) = x \prod_{i=1}^n (x^2 - i^2)$

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RESUME

The Liénard equation is a subject that has occupied mathematicians for almost a century. In spite of this, we are still dealing with the same rare examples, including that of the Van der Pol equation. Mathematicians are lacking a wide collection of cases from which to test or draw some hypotheses.

We present algorithms that make it very easy to botanize many new specimens. This should energize the subject.

The usual approach used for the search for periodic solutions is that of analysis. In general it deals only with the existence of periodic solutions very close to the origin for functions containing a very small forced oscillation.

We deal with functions without forced oscillation and identify and locate periodic solutions at even very long distances from the origin.

The tools of analysis that are usually used require the mastery of elaborate techniques and it is not easy to make the link between the analytical results and the appearance of the trajectories.

We use elementary mathematical tools and a geometric approach completely in symbiosis with the appearance of trajectories.

The numerical search methods for periodic solutions do not allow us to know whether we have explored sufficiently far from the origin or limit the time of calculation to the strict necessary.

We give a method to determine very simply, for a very wide range of functions, the bounded space in which all the periodic solutions are located.

The Grail underlying the study of the Liénard equation is that of the resolution of the 16th Hilbert problem.

We have not solved it however we show that the number of periodic solutions is at least $\frac{\text{degree } F(x)-1}{2}$

MAIN RESULTS

The main results obtained are:

- 1) For $F(x) = x \prod_{i=1}^n (x^2 - i^2)$ there are at least n periodic solutions namely at least one periodic solution crossing $[-i - 1; -i]_{Ox}$ and $[i; i + 1]_{Ox}$ for all $1 \leq i \leq n$ and at least one crossing $]-\infty; -n]_{Ox}$ and $[n; \infty[_{Ox}$

$$2) \text{ For } F(x) = \begin{cases} -\frac{|x|}{x} n^{|x|+1} |\sin(\pi x)|^k & \text{si } x \in \cup_i \text{ impair } [i; i + 1[\\ 0 & \text{si } x = 0 \\ \frac{|x|}{x} n^{|x|+1} |\sin(\pi x)|^k & \text{si } x \in \cup_i \text{ pair } [i; i + 1[\end{cases} \quad \text{where}$$

$i \in \mathbb{Z}$ and $k, n \in \mathbb{R}^*$ and $n \geq 2$ there are an infinity of periodic solutions including at least one that cuts $[-i - 1; -i]_{Ox}$ et $[i; i + 1]_{Ox}$ for all $i \in \mathbb{Z}^*$

We give other examples.

- 3) For $F(x) = \sin(\pi x)$ there are at least 10^6 periodic solutions with at least one that cuts $[-i - 1; -i]_{Ox}$ et $[i; i + 1]_{Ox}$ for all $1 < i \leq 10^6$ In reality there is very probably an infinity of periodic solutions.

We give other examples.

- 4) When $F(x)$ is a polynomial of odd degree then the potential periodic solutions are included in a limited space around the origin and we give two general methods to determine this space.

By example for $F(x) = nx^3 - mx$ when $m > 0$ and $n > 0$ the possible periodic solutions are located in the space

$$\left[-e^{-1} \frac{4}{3} (2 + m) \sqrt{\frac{2+m}{3n}}; e^{-1} \frac{4}{3} (2 + m) \sqrt{\frac{2+m}{3n}} \right]_{Ox} \quad X \quad \left[-\frac{2}{3} (2 + m) \sqrt{\frac{2+m}{3n}}; \frac{2}{3} (2 + m) \sqrt{\frac{2+m}{3n}} \right]_{Oy}$$

and do not cross $\left[-\sqrt{\frac{m}{n}}; \sqrt{\frac{m}{n}} \right]_{Ox}$ or $\left[\min \left\{ -\frac{2m}{3} \sqrt{\frac{m}{3n}}; -\sqrt{\frac{m}{n}} \right\}; \max \left\{ \frac{2m}{3} \sqrt{\frac{m}{3n}}; \sqrt{\frac{m}{n}} \right\} \right]_{Oy}$

- 5) When $F(x)$ is a polynomial of even degree the potential periodic solutions are crossing a limited interval of Oy_+ and we give a general method to determine this space.

By example for $F(x) = mx^{2n}$ the potential periodic solutions are crossing $\left] - \left(2 - \frac{1}{n}\right) \left(\frac{1}{nm}\right)^{\frac{1}{2n-1}}; 0 \right]_{Oy}$

- 6) The number of periodic solutions for $F(x)$ is not equal to the number of periodic solutions for $\lambda F(x)$ when $\lambda \in \mathbb{R}/\{1\}$

- 7) We present a «Trap» a «Fishing zone» a «Fishing line» un «Fishing net» and un «Comb» which are polynomial algorithms which detect the presence of periodic solutions and where applicable give a precise location of them. These methods allow to botanize all kind of new configurations of periodic solutions included some non-symmetrical $F(x)$

- 8) The study of periodic solutions for polynomials factorizable on \mathbb{R} comes down to the study of periodic solutions for polynomials of form $F(x) = \lambda x \prod_{i=1}^n (x - c_i)$ where $\lambda \neq 0$ and $c_i \in [-1; 1]$ for all $1 \leq i \leq n$

- 9) If $F(x)$ is a polynomial of degree n then the probability of not having any periodic solutions is greater or equal to
$$\begin{cases} \left(\frac{1}{2}\right)^{\frac{n-2}{2}} & \text{if } n \geq 2 \text{ and even} \\ \left(\frac{1}{2}\right)^{\frac{n-1}{2}} & \text{if } n \geq 3 \text{ and odd} \end{cases}$$

In particular if $F(x) = a_2x^2 + a_1x$ when $a_1 \neq 0$ and $a_2 \neq 0$ there is no periodic solution.

TABLE OF CONTENTS

CHAPTER 1

We establish a relation between functions that frame $F(x)$ and the relative position of their respective trajectories when they are issued from the same point.

We use the O_y symmetry of $F(x)$ to determine the criteria for the existence of periodic solutions.

In case $F(x)$ is a polynomial we determine the existence of periodic solutions in function of its coefficients.

We establish function equivalency classes when they are factorizable polynomials on \mathbb{R}

CHAPTER 2

We determine criteria for $F(x)$ so that all periodic solutions cut a closed interval of O_y (this is the case for polynomials of even degree) or are in a bounded space around the origin (this is the case for polynomials of odd degree).

We give two algorithms for this purpose and apply them to the classical examples.

We give an example of a function such that the number of periodic solutions for $F(x)$ is different from that for $\lambda F(x)$

CHAPTER 3

When cutting O_y the trajectories issued from a finite set of points define a partition of O_y that we use to define a linear application. And the study of this application makes it easy to detect and locate periodic solutions.

If for two functions that frame $F(x)$ there is a periodic solution, we prove that there is a periodic solution for $F(x)$ and that this periodic solution is located between the other two periodic solutions.

We give an algorithm based on that and apply it on the classical examples.

CHAPTER 4

We give an algorithm which consists in framing $F(x)$ by two stair functions and applying the results of Chapter 3 to these three functions.

We apply it to the classic examples discussed in Chapter 3.

CHAPTER 5

We give an algorithm that consists of framing $F(x)$ with two « comb » functions.

CHAPTER 6

We determine criteria so that there is at least one periodic solution that goes through the intervals $[-k-1; -k]$ and $[k; k+1]$

CHAPTER 7

We apply the results of Chapter 6 to $F(x) = x \prod_{i=1}^n (x^2 - i^2)$ and prove that there are at least n periodic solutions.

To facilitate the reading we have included in the Appendix various results which are necessary for the demonstrations, but which are heavy and daunting without having particular difficulties or conceptual interests.

CHAPTER 8

We apply the results of Chapter 6 to

$$F(x) = \begin{cases} -\frac{|x|}{x} n^{[|x|+1]} |\sin(\pi x)|^k & \text{si } x \in \cup_{i \text{ impair}} [i; i+1[\\ 0 & \text{si } x = 0 \\ \frac{|x|}{x} n^{[|x|+1]} |\sin(\pi x)|^k & \text{si } x \in \cup_{i \text{ pair}} [i; i+1[\end{cases} \quad \text{and prove that there are}$$

an infinity of periodic solutions.

CHAPTER 9

We apply the results of Chapter 6 to $F(x) = \sin(\pi x)$ and numerically determine that there are at least 10^6 periodic solutions.