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PURE MATHEMATICS | RESEARCH ARTICLE

Generalised colouring sums of graphs

Johan Kok¹, N.K. Sudev^{2*} and K.P. Chithra³

Abstract: The notion of the b -chromatic number of a graph attracted much research interests and recently a new concept, namely the b -chromatic sum of a graph, denoted by $\varphi'(G)$, has also been introduced. Motivated by the studies on b -chromatic sum of graphs, in this paper we introduce certain new parameters such as χ -chromatic sum, χ^+ -chromatic sum, b^+ -chromatic sum, π -chromatic sum and π^+ -chromatic sum of graphs. We also discuss certain results on these parameters for a selection of standard graphs.

Subjects: Advanced Mathematics; Combinatorics; Discrete Mathematics; Mathematics Statistics; Science

Keywords: chromatic number; χ -chromatic sum; χ^+ -chromatic sum; b -chromatic number; b -chromatic sum; b^+ -chromatic sum; Thue chromatic number; π -chromatic sum; π^+ -chromatic sum

AMS subject classifications: 05C62; 05C05; 05C20; 05C38

1. Introduction

For general notations and concepts in graph theory and digraph theory, we refer to Bondy and Murty (1976), Chartrand and Lesniak (2000), Chartrand and Zhang (2009), Gross and Yellen (2006), Harary (1969), West (2001). Unless mentioned otherwise, all graphs mentioned in this paper are non-trivial, simple, connected, finite and undirected graphs.

Graph colouring has become a fertile research area since its introduction in the second half of nineteenth century. It has numerous theoretical and practical applications. Let us first recall the fact that in a *proper colouring* of a graph G , no two adjacent vertices in G can have the same colour. The

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PUBLIC INTEREST STATEMENT

Graph colouring attracted wide interest among researchers since its introduction in the second half of the nineteenth century. A number of interesting extremal graph theoretic problems were researched well. Colouring sums are more recent and allows for applications where colours may be technologies of kind with some relation between the distinct technologies. It is envisaged that colour products and other mathematical relations between colours will naturally follow as enhanced research fields. It is foreseen that the modelling of metabolic or artificial intelligent structures as “colours” within larger real or virtual living structures of which certain components are modelled as graphs will reveal interesting applications. Colouring sums are extremely useful in many practical problems in project management, communication, routing and transportation, assignments, distributions etc.

minimum number of colours in a proper colouring of a graph G is called the *chromatic number* of G , denoted by $\chi(G)$.

Consider a *proper k -colouring* of a graph G and denote the set of k colours by $C = \{c_1, c_2, c_3, \dots, c_k\}$. Also, consider the disjoint subsets of $V(G)$, defined by $V_{c_i} = \{v_j: v_j \mapsto c_i, v_j \in V(G), c_i \in C, 1 \leq i \leq k\}$. Clearly, we can see that $V(G) = \bigcup_{i=1}^k V_{c_i}$.

The notion of the b -colouring of a graph and the parameter *b -chromatic number*, $\varphi(G)$, of a graph $G(V, E)$, has been introduced in Irving and Manlove (1999) as follows. Let G be a graph on n vertices, say $v_1, v_2, v_3, \dots, v_n$. The *b -chromatic number* of G is defined as the maximum number k of colours that can be used to colour the vertices of G , such that we obtain a proper colouring and each colour i , with $1 \leq i \leq k$, has at least one element x_i which is adjacent to a vertex of every colour j , $1 \leq j \neq i \leq k$. Such a colouring is called a *b -colouring* of G (see Effatin & Kheddouci, 2003; Irving & Manlove, 1999).

The concept of b -chromatic number has attracted much attention and many studies have been made on this parameter (see Effatin & Kheddouci, 2003; Irving & Manlove, 1999; Kok & Sudev, in press; Kouider & Mahéo, 2002; Vaidya & Isaac, 2014, 2015; Vivin & Vekatachalam, 2015).

2. General colouring sum of graphs

The notion of the *b -chromatic sum* of a given graph G , denoted by $\varphi'(G)$, has been introduced in Lisna and Sunitha (2015) as the minimum of sum of colours $c(v)$ of v for all $v \in V$ in a b -colouring of G using $\varphi(G)$ colours. Some results on b -chromatic sums proved in Lisna and Sunitha (2015), which are relevant and useful results in our present study, are listed below.

THEOREM 2.1 (Lisna & Sunitha, 2015) *The b -chromatic sum of a path P_n , $n \geq 2$ is*

$$\varphi'(P_n) = \begin{cases} \frac{3}{2}(n-1) + 3, & \text{if } n \geq 5, n \text{ is odd,} \\ \frac{3n}{2} + 1, & \text{if } n \geq 6, n \text{ is even,} \\ 4, & \text{if } n = 3, \\ \frac{3n}{2}, & \text{if } n \in \{2, 4\}. \end{cases}$$

THEOREM 2.2 Lisna & Sunitha, 2015 *The b -chromatic sum of a cycle C_n is given by*

$$\varphi'(C_n) = \begin{cases} \frac{3n}{2} + 3, & \text{if } n \text{ is even, } n \neq 4, \\ \frac{3}{2}(n-1) + 3, & \text{if } n \text{ is odd,} \\ 6, & \text{if } n = 4. \end{cases}$$

THEOREM 2.3 Lisna & Sunitha, 2015 *The b -chromatic sum of a wheel graph W_{n+1} is*

$$\varphi'(W_{n+1}) = \begin{cases} \frac{3(n-1)}{2} + 7, & \text{if } n \text{ is odd,} \\ \frac{3n}{2} + 7, & \text{if } n \text{ is even, } n \neq 4, \\ 9, & \text{if } n = 4. \end{cases}$$

THEOREM 2.4 (Lisna & Sunitha, 2015) *For a complete bipartite graph $K_{m,n}$ assume without loss of generality that $m \geq n$, then $\varphi'(K_{m,n}) = m + 2n$.*

This interesting new invariant motivates us for studying similar concepts in graph colouring. This leads us to define the concept of the general colouring sum of graphs as follows.

Definition 2.5 Let $C = \{c_1, c_2, c_3, \dots, c_k\}$ allows a b -colouring S of a given graph G . Clearly, there are $k!$ ways of allocating the colours to the vertices of G . The *colour weight* of colour, denoted by $\theta(c_i)$, is the number of times a particular colour c_i is allocated to vertices. Then, the *colouring sum* of a colouring S of a given graph G , denoted by $\omega(S)$, is defined to be $\omega(S) = \sum_{i=1}^k i \theta(c_i)$.

In view of the above definition, the b -chromatic sum of a graph G can be viewed as $\varphi'(G) = \min \left\{ \sum_{i=1}^k i \theta(c_i) \right\}$, where this sum varies over all b -colourings of G .

In view of Definition 2.5, in this paper we introduce certain other colouring sums of graphs similar to the b -chromatic sum of graphs.

3. χ -Chromatic sum of certain graphs

The notion of the χ -chromatic sum of a graph G with respect to a proper k -colouring of G is introduced as follows.

Definition 3.1 Let $C = \{c_1, c_2, \dots, c_k\}$ be a proper colouring of a graph G . Then, the χ -chromatic sum of G , denoted by $\chi'(G)$, is defined as $\chi'(G) = \min \left\{ \sum_{i=1}^k i \theta(c_i) \right\}$ where the sum varies over all minimum proper colourings of G .

In the following discussion, we investigate the χ -chromatic sum of certain fundamental graph classes. First, we determine the χ -chromatic sum of path graphs in the following theorem.

THEOREM 3.2 The χ -chromatic sum of a path P_n is given by

$$\chi'(P_n) = \begin{cases} 1, & \text{if } n = 1, \\ \frac{3n}{2}, & \text{if } n \text{ is even,} \\ \frac{3n-1}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

Proof Being a bipartite graph, the vertices of a path graph P_n can be coloured using two colours, say c_1 and c_2 . Then, we need to consider the following cases.

- (1) Assume that $n = 1$. Then, $P_n \cong K_1$ with a single vertex say v_1 . Colour this vertex by the colour c_1 . Hence, $\theta(c_1) = 1$. Therefore, $\chi'(P_n) = 1$.
- (2) Let n be an even integer. Then, the vertices of path P_n can be coloured alternatively by the colours c_1 and c_2 and hence $\theta(c_1) = \theta(c_2) = \frac{n}{2}$. Therefore, $\chi'(P_n) = 1 \cdot \frac{n}{2} + 2 \cdot \frac{n}{2} = \frac{3n}{2}$.
- (3) Let $n > 1$ be an odd integer. Without loss of generality, label the vertices of P_n with odd subscripts by the colour c_1 and the vertices with even subscripts by the colour c_2 . Then, $\theta(c_1) = \frac{n+1}{2}$ and $\theta(c_2) = \frac{n-1}{2}$. Therefore, $\chi'(P_n) = 1 \cdot \frac{n+1}{2} + 2 \cdot \frac{n-1}{2} = \frac{3n-1}{2}$. \square

In a similar way, the χ -chromatic sum of a cycle graph C_n can be determined as follows.

THEOREM 3.3 The χ -chromatic sum of a cycle C_n is $\chi'(C_n) = 3 \lceil \frac{n}{2} \rceil$.

Proof Let C be a proper colouring of the cycle C_n . If n is even, C must contain at least two colours, say c_1 and c_2 and if n is odd, then C must contain at least three colours, say c_1, c_2 and c_3 . Then, we consider the following cases.

- (1) Let n be an odd integer. Now, we can assign the colour c_1 to the vertices having odd subscripts other than n , the colour c_2 to the vertices having even subscripts and the colour c_3 to the vertex v_n . Hence $\theta(c_1) = \theta(c_2) = \frac{n-1}{2}$ and $\theta(c_3) = 1$. Therefore, $\chi'(G) = 1 \cdot \frac{n-1}{2} + 2 \cdot \frac{n-1}{2} + 3 \cdot 1 = 3 \cdot \frac{n+1}{2}$.
- (2) Let n be an even integer. Then, as explained in the previous result, we can assign the colour c_1 to the vertices having odd subscripts and the colour c_2 to the vertices having even subscripts. Hence $\theta(c_1) = \theta(c_2) = \frac{n}{2}$. Therefore, $\chi'(C_n) = 1 \cdot \frac{n}{2} + 2 \cdot \frac{n}{2} = 3 \cdot \frac{n}{2}$. Combining the above two cases, we have $\chi'(C_n) = 3 \cdot \lceil \frac{n}{2} \rceil$. \square

A wheel graph, denoted by W_{n+1} , is defined to be the join of a cycle C_n and a trivial graph K_1 . That is, $W_{n+1} = C_n + K_1$. The χ -chromatic sum of a wheel graph is determined in the following theorem.

THEOREM 3.4 The χ -chromatic sum of a wheel graph W_{n+1} is given by

$$\chi'(W_{n+1}) = \begin{cases} \frac{3n+11}{2}, & \text{if } n \text{ is odd,} \\ \frac{3n+6}{2}, & \text{if } n \text{ is even.} \end{cases}$$

Proof Let us denote the central vertex of the wheel W_{n+1} by v and the vertices of the outer cycle of W_{n+1} by $v_1, v_2, v_3, \dots, v_n$. Let C be a minimal proper colouring of W_{n+1} . Then, C must contain three colours, say c_1, c_2, c_3 , if n is even and it must contain four colours, say c_1, c_2, c_3, c_4 , if n is odd. Hence, we have the following two cases.

- (1) Let n be an even integer. Then, in the outer cycle, $\frac{n}{2}$ vertices have colour c_1 and the other $\frac{n}{2}$ vertices have the colour c_2 . But the central vertex being adjacent to all vertices of the outer cycle must be coloured using a new colour say c_3 . Therefore, $\theta(c_1) = \theta(c_2) = \frac{n}{2}$ and $\theta(c_3) = 1$. Hence, $\chi'(G) = 1 \cdot \frac{n}{2} + 2 \cdot \frac{n}{2} + 3 = \frac{3n+6}{2}$.
- (2) Let n be an odd integer. Then, in the outer cycle C_n , $\frac{n-1}{2}$ vertices have colour c_1 and $\frac{n-1}{2}$ vertices have the colour c_2 and the remaining one vertex has the colour c_3 . As mentioned in the above case, the central vertex v must be coloured using a new colour say c_4 . Therefore, $\theta(c_1) = \theta(c_2) = \frac{n-1}{2}$ and $\theta(c_3) = \theta(c_4) = 1$ and hence $\chi'(G) = 1 \cdot \frac{n-1}{2} + 2 \cdot \frac{n-1}{2} + 3 + 4 = \frac{3n+11}{2}$. \square

The following result describes the χ -chromatic sum of a complete graph K_n .

Proposition 3.5 The χ -chromatic sum of a complete graph K_n is $\chi'(K_n) = \frac{n(n+1)}{2}$.

Proof We know that in a proper colouring of K_n , every vertex has distinct colours. That is, $\chi(K_n) = n$. Therefore, $\theta(c_i) = 1$, for all $1 \leq i \leq n$. Hence, we have $\chi'(K_n) = \sum_{i=1}^n i = \frac{n(n+1)}{2}$. \square

The χ -chromatic sum of a complete bipartite graph is determined in the following result.

Proposition 3.6 The χ -chromatic sum of a complete bipartite graph $K_{m,n}$, $m \geq n$ is $\chi'(K_{m,n}) = m + 2n$.

Proof Assume that G be the complete bipartite graph with a bipartition (X, Y) such that $|X| \geq |Y|$. As a bipartite graph, G is 2-colourable. Since $|X| \geq |Y|$, label every vertex in X by the colour c_1 and every vertex of Y by the colour c_2 . Hence, $\theta(c_1) = |X|$ and $\theta(c_2) = |Y|$. Therefore, $\chi'(G) = |X| + 2|Y|$. \square

Let us now recall the definition of a Rasta graph defined in Kok, Sudev, and Sudev (in press) as follows.

Definition 3.7 (Kok et al., in press) For a l -term sum set $\{t_1, t_2, t_3, \dots, t_l\}$ with $t_1 > t_2 > t_3 > \dots > t_l > 1$, define the directed graph $G^{(l)}$ with vertices $V(G^{(l)}) = \{v_{ij} : 1 \leq j \leq t_i, 1 \leq i \leq l\}$ and the arcs, $A(G^{(l)}) = \{(v_{ij}, v_{(i+1),m}) : 1 \leq i \leq (l-1), 1 \leq j \leq t_i \text{ and } 1 \leq m \leq t_{(i+1)}\}$.

In Kok and Sudev (in press), it is shown that for a Rasta graph R corresponding to the underlying graph of $G^{(l)}$ the chromatic number $\varphi(R) = 2$. Assume, without loss of generality, that $\sum_{i=1}^{\lfloor \frac{l}{2} \rfloor} t_{(2i-1)} \geq \sum_{i=1}^{\lfloor \frac{l}{2} \rfloor} t_{2i}$ if l is even and $\sum_{i=1}^{\lfloor \frac{l}{2} \rfloor} t_{(2i-1)} \geq \sum_{i=1}^{\lfloor \frac{l}{2} \rfloor} t_{2i}$ if l is odd. Then, the χ -chromatic sum of R is determined in the following theorem.

THEOREM 3.8 The χ -chromatic sum of a Rasta graph R is given by

$$\chi'(R) = \begin{cases} \sum_{i=1}^{\lfloor \frac{l}{2} \rfloor} t_{(2i-1)} + 2 \sum_{i=1}^{\lfloor \frac{l}{2} \rfloor} t_{2i}, & \text{if } l \text{ is even,} \\ \sum_{i=1}^{\lfloor \frac{l}{2} \rfloor} t_{(2i-1)} + 2 \sum_{i=1}^{\lfloor \frac{l}{2} \rfloor} t_{2i}, & \text{if } l \text{ is odd.} \end{cases}$$

Proof

- (1) Let l be an even integer. Since all vertices corresponding to $t_{(2i-1)}$, $1 \leq i \leq \frac{l}{2}$ are non-adjacent and hence we can colour these vertices by c_1 . Also, the remaining vertices, corresponding to t_{2i} , $1 \leq i \leq \frac{l}{2}$ are also non-adjacent among themselves and these vertices can be coloured using the colour c_2 . That is, $\theta(c_1) = \sum_{i=1}^{\frac{l}{2}} t_{(2i-1)}$ and $\theta(c_2) = \sum_{i=1}^{\frac{l}{2}} t_{2i}$. Therefore, in this case

$$\chi'(G) = \sum_{i=1}^{\frac{l}{2}} t_{(2i-1)} + 2 \sum_{i=1}^{\frac{l}{2}} t_{2i}$$

- (2) Let l be an odd integer. Then, as explained in the above case, the $\lceil \frac{l}{2} \rceil$ vertices corresponding to $t_{(2i-1)}$; $1 \leq i \leq \lceil \frac{l}{2} \rceil$ are non-adjacent among themselves and hence we can colour these vertices by c_1 . The remaining $\lfloor \frac{l}{2} \rfloor$ vertices corresponding to t_{2i} ; $1 \leq i \leq \lfloor \frac{l}{2} \rfloor$ are also non-adjacent among

themselves and hence we can colour these vertices by c_2 . Therefore, $\theta(c_1) = \sum_{i=1}^{\lceil \frac{l}{2} \rceil} t_{(2i-1)}$ and $\theta(c_2) = \sum_{i=1}^{\lfloor \frac{l}{2} \rfloor} t_{2i}$ and hence $\chi'(G) = \sum_{i=1}^{\lceil \frac{l}{2} \rceil} t_{(2i-1)} + 2 \sum_{i=1}^{\lfloor \frac{l}{2} \rfloor} t_{2i}$. □

4. The χ^+ -chromatic sum of certain graphs

We now define a new colouring sum, namely χ^+ -chromatic sum of a given graph G as follows.

Definition 4.1 Let $C = \{c_1, c_2, \dots, c_k\}$ be a proper colouring of a graph G . Then, the χ^+ -chromatic sum of a graph G , denoted by $\chi^+(G)$, is defined as $\chi^+(G) = \max \left\{ \sum_{i=1}^k i \theta(c_i) \right\}$, where the sum varies over all minimum proper colourings of G .

Analogous to the studies on χ -chromatic sum of certain graphs, here we study the χ^+ -chromatic sum of the corresponding graphs.

THEOREM 4.2 For $n \geq 1$, the χ^+ -chromatic sum of a path P_n is given by

$$\chi^+(P_n) = \begin{cases} 1, & \text{if } n = 1, \\ \frac{3n}{2}, & \text{if } n \text{ is even,} \\ \frac{3n+1}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

Proof If $n = 1$, we can assign c_1 to its unique vertex, which shows that $\chi^+(P_n) = 1$. Hence, let $n > 1$. As stated earlier, every path P_n , $n \geq 2$ is 2-colourable. Then, we have to consider the following cases.

- (1) If n is even, as mentioned in Theorem 3.2, the vertices can be coloured alternatively by the colours c_1 and c_2 and hence in this case, $\chi^+(P_n) = \frac{3n}{2}$.
- (2) If n is odd, then the mutually non-adjacent $\frac{n-1}{2}$ vertices are coloured by c_1 and the remaining mutually non-adjacent $\frac{n+1}{2}$ vertices can be coloured by the colour c_2 . Therefore, $\chi^+(P_n) = 1 \cdot \frac{n-1}{2} + 2 \cdot \frac{n+1}{2} = \frac{3n+1}{2}$. This completes the proof. □

The following is an immediate consequence of Theorem 3.2 and Theorem 4.2.

COROLLARY 4.3 For a path P_n , $n \geq 1$ it follows that, $\chi^+(P_n) = \chi'(P_n)$ if $n = 1$ or even, else $\chi^+(P_n) = \chi'(P_n) + 1$.

In the following result, let us determine the χ^+ -chromatic sum of cycles.

THEOREM 4.4 The χ^+ -chromatic sum of a cycle C_n is given by

$$\chi^+(C_n) = \begin{cases} \frac{3n}{2}, & \text{if } n \text{ is even,} \\ \frac{5n-3}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

Proof As stated earlier, if n is even, then C_n is 2-colourable and if n is odd, C_n is 3-colourable. Then, we have to consider the following cases.

- (1) Let n be an even integer. Then, the vertices of C_n can be alternatively coloured by two colours c_1 and c_2 . We can see that exactly $\frac{n}{2}$ vertices in C_n have the colours c_1 and c_2 each. Therefore, $\theta(c_1) = \theta(c_2) = \frac{n}{2}$. Therefore, $\chi^+(C_n) = \frac{3n}{2}$.
- (2) Let n be an odd integer. Then, we can assign colour c_3 to $\frac{n-1}{2}$ vertices, colour c_2 to $\frac{n-1}{2}$ vertices and colour c_1 to one vertex, which provides a 3-colouring such that $\theta(c_1) = 1, \theta(c_2) = \theta(c_3) = \frac{n-1}{2}$. Therefore, $\chi^+(C_n) = 5 \cdot \frac{n-1}{2} + 1 = \frac{5n-3}{2}$. \square

The following theorem describes the χ^+ -chromatic sum of a wheel graph W_{n+1}

THEOREM 4.5 The χ^+ -chromatic sum of a wheel graph W_{n+1} is given by

$$\chi^+(W_{n+1}) = \begin{cases} \frac{5n+2}{2}, & \text{if } n \text{ is even,} \\ \frac{7n-1}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

Proof Let $v_1, v_2, v_3, \dots, v_n$ be the vertices of the outer cycle the wheel graph and v be its central vertex. We have already mentioned in Theorem 3.4 that if n is even, then W_{n+1} is 3-colourable and if n is odd, then W_{n+1} is 4-colourable. Then, we have the following cases.

- (1) Let n be an even integer. Then, we can assign the colour c_3 to $\frac{n}{2}$ vertices of the outer cycle, the colour c_2 to the remaining $\frac{n}{2}$ vertices of the outer cycle and the colour c_1 to the central vertex. Hence, $\theta(c_3) = \theta(c_2) = \frac{n}{2}$ and $\theta(c_1) = 1$. Therefore, $\chi^+(W_{n+1}) = 3 \cdot \frac{n}{2} + 2 \cdot \frac{n}{2} + 1 = \frac{5n+2}{2}$.
- (2) Let n be an odd integer. Then, we can assign colour c_3 to the $\frac{n-1}{2}$ non-adjacent vertices, assign colour c_2 to the $\frac{n-1}{2}$ non-adjacent vertices, colour c_3 for the remaining single vertex and colour c_4 to the central vertex, so that we get $\theta(c_3) = \theta(c_4) = \frac{n-1}{2}, \theta(c_2) = 1$ and $\theta(c_1) = 1$. Therefore, we have $\chi^+(W_{n+1}) = 4 \cdot \frac{n-1}{2} + 3 \cdot \frac{n-1}{2} + 2 \cdot 1 + 1 \cdot 1 = \frac{7n-1}{2}$. \square

The following result is an obvious and straightforward result on the χ^+ -chromatic sum of complete graphs.

Proposition 4.6 The χ^+ -chromatic sum of a complete graph K_n is given by $\chi^+(K_n) = \chi'(G) = \frac{n(n+1)}{2}$.

Proof Note that $\chi(K_n) = n$ and hence as mentioned in Theorem 3.8, all vertices have distinct colours. That is, we have $\theta(c_i) = 1$; for all $1 \leq i \leq n$. Hence, $\chi^+(K_n) = \sum_{i=1}^n i = \frac{n(n+1)}{2}$. \square

An obvious and straightforward result on the χ^+ -chromatic sum of complete bipartite graphs is given below.

THEOREM 4.7 Consider the χ^+ -chromatic sum of a complete bipartite graph $K_{m,n}$, $m \geq n \geq 1$, $\chi^+(K_{m,n}) = 2m + n$.

Proof Since $n \geq m$ the maximum sum is obtained by allocating colour c_2 to the n non-adjacent vertices and c_1 to the m non-adjacent vertices. So $\theta(c_1) = n$ and $\theta(c_2) = m$. Therefore, $\chi^+(K_{m,n}) = 2m + n$. \square

The χ^+ -chromatic sum of Rasta graph can be determined as in the following theorem.

THEOREM 4.8 The χ^+ -chromatic sum of Rasta graph R is given by

$$\chi^+(R) = \begin{cases} 2 \sum_{i=1}^{\lfloor \frac{l}{2} \rfloor} t_{(2i-1)} + \sum_{i=1}^{\lfloor \frac{l}{2} \rfloor} t_{2i}, & \text{if } l \text{ is even,} \\ 2 \sum_{i=1}^{\lfloor \frac{l}{2} \rfloor} t_{(2i-1)} + \sum_{i=1}^{\lfloor \frac{l}{2} \rfloor} t_{2i}, & \text{if } l \text{ is odd.} \end{cases}$$

Proof

- (1) Let l be an even integer. Since all $\frac{n}{2}$ vertices, corresponding to $t_{(2i-1)}$, for all $1 \leq i \leq \frac{l}{2}$ are non-adjacent, these vertices can be coloured using the colour c_2 . By the same reason, the colour c_1 is allocated to the vertices corresponding to t_{2i} , $1 \leq i \leq \frac{l}{2}$. Hence, $\theta(c_1) = \sum_{i=1}^{\frac{l}{2}} t_{2i}$ and $\theta(c_2) = \sum_{i=1}^{\lceil \frac{l}{2} \rceil} t_{(2i-1)}$. Hence, $\chi^+(R) = 2 \sum_{i=1}^{\frac{l}{2}} t_{(2i-1)} + \sum_{i=1}^{\frac{l}{2}} t_{2i}$ for the even values of n .
- (2) If l is an odd integer, then the $\frac{n+1}{2}$ mutually non-adjacent vertices can be coloured using c_2 and the remaining $\frac{n-1}{2}$ mutually non-adjacent vertices can be coloured using c_1 . Hence, $\chi^+(R) = 2 \sum_{i=1}^{\lceil \frac{l}{2} \rceil} t_{(2i-1)} + \sum_{i=1}^{\lfloor \frac{l}{2} \rfloor} t_{2i}$, for the odd values of n . □

5. b^+ -Chromatic Sum of Certain Graphs

Analogous to the χ -chromatic sum and χ^+ -chromatic sum of graphs, we can also define the b^+ chromatic sum as follows.

Definition 5.1 The b^+ -chromatic sum of a graph G , denoted by $\phi^+(G)$, is defined as $\phi^+(G) = \max\{\sum_{i=1}^k i \theta(c_i)\}$, where the sum varies over a minimal b -colouring using $\phi(G)$ colours.

Now, for determining the respective values of ϕ^+ for different graph classes, we use the proof techniques followed in Lisna and Sunitha (2015). Reversing the colouring pattern explained in Lisna and Sunitha (2015), we work out the b^+ -chromatic sum of given graph classes. Hence, we have the following results.

THEOREM 5.2 The b^+ -chromatic sum of a path $P_n, n \geq 2$ is given by

$$\phi^+(P_n) = \begin{cases} \frac{5n-3}{2}, & \text{if } n \geq 5, n \text{ is odd,} \\ \frac{5n-2}{2}, & \text{if } n \geq 6, n \text{ is even,} \\ 5, & \text{if } n = 3, \\ \frac{3n}{2}, & \text{if } n \in \{2, 4\}. \end{cases}$$

Proof We know that a b -colouring of a path P_n requires at most three colours. If $1 < n \leq 4$, the b -chromatic number of P_n is 2. In this context, the following cases are to be considered.

- (1) Let n be even. That is, $n = 2, 4$. If $n = 2$, then, one of its two vertices has colour c_1 and the other vertex has colour c_2 . Hence, the b^+ -chromatic sum of P_2 is $2 \cdot 1 + 1 \cdot 1 = 3$. If $n = 4$, Let $C_1 = \{v_2, v_4\}$ and $C_2 = \{v_1, v_3\}$ be the colour classes of the colours c_1 and c_2 , respectively, so that $C = \{c_1, c_2\}$ is a b -colouring of P_n . Then, the b^+ -chromatic sum of P_4 is given by $2 \cdot 2 + 1 \cdot 2 = 6$. Combining these two cases, it follows that $\phi^+(P_n) = \frac{3n}{2}$, for $n = 2, 4$.
- (2) Let $n = 3$. Then, let $C_1 = \{v_2\}$ and $C_2 = \{v_1, v_3\}$, so that $C = \{c_1, c_2\}$ is a b^+ -colouring of P_n . Then, the b^+ -chromatic sum of P_4 is given by $2 \cdot 2 + 1 \cdot 1 = 5$. If $n \geq 5$, the b -chromatic number of a path P_n is 3. Hence, we have to consider the following cases.
- (3) Let $n \geq 5$ and n be odd. Now, let $C = \{c_1, c_2, c_3\}$ be a colouring on P_n such that $C_1 = \{v_3\}$ be the colour class of the colour c_1 , $C_2 = \{v_2, v_5, v_7, \dots, v_n\}$ be the colour class of the colour c_2 and $C_3 = \{v_1, v_4, v_6, \dots, v_{n-1}\}$ be the colour class of colour c_3 . Clearly, this colouring is a b^+ -colouring of P_n . Then, we have $\theta(c_1) = 1, \theta(c_2) = \frac{n-1}{2}$ and $\theta(c_3) = \frac{n-1}{2}$. Hence, for $n \geq 5$ and n is odd, $\phi^+(P_n) = \frac{3}{2}(n-1) + \frac{2}{2}(n-1) + 1 = \frac{5n-3}{2}$.
- (4) Let $n \geq 5$ and n be even. Here, assume that $C = \{c_1, c_2, c_3\}$ be a colouring on P_n such that the colour classes C_1, C_2 and C_3 are exactly as defined in the previous case. This colouring is obviously a b^+ colouring of P_n . Then, it follows that $\theta(c_1) = 1, \theta(c_2) = \frac{n-2}{2}$ and $\theta(c_3) = \frac{n}{2}$. Hence, for $n \geq 6, k$ is even, $\phi^+(P_n) = 3 \cdot \frac{n}{2} + 2 \cdot \frac{n-2}{2} + 1 = \frac{5n-2}{2}$. □

Similarly, the b^+ -chromatic sum of a cycle C_n is determined in the following theorem.

THEOREM 5.3 *The b^+ -chromatic sum of a cycle C_n is given by*

$$\varphi^+(C_n) = \begin{cases} 6, & \text{if } n = 4, \\ \frac{5n-3}{2}, & \text{if } n \text{ is odd,} \\ \frac{5n-6}{2}, & \text{if } n \text{ is even, } n \neq 4. \end{cases}$$

Proof First, let $n = 4$. It is to be noted that the b -chromatic number of the cycle C_4 is 2, where the vertices v_1 and v_3 have colour c_1 and the vertices v_2 and v_4 have the colour c_2 . Therefore, the b^+ -chromatic sum of C_4 is $2 \cdot 2 + 2 \cdot 1 = 6$.

Next, assume that $n \neq 4$. We know that the b -chromatic number of a cycle $C_n, n \neq 4$ is 3. Let $C = \{c_1, c_2, c_3\}$ be a b -colouring of a given cycle C_n . Here, we have to consider the following cases.

- (1) Let n be odd. Now a b -colouring which forms the colour classes, $C_1 = \{v_3\}, C_2 = \{v_1, v_4, v_6, \dots, v_{n-1}\}$ and $C_3 = \{v_2, v_5, v_7, \dots, v_n\}$, yield the desirable b -colouring such that $\theta(c_1) = 1, \theta(c_2) = \frac{n-1}{2}$ and $\theta(c_3) = \frac{n-1}{2}$. Therefore, here the b^+ -chromatic sum is given by $3 \cdot \frac{n-1}{2} + 2 \cdot \frac{n-1}{2} + 1 \cdot 1 = \frac{5n-3}{2}$.
- (2) Let n be even. Now, a b -colouring which forms the colour classes, $C_1 = \{v_3, v_n\}, C_2 = \{v_1, v_4, v_6, \dots, v_{n-2}\}$ and $C_3 = \{v_2, v_5, v_7, \dots, v_{n-1}\}$, yield the desirable b -colouring such that $\theta(c_1) = 2, \theta(c_2) = \frac{n-2}{2}$ and $\theta(c_3) = \frac{n-2}{2}$. Therefore, we have $\varphi^+(C_n) = 3 \cdot \frac{n-2}{2} + 2 \cdot \frac{n-2}{2} + 2 = \frac{5n-6}{2}$. This completes the proof. \square

Now, the b^+ -chromatic sum of a wheel graph W_{n+1} is determined in the following result.

THEOREM 5.4 *The b^+ -chromatic sum of a wheel graph W_{n+1} is given by*

$$\varphi^+(W_{n+1}) = \begin{cases} 11, & \text{if } n = 4, \\ \frac{7n-1}{2}, & \text{if } n \text{ is odd,} \\ \frac{7n-4}{2}, & \text{if } n \text{ is even, } n \neq 4. \end{cases}$$

Proof We have already stated that the b -chromatic number of the cycle C_4 is 3. Therefore, a b -colouring of $W_5 = C_4 + K_1$ must contain 3 colours, say c_1, c_2 and c_3 . Let the corresponding colour classes be $C_1 = \{v\}, C_2 = \{v_1, v_3\}$ and $C_3 = \{v_2, v_4\}$, where v is the central vertex of the wheel graph. Then, $\theta(c_1) = 1, \theta(c_2) = 2$ and $\theta(c_3) = 2$. Hence, $\varphi^+(W_5) = 1 \cdot 1 + 2 \cdot 2 + 3 \cdot 2 = 11$. Next, assume that $n \neq 4$. Then, every b -colouring of W_{n+1} must contain 4 colours. Let $C = \{c_1, c_2, c_3, c_4\}$ be the required colouring of G . Then, we have to consider the following cases.

- (1) Assume that n is odd. Then, colour the vertices of W_{n+1} using the colours in C in such a way that the corresponding colour classes are $C_1 = \{v\}, C_2 = \{v_3\}, C_3 = \{v_1, v_4, v_6, \dots, v_{n-1}\}$ and $C_4 = \{v_2, v_5, v_7, \dots, v_n\}$. Therefore, we have $\theta(c_1) = \theta(c_2) = 1$ and $\theta(c_3) = \frac{n-1}{2}$ and $\theta(c_4) = \frac{n-1}{2}$. Then, we have $\varphi^+(W_{n+1}) = 4 \cdot \frac{n-1}{2} + 3 \cdot \frac{n-1}{2} + 2 + 1 = \frac{7n-1}{2}$.
- (2) Assume that n is even. Colour the vertices of W_{n+1} in such a way that the corresponding colour classes are $C_1 = \{v\}, C_2 = \{v_3, v_n\}, C_3 = \{v_1, v_4, v_6, \dots, v_{n-2}\}$ and $C_4 = \{v_2, v_5, v_7, \dots, v_{n-1}\}$. Then, we have $\theta(c_1) = 1, \theta(c_2) = 2$ and $\theta(c_3) = \theta(c_4) = \frac{n-2}{2}$. Hence, $\varphi^+(W_{n+1}) = 4 \cdot \frac{n-2}{2} + 3 \cdot \frac{n-2}{2} + 2 \cdot 2 + 1 = \frac{7n-4}{2}$. \square

The following theorem describes the φ^+ -chromatic number of a complete bipartite graph.

THEOREM 5.5 *The b^+ -chromatic sum of a complete bipartite graph $K_{m,n}, m \geq n$ is $\varphi^+(K_{m,n}) = 2m + n$.*

Proof The result follows directly from the proof of Theorem 4.7. \square

The b -chromatic sum and the b^+ -chromatic sum of Rasta Graph R is determined in the theorem given below.

THEOREM 5.6 The b -chromatic sum of a Rasta graph R is given by

$$\varphi'(R) = \begin{cases} \sum_{i=1}^{\lfloor \frac{l}{2} \rfloor} t_{(2i-1)} + 2 \sum_{i=1}^{\lfloor \frac{l}{2} \rfloor} t_{2i}, & \text{if } l \text{ is even,} \\ \sum_{i=1}^{\lfloor \frac{l}{2} \rfloor} t_{(2i-1)} + 2 \sum_{i=1}^{\lfloor \frac{l}{2} \rfloor} t_{2i}, & \text{if } l \text{ is odd,} \end{cases}$$

and the b^+ -chromatic sum of R is given by

$$\varphi^+(R) = \begin{cases} 2 \sum_{i=1}^{\lfloor \frac{l}{2} \rfloor} t_{(2i-1)} + \sum_{i=1}^{\lfloor \frac{l}{2} \rfloor} t_{2i}, & \text{if } l \text{ is even,} \\ 2 \sum_{i=1}^{\lfloor \frac{l}{2} \rfloor} t_{(2i-1)} + \sum_{i=1}^{\lfloor \frac{l}{2} \rfloor} t_{2i}, & \text{if } l \text{ is odd.} \end{cases}$$

Proof The proof follows directly from the proofs of Theorem 3.8 and 4.8. □

6. Two Thue chromatic sums of a path

A finite sequence $S = (q_1, q_2, q_3, \dots, q_t)$ of symbols of any alphabet is known to be *non-repetitive* if for all its subsequences $(r_1, r_2, r_3, \dots, r_{2m})$; $1 \leq m \leq \frac{t}{2}$, the condition $r_i \neq r_{2i}, \forall 1 \leq i \leq m$, holds.

Let G be a simple undirected graph on n vertices and let a minimum set of colours C allow a proper vertex colouring of G . If the sequence of vertex colours of any path of even and finite length in G is non-repetitive, then this proper colouring is said to be a *Thue colouring* of G (see Alon, Grytczuk, Hauszczak & Riordan, 2002).

The *Thue chromatic number* of G , denoted $\pi(G)$, is defined as the minimum number of colours required for a Thue colouring of G .

It is known that $\pi(P_1) = 1, \pi(P_2) = \pi(P_3) = 2$ and for $n \geq 4, \pi(P_n) = 3$. Determining $\pi'(P_n)$ is a hard problem, hence the problem is very hard for graphs in general.

The following lemma is the motivation for our further discussions in this paper.

LEMMA 6.1 Škrabul'áková, *in press* Up to equivalence, there is exactly one non-repetitive 3-colouring of the cycle C_{11} .

In view of this lemma, we restrict our further discussion to the path P_{11} . Let the vertices of P_n be labelled from left to right to be $v_1, v_2, v_3, \dots, v_{11}$. Recall that the *colouring sum* of a colouring S is defined by $\omega(S) = \sum_{i=1}^k i \theta(c_i)$. The possible minimum Thue colourings of P_{11} are listed below.

- (1) $S_1 = (c_1, c_2, c_1, c_3, c_1, c_2, c_3, c_1, c_3, c_2, c_3)$ and $\omega(S_1) = 22$
- (2) $S_2 = (c_1, c_2, c_1, c_3, c_1, c_2, c_3, c_2, c_1, c_2, c_3)$ and $\omega(S_2) = 21$
- (3) $S_3 = (c_1, c_2, c_1, c_3, c_2, c_1, c_2, c_3, c_2, c_1, c_3)$ and $\omega(S_3) = 21$
- (4) $S_4 = (c_1, c_2, c_1, c_3, c_2, c_3, c_1, c_3, c_2, c_1, c_3)$ and $\omega(S_4) = 22$
- (5) $S_5 = (c_1, c_2, c_1, c_3, c_1, c_2, c_3, c_1, c_3, c_2, c_1)$ and $\omega(S_5) = 20$
- (6) $S_6 = (c_1, c_2, c_1, c_3, c_2, c_3, c_1, c_3, c_2, c_1, c_2)$ and $\omega(S_6) = 21$

(7) $S_7 = (c_2, c_1, c_2, c_3, c_2, c_1, c_3, c_2, c_3, c_1, c_3)$ and $\omega(S_7) = 23$

(8) $S_8 = (c_2, c_1, c_2, c_3, c_2, c_1, c_3, c_1, c_2, c_1, c_3)$ and $\omega(S_8) = 21$

(9) $S_9 = (c_2, c_1, c_2, c_3, c_1, c_2, c_1, c_3, c_1, c_2, c_3)$ and $\omega(S_9) = 21$

(10) $S_{10} = (c_2, c_1, c_2, c_3, c_1, c_3, c_2, c_3, c_1, c_2, c_3)$ and $\omega(S_{10}) = 23$

(11) $S_{11} = (c_2, c_1, c_2, c_3, c_2, c_1, c_3, c_2, c_3, c_1, c_2)$ and $\omega(S_{11}) = 22$

(12) $S_{12} = (c_2, c_1, c_2, c_3, c_1, c_3, c_2, c_3, c_1, c_2, c_1)$ and $\omega(S_{12}) = 21$

(13) $S_{13} = (c_3, c_2, c_3, c_1, c_3, c_2, c_1, c_3, c_1, c_2, c_1)$ and $\omega(S_{13}) = 22$

(14) $S_{14} = (c_3, c_2, c_3, c_1, c_3, c_2, c_1, c_2, c_3, c_2, c_1)$ and $\omega(S_{14}) = 23$

(15) $S_{15} = (c_3, c_2, c_3, c_1, c_2, c_3, c_2, c_1, c_2, c_3, c_1)$ and $\omega(S_{15}) = 23$

(16) $S_{16} = (c_3, c_2, c_3, c_1, c_2, c_1, c_3, c_1, c_2, c_3, c_1)$ and $\omega(S_{16}) = 22$

(17) $S_{17} = (c_3, c_2, c_3, c_1, c_3, c_2, c_1, c_3, c_1, c_2, c_3)$ and $\omega(S_{17}) = 24$

(18) $S_{18} = (c_3, c_2, c_3, c_1, c_2, c_1, c_3, c_1, c_2, c_3, c_2)$ and $\omega(S_{18}) = 23$

(19) $S_{19} = (c_1, c_3, c_1, c_2, c_1, c_3, c_2, c_1, c_2, c_3, c_2)$ and $\omega(S_{19}) = 21$

(20) $S_{20} = (c_1, c_3, c_1, c_2, c_1, c_3, c_2, c_3, c_1, c_3, c_2)$ and $\omega(S_{20}) = 22$

(21) $S_{21} = (c_1, c_3, c_1, c_2, c_3, c_1, c_3, c_2, c_3, c_1, c_2)$ and $\omega(S_{21}) = 22$

(22) $S_{22} = (c_1, c_3, c_1, c_2, c_3, c_2, c_1, c_2, c_3, c_1, c_2)$ and $\omega(S_{22}) = 21$

(23) $S_{23} = (c_1, c_3, c_1, c_2, c_1, c_3, c_2, c_1, c_2, c_3, c_1)$ and $\omega(S_{23}) = 20$

(24) $S_{24} = (c_1, c_3, c_1, c_2, c_3, c_2, c_1, c_2, c_3, c_1, c_3)$ and $\omega(S_{24}) = 22$ From the above list, we note that $\pi'(P_{11}) = 20$ and $\pi^+(P_{11}) = 24$. We strongly believe that the next conjecture holds. Conjecture 6.2 For a path P_n , $n \geq 4$, there is a unique permutation over all proper b -colourings for which $\varphi^+(P_n)$ is obtained, and exactly two permutations for which $\varphi'(P_n)$ is obtained.

The following general result is of importance for all variations of colouring sums discussed thus far. It holds for improper colourings as well. A general colouring which meets some general colouring index is called the ϑ -chromatic number of G and denoted, $\vartheta(G)$.

Theorem 6.3 (Makungu's Theorem¹) *If the set of colours $C = \{c_j: 1 \leq j \leq k\}$ allows a general colouring, $S: f(v_i) = c_p$, $l \in \{1, 2, 3, \dots, k\}$ of G , such that $\omega(S) = \vartheta'(G) = \min\{\sum_{i=1}^k i \cdot \theta(c_i): \forall S\text{-colourings of } G\}$, then $\vartheta^+(G) = \sum_{i=1}^k i \cdot \theta(c_{(k+1)-i})$.*

Proof Since for $a_1 \geq a_2$ it follows that $1 \cdot a_1 + 2 \cdot a_2 \leq 2 \cdot a_1 + 1 \cdot a_2$, it follows through immediate induction that if $a_1 \geq a_2 \geq a_3 \geq \dots \geq a_k$ then for permuted one-on-one allocations of the elements in $b_i \in \{1, 2, 3, \dots, k\}$ to form $\sum_{i=1}^k a_i b_i$ we have, $\min\{\sum_{i=1}^k a_i b_i\} = \sum_{i=1}^k i \cdot a_i$ and $\max\{\sum_{i=1}^k a_i b_i\} = \sum_{i=1}^k i \cdot a_{(k+1)-i}$. Hence, if a ϑ -colouring of G is allowed by $C = \{c_1, c_2, c_3, \dots, c_k\}$ such that, $\theta(c_1) \geq \theta(c_2) \geq \theta(c_3) \geq \dots \geq \theta(c_k)$ then, $\vartheta'(G) = \sum_{i=1}^k i \cdot \theta(c_i)$ and $\vartheta^+(G) = \sum_{i=1}^k i \cdot \theta(c_{(k+1)-i})$. □

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Note

1 The first author dedicates this theorem to Makungu Mathebula and he hopes this young lady will grow up with a deep fondness for mathematics.

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