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Multivariate abrupt change detectors

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Contents

1	Introduction	2
2	Hypothesis testing framework	2
3	Detectors' decision functions	2
3.1	Bayesian Information Criterion (BIC)	2
3.1.1	Criterion derivation for a multivariate case	3
3.1.2	Criterion derivation for a scalar case	4
3.2	CUMulative Sum (CUSUM) algorithm	4
3.2.1	Criterion derivation for a multivariate case	4
3.2.2	Criterion derivation for a scalar case	6
3.3	Hotelling T ² test	6
3.3.1	Criterion derivation for a multivariate case	6
3.3.2	Criterion derivation for a scalar case: the Student t-test	8

1 Introduction

In this report, we aim at deriving three abrupt change detectors' decision functions: the Bayesian Information Criterion (BIC), the Cumulative Sum (CUSUM) and the Hotelling T^2 test. The three expressions are given by considering both univariate and multivariate cases. We first start by mathematically formulating the detection problem.

2 Hypothesis testing framework

Three detection algorithms capable of determining step-changes in signals are studied in this report. These changes can be detected by comparing the mean of the current observations with the mean of previous observations. Let $X_{n,p} = (x_{n_a}, x_{n_a+1}, \dots, x_m, \dots, x_n)$ with $x_m \in \mathbb{R}^p$, $\forall m \in \{n_a, \dots, n\}$, be a realization of a Gaussian process corresponding to a $(p \times w)$ matrix of the last $w = n - n_a + 1$ samples of a p -dimensional time-series at the current time instant n . Each signal sample x_m corresponds to a vector of p features such as $x_m = (x_{m,1}, \dots, x_{m,j}, \dots, x_{m,p})^T$, where $x_{m,j}$ is the value of feature $j \in \{1, \dots, p\}$ at time instant m . Each vector x_m is assumed to follow a multivariate Gaussian distribution $\mathcal{N}_p(\mu, \Sigma)$, where $\mu \in \mathbb{R}^p$ is the mean vector and $\Sigma \in \mathbb{R}^{p \times p}$ is the semidefinite covariance matrix. An abrupt change occurring at a change time instant $n_a < n_c < n$ is modeled by an instantaneous modification of the statistical parameters (i.e. mean vector and/or covariance matrix). Two hypotheses must be considered:

$$H_0 : x_{n_a}, \dots, x_n \sim \mathcal{N}_p(\mu_0, \Sigma_0) \quad (1)$$

$$H_1 : \begin{aligned} x_{n_a}, \dots, x_{n_b} &\sim \mathcal{N}_p(\mu_{1a}, \Sigma_{1a}) \\ x_{n_c}, \dots, x_n &\sim \mathcal{N}_p(\mu_{1b}, \Sigma_{1b}) \end{aligned} \quad (2)$$

with $n_b = n_c - 1$. The “without change” hypothesis H_0 supposes that, on both sides of n_c , the signal follows the same distribution (1), that is a multivariate Gaussian distribution with a mean vector μ_0 and a covariance matrix Σ_0 . On the opposite, the “with change” hypothesis H_1 supposes that a change of distribution occurs at n_c (2). Before the change, the signal samples x_{n_a}, \dots, x_{n_b} follow a multivariate Gaussian distribution with a mean vector μ_{1a} and a covariance matrix Σ_{1a} . After the change, x_{n_c}, \dots, x_n follow a multivariate Gaussian distribution with a mean vector corresponding to $\mu_{1b} \neq \mu_{1a}$ and a covariance matrix equal to $\Sigma_{1b} \neq \Sigma_{1a}$. A detector performs a hypothesis test for each potential change point in a signal. An on-line approach is followed by using a sliding window over the signal. Therefore, at each time instant n , a decision between H_0 and H_1 (i.e. a decision “to reject H_0 in favor of H_1 ”) is made by comparing a decision function g_n to an a dimensional threshold h [1].

$$\text{decide } H_1 \quad \text{if } g_n > h \quad (3)$$

$$\text{decide } H_0 \quad \text{if } g_n \leq h \quad (4)$$

In what follows three abrupt change detectors in their univariate and multivariate versions are studied: the BIC, the CUSUM and the Hotelling T^2 test.

3 Detectors' decision functions

In what follows, each decision function is presented in both, the multivariate and the univariate cases. For each detector, we assume that the samples x_m are taken from independent and identically distributed (*i.i.d*) Gaussian random vectors. It is also assumed that an abrupt change may occur at $n_c = n - \frac{w}{2} + 1$, the estimated covariance matrices are invertible and the sliding window length is $w \geq 2(p + 1)$.

3.1 Bayesian Information Criterion (BIC)

The Bayesian Information Criterion (BIC) of $X_{n,p}$ under hypothesis H_i , $i \in \{0, 1\}$ is defined as a maximum likelihood criterion penalized by the model complexity [3, 4], proportional to the number M_i of free parameters to be estimated:

$$\text{BIC}_n(H_i) = \max_{\mu, \Sigma} \ln(\mathcal{L}_{n,i}) - \frac{\lambda}{2} M_i \ln(w) \quad (5)$$

where $\mathcal{L}_{n,i}$ is the data likelihood function under hypothesis H_i , $i \in \{0, 1\}$ defined as the joint Probability Density Function (PDF) of the observed data and considered as a function of the statistical parameters μ and Σ . The scalar λ is a penalty factor, ideally equal to 1 [3].

3.1.1 Criterion derivation for a multivariate case

For a multi dimensional signal, under H_0 , the number of free parameters M_0 in (5) corresponds to the sum of the dimension p of the mean vector μ_0 plus the $p(p+1)/2$ variances and covariances to be estimated from the symmetric $(p \times p)$ covariance matrix Σ_0 , resulting in $M_0 = p(p+3)/2$. Under H_1 , the number of free parameters are the p ones from the $(p \times 1)$ mean vectors μ_{1a} , and the p ones from the $(p \times 1)$ mean vector μ_{1b} plus the $p(p+1)/2$ ones from the $(p \times p)$ covariance matrices Σ_{1a} , and the $p(p+1)/2$ ones from Σ_{1b} , so $M_1 = p(p+3) = 2M_0$.

Assuming that the samples x_m are taken from *i.i.d* Gaussian random vectors, the likelihood functions of $X_{n,p}$ under H_0 and H_1 correspond to the joint PDFs which are, by independance of the x_m values, equal to the product of the samples' PDFs, such as:

$$\mathcal{L}_{n,0} = \prod_{m=n_a}^n p_{\mu_0, \Sigma_0}(x_m) \quad (6)$$

$$= \prod_{m=n_a}^n \frac{1}{(2\pi)^{\frac{p}{2}}} \det(\Sigma_0)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(x_m - \mu_0)^T \Sigma_0^{-1}(x_m - \mu_0)\right) \quad (7)$$

$$\mathcal{L}_{n,1} = \prod_{m=n_a}^{n_b} p_{\mu_{1a}, \Sigma_{1a}}(x_m) \prod_{m=n_c}^n p_{\mu_{1b}, \Sigma_{1b}}(x_m) \quad (8)$$

$$= \prod_{m=n_a}^{n_b} \frac{1}{(2\pi)^{\frac{p}{2}}} \det(\Sigma_{1a})^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(x_m - \mu_{1a})^T \Sigma_{1a}^{-1}(x_m - \mu_{1a})\right) \quad (9)$$

$$\times \prod_{m=n_c}^n \frac{1}{(2\pi)^{\frac{p}{2}}} \det(\Sigma_{1b})^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(x_m - \mu_{1b})^T \Sigma_{1b}^{-1}(x_m - \mu_{1b})\right) \quad (10)$$

The two expressions are maximized when considering the Maximum Likelihood Estimators (MLEs) of the mean vectors and covariance matrices [3, 4]:

$$(\hat{\mu}_0, \hat{\Sigma}_0) = \underset{\mu_0, \Sigma_0}{\operatorname{argmax}} \mathcal{L}_{n,0} \quad \text{and} \quad (\hat{\mu}_{1a}, \hat{\mu}_{1b}, \hat{\Sigma}_{1a}, \hat{\Sigma}_{1b}) = \underset{\mu_{1a}, \mu_{1b}, \Sigma_{1a}, \Sigma_{1b}}{\operatorname{argmax}} \mathcal{L}_{n,1} \quad (11)$$

such as:

$$\hat{\mu}_{1a} = \frac{1}{n_c - n_a} \sum_{m=n_a}^{n_b} x_m, \quad \hat{\Sigma}_{1a} = \frac{1}{n_c - n_a} \sum_{m=n_a}^{n_b} (x_m - \hat{\mu}_{1a})(x_m - \hat{\mu}_{1a})^T \quad (12)$$

$$\hat{\mu}_{1b} = \frac{1}{n - n_b} \sum_{m=n_c}^n x_m, \quad \hat{\Sigma}_{1b} = \frac{1}{n - n_b} \sum_{m=n_c}^n (x_m - \hat{\mu}_{1b})(x_m - \hat{\mu}_{1b})^T \quad (13)$$

$$\hat{\mu}_0 = \frac{1}{w} \sum_{m=n_a}^n x_m = \frac{n_c - n_a}{w} \hat{\mu}_{1a} + \frac{n - n_b}{w} \hat{\mu}_{1b}, \quad \hat{\Sigma}_0 = \frac{1}{w} \sum_{m=n_a}^n (x_m - \hat{\mu}_0)(x_m - \hat{\mu}_0)^T \quad (14)$$

Then, we have:

$$\text{BIC}_n(H_0) = \ln(\mathcal{L}_{n,0}) - \frac{\lambda}{4} p(p+3) \ln(w) \quad (15)$$

$$= \ln\left(\left(\frac{1}{(2\pi)^p \det(\hat{\Sigma}_0)}\right)^{\frac{w}{2}} \exp\left(-\frac{1}{2} \sum_{m=n_a}^n (x_m - \hat{\mu}_0)^T \hat{\Sigma}_0^{-1} (x_m - \hat{\mu}_0)\right)\right) - \frac{\lambda}{4} p(p+3) \ln(w) \quad (16)$$

$$= -\frac{1}{2} \sum_{m=n_a}^n (x_m - \hat{\mu}_0)^T \hat{\Sigma}_0^{-1} (x_m - \hat{\mu}_0) - \frac{wp}{2} \ln(2\pi) - \frac{w}{2} \ln(\det(\hat{\Sigma}_0)) - \frac{\lambda}{4} p(p+3) \ln(w) \quad (17)$$

and

$$\text{BIC}_n(H_1) = \ln(\mathcal{L}_{n,1}) - \frac{\lambda}{2} p(p+3) \ln(w) \quad (18)$$

$$= \ln\left(\left(\frac{1}{(2\pi)^p \det(\hat{\Sigma}_{1a})}\right)^{\frac{n_c - n_a}{2}} \left(\frac{1}{(2\pi)^p \det(\hat{\Sigma}_{1b})}\right)^{\frac{n - n_b}{2}} \exp\left(-\frac{1}{2} \left(\sum_{m=n_a}^{n_b} (x_m - \hat{\mu}_{1a})^T \hat{\Sigma}_{1a}^{-1} (x_m - \hat{\mu}_{1a})\right) + \sum_{m=n_c}^n (x_m - \hat{\mu}_{1b})^T \hat{\Sigma}_{1b}^{-1} (x_m - \hat{\mu}_{1b})\right)\right) - \frac{\lambda}{2} p(p+3) \ln(w) \quad (19)$$

$$(20)$$

Since:

$$\sum_{m=n_a}^n (x_m - \hat{\mu}_0)^T \hat{\Sigma}_0^{-1} (x_m - \hat{\mu}_0) = \sum_{m=n_a}^n \text{tr} \left[(x_m - \hat{\mu}_0)^T \hat{\Sigma}_0^{-1} (x_m - \hat{\mu}_0) \right] \quad (21)$$

$$= \sum_{m=n_a}^n \text{tr} \left[\hat{\Sigma}_0^{-1} (x_m - \hat{\mu}_0) (x_m - \hat{\mu}_0)^T \right] \quad (22)$$

$$= \text{tr} \left[\hat{\Sigma}_0^{-1} \sum_{m=n_a}^n (x_m - \hat{\mu}_0) (x_m - \hat{\mu}_0)^T \right] \quad (23)$$

$$= \text{tr} [w\mathbf{I}] = wp \quad (24)$$

likewise for $\sum_{m=n_a}^{n_b} (x_m - \hat{\mu}_{1a})^T \hat{\Sigma}_{1a}^{-1} (x_m - \hat{\mu}_{1a})$ and $\sum_{m=n_c}^n (x_m - \hat{\mu}_{1b})^T \hat{\Sigma}_{1b}^{-1} (x_m - \hat{\mu}_{1b})$, then the BIC values under H_0 and H_1 are respectively:

$$\text{BIC}_n(H_0) = -\frac{wp}{2} \ln(2\pi) - \frac{w}{2} \ln(\det(\hat{\Sigma}_0)) - \frac{wp}{2} - \frac{\lambda}{4} p(p+3) \ln(w) \quad (25)$$

$$\text{BIC}_n(H_1) = -\frac{wp}{2} \ln(2\pi) - \frac{(n_c - n_a)}{2} \ln(\det(\hat{\Sigma}_{1a})) - \frac{(n - n_b)}{2} \ln(\det(\hat{\Sigma}_{1b})) - \frac{wp}{2} - \frac{\lambda}{2} p(p+3) \ln(w) \quad (26)$$

The BIC variation for a given value of n_c is given by:

$$\Delta \text{BIC}_n = \text{BIC}_n(H_1) - \text{BIC}_n(H_0) \quad (27)$$

$$= \frac{w}{2} \ln(\det(\hat{\Sigma}_0)) - \frac{(n_c - n_a)}{2} \ln(\det(\hat{\Sigma}_{1a})) - \frac{(n - n_b)}{2} \ln(\det(\hat{\Sigma}_{1b})) - \frac{\lambda}{4} p(p+3) \ln(w) \quad (28)$$

If $\Delta \text{BIC}_n > 0$, the model of two Gaussians is favored (i.e. the signal can be segmented into two parts at n_c). Consequently, the decision function can be expressed as:

$$\begin{aligned} & \begin{matrix} H_1 \\ g_n \geq h \\ H_0 \end{matrix} \quad \text{with} \quad g_n = \Delta \text{BIC}_n, \end{aligned} \quad (29)$$

$$= \ln \left(\frac{\det(\hat{\Sigma}_0)^{\frac{w}{2}}}{\det(\hat{\Sigma}_{1a})^{\frac{n_c - n_a}{2}} \det(\hat{\Sigma}_{1b})^{\frac{n - n_b}{2}}} \right) - \frac{\lambda}{4} p(p+3) \ln(w) \quad (30)$$

Since we consider that an abrupt change occurs at $n_c = n - \frac{w}{2} + 1$, the decision function becomes:

$$\begin{aligned} & \begin{matrix} H_1 \\ g'_n \geq h' \\ H_0 \end{matrix} \quad \text{with} \quad g'_n = \ln \left(\frac{\det(\hat{\Sigma}_0)^2}{\det(\hat{\Sigma}_{1a}) \det(\hat{\Sigma}_{1b})} \right) \quad \text{and} \quad h' = \frac{4}{w} h + \frac{\lambda}{w} p(p+3) \ln(w) \end{aligned} \quad (31)$$

3.1.2 Criterion derivation for a scalar case

For a one dimensional signal, the number of free parameters under H_0 corresponds to μ_0 and σ_0 , so $M_0 = 2$. Under H_1 , the number of free parameters is $\mu_{1a}, \sigma_{1a}, \mu_{1b}, \sigma_{1b}$, so $M_1 = 4$. The BIC criterion derivation for a one dimensional signal is given in [?] and corresponds to

$$\begin{aligned} & \begin{matrix} H_1 \\ g'_n \geq h' \\ H_0 \end{matrix} \quad \text{with} \quad g'_n = \ln \left(\frac{\hat{\sigma}_0^2}{\hat{\sigma}_{1a} \hat{\sigma}_{1b}} \right), \quad h' = \frac{4}{w} (h + \lambda \ln(w)) \end{aligned} \quad (32)$$

3.2 CUMulative Sum (CUSUM) algorithm

The CUSUM algorithm is a statistical test for the detection of a mean change in a Gaussian process [1]. It involves the calculation of a cumulative sum and works by tracking its deviations from a threshold value.

3.2.1 Criterion derivation for a multivariate case

In the case of a multi dimensional signal, when considering (1) and (2), in order to derive the CUSUM decision function, it is assumed that under H_0 , $\mu_0 = \mu_{1a}$ and $\Sigma_0 = \Sigma_{1a}$. Under H_1 , $\Sigma_{1b} = \Sigma_{1a}$. The decision rule is

based on a maximization of the log-likelihood ratio such as:

$$\ln \frac{H_1}{H_0} \underset{\geq}{\overset{\geq}{\leq}} h, \quad \text{with} \quad g_n = \sum_{m=n_c}^n s_m \quad (33)$$

Indeed, the log likelihood ration over n_c corresponds to the cumulative sum of values s_m such that:

$$\ln \left(\frac{\mathcal{L}_{n,1}}{\mathcal{L}_{n,0}} \right) = \ln \left(\frac{\prod_{m=n_a}^{n_b} p_{\mu_{1a}, \Sigma_{1a}}(x_m) \prod_{m=n_c}^n p_{\mu_{1b}, \Sigma_{1a}}(x_m)}{\prod_{m=n_a}^n p_{\mu_{1a}, \Sigma_{1a}}(x_m)} \right) \quad (34)$$

$$= \sum_{m=n_c}^n \ln \left(\frac{p_{\mu_{1b}, \Sigma_{1a}}(x_m)}{p_{\mu_{1a}, \Sigma_{1a}}(x_m)} \right) \quad (35)$$

$$= \sum_{m=n_c}^n s_m \quad (36)$$

Because x_m are i.i.d $\mathcal{N}_p(\mu, \Sigma)$, s_m becomes:

$$s_m = \ln \left(\frac{(2\pi)^{-\frac{p}{2}} \det(\Sigma_{1a})^{-\frac{1}{2}} \exp(-\frac{1}{2}(x_m - \mu_{1b})^T \Sigma_{1a}^{-1} (x_m - \mu_{1b}))}{(2\pi)^{-\frac{p}{2}} \det(\Sigma_{1a})^{-\frac{1}{2}} \exp(-\frac{1}{2}(x_m - \mu_{1a})^T \Sigma_{1a}^{-1} (x_m - \mu_{1a}))} \right) \quad (37)$$

$$= \ln \left(\frac{\exp(-\frac{1}{2}(x_m - \mu_{1b})^T \Sigma_{1a}^{-1} (x_m - \mu_{1b}))}{\exp(-\frac{1}{2}(x_m - \mu_{1a})^T \Sigma_{1a}^{-1} (x_m - \mu_{1a}))} \right) \quad (38)$$

$$= \frac{1}{2}(x_m - \mu_{1a})^T \Sigma_{1a}^{-1} (x_m - \mu_{1a}) - \frac{1}{2}(x_m - \mu_{1b})^T \Sigma_{1a}^{-1} (x_m - \mu_{1b}) \quad (39)$$

$$= -\frac{1}{2}x_m^T \Sigma_{1a}^{-1} \mu_{1a} - \frac{1}{2}\mu_{1a}^T \Sigma_{1a}^{-1} x_m + \frac{1}{2}\mu_{1a}^T \Sigma_{1a}^{-1} \mu_{1a} + \frac{1}{2}x_m^T \Sigma_{1a}^{-1} \mu_{1b} + \frac{1}{2}\mu_{1b}^T \Sigma_{1a}^{-1} x_m - \frac{1}{2}\mu_{1b}^T \Sigma_{1a}^{-1} \mu_{1b} \quad (40)$$

Since:

$$x_m^T \Sigma_{1a}^{-1} \mu_{1b} = \mu_{1b}^T \Sigma_{1a}^{-1} x_m \quad \text{and} \quad x_m^T \Sigma_{1a}^{-1} \mu_{1a} = \mu_{1a}^T \Sigma_{1a}^{-1} x_m \quad (41)$$

Then:

$$s_m = \frac{1}{2} [2\mu_{1b}^T \Sigma_{1a}^{-1} x_m - 2\mu_{1a}^T \Sigma_{1a}^{-1} x_m + (\mu_{1a}^T \Sigma_{1a}^{-1} \mu_{1a} - \mu_{1b}^T \Sigma_{1a}^{-1} \mu_{1b})] \quad (42)$$

$$= (\mu_{1b} - \mu_{1a})^T \Sigma_{1a}^{-1} x_m - \frac{1}{2}(\mu_{1b} + \mu_{1a})^T \Sigma_{1a}^{-1} (\mu_{1b} - \mu_{1a}) \quad (43)$$

$$= (\mu_{1b} - \mu_{1a})^T \Sigma_{1a}^{-1} \left(x_m - \frac{\mu_{1b} + \mu_{1a}}{2} \right) \quad (44)$$

To calculate s_m , all the unknowns in its expression, such as the mean vectors $\hat{\mu}_{1b}$, $\hat{\mu}_{1a}$ and the covariance matrix $\hat{\Sigma}_{1a}$, are replaced by their MLEs expressed in (12), (13). Then, according to (33), the CUSUM decision function becomes:

$$g_n \underset{\geq}{\overset{\geq}{\leq}} h \quad \text{with} \quad g_n = (\hat{\mu}_{1b} - \hat{\mu}_{1a})^T \hat{\Sigma}_{1a}^{-1} \left(\sum_{m=n_c}^n x_m - \frac{\hat{\mu}_{1b} + \hat{\mu}_{1a}}{2} \right) \quad (45)$$

$$= (\hat{\mu}_{1b} - \hat{\mu}_{1a})^T \hat{\Sigma}_{1a}^{-1} \left((n_c - n + 1)\hat{\mu}_{1b} - (n_c - n + 1)\frac{\hat{\mu}_{1b} + \hat{\mu}_{1a}}{2} \right) \quad (46)$$

$$= \frac{(n - n_b)}{2} (\hat{\mu}_{1b} - \hat{\mu}_{1a})^T \hat{\Sigma}_{1a}^{-1} (\hat{\mu}_{1b} - \hat{\mu}_{1a}) \quad (47)$$

If considering an abrupt change occurring inside a sliding window of w samples at $n_c = n - \frac{w}{2} + 1$, then the decision function can be expressed as follows, when considering (14) and (12):

$$g'_n \underset{\geq}{\overset{\geq}{\leq}} h' \quad \text{where} \quad g_n = (\hat{\mu}_{1b} - \hat{\mu}_{1a})^T \hat{\Sigma}_{1a}^{-1} (\hat{\mu}_{1b} - \hat{\mu}_{1a}) \quad \text{and} \quad h' = \frac{4}{w} h \quad (48)$$

3.2.2 Criterion derivation for a scalar case

The CUSUM criterion derivation for a one dimensional signal is given in [?] and corresponds to:

$$g_n = \sum_{m=n_c}^n s_m \underset{H_0}{\overset{H_1}{\geq}} h \quad \text{with} \quad s_m = (n - n_b) \frac{(\hat{\mu}_{1b} - \hat{\mu}_{1a})^2}{2\hat{\sigma}_{1a}^2} \quad (49)$$

Since we aim at detecting a mean change at $n_c = n - \frac{w}{2} + 1$, the CUSUM decision function corresponds to:

$$g'_n = \frac{(\hat{\mu}_{1b} - \hat{\mu}_{1a})^2}{\hat{\sigma}_{1a}^2} \underset{H_0}{\overset{H_1}{\geq}} h' \quad \text{where} \quad h' = \frac{4}{w} h \quad (50)$$

3.3 Hotelling T² test

When testing for the difference between the means of two normally distributed samples with unknown variances but assumed equal, the most commonly used statistical test is the Hotelling T² test [5]. It aims at quantifying the difference between two Normal distributions using the mean and variance in the data.

3.3.1 Criterion derivation for a multivariate case

The Hotelling T² test is the multivariate extension of the Student's t-test [6]. In a Hotelling T² test, the difference between the $(p \times 1)$ mean vectors of two samples is considered [7]. In order to formulate the Hotelling T² test, we need to derive the maximum likelihood ratio test for both hypotheses H₀ and H₁. The likelihood ratio corresponds to:

$$\Lambda_n = \frac{\mathcal{L}_{n,1}}{\mathcal{L}_{n,0}} \quad (51)$$

which is maximized when considering the MLEs in (14), (12) and (13), such as:

$$\Lambda_n = \frac{\left(\frac{1}{(2\pi)^p \det(\hat{\Sigma}_1)} \right)^{\frac{w}{2}} \exp \left(-\frac{1}{2} \left(\sum_{m=n_a}^{n_b} (x_m - \hat{\mu}_{1a})^T \hat{\Sigma}_1^{-1} (x_m - \hat{\mu}_{1a}) + \sum_{m=n_c}^n (x_m - \hat{\mu}_{1b})^T \hat{\Sigma}_1^{-1} (x_m - \hat{\mu}_{1b}) \right) \right)}{\left(\frac{1}{(2\pi)^p \det(\hat{\Sigma}_0)} \right)^{\frac{w}{2}} \exp \left(-\frac{1}{2} \sum_{m=n_a}^n (x_m - \hat{\mu}_0)^T \hat{\Sigma}_0^{-1} (x_m - \hat{\mu}_0) \right)} \quad (52)$$

where:

$$\hat{\Sigma}_1 = \frac{1}{w} \left(\sum_{m=n_a}^{n_b} (x_m - \hat{\mu}_{1a})(x_m - \hat{\mu}_{1a})^T + \sum_{m=n_c}^n (x_m - \hat{\mu}_{1b})(x_m - \hat{\mu}_{1b})^T \right) \quad (53)$$

$$= \frac{n_c - n_a}{w} \hat{\Sigma}_{1a} + \frac{n - n_b}{w} \hat{\Sigma}_{1b} \quad (54)$$

$$(55)$$

Then (52) can be simplified using (24), such as:

$$\Lambda_n = \left(\frac{\det(\hat{\Sigma}_0)}{\det(\hat{\Sigma}_1)} \right)^{\frac{w}{2}} \quad (56)$$

$$= \left(\frac{\det \left[\frac{1}{w} \sum_{m=n_a}^n (x_m - \hat{\mu}_0)(x_m - \hat{\mu}_0)^T \right]}{\det \left[\frac{1}{w} \left(\sum_{m=n_a}^{n_b} (x_m - \hat{\mu}_{1a})(x_m - \hat{\mu}_{1a})^T + \sum_{m=n_c}^n (x_m - \hat{\mu}_{1b})(x_m - \hat{\mu}_{1b})^T \right) \right]} \right)^{\frac{w}{2}} \quad (57)$$

The above expression can be simplified, indeed:

$$\sum_{m=n_a}^{n_b} (x_m - \hat{\mu}_0)(x_m - \hat{\mu}_0)^T = \sum_{m=n_a}^{n_b} (x_m - \hat{\mu}_{1a})(x_m - \hat{\mu}_{1a})^T + (n_c - n_a)(\hat{\mu}_{1a} - \hat{\mu}_0)(\hat{\mu}_{1a} - \hat{\mu}_0)^T \quad (58)$$

$$= \sum_{m=n_a}^{n_b} (x_m - \hat{\mu}_{1a})(x_m - \hat{\mu}_{1a})^T + \frac{(n_c - n_a)(n - n_b)^2}{w^2} (\hat{\mu}_{1a} - \hat{\mu}_{1b})(\hat{\mu}_{1a} - \hat{\mu}_{1b})^T \quad (59)$$

Likewise, we have:

$$\sum_{m=n_c}^n (x_m - \hat{\mu}_0)(x_m - \hat{\mu}_0)^T = \sum_{m=n_c}^n (x_m - \hat{\mu}_{1b})(x_m - \hat{\mu}_{1b})^T + \frac{(n - n_b)(n_c - n_a)^2}{w^2} (\hat{\mu}_{1b} - \hat{\mu}_{1a})(\hat{\mu}_{1b} - \hat{\mu}_{1a})^T \quad (60)$$

According to (54), we obtain:

$$\hat{\Sigma}_0 = \hat{\Sigma}_1 + \frac{(n_c - n_a)(n - n_b)}{w^2} (\hat{\mu}_{1b} - \hat{\mu}_{1a})(\hat{\mu}_{1b} - \hat{\mu}_{1a})^T \quad (61)$$

Therefore, when substituting (61) in (57), we have:

$$\mathbf{\Lambda}_n = \left(\frac{\det(\hat{\Sigma}_1 + \frac{(n_c - n_a)(n - n_b)}{w^2} (\hat{\mu}_{1b} - \hat{\mu}_{1a})(\hat{\mu}_{1b} - \hat{\mu}_{1a})^T)}{\det(\hat{\Sigma}_1)} \right)^{\frac{w}{2}} \quad (62)$$

$$= \left(\frac{\frac{w^2}{(n_c - n_a)(n - n_b)} \det(\hat{\Sigma}_1 + \frac{(n_c - n_a)(n - n_b)}{w^2} (\hat{\mu}_{1b} - \hat{\mu}_{1a})(\hat{\mu}_{1b} - \hat{\mu}_{1a})^T)}{\frac{w^2}{(n_c - n_a)(n - n_b)} \det(\hat{\Sigma}_1)} \right)^{\frac{w}{2}} \quad (63)$$

$$= \left(\frac{\det(\hat{\Sigma}_1) \left(\frac{w^2}{(n_c - n_a)(n - n_b)} + (\hat{\mu}_{1b} - \hat{\mu}_{1a})^T \hat{\Sigma}_1^{-1} (\hat{\mu}_{1b} - \hat{\mu}_{1a}) \right)}{\frac{w^2}{(n_c - n_a)(n - n_b)} \det(\hat{\Sigma}_1)} \right)^{\frac{w}{2}} \quad (64)$$

$$= \left(1 + \frac{(n_c - n_a)(n - n_b)}{w^2} (\hat{\mu}_{1b} - \hat{\mu}_{1a})^T \hat{\Sigma}_1^{-1} (\hat{\mu}_{1b} - \hat{\mu}_{1a}) \right)^{\frac{w}{2}} \quad (65)$$

$$= (1 + g_n)^{\frac{w}{2}} \quad (66)$$

Note, when considering $u = \frac{w^2}{(n_c - n_a)(n - n_b)}$, $V = \hat{\Sigma}_1$, $\vec{w} = (\hat{\mu}_{1b} - \hat{\mu}_{1a})$:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} u & \vec{w}^T \\ -\vec{w} & V \end{bmatrix} \quad (67)$$

$$\det(A) = \begin{cases} \det(A_{11}) \det(A_{22} - A_{21} A_{11}^{-1} A_{12}) \\ \det(A_{22}) \det(A_{11} - A_{12} A_{22}^{-1} A_{21}) \end{cases} = \begin{cases} u \det(V + \frac{1}{u} \vec{w} \vec{w}^T) \\ \det(V) (u + \vec{w}^T V^{-1} \vec{w}) \end{cases} \quad (68)$$

Thus,

$$u \det(V + \frac{1}{u} \vec{w} \vec{w}^T) = \det(V) (u + \vec{w}^T V^{-1} \vec{w}) \quad (69)$$

$$\frac{\det(V + \frac{1}{u} \vec{w} \vec{w}^T)}{\det(V)} = 1 + \frac{\vec{w}^T V^{-1} \vec{w}}{u} \quad (70)$$

Informally, this likelihood ratio aims at measuring the plausibility of H_0 relative to H_1 . Therefore, if the likelihood ratio is sufficiently small, we might be inclined to reject H_0 . According to Neyman Pearson Lemma [6], this is made possible by setting (66) less than $\gamma \in [0, 1]$. Indeed, under H_0 , we have $\mu_{1a} = \mu_{1b}$, so $\mathbf{\Lambda}_n = 1$ and under H_1 , we have $\mu_{1a} \neq \mu_{1b}$, so $\mathbf{\Lambda}_n > 1$. This leads to:

$$g_n > \gamma^{\frac{2}{w}} - 1 = h \quad (71)$$

g_n corresponds to the decision function that is compared to an adjusted threshold h such as:

$$g_n \underset{H_0}{\overset{H_1}{\geq}} h \quad \text{with} \quad g_n = \left(\frac{w}{n_c - n_a} + \frac{w}{n - n_b} \right)^{-1} (\hat{\mu}_{1b} - \hat{\mu}_{1a})^T \hat{\Sigma}_1^{-1} (\hat{\mu}_{1b} - \hat{\mu}_{1a}) \quad (72)$$

According to (54), the decision function g_n can also be expressed as:

$$g'_n \underset{H_0}{\overset{H_1}{\geq}} h' = h \times \frac{w^2}{(n_c - n_a)(n - n_b)} \quad (73)$$

$$\text{with} \quad g'_n = (\hat{\mu}_{1b} - \hat{\mu}_{1a})^T \left(\frac{n_c - n_a}{w} \hat{\Sigma}_{1a} + \frac{n - n_b}{w} \hat{\Sigma}_{1b} \right)^{-1} (\hat{\mu}_{1b} - \hat{\mu}_{1a}) \quad (74)$$

If an abrupt change occurs inside a sliding window of w samples at $n_c = n - \frac{w}{2} + 1$, the Hotelling's T^2 decision rules becomes:

$$g''_n \underset{H_0}{\overset{H_1}{\geq}} h'' = h' \times 2 \quad \text{with} \quad g'_n = (\hat{\mu}_{1b} - \hat{\mu}_{1a})^T (\hat{\Sigma}_{1a} + \hat{\Sigma}_{1b})^{-1} (\hat{\mu}_{1b} - \hat{\mu}_{1a}) \quad (75)$$

3.3.2 Criterion derivation for a scalar case: the Student t-test

The student's t-test is a generalization of Hotelling's T^2 statistic, used in univariate hypothesis testing. As for the multivariate case, we need to derive the maximum likelihood ratio test which corresponds to:

$$\Lambda_n = \frac{\mathcal{L}_{n,1}}{\mathcal{L}_{n,0}} = \left(\frac{\hat{\sigma}_0^2}{\hat{\sigma}_1^2} \right)^{\frac{w}{2}} \quad (76)$$

$$= \left(\frac{\frac{1}{w} \left(\sum_{m=n_a}^{n_b} (x_m - \hat{\mu}_0)^2 + \sum_{m=n_c}^n (x_m - \hat{\mu}_0)^2 \right)}{\frac{1}{w} \left(\sum_{m=n_a}^{n_b} (x_m - \hat{\mu}_{1a})^2 + \sum_{m=n_c}^n (x_m - \hat{\mu}_{1b})^2 \right)} \right)^{\frac{w}{2}} \quad (77)$$

The above expression can be simplified, indeed:

$$\sum_{m=n_a}^{n_b} (x_m - \hat{\mu}_0)^2 = \sum_{m=n_a}^{n_b} ((x_m - \hat{\mu}_{1a}) + (\hat{\mu}_{1a} - \hat{\mu}_0))^2 \quad (78)$$

$$= \sum_{m=n_a}^{n_b} (x_m - \hat{\mu}_{1a})^2 + (n_c - n_a)(\hat{\mu}_{1a} - \hat{\mu}_0)^2 \quad (79)$$

$$= \sum_{m=n_a}^{n_b} (x_m - \hat{\mu}_{1a})^2 + (n_c - n_a) \left(\hat{\mu}_{1a} - \frac{(n_c - n_a)\hat{\mu}_{1a} + (n - n_b)\hat{\mu}_{1b}}{w} \right)^2 \quad (80)$$

$$= \sum_{m=n_a}^{n_b} (x_m - \hat{\mu}_{1a})^2 + \frac{(n_c - n_a)(n - n_b)^2}{w^2} (\hat{\mu}_{1a} - \hat{\mu}_{1b})^2 \quad (81)$$

As for (78), we have:

$$\sum_{m=n_c}^n (x_m - \hat{\mu}_0)^2 = \sum_{m=n_c}^n (x_m - \hat{\mu}_{1b})^2 + \frac{(n_c - n_a)^2 (n - n_b)}{w^2} (\hat{\mu}_{1b} - \hat{\mu}_{1a})^2 \quad (82)$$

and $\hat{\sigma}_0^2 = \frac{1}{w} \sum_{m=n_a}^n (x_m - \hat{\mu}_0)^2$ can also be expressed as follows:

$$\begin{aligned} \hat{\sigma}_0^2 &= \frac{1}{w} \left(\sum_{m=n_a}^{n_b} (x_m - \hat{\mu}_{1a})^2 + \sum_{m=n_c}^n (x_m - \hat{\mu}_{1b})^2 \right) + \frac{(n_c - n_a)(n - n_b)^2}{w^3} (\hat{\mu}_{1b} - \hat{\mu}_{1a})^2 \\ &\quad + \frac{(n_c - n_a)^2 (n - n_b)}{w^3} (\hat{\mu}_{1b} - \hat{\mu}_{1a})^2 \end{aligned} \quad (83)$$

By analogy with (54), we have:

$$\hat{\sigma}_1^2 = \frac{1}{w} \left(\sum_{m=n_a}^{n_b} (x_m - \hat{\mu}_{1a})^2 + \sum_{m=n_c}^n (x_m - \hat{\mu}_{1b})^2 \right) \quad (84)$$

$$= \frac{n_c - n_a}{w} \hat{\sigma}_{1a}^2 + \frac{n - n_b}{w} \hat{\sigma}_{1b}^2 \quad (85)$$

$$\text{where } \hat{\sigma}_{1a}^2 = \frac{1}{n_c - n_a} \sum_{m=n_a}^{n_b} (x_m - \hat{\mu}_{1a})^2 \quad \text{and} \quad \hat{\sigma}_{1b}^2 = \frac{1}{n - n_b} \sum_{m=n_c}^n (x_m - \hat{\mu}_{1b})^2 \quad (86)$$

Where $\hat{\sigma}_1^2$ is commonly referred as the pooled variance [7]. Therefore,

$$\hat{\sigma}_0^2 = \hat{\sigma}_1^2 + \frac{(n_c - n_a)(n - n_b)}{w^2} (\hat{\mu}_{1b} - \hat{\mu}_{1a})^2 \quad (87)$$

Thus, when substituting (87) in (77), we have:

$$\Lambda_n = \left(\frac{\hat{\sigma}_1^2 + \frac{(n_c - n_a)(n - n_b)}{w^2} (\hat{\mu}_{1b} - \hat{\mu}_{1a})^2}{\hat{\sigma}_1^2} \right)^{\frac{w}{2}} \quad (88)$$

$$= \left(1 + \frac{(n_c - n_a)(n - n_b)}{w^2 \hat{\sigma}_1^2} \frac{(\hat{\mu}_{1b} - \hat{\mu}_{1a})^2}{\hat{\sigma}_1^2} \right)^{\frac{w}{2}} \quad (89)$$

$$= (1 + g_n)^{\frac{w}{2}} \quad (90)$$

g_n corresponds to the Student's t-test and can be also expressed as:

$$g_n = \left(\frac{w}{n_c - n_a} + \frac{w}{n - n_b} \right)^{-1} \frac{(\hat{\mu}_{1b} - \hat{\mu}_{1a})^2}{\hat{\sigma}_1^2} \quad (91)$$

By setting the likelihood ratio less than $\lambda \in [0, 1]$ according to Neyman Pearson Lemma, we obtain:

$$(1 + g_n)^{\frac{w}{2}} < \gamma \Rightarrow g_n > \gamma^{\frac{2}{w}} - 1 = h \quad (92)$$

The Student t-test can be expressed as:

$$g'_n \underset{H_0}{\overset{H_1}{\geq}} h' \quad \text{with} \quad g'_n = g_n \times \frac{w^2}{(n_c - n_a)(n - n_b)} = \frac{(\hat{\mu}_{1b} - \hat{\mu}_{1a})^2}{\hat{\sigma}_1^2} \quad (93)$$

$$= \frac{(\hat{\mu}_{1b} - \hat{\mu}_{1a})^2}{\frac{n_c - n_a}{w} \hat{\sigma}_{1a}^2 + \frac{n - n_b}{w} \hat{\sigma}_{1b}^2} \quad (94)$$

$$\text{and} \quad h' = h \times \frac{w^2}{(n_c - n_a)(n - n_b)} \quad (95)$$

with h' is a threshold value adjusted according to some desired decision probabilities. When an abrupt change is occurring at $n_c = n - \frac{w}{2} + 1$, the Student's t-test decision rules becomes:

$$g''_n = \frac{(\hat{\mu}_{1b} - \hat{\mu}_{1a})^2}{\hat{\sigma}_{1a}^2 + \hat{\sigma}_{1b}^2} \underset{H_0}{\overset{H_1}{\geq}} h'' = 2h' \quad (96)$$

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