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Polynomial description of acoustic modes in fluid ellipsoids

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Résumé. Les modes acoustiques sont couramment utilisés pour déterminer les propriétés physiques de l'intérieur des planètes et des étoiles. La méthode d'imagerie par vélocimétrie acoustique a été récemment introduite en mécanique des fluides, en complément des méthodes d'imagerie existantes. Elle consiste à reconstruire l'écoulement à partir de la levée de dégénérescence des modes acoustiques, observée expérimentalement (par exemple) en présence d'un profil de rotation. Ainsi, des algorithmes d'inversion sont nécessaires pour reconstruire indirectement l'écoulement en rotation dans des expériences en géométrie quasi-sphérique. Cependant, les objets célestes et les expériences ne sont pas rigoureusement sphériques, mais plutôt ellipsoïdaux (au premier ordre). Les effets topographiques et la rotation doivent alors être pris en compte correctement dans le modèle physique. Les méthodes analytiques et numériques existantes (en géométrie déformée) ne peuvent plus alors être couplées facilement et efficacement aux algorithmes d'inversion. Pour remédier à ce problème, nous avons développé une nouvelle méthode de Galerkin semi-analytique en ellipsoïdes tri-axes. Elle repose sur une décomposition polynomiale explicite, en coordonnées cartésiennes, des écoulements potentiels en ellipsoïdes. Nous illustrons la méthode en considérant les modes acoustiques d'un fluide compressible, uniforme et homentropique, avec une condition de Dirichlet homogène à la paroi (condition isobare). Nous validons nos résultats avec des simulations numériques réalisées avec le logiciel COMSOL. Nous étendrons cette méthode novatrice afin de prendre en compte des effets physiques supplémentaires, afin de décrire plus précisément les conditions expérimentales.

Abstract. Acoustic eigenmodes are often used to passively infer physical properties of the interiors of rotating planets and stars. Acoustic wave imaging has been also introduced in fluid dynamics. The experimental technique consists in observing and analysing the splitting in frequency of the acoustic modes, by solving an inverse problem. For instance, the presence of an (unknown) rotational profile disturbs the acoustic spectrum. Moreover, a topographic splitting is often superimposed on the rotational splitting. Indeed, rotating celestial objects are rather ellipsoidal than spherical at the leading order, as well as laboratory experiments which are weakly non-spherical (due to mechanical deformations). Rotation and topography should be taken into account simultaneously, but the acoustic problem does not admit exact solutions in the presence of rotation. Fully numerical solutions are often computed. However, they cannot be easily combined with inversion schemes to yield robust results. Hence, the usual approach is to consider small perturbations to non-rotating solutions. As an alternative, we present a new description of the compressible modes, relying on the method of weighted residuals. Here, we present a new spectral method to describe potential flows in triaxial ellipsoids. It relies on an explicit Galerkin expansion, made of global polynomials expressed in Cartesian coordinates. We illustrate this method by considering the canonical situation of diffusionless acoustic modes in an homogeneous, compressible and homentropic fluid ellipsoid at rest. The pressure satisfies a Dirichlet condition (sound soft boundary). We validate our results against fully numerical simulations performed with the commercial software COMSOL, showing a perfect quantitative agreement. This new method will be extended to include additional effects and alternative boundary conditions.

1 Introduction

Studying the interior conditions of planets and stars is a fundamental research topic. Several passive imaging techniques have been developed for this purpose. The most commonly used is acoustic wave imaging. For instance, this method has been used in the Sun to reconstruct its internal differential rotation profile [1]. Indeed, the eigenfrequencies of acoustic modes are affected by rotational fluid motions, yielding an observable rotational splitting of the modes [2]. The same approach has been introduced in experimental fluid dynamics [3]. It aims at probing the global dynamics of rotating flows in quasi-spherical resonators. Indeed, conventional fluid velocimetry techniques (e.g. Doppler or Particle Image Velocimetry) give partial information on the flow structure. Moreover, the fluid must be seeded with neutrally buoyant

tracer particles to act as scatterers, which is expensive and not always efficient (tracers may float or sink over time).

The method of acoustic mode velocimetry is an inverse problem, formally written as $\mathbf{d} = \mathcal{G}(\mathbf{m}) + \boldsymbol{\epsilon}$, where \mathbf{d} is the data vector (the frequencies of acoustic modes in a container), \mathbf{m} contains the parameters of the model, \mathcal{G} is the physical operator of the problem (the wave equation) and $\boldsymbol{\epsilon}$ some uncertainties (in the model and/or the data). The forward problem is the determination of the eigenfrequencies of acoustic modes \mathbf{d} , given a set of parameters \mathbf{m} . Because the latter is *a priori* unknown, it is often estimated by using iterative methods. The inverse problem is rather complicated, involving variational principles e.g. in the presence of rotation [4]. Hence, forward models as efficient as possible are desirable to develop versatile and efficient inverse schemes. Acoustic modes do admit analytic solutions in non-rotating spheres. However, rotation and topography should be taken into account. Indeed, rotating celestial bodies and experiments are rather ellipsoidal at the leading order. Several methods have been employed to go beyond the spherical geometry, e.g. relying on perturbation methods [5], transcendental functions in ellipsoidal coordinates [6–8] or numerical simulations [9, 10]. Yet, none of them can easily be coupled with inverse schemes to yield fast and accurate results.

In the following, we focus on the forward problem of acoustic modes. We present a new spectral method to compute acoustic modes in arbitrary triaxial ellipsoids. The paper is divided as follows. In Section 2, we introduce our new method for a simple acoustics test case. Then in Section 3, we present the results obtained with the new method. We also benchmark them against three-dimensional simulations performed with the commercial software COMSOL. Finally, Section 4 ends the paper with a short conclusion and draws some perspectives.

2 Modelling

2.1 Linearized problem

The acoustic wave equation is obtained by expanding the field variables (velocity, density, pressure) as the sum of background quantities and acoustic perturbations. Several forms of the acoustic equation can be deduced from the full Navier–Stokes equations, depending on the background state. We refer the reader to Campos [11] for an exhaustive review. To present the new polynomial expansion, we tackle here the simplest physical situation.

We consider a compressible fluid, enclosed within a full triaxial ellipsoid of semi-axes (a, b, c) and volume $\mathcal{V} = 4\pi abc/3$. The ellipsoid has a boundary $\partial\mathcal{V}$, expressed in Cartesian coordinates (x, y, z) by $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. The fluid is homentropic and characterized by a background density ρ_0 and pressure P_0 . They are assumed to be spatially uniform, though the new method can account for arbitrary $[\rho_0, P_0]$. Then, we expand the velocity field from a state of rest as $\mathbf{u}(\mathbf{r}, t) = \nabla\Phi(\mathbf{r}, t)$, where $\Phi(\mathbf{r}, t)$ is the velocity potential (irrotational flow). In the absence of any diffusive effect (e.g. viscosity or thermal conduction) and body force, the velocity potential satisfies the wave equation

$$\frac{\partial^2\Phi}{\partial t^2} = c_0^2 \nabla^2\Phi, \quad (1)$$

where $c_0 = \sqrt{\gamma P_0/\rho_0}$ is the sound speed in the background state (with γ the adiabatic index). We seek modal solutions of equation (1) of the form $\Phi(\mathbf{r}, t) = \widehat{\Phi}(\mathbf{r}) \exp(-i\omega t)$, where $\omega \in \mathbb{R}$ is the angular frequency and $\widehat{\Phi}(\mathbf{r})$ the eigenfunction solution of Helmholtz equation

$$\mathcal{L}(\widehat{\Phi}) = 0, \quad \mathcal{L} = \omega^2 + c_0^2 \nabla^2. \quad (2)$$

Finally, Equation (2) is supplemented by an appropriate boundary condition on $\widehat{\Phi}$ (see Subsection 2.2).

2.2 The new Galerkin expansion

We solve the above eigenvalue problem by using the method of weighted residuals [12]. To do so, we introduce a new spectral Galerkin expansion in triaxial ellipsoids. We expand the velocity potential onto global polynomial elements $\{\phi_l(\mathbf{r})\}_l$ as

$$\widehat{\Phi}(\mathbf{r}) = \sum_{l=1}^{\infty} \alpha_l \phi_l(\mathbf{r}), \quad (3)$$

where $\{\alpha_l\}_l$ are modal coefficients. The polynomial elements $\{\phi_l(\mathbf{r})\}_l$ are made of linear combinations of Cartesian monomials $\{x^i y^j z^k\}_{i+j+k < \infty}$, such that any element $\phi_l(\mathbf{r})$ satisfies exactly the boundary condition for $\widehat{\Phi}(\mathbf{r})$. Expansion (3) is similar to expansions used in the finite-element method (FEM). However, compared to the traditional FEM, we emphasize that our basis elements $\{\phi_l\}_l$ are global polynomials, infinitely continuously differentiable in ellipsoids. Therefore, the mathematical completeness of decomposition (3) is ensured by using the Weierstrass approximation theorem [13]. Hence, this is a rigorous spectral decomposition in ellipsoids.

In practice, we arbitrary truncate the series at the polynomial degree $n \geq 2$ (see later). Then, we substitute the truncated series into Helmholtz equation (2), yielding a finite-dimensional, quadratic eigenvalue problem for the state vector $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots)^\top$, i.e.

$$\mathbf{L}\boldsymbol{\alpha} = 0, \quad L_{ij} = \int_{\mathcal{V}} \widehat{\Phi}_i \cdot \mathcal{L}(\widehat{\Phi}_j) d\mathcal{V}. \quad (4)$$

In Equation (4), the Galerkin projections are computed analytically by using the three-dimensional integral for any Cartesian monomial $x^i y^j z^k$ within an ellipsoid

$$\int_{\mathcal{V}} x^i y^j z^k d\mathcal{V} = \begin{cases} 0 & \text{if } i, j \text{ or } k \text{ is odd,} \\ \frac{2a^{i+1} b^{j+1} c^{k+1}}{3+i+j+k} \beta\left(\frac{i+j}{2} + 1, \frac{k+1}{2}\right) \beta\left(\frac{i+1}{2}, \frac{j+1}{2}\right) & \text{if } i, j \text{ and } k \text{ are even,} \end{cases} \quad (5)$$

where $\beta(i, j)$ is the transcendental beta function defined as a function of the Gamma function $\Gamma(i)$ by

$$\beta(i, j) = \frac{\Gamma(i)\Gamma(j)}{\Gamma(i+j)}, \quad \Gamma\left(i + \frac{1}{2}\right) = \frac{(2i)!}{2^{2i} i!} \sqrt{\pi}. \quad (6)$$

The explicit form of the polynomial basis depends on the chosen boundary condition. To illustrate the new method, we consider the vanishing Dirichlet boundary condition for the potential (sound soft boundary), i.e. $\widehat{\Phi} = 0$ on $\partial\mathcal{V}$. For this boundary condition, an appropriate polynomial basis of maximum degree n for the velocity field is

$$\widehat{\Phi} = \sum_{l=1}^N \alpha_l \phi_l, \quad \phi_l(x, y, z) = \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right) x^i y^j z^k \quad (0 \leq i+j+k \leq n-2). \quad (7)$$

The number of basis elements is $N = n(n+1)(n-1)/6$. Note that the basis functions $\{\widehat{\Phi}_l\}_l$ can be normalized with respect to the real scalar product based on formula (5).

3 Proof-of-concept

We have implemented this method in Python as a proof-of-concept. For the validation, we have solved Equation (4) in dimensionless form, by considering a as the unit of length and a/c_0 as the unit of time.

Bessel		$n = 5$			$n = 15$			COMSOL		$n = 5$			$n = 15$		
(k, l)	$f_{k,l}$	f	Rel. err. (%)	f	Rel. err. (%)	f	Rel. err. (%)	f	f	Rel. err. (%)	f	Rel. err. (%)	f	Rel. err. (%)	
(1, 0)	0.500	0.5	10^{-2}	0.5	10^{-8}	0.7340	0.7346	10^{-1}	0.7340	10^{-6}	0.7340	10^{-6}	0.7340	10^{-6}	
(1, 1)	0.715	0.716	10^{-1}	0.715	10^{-8}	0.9254	0.9278	10^{-1}	0.9254	10^{-7}	0.9254	10^{-7}	0.9254	10^{-7}	
(2, 1)	1.229	1.217	10^0	1.229	10^{-8}	0.9879	0.990	10^{-1}	0.9879	10^{-7}	0.9879	10^{-7}	0.9879	10^{-7}	
(3, 1)	1.735	\emptyset	\emptyset	1.7354	10^{-4}	1.1272	\emptyset	\emptyset	1.1272	10^{-7}	1.1272	10^{-7}	1.1272	10^{-7}	

(a) Sphere

(b) Triaxial ellipsoid

Table 1. Comparison of acoustic frequencies $f = \omega/(2\pi)$ and relative error (a) in a sphere between theory and the polynomial approach and (b) in a strongly deformed ellipsoid ($a = 1, b = 0.8, c = 0.5$) between the polynomial approach and COMSOL.

3.1 Benchmarks in spheres and ellipsoids

To benchmark our polynomial method, we first consider a sphere ($a = b = c = 1$). Helmholtz equation (2) admits exact solutions in spherical coordinates (r, θ, ϕ) . These solutions are $\Phi_{k,l} \propto j_l(r\omega_{k,l})Y_l^m(\theta, \phi)$ where $Y_l^m(\theta, \phi)$ is the spherical harmonics of degree l and order m and $j_l(r\omega_{k,l})$ the associated spherical Bessel function $j_l(r\omega_{k,l})$. For the vanishing Dirichlet condition, the acoustic frequency $f_{k,l} = \omega_{k,l}/(2\pi)$ is the k^{th} root of the associated spherical Bessel function. Because of the spherical symmetry, the modes with different m are degenerated (i.e. have the same frequency). The fundamental frequency is exactly $f_{1,0} = 1/2$, since $j_{k,0}$ is the cardinal sine function (its zeros are multiple of π). The second frequency is approximatively $f_{1,1} = 0.715$ and so on. Note that Bessel functions do not admit explicit expansions involving a finite number of Cartesian polynomials. This clearly shows that our polynomial method is semi-analytical, in contrast with rotational modes [14, 15]. We must truncate the series (3) at a sufficient degree n to have a good convergence for the modes we are interested in. However, even spherical solutions in this canonical case are not entirely explicit, since $\omega_{k,l}$ must be generally computed iteratively (for high order modes) or approximated by using asymptotic expansions [16].

We show in Table 1 (a) the comparison between theoretical (i.e. using Bessel functions) and numerical values for a few modes, by varying the truncation degree n . For $n = 5$, the first modes are already well described by the polynomial expansion.

The mode (3, 1) does not appear in the acoustic spectrum at this degree. Thus, this is necessary to increase the polynomial degree to describe it. Then, this mode is accurately well described at $n = 15$, with extremely small relative departure from the theoretical frequency.

To go beyond the spherical case, exact analytical solutions do exist in ellipsoids by using separation of variables [6–8]. However, they rely on transcendental ellipsoidal functions, which cannot be easily manipulated. Instead, Mehl [5, 17] obtained approximate solutions by considering perturbations from the spherical geometry, but large ellipsoidal deformations cannot be accurately described. Thus, we validate our results against numerical computations performed with the commercial software COMSOL. The latter code solves Equation (2) with the FEM method (Lagrange elements of order 5). In spheres, COMSOL yields the theoretical eigenfrequencies with a relative tolerance always smaller than 10^{-5} % (depending on the numerical mesh). Similarly, COMSOL will provide fairly accurate benchmarks in deformed ellipsoids. The quantitative comparison on the frequencies is given in Table 1 (b) for an given triaxial configuration. At $n = 5$, only the first modes are qualitatively described (but with a relative tolerance smaller than 1%). Then, these modes are extremely well described at $n = 15$.

3.2 Topographic splitting

After having quantitatively validated the polynomial method, we investigate the topographic splitting. We show in in Figure 1 the splitting of the first few modes in spheroids. Note that we obtain a perfect quantitative agreement between the polynomial solutions and COMSOL. Then, most of the modes undergo a topographic splitting, e.g. involving higher order azimuthal numbers ($m = \pm 1, m = \pm 2, \dots$ in spheroids)

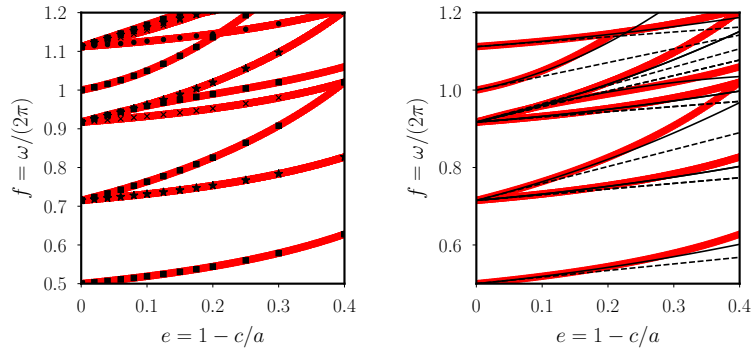


Figure 1. Topographic splitting of the acoustic spectrum in spheroids ($a = b = 1$) as a function of the polar flattening $e = 1 - c/a$. Comparison with COMSOL (left) and with asymptotic scalings (right). Red (thick) curves: polynomial modes ($n = 15$). Black squares: COMSOL ($m = 0$). Black stars: COMSOL ($m = \pm 1$). Black crosses: COMSOL ($m = \pm 2$). Black points: COMSOL ($m = \pm 3$). Dashed curves: linear in e . Solid curves: quadratic in e .

and crossing of several branches. Moreover, we observe that the frequency does not scale linearly with the deformation [5], and even for moderate finite deformations for a few modes. Hence, second-order topographic effects [17] should be considered to reasonably cover a wide range of deformation.

Finally, we illustrate two acoustic modes in a strongly oblate spheroid ($a = b \neq c$) in Figure 2. The spheroid has a flattened vertical axis $c = 0.6$. This value corresponds to the observed flattening of the star Archenar [18]. This is one of the least spherical stars in the Milky Way. Indeed, it spins so rapidly that its shape is a strongly flattened oblate spheroid. The two modes have the same azimuthal number $m = 1$ but different radial complexities, which are well recovered by the polynomial approximation.

4 Conclusion and perspectives

We have presented a new semi-analytical approach to compute acoustic modes in arbitrary triaxial ellipsoids. This method relies on an analytical Galerkin method, using global Cartesian polynomials of maximum degree n satisfying the appropriate boundary condition. As a proof-of-concept of the method, we have computed the acoustic modes with the vanishing Dirichlet condition (sound soft boundary) in ellipsoids. We have quantitatively benchmarked this method against theoretical and FEM computations, exhibiting a very good quantitative agreement.

Compared to the existing alternative methods, this new technique is worthy of interest. Notably, it has a great numerical potential for modal acoustic velocimetry technique. Indeed, it can be implemented in open-source programming languages (e.g. here Python), that can run on laptops and supercomputers (if memory bound computations at very large n are required). Then, the forward problem can be easily coupled with built-in inverse methods (e.g. in Python [19]). The forward problem of acoustic waves is also easier to solve, because no approximation is made on topographic effects (which avoids the use of perturbation methods). Additionally, the post-processing is made easier by using Cartesian coordinates (rather than ellipsoidal ones). Therefore, we advocate to use this method for future applications of modal acoustic velocimetry. This technique seems to be very promising, in particular for liquid-metal experiments [20] for which direct optical methods are unsuitable and ultrasonic techniques challenging.

Finally, it turns out that the polynomial method is also useful beyond the idealized acoustic case considered here. Indeed, we have extended the Galerkin polynomial method presented here, notably by considering a vanishing Neumann condition on the pressure (sound hard boundary, i.e. non-penetration condition). This is a more relevant condition for a fluid bounded in a rigid container. Moreover, the Galerkin method can also accommodate additional effects, such as rotation or density variations. Accounting for these effects in the direct problem may also greatly simplify the inverse problem, for future experimental and planetary applications.

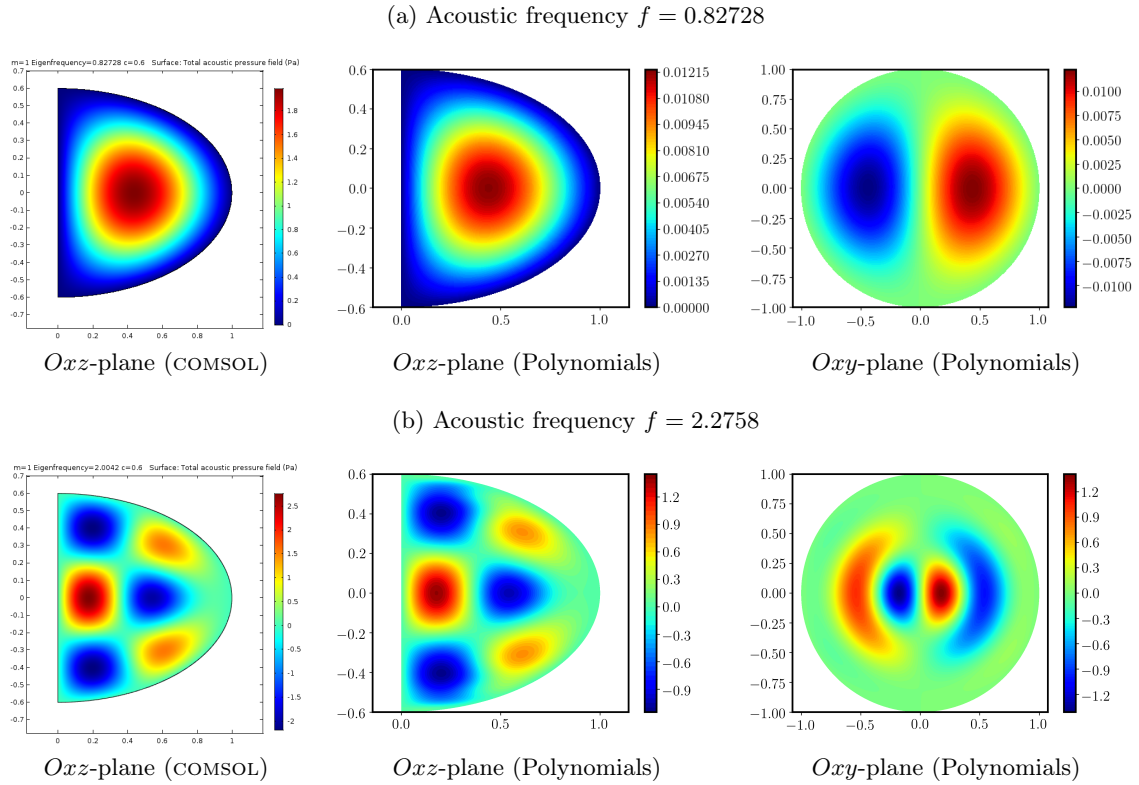


Figure 2. Potential field $\hat{\Phi}(\mathbf{r})$ for two large-scale acoustic modes in an oblate spheroid, computed with COMSOL and the polynomial method ($n = 20$). Ellipsoidal geometry $a = b = 1$ and $c = 0.65$.

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