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A study on total irregularities of certain graphs and digraphs

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Abstract: The total irregularity of a simple undirected graph G is denoted by $irr_t(G)$ and is defined as $irr_t(G) = \frac{1}{2} \sum_{u,v \in V(G)} |d(u) - d(v)|$. In this paper, we introduce the notion of edge-transformation in relation to total irregularity of simple graphs with at least one cut edge as well as an edge-joint between two graphs. We also introduce the notion of total irregularity with respect to in-degree and out-degree in directed graphs. We also introduce the concept of total irregularity in respect of in-degree and out-degree in simple directed graphs. These invariants are called total in-irregularity and total out-irregularity, respectively. In this paper, we initiate a study on these parameters of given simple undirected graphs and simple digraphs.

Subjects: Advanced Mathematics; Combinatorics; Discrete Mathematics; Mathematics & Statistics; Science

Keywords: total irregularity; branch-transformation; edge-transformation; edge-joint; total in-irregularity; total out-irregularity

Mathematics Subject classifications: 05C07; 05C20; 05C38; 05C75

1. Introduction

For general notations and concepts in graph theory, we refer to Bondy and Murty (1976), Harary (1969), West (2001) and for digraph theory, we further refer to Chartrand and Lesniak (2000), Jensen

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Johan Kok is registered with the South African Council for Natural Scientific Professions (South Africa) as a professional scientists in both Physical Science and Mathematical Science. His main research areas are in Graph Theory and the reconstruction of motor vehicle collisions. He has been endorsed by international peers as skilled in a wide range of combinatorica disciplines. Johan has a keen interest in mathematics education as well.

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PUBLIC INTEREST STATEMENT

Major real world applications of irregularities of graphs are in chemical graph theory, biological and economical systems. Results stemming from operations between graphs or structurally changing a graph, are enhanced through this study related to branch-transformation and the introduction of an edge joint. The immediate application of edge-joint is to gain an understanding of stringing of graphs and many biological structures can be modelled as graphs stringed sequentially. With regards to directed graphs the field of study is wide open. Directed graphs allow the analysis of the influence a vertex (the tail) has over a neighbour (the head). Applications lies in the initial modelling of directed graphs as null-graphs with vertices carrying floating out-arcs seeking heads. A vertex with zero floating out-arcs can only be a head of one or more vertices, hence resembles a black hole in cosmic space. Modelling cosmic systems as null-graphs with vertices carrying floating out-arcs seeking heads is regarded as a promising new application.

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and Gutin (2007). All graphs mentioned in this paper are simple, connected and finite graphs, unless mentioned otherwise. Also, except for Section 4, all the graphs mentioned here are undirected graphs.

A graph G is said to be *regular* if the degree of all vertices are equal. A graph that is not regular is called an *irregular graph*. The total irregularity of a given simple connected graph is defined in Albertson (1997) as follows.

Definition 1.1 (Albertson, 1997) The *imbalance* of an edge $e = uv$ in a given graph G is defined as $|d(u) - d(v)|$. The *total irregularity* of a graph G, denoted by $irr_t(G)$, is defined as $irr_t(G) = \frac{1}{2} \sum_{u,v \in V(G)} |d(u) - d(v)|$.

If the vertices of a graph G on n vertices are labelled as $v_i, i = 1, 2, 3, \ldots, n$, then the definition may be $irr_t(G) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} |d(v_i) - d(v_j)| = \sum_{i=1}^{n} \sum_{j=i+1}^{n} |d(v_i) - d(v_j)|$ or $\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} |d(v_i) - d(v_j)|$. For a graph on a singular vertex (*1-null* graph or K_1), we define $irr_t(G) = 0$. Clearly, $irr_t(G) = 0$ if and only if G is regular.

If an edge e is a cut-edge of the graph G and a component of $G - e$ is a tree T, then T is called a *hanging tree* of G. The notion of *branch-transformation* of a graph has been introduced in Zhu, You, and Yang (in press) as follows.

Definition 1.2 (Zhu et al., in press) Let G be a graph with at least two pendant vertices. Without loss of generality, let u be a vertex of G with $d_G(u) \geq 3$, T be a hanging tree of G connecting to u with $|V(T)| \geq 1$ and v be a pendant vertex of G with $v \notin T$. Let G' be the graph obtained from G by deleting T from vertex u and attaching it to vertex v. We call the transformation from G to G' a *branch-transformation* on G from vertex u to vertex v.

Certain studies on irregularities and total irregularities of given graphs and the properties graphs related to these irregularities have been done in Abdo, Brandt, and Dimitrov (2014), Abdo, Cohen, and Dimitrov (in press), Abdo and Dimitrov (2014), Albertson (1997), Dimitrov and Škrekovski (2015), Henning and Rautenbach (2007), Zhu et al. (in press). There are many specific results in respect of cut vertices and cut-edges in the studies on various concepts in graph theory. Many applications rely on either the existence of cut-vertices or cut edges as well. Where *stringing* of graphs is required through linking graphs pairwise through adding a single edges between pairs of vertices, multiple cut-edges exist in the resultant stringed graph. The order of stringing may, in some instances, not obey the commutative property with respect to certain invariants. For directed graphs, orientated stringing is generally more complex and may require extremal graph theoretic analysis.

Motivated from these observations, in this paper, we introduce the notion of edge-transformation in relation to total irregularity of simple graphs with at least one cut edge as well as an edge-joint between two graphs. We also introduce the notion of total irregularity with respect to in-degree and out-degree in directed graphs and initiate a study on certain types of total irregularities of given classes of directed and undirected graphs.

2. Total irregularity resulting from edge-joints

Consider a graph G on n vertices with two connected components G_1 and G_2. Therefore, $G = G_1 \cup G_2$. Hence, the total irregularity of G is given by $irr_t(G) = irr_t(G_1) + irr_t(G_2) + \sum_{i=1}^{r} \sum_{j=1}^{s} |d(u_i) - d(v_j)|$, where $u_i \in V(G_1), v_j \in V(G_2)$ and $r = |V(G_1)|$ and $s = |V(G_2)|$.

The concept of an edge-joint between two simple undirected graphs G and H is defined below.

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Definition 2.1 The *edge-joint* of two graphs G and H is the graph obtained by adding one edge, say uv, where $u \in V(G), v \in V(H)$, and is denoted by $G \rightsquigarrow_{uv} H$.

Remark 2.2 It is to be noted that $G \rightsquigarrow_{uv} H = G \cup H + uv$ and $G \rightsquigarrow_{uv} H \simeq H \rightsquigarrow_{vu} G$.

Let G be a graph on n vertices with two connected components G_1 and G_2 whose vertex sets are $V(G_1) = \{u_i : 1 \le i \le r\}$ and $V(G_2) = \{v_j : 1 \le j \le s\}$. We fix the vertices u_1 from G_1 and v_1 from G_2. Now, we define the vertex subsets $V_1 = \{u_x : d_{G_1}(u_x) \le d_{G_1}(u_1), x \ne 1\}; V_2 = \{u_y : d_{G_1}(u_y) > d_{G_1}(u_1)\}$ and let $|V_1| = a$ and $|V_2| = b$. Then, choose $V_3 = \{v_x : d_{G_2}(v_x) \le d_{G_1}(u_1)\}$ and $V_4 = \{v_y : d_{G_2}(v_y) > d_{G_1}(u_1)\}$, where $|V_3| = a^*$ and $|V_4| = b^*$. Similarly, let $V_5 = \{v_z : d_{G_2}(v_z) \le d_{G_2}(v_1),\ z \ne 1\}$ and $V_6 = \{v_w : d_{G_2}(v_w) > d_{G_2}(v_1)\}$ where $|V_5| = c$ and $|V_6| = d$ and choose $V_7 = \{u_z : d_{G_1}(u_z) \le d_{G_2}(v_1)\}$ and $V_8 = \{u_w : d_{G_1}(u_w) > d_{G_2}(v_1)\}$ where $|V_7| = c^*$ and $|V_8| = d^*$.

Define the variables $b = r - a, d = s - c = n - r - c, b^* = r - a^*$ and $d^* = s - c^* = n - r - c^*$.

THEOREM 2.3 *Let G be a graph on n vertices with two connected components G_1 and G_2, where $V(G_1) = \{u_i : 1 \le i \le r\}$ and $V(G_2) = \{v_j : 1 \le j \le s\}$. Let $G' = G_1 \rightsquigarrow_{u_1 v_1} G_2$. Then, we have* $irr_t(G') = irr_t(G_1) + irr_t(G_2) + \sum_{i=1}^{r}\sum_{j=1}^{s} |d_{G_1}(u_i) - d_{G_2}(v_j)| + 2n - 2(b + b^* + d + d^*) - 2$ *or* $irr_t(G') = irr_t(G_1) + irr_t(G_2) + \sum_{i=1}^{r}\sum_{j=1}^{s} |d_{G_1}(u_i) - d_{G_2}(v_j)| + 2(a + a^* + c + c^*) - 2n + 2.$

Proof Clearly, for the graph $G = G_1 \cup G_2$, we have $irr_t(G) = irr_t(G_1) + irr_t(G_2) + \sum_{i=1}^{r}\sum_{j=1}^{s} |d_{G_1}(u_i) - d_{G_2}(v_j)|$ with $|V(G_1)| = r$ and $|V(G_2)| = s$.

By increasing $d_{G_1}(u_1)$ by 1 we increase the partial sum $\sum^{a}_{\substack{j=1 \\ w_j \in V_1}} |d_{G_1}(u_1) - d_{G_1}(w_j)|$ by exactly $(a-1)$. It also reduces the partial sum $\sum^{b}_{\substack{j=1 \\ w_j \in V_2}} |d_{G_1}(u_1) - d_{G_1}(w_j)|$ by exactly b. It also increases the partial sum $\sum^{a^*}_{\substack{j=1 \\ w_j \in V_3}} |d_{G_1}(u_1) - d_{G_2}(w_j)|$ by exactly a^* and decreases the partial sum $\sum^{b^*}_{\substack{j=1 \\ w_j \in V_4}} |d_{G_1}(u_1) - d_{G_2}(w_j)|$ by exactly b^*. Furthermore, by increasing $d_{G_2}(v_1)$ by 1, we increase the partial sum $\sum^{c}_{\substack{j=1 \\ w_j \in V_5}} |d_{G_2}(v_1) - d_{G_2}(w_j)|$ by exactly $(c-1)$. It also reduces the partial sum $\sum^{d}_{\substack{j=1 \\ w_j \in V_6}} |d_{G_1}(u_1) - d_{G_2}(w_j)|$ by exactly d. It also increases the partial sum $\sum^{c^*}_{\substack{j=1 \\ w_j \in V_7}} |d_{G_1}(u_1) - d_{G_1}(w_j)|$ by exactly c^* and decreases the partial sum $\sum^{d^*}_{\substack{j=1 \\ w_j \in V_8}} |d_{G_1}(u_1) - d_{G_1}(w_j)|$ by exactly d^*.

Hence, we have an interim result as follows.

$$\begin{aligned} irr_t(G') &= irr_t(G_1) + irr_t(G_2) + \sum_{i=1}^{r}\sum_{j=1}^{s} |d_{G_1}(u_i) - d_{G_2}(v_j)| + (a-1) - b + a^* - b^* + (c-1) - d + c^* - d^* \\ &= irr_t(G_1) + irr_t(G_2) + \sum_{i=1}^{r}\sum_{j=1}^{s} |d_{G_1}(u_i) - d_{G_2}(v_j)| + (a-b) + (a^* - b^*) + (c-d) + (c^* - d^*) - 2. \end{aligned}$$

By substituting the variables b, d, b^* and d^* as defined in Definition the final result is as follows.

$$irr_t(G') = irr_t(G_1) + irr_t(G_2) + \sum_{i=1}^{r}\sum_{j=1}^{s} |d_{G_1}(u_i) - d_{G_2}(v_j)| + 2n - 2(b + b^* + d + d^*) - 2, \quad \textit{or;}$$

$irr_t(G') = irr_t(G_1) + irr_t(G_2) + \sum_{i=1}^{r}\sum_{j=1}^{s} |d_{G_1}(u_i) - d_{G_2}(v_j)| + 2(a + a^* + c + c^*) - 2n + 2$, follows. □

Clearly $irr_t(G')$ is edge dependent in general but we have the following Corollary.

COROLLARY 2.4 Let the degree sequence of graphs G_1 and G_2 be $(d_{G_1}(u_1) \le d_{G_1}(u_2) \le d_{G_1}(u_3) \le \ldots \le d_{G_1}(u_n))$ and $(d_{G_2}(v_1) \le d_{G_2}(v_2) \le d_{G_2}(v_3) \le \ldots \le d_{G_2}(v_m))$, respectively. If $d_{G_1}(u_i) = d_{G_2}(v_j)$ for some i, j and $d_{G_1}(u_k) = d_{G_2}(v_l)$ for some k, l and $G' = G_1 \rightsquigarrow_{u_i v_l} G_2$ and $G'' = G_1 \rightsquigarrow_{u_k v_j} G_2$ then, $irr_t(G') = irr_t(G'')$.

Proof Begin the proof by choosing any vertex degree value t_1 in the degree sequence of G_1 and identify largest vertex index say, i for which $d_{G_1}(u_i) = t_1$. Similarly, choose any vertex degree value t_2 in the degree sequence of G_2 and identify largest vertex index say, l for which $d_{G_2}(v_l) = t_2$. Here, we have to consider the following cases.

Case-1: With respect to $G' = G_1 \rightsquigarrow_{u_i v_l} G_2$, set the values as follows.

(i) $|V_1| = a = i - 1$,

(ii) $|V_2| = b = n - i$,

(iii) $|V_3| = a^* = j$,

(iv) $|V_4| = b^* = m - j$,

(v) $|V_5| = c = l - 1$,

(vi) $|V_6| = d = m - l$,

(vii) $|V_7| = c^* = k$,

(viii) $|V_8| = d^* = n - k$.

Therefore, we have
$2(n + m) - 2((n - i) + (m - j) + (m - l) + (n - k)) - 2 = 2(i + j + k + l - (n + m)) - 2$.

Case-2: In respect of $G'' = G_1 \rightsquigarrow_{u_k v_j} G_2$, set the values as follows.

(i) $|V_1| = a = k - 1$,

(ii) $|V_2| = b = n - k$,

(iii) $|V_3| = a^* = l$,

(iv) $|V_4| = b^* = m - l$,

(v) $|V_5| = c = j - 1$,

(vi) $|V_6| = d = m - j$,

(vii) $|V_7| = c^* = i$,

(viii) $|V_8| = d^* = n - i$.

Therefore, here we have
$2(n + m) - 2((n - k) + (m - l) + (m - j) + (n - i)) - 2 = 2(k + l + j + i - (n + m)) - 2$.

Since Case-1 and Case-2 yield the same result, the result $irr_t(G') = irr_t(G'')$ follows from Theorem 2.3. □

An immediate consequence of Corollary 2.4 is that for regular graphs G_1 and G_2 we have $irr_t(G_1 \rightsquigarrow_{vu} G_2)$ $u \in V(G_1) v \in V(G_2)$ is a constant. This result is proved in the following proposition.

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Proposition 2.5 For the regular graphs G_1, G_2 on n, m vertices, respectively, with $d_{G_1}(u) \geq d_{G_2}(v)$ we have

$$irr_t(G_1 \rightsquigarrow_{uv} G_2) = \begin{cases} 2(n+m)-2, & \text{if } d_{G_1}(u) = d_{G_2}(v), \\ nm|d_{G_1}(u) - d_{G_2}(v)| + 2(n-1), & \text{if } d_{G_1}(u) > d_{G_2}(v). \end{cases}$$

Proof The proof follows immediately from Corollary 2.4. □

We note that if G_1 and G_2 are of equal k-regularity, then $irr_t(G_1 \rightsquigarrow_{uv} G_2)$ is independent of the k-*degree* of the vertices.

3. Total irregularity due to edge-transformation

Consider a graph G on $n = l_1 + l_2$ vertices and a cut edge u_1v_1. Let $G = (G_1 \cup G_2) + u_1v_1$, $u_1 \in V(G_1) = \{u_i : 1 \leq i \leq l_1\}$ and $v_1 \in V(G_2) = \{v_i : 1 \leq i \leq l_2\}$. Edge-transformation with respect to u_1 will be the graph $G^{u_iv_1}$ obtained by deleting the edge u_1v_1 and adding the edge u_iv_1 for any $i \neq 1$. We call G_1 the *master graph* and G_2 the *slave graph*.

Let us now introduce the notion of edge-transformation partitioning of a vertex set of a given graph as follows.

Definition 3.1 The *edge-transformation partitioning* of the vertex set $V(G)$ of a graph G on n vertices with at least one cut edge say u_1v_1, is defined to be $V_h = \{u_i, v_k : d_{G_1}(u_i) = d_{G_1}(u_1) - 1$ and $d_{G_2}(v_k) = d_{G_1}(u_1) - 1\} \cup \{u_1\}, h = |V_h|$, and $V_s = \{u_i, v_k : d_{G_1}(u_i) > d_{G_1}(u_1) - 1$ and $d_{G_2}(v_k) > d_{G_1}(u_1) - 1\}, s = |V_s|$ and $V_t = \{u_i, v_k : d_{G_1}(u_i) < d_{G_1}(u_1) - 1$ and $d_{G_2}(v_k) < d_{G_1}(u_1) - 1\}, t = |V_t|$.

Invoking Definition 3.1, we now define certain vertex sets in G as given below. Let $V_{s_1} = \{u_j, v_k : d_{G_1}(u_j) \leq d_{G_1}(u_i)$ and $d_{G_2}(v_k) \leq d_{G_1}(u_i)$ and $u_j, v_k \in V_s\}, m = |V_{s_1}|$, and $V_{s_2} = \{u_j, v_k : d_{G_1}(u_j) > d_{G_1}(u_i)$ and $d_{G_2}(v_k) > d_{G_1}(u_i)$ and $u_j, v_k \in V_s\}, l = |V_{s_2}|$, and $V_{t_1} = \{u_j, v_k : d_{G_1}(u_j) \leq d(u_i)$ and $d_{G_2}(v_k) \leq d_{G_1}(u_i)$ and $u_j, v_k \in V_t\}, m_1 = |V_{t_1}|$, and $V_{t_2} = \{u_j, v_k : d_{G_1}(u_j) > d_{G_1}(u_i)$ and $d_{G_2}(v_k) > d_{G_1}(u_i)$ and $u_j, v_k \in V_t\}, l_1 = |V_{t_2}|$.

In view of the above notions, we propose the following theorem.

THEOREM 3.2 *For a graph G with a cut edge u_1v_1, let $G - u_1v_1 = G_1 \cup G_2$. After edge-transformation in respect of v_1 we have*

$$irr_t(G^{u_iv_1}) = \begin{cases} irr_t(G), & \text{if } d_{G_1}(u_i) = d_{G_1}(u_1) - 1, \\ irr_t(G) + 2m, & \text{if } d_{G_1}(u_i) > d_{G_1}(u_1) - 1, \\ irr_t(G) - 2(h + l_1), & \text{if } d_{G_1}(u_i) < d_{G_1}(u_1) - 1. \end{cases}$$

Proof If $d_{G_1}(u_i) = d_{G_1}(u_1) - 1$, then reducing $d_{G_1}(u_1)$ by 1, reduces the partial sum $\sum^{h}_{\substack{j=1 \\ w_j \in V_h}} |d_{G_1}(u_1) - d(w_j)|$ by exactly $(h-1)$. It increases the partial sum $\sum^{s}_{\substack{j=1 \\ w_j \in V_s}} |d_{G_1}(u_1) - d(w_j)|$ by exactly s and finally it reduces the the partial sum $\sum^{t}_{\substack{j=1 \\ w_j \in V_t}} |d_{G_1}(u_1) - d(w_j)|$ by exactly t.

Case 1: By increasing $d_{G_1}(u_i), u_i \in V_h$ by 1, the partial sum $\sum^{h}_{\substack{j=1 \\ w_j \in V_h}} |d_{G_1}(u_i) - d(w_j)|$ increases by exactly $(h-1)$. It decreases the partial sum $\sum^{s}_{\substack{j=1 \\ w_j \in V_s}} |d_{G_1}(u_i) - d(w_j)|$ by exactly s and

finally it increases the the partial sum $\sum_{\substack{j=1 \\ w_j \in V_t}}^{t} |d_{G_1}(u_i) - d(w_j)|$ by exactly t. Hence, the result, $irr_t(G^{u_iv_1}) = irr_t(G) - (h-1) + s - t + ((h-1) - s + t) = irr_t(G)$ follows.

Case 2: By increasing $d_{G_1}(u_i), u_i \in V_s$ by 1, the partial sum $\sum_{\substack{j=1 \\ w_j \in V_h}}^{h} |d_{G_1}(u_i) - d(w_j)|$ increases by exactly h. It changes the partial sum $\sum_{\substack{j=1 \\ w_j \in V_s}}^{s} |d_{G_1}(u_i) - d(w_j)|$ by exactly $(m-1) - l$ and finally it increases the the partial sum $\sum_{\substack{j=1 \\ w_j \in V_t}}^{t} |d_{G_1}(u_i) - d(w_j)|$ by exactly t. Hence, the result, $irr_t(G^{u_iv_1}) = irr_t(G) - (h-1) + s - t + h + (m-1) - l + t = irr_t(G) + 2m$ follows.

Case 3: By increasing $d_{G_1}(u_i), v_i \in V_t$ by 1, the partial sum $\sum_{\substack{j=1 \\ w_j \in V_t}}^{h} |d_{G_1}(u_i) - d(w_j)|$ decreases by exactly h. It decreases the partial sum $\sum_{\substack{j=1 \\ w_j \in V_t}}^{s} |d_{G_1}(u_i) - d(w_j)|$ by exactly s and finally it changes the the partial sum $\sum_{\substack{j=1 \\ w_j \in V_t}}^{t} |d_{G_1}(u_i) - d(w_j)|$ by exactly $(m_1 - 1) - l_1$.
Hence, the result $irr_t(G^{u_iv_1}) = irr_t(G) - (h-1) + s - t - (h-1) - 2 - s + (m_1 - 1) - l_1 = irr_t(G) - 2(h + l_1)$ follows. □

It is to be noted Theorem 3.2 provides an alternate proof for the following that lemma provided in Zhu et al. (in press).

LEMMA 3.3 (Zhu et al., in press) *Let G' be the graph obtained from G by branch-transformation from u to v. Then $irr_t(G) > irr_t(G')$.*

Theorem 3.2 can be extended to multi graphs also as explained in the following result.

COROLLARY 3.4 If multiple edges or loops are allowed in the graph or if edge-transformation is performed in a simple graph without a cut edge to give $G^{w_iv_1}_{w_i \in G}$, then we have

$$irr_t(G^{u_iv_1}) = \begin{cases} irr_t(G), & \text{if } d_{G_1}(u_i) = d_{G_1}(u_1) - 1, \\ irr_t(G) + 2m, & \text{if } d_{G_1}(u_i) > d_{G_1}(u_1) - 1, \\ irr_t(G) - 2(h + l_1), & \text{if } d_{G_1}(u_i) < d_{G_1}(u_1) - 1. \end{cases}$$

Proof The proof of this theorem follows immediately as a consequence of Theorem 3.2. □

4. Total irregularities of directed graphs

In this section, we extend the concept of total irregularities of graphs mentioned in above sections to directed graphs. Since the edges of a digraph D are directed edges and the vertices of D has two types of degrees, in-degrees and out-degrees, we need to define two types of total irregularities for a digraph, which are called total in-degree irregularities and total out-degree irregularities.

Let the vertices of a simple directed graph $D^{\rightarrow}$ on n vertices be labelled as v_i; $i = 1, 2, 3, \ldots, n$ and let $d^+_{D^{\rightarrow}}(v_i) = d^+(v_i)$ and $d^-_{D^{\rightarrow}}(v_i) = d^-(v_i)$. Then, the notion of total in-irregularity of a given directed graph is introduced as follows.

Definition 4.1 The *total in-irregularity* of a directed graph D with respect to the in-degree of all vertices of D, denoted by $irr^-_t(D^{\rightarrow})$, is defined as $irr^-_t(D^{\rightarrow}) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} |d^-(v_i) - d^-(v_j)| = \sum_{i=1}^{n}\sum_{j=i+1}^{n} |d^-(v_i) - d^-(v_j)|$

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or $\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}|d^-(v_i)-d^-(v_j)|$.

Similarly, the total out-irregularity of a digraph can also be defined as follows.

Definition 4.2 The *total out-irregularity* of a directed graph D with respect to the out-degree of all vertices of D, denoted by $irr_t^+(D^{\rightarrow})$, is defined as $irr_t^+(G^{\rightarrow}) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}|d^+(v_i)-d^+(v_j)| = \sum_{i=1}^{n}\sum_{j=i+1}^{n}|d^+(v_i)-d^+(v_j)|$ or $\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}|d^+(v_i)-d^+(v_j)|$.

Re-orientation of an arc or arc-transformation of an arc will find application in most classical applications of directed graphs like tournaments, transportation problems, flow analysis or alike.

4.1. Total irregularities of directed paths and cycles

The total in-irregularity and the total out-irregularity of a directed path are determined in the following proposition.

Proposition 4.3 For a directed path $P_n^{\rightarrow}$ which is consecutively directed from left to right for which vertices v_1, v_n are called the *start-vertex* and the *end-vertex*, respectively, we have

(i) $irr_t^-(P_n^{\rightarrow}) = irr_t^+(P_n^{\rightarrow}) = n-1$,

(ii) $$irr_t^-(P_n^{\rightarrow}) = \begin{cases} n-1, & \text{if the orientation of } (v_1, v_2) \text{ is reversed,} \\ 3n-5, & \text{if the orientation of } (v_i, v_{i+1}), 2 \le i \le (n-1) \\ & \text{is reversed} \end{cases}$$

(iii) $$irr_t^+(P_n^{\rightarrow}) = \begin{cases} n-1, & \text{if the orientation of } (v_{n-1}, v_n) \text{ is reversed,} \\ 3n-5, & \text{if the orientation of } (v_i, v_{i+1}), \le i \le n-2 \\ & \text{is reversed.} \end{cases}$$

Proof The proof is obvious from the definition of total in-irregularity and total out-irregularity of a given digraph. □

The total in-irregularity and the total out-irregularity of a directed cycle are determined in the following proposition.

Proposition 4.4 For a directed cycle $C_n^{\rightarrow}$ which is consecutively directed clockwise we have

(i) $irr_t^-(C_n^{\rightarrow}) = irr_t^+(C_n^{\rightarrow}) = 0$,

(ii) $irr_t^-(C_n^{\rightarrow}) = irr_t^+(C_n^{\rightarrow}) = 2(n-1)$, if we reverse the orientation of any arc.

Proof The proof is obvious from the definition of total in-irregularity and total out-irregularity of a given digraph. □

Through a simple change of Definition 3.1 the in-arc-transformation partitioning in respect of v_1 and the out-arc-transformation partitioning in respect of v_1 can be defined.

Definition 4.5 The *in-arc-transformation partitioning* with respect to a vertex v_i of the vertex set $V(G)$ of a simple connected directed graph $G^{\rightarrow}$ on n vertices is defined to be $V_h = \{v_i : d^-(v_i) = (d^-(v_1)-1)\} \cup \{v_1\}, h = |V_h|$, and $V_s = \{v_i : d^-(v_i) > (d^-(v_1)-1)\}, s = |V_s|$ and $V_t = \{v_i : d^-(v_i) < (d^-(v_1)-1)\}, t = |V_t|$.

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In view of Definition 4.5, we define some vertex sets of a given digraph are defined as follows.

(i) $V_{s_1} = \{v_j : d^-(v_j) \le d^-(v_i), v_j \in V_s\}, m = |V_{s_1}|$,
(ii) $V_{s_2} = \{v_j : d^-(v_j) > d^-(v_i), v_j \in V_s\}, l = |V_{s_2}|$,
(iii) $V_{t_1} = \{v_j : d^-(v_j) \le d^-(v_i), v_j \in V_t\}, m_1 = |V_{t_1}|$,
(iv) $V_{t_2} = \{v_j : d^-(v_j) > d^-(v_i), v_j \in V_t\}, l_1 = |V_{t_2}|$.

Definition 4.6 The *out-arc-transformation partitioning* with respect to a vertex v_i of the vertex set $V(G^{\rightarrow})$ of a simple connected directed graph $G^{\rightarrow}$ on n vertices is defined to be $V_{h^*} = \{v_i : d^+(v_i) = (d^+(v_1) - 1)\} \cup \{v_1\}, h^* = |V_{h^*}|$, and $V_{s^*} = \{v_i : d^+(v_i) > (d^+(v_1) - 1)\}, s^* = |V_{s^*}|$ and $V_{t^*} = \{v_i : d^+(v_i) < (d^+(v_1) - 1)\}, t^* = |V_{t^*}|$.

In view of Definition 4.5, we define some vertex sets of a given digraph are defined as follows.

(i) $V_{s_1^*} = \{v_j : d^+(v_j) \le d^+(v_i), v_j \in V_s\}, m^* = |V_{s_1^*}|$,
(ii) $V_{s_2^*} = \{v_j : d^+(v_j) > d^+(v_i), v_j \in V_s\}, l^* = |V_{s_2^*}|$,
(iii) $V_{t_1^*} = \{v_j : d^+(v_j) \le d^+(v_i), v_j \in V_t\}, m_1^* = |V_{t_1^*}|$,
(iv) $V_{t_2^*} = \{v_j : d^+(v_j) > d^+(v_i), v_j \in V_t\}, l_1^* = |V_{t_2^*}|$.

Analogous to Theorem 3.2, we propose the following result.

Proposition 4.7 Consider a simple connected directed graph G. After in-arc-transformation in respect of v_1 we have

(i)
$$irr_t^-(G^{v_i u_1}) = \begin{cases} irr_t^-(G), & \text{if } d^-(v_i) = d^-(v_1) - 1 \\ irr_t^-(G) + 2m, & \text{if } d^-(v_i) > d^-(v_1) - 1, \\ irr_t^-(G) - 2(h + l_1), & \text{if } d^-(v_i) < d^-(v_1) - 1 \end{cases}$$

and

(ii)
$$irr_t^+(G^{v_i u_1}) = \begin{cases} irr_t^+(G), & \text{if } d^+(v_i) = d^+(v_1) - 1, \\ irr_t^+(G) + 2m^*, & \text{if } d^+(v_i) > d^+(v_1) - 1, \\ irr_t^+(G) - 2(h^* + l_1^*), & \text{if } d^+(v_i) < d^+(v_1) - 1. \end{cases}$$

Proof The proof is similar to Theorem 3.2. □

4.2. Total irregularities of directed complete graphs

In this section, we initiate a study on the two types of irregularities of directed complete graphs. Consider a complete undirected graph K_n and label the vertices $v_1, v_2, v_3, \ldots, v_n$. Assign direction the edges of K_n to get a directed graph, with K_n as its underlying graph, in such a way that the edge $v_i v_j$ becomes the arc (v_i, v_j) of this directed graph if $i < j$. We denote this directed graph by $K_n^{\rightarrow}$. The following lemma discusses the two types of irregularities of $K_n^{\rightarrow}$.

LEMMA 4.8 *For the directed complete graph $K_n^{\rightarrow}$, the total irregularities are given by* $irr_t^+(K_n^{\rightarrow}) = irr_t^-(K_n^{\rightarrow}) = \sum_{i=1}^{n-1} \sum_{j=1}^{i} j = \frac{1}{6} n(n^2 - 1)$.

Proof The orientation results in an in-degree sequence $(0, 1, 2, \ldots, (n-1))$ and an out-degree sequence $(n-1, n-2, n-3, \ldots, 0)$. Choose the k-th entry of the in-degree sequence. We know that the k-th term is given by $\sum_{j=k+1}^{n} |d^-(v_k) - d^-(v_j)| = \sum_{i=1}^{n-(k+1)} i$. Also, we have $irr_t^- = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} |d^-(v_i) - d^-(v_j)|$ and

hence $irr_t^-(\vec{K_n}) = \sum_{i=1}^{n-1} i + \sum_{i=1}^{n-2} i + \ldots + \sum_{i=1}^{n-(n-1)} i = \sum_{i=1}^{n-1}\sum_{j=1}^{i} j = \frac{1}{6}n(n^2-1)$. Furthermore, since the out-degree sequence is a mirror image of the in-degree sequence and $irr_t^+ = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} |d^+(v_i) - d^+(v_j)|$, the result follows similarly. □

A general application of this study can be the following.

Consider any connected undirected graph G on n vertices and label its vertices randomly by $v_1, v_2, v_3, \ldots, v_n$. Assign direction to the edges of the graph G to be arcs according to the condition mentioned above and refer to the directed graph as the *root directed graph*, $\vec{G}_{root-graph}$. Then, calculate both $irr_t^+(\vec{G}_{root-graph})$ and $irr_t^-(\vec{G}_{root-graph})$. In a *derivative graph* $\vec{G}_{derivative}$ identify all arcs which were re-oriented or subjected to arc-transformation and apply the applicable results to recursively determine the total in-irregularity and total out-irregularity.

Consider the complete bipartite graph $K_{(m,n)}$ and call the m vertices in the first bipartition by *left-side vertices* and the n vertices in the second bipartition by *right-side vertices*. Assign directions to the edges of $K_{m,n}$ strictly from *left-side vertices* to *right-side vertices* to obtain $K_{m,n}^{l\to r}$.

Proposition 4.9 For the directed graph $K_{m,n}^{l\to r}$, we have $irr_t^-(K_{m,n}^{l\to r}) = m^2 n$ and $irr_t^+(K_{m,n}^{l\to r}) = mn^2$.

Proof The orientation of the directed complete bipartite graph $K_{m,n}^{l\to r}$ results in the in-degree sequence $(\underbrace{0,0,\ldots,0}_{m-entries},\underbrace{m,m,\ldots,m}_{n-enties})$ and the out-degree sequence $(\underbrace{n,n,\ldots,n}_{m-entries},\underbrace{0,0,\ldots,0}_{n-enties})$. Here, we have the following cases.

Case 1: For the above-mentioned in-degree sequence of $K_{m,n}^{l\to r}$, we have the sum $\sum_{i=1}^{(m+n)-1}\sum_{j=(i+1)}^{(m+n)} |d^-(v_i) - d^-(v_j)|$ results in the value m, (mn times) and 0, ($(m+n)-2$ times). Hence, $irr_t^-(K_{(m,n)}^{l\to r}) = m^2 n$.

Case 2: For the above-mentioned out-degree sequence of $K_{m,n}^{l\to r}$, we have the sum $\sum_{i=1}^{(m+n)-1}\sum_{j=(i+1)}^{(m+n)} |d^+(v_i) - d^+(v_j)|$ results in the value n, (mn times) and 0, ($(m+n)-2$) times). Hence, $irr_t^+(K_{(1,n)}^{l\to r}) = mn^2$. This completes the proof. □

Invoking from Proposition 4.9, we note that for the directed bipartite graph $K_{1,n}^{l\to r}$, we have $irr_t^-(K_{1,n}^{l\to r}) = n$ and $irr_t^+(K_{(1,n)}^{l\to r}) = n^2$ and $irr_t^-(K_{m,1}^{l\to r}) = m^2$ and $irr_t^+(K_{m,1}^{l\to r}) = m$.

The following is a challenging and interesting problem in this context.

Problem 4.10 Describe an efficient algorithm to determine $irr_t^-(\vec{G}_{derivative})$ and $irr_t^+(\vec{G}_{derivative})$ from $irr_t^-(\vec{G}_{root-graph})$ and $irr_t^+(\vec{G}_{root-graph})$.

5. Conclusion

In this paper, we have studied certain types of total irregularities of certain graphs and digraphs. More problems in this area still remain unsettled. More studies on different types of irregularities for different graph classes, graph operations, graph products and on certain associated graphs such as line graphs and total graphs of given graphs and digraphs remain open. All these facts indicate that there is a wide scope for further investigations in this area.

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