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# Restricted Tutte polynomials for some periodic oriented forests on infinite square lattice

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**Abstract.** For any finite graph, the Tutte polynomial is the generating function of spanning trees counted by their numbers of active external, respectively internal, edges. We consider two restrictions of this definition, either summing over a subset of spanning trees or counting only the activities in a subset of edges. Adding to the (infinite) square lattice one projective vertex in a (rational) direction  $\vec{\theta}$ , we define the restricted Tutte polynomial  $T_{\vec{\theta}, W \times H}(q, t)$  summing over some periodic spanning forests of period  $W \times H$  and considering only activities on edges of the fundamental domain. Those polynomials are symmetric in  $q$  and  $t$  by self-duality of square lattice. Our main result is a family of bijections indexed by a finite number of  $\vec{\theta}$  proving that  $(T_{\vec{\theta}, W \times H}(q, 1))_{\vec{\theta}}$  does not depend on  $\vec{\theta}$ . Auto-duality preserving the number of trees per period and their common slope, we obtain refinements  $(T_{\vec{\theta}, W \times H}(w, z; q, t))_{\vec{\theta}}$  still symmetric in  $q$  and  $t$ .

**Keywords:** Sandpile model, Square lattice, Tutte polynomial, Symmetric  $q, t$ -numbers

This work is motivated by results on the sandpile model presented in [Section 3](#) at the end of this document. We focus first on the combinatorial result on an analogue of Tutte polynomial for the infinite square lattice.

## 1 Tutte polynomial

For any finite connected graph  $G = (V, E)$ , the Tutte polynomial [\[Tut54\]](#) is a classical graph invariant defined as follow:

$$T_G(q, t) := \sum_T q^{\text{ext}(T)} t^{\text{int}(T)}$$

where  $T$  runs over the set  $\mathcal{T}_G$  of spanning trees of  $G$  and  $\text{ext}(T)$ , respectively  $\text{int}(T)$ , is the soon defined external, respectively internal, (Tutte) activity. Tutte activities depends on a arbitrary permutation of the edges of  $G$  also denoted as a total order  $<_E$ . There are  $|E|!$

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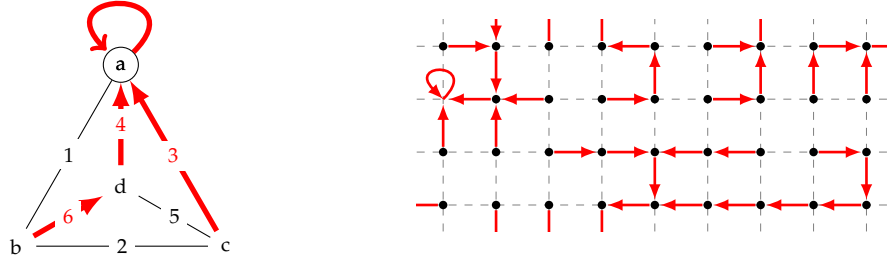
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such orders/permutations. An external edge  $e \notin T$  of the spanning tree  $T$  is (*externally*) *active* if minimal according to  $<_E$  in the fundamental cycle that is the cycle in  $T \cup \{e\}$ . An internal edge  $e \in T$  is (*internally*) *active* if minimal according to  $<_E$  in the fundamental cutset that is the set of edges between the two connected components of  $T - \{e\}$ . The external, respectively internal, activity is the number of external, respectively internal, active edges. The activities of a given tree may change with the order  $<_E$  but the Tutte polynomial does not.

The proof of this invariant relies on the effect of interchanging two consecutive edges in  $<_E$ . While Tutte shows this results taking account both activities, we propose a lightened proof for its marginal  $T_G(q, 1)$ . We consider the (elementary) transposition  $\tau_i$  that exchanges the two consecutive edges  $e_i, e_{i+1}$  in the order  $<_E := (e_i)_{i=1, \dots, |E|}$  leading to  $<_{\tau_i, E} := e_1, \dots, e_{i-1}, e_{i+1}, e_i, e_{i+2}, \dots$ . A pair of edges  $(e_j, e_k)$  is a *critical pair* in tree  $T$  for the order  $<_E$  if  $e_j$  is an external edge,  $e_k$  belongs to the fundamental cycle of  $e_j$ , denoted by  $C_{e_j}(T)$ , and  $e_j$  and  $e_k$  are the two minimal edges in this cycle. We define a map  $\Phi_{\tau_i, <_E}$  on spanning trees by  $\Phi_{\tau_i, <_E}(T) = (T - \{e_i\}) \cup \{e_{i+1}\}$  if  $(e_i, e_{i+1})$  is a critical pair for order  $<_E$ ,  $\Phi_{\tau_i, <_E}(T) = (T - \{e_{i+1}\}) \cup \{e_i\}$  if  $(e_{i+1}, e_i)$  is a critical pair for order  $<_E$  and  $\Phi_{\tau_i, <_E}(T) = T$  otherwise. When non trivial, this update is a case of the exchange property of graphic matroids.

**Lemma 1.** *For any order  $<_E$ , any elementary transposition  $\tau_i = (i, i + 1)$ , the involution  $\Phi_{\tau_i, <_E}$  maps a tree  $T$  of external activity  $k$  for order  $<_E$  to a tree  $\Phi_{\tau_i, <_E}(T)$  with the same external activity  $k$  for order  $<_{\tau_i, E}$ .*

*Proof.* Following Tutte proof's beginning, the activities are preserved by the exchange when  $e_i$  and  $e_{i+1}$  are both internal or both external. Otherwise, there are still preserved if the internal edge is not on the fundamental cycle of the external edge. In the later case, if there are not both minimal in the fundamental cycle, there is an internal edge that is smaller than both in the cycle since there are consecutive in  $<_E$ . Then, the exchange preserves the external activity. It remains the case where the edges are the two minimal edges of the cycle. Suppose  $(e_i, e_{i+1})$  is a critical pair without loss of generality. Then  $\Phi_{\tau_i, <_E}$  maps the tree  $T$  to the tree  $T' := T \Delta \{e_i, e_{i+1}\}$ , the symmetric difference of edges. This map exchanges  $e_i$  and  $e_{i+1}$  in the tree and in the order. Then  $C_{e_i}(T)$  and  $C_{e_{i+1}}(T')$  are identical, so the activities of the external edges  $e_i$  or  $e_{i+1}$  in  $T$  or  $T \Delta \{e_i, e_{i+1}\}$  are the same. An internal edge  $e \notin \{e_i, e_{i+1}\}$  of  $T$  remains internal in  $T'$  so does not change external activity. It remains to consider the external edge  $e$  not in  $\{e_i, e_{i+1}\}$ . We consider the two connected components of the vertices on the cycle  $C_{e_i}(T)$  after the deletion of  $e_i$  and  $e_{i+1}$  then discuss if the fundamental cycle  $C_e(T)$  of the external edge  $e$  contains one vertex of each component or not. If it does not, this cycle is the same in  $T$  and  $T'$  so activity is preserved. If it does, this cycle  $C_e(T)$  in  $T$  becomes  $C_e(T') := C_e(T) \Delta C_{e_i}(T)$  in  $T'$ , hence all the new edges in  $C_e(T')$  belongs to  $C_{e_i}(T)$ . Hence the two exchanged set of edges in the fundamental cycles of  $e$  in  $T$  and  $T'$  are  $C_e(T)/C_{e_i}(T)$  and  $C_{e_i}(T)/C_e(T)$



**Figure 1:** Example of weakly acyclic PARENTHOOD function on  $K_4$  (left) and  $\mathbb{Z}^2$  (right)

whose minimal edges are  $e_i$  or  $e_{i+1}$ . Since  $e_i$  and  $e_{i+1}$  remain consecutive in the order, it means that the activity of  $e$  is the same in  $T$  and  $T'$ .  $\square$

## 1.1 Spanning tree as PARENTHOOD function

Let  $G = (V, E)$  be a finite simple connected graph. We denote the neighborhood of  $v \in V$  noted  $\mathcal{V}(v)$  the set of vertices adjacent to  $v$  in  $G$ . We call PARENTHOOD an endofunction  $R$  that maps for any vertex  $v \in V$  a vertex in  $\mathcal{V}(v) \cup \{v\}$ . This definition corresponds to oriented cycle rooted spanning forests (OCRSF) [Ken17], and uses the formalism from infinite matroids in graphs [BD11].

Note that a rooted spanning tree is a PARENTHOOD where each vertex maps to its father in the tree and where the root maps to itself. A  $k$ -cycle in  $R$  is a sequence of vertices  $(v_1, v_2, \dots, v_k)$  such that  $v_{i+1} = R(v_i)$  for  $1 \leq k-1$  and  $v_1 = R(v_k)$ . Then, rooted spanning trees match with the PARENTHOOD function with exactly one cycle which size is 1, coding the root (see Figure 1). The bijection from PARENTHOOD functions to spanning trees consist on removing the loop on the root and removing the orientation. The rooted spanning forest are the functions which cycles have size 1.

**Definition 1.** A PARENTHOOD function is said weakly acyclic if all its cycles have length 1.

We place ourself in the case functions have only one 1-cycles. For any  $u \in V$ , we denote by  $\text{Ray}(u)$  the ray of  $u$  in  $R$ , that is the set of edges in the orbite of  $u$  in  $R$ :  $(u, R(u), R^2(u), \dots)$ . Then, the fundamental cycle of an external edge  $uv$  is the symmetrical difference of  $\text{Ray}(u)$  and  $\text{Ray}(v)$ :  $C_{uv} = \text{Ray}(u) \Delta \text{Ray}(v)$ .

We can extend the definition of fundamental cycle for rooted spanning forests using this definition.

**Definition 2.** Let  $G = (V, E)$  be a finite simple connected graph,  $R$  an weakly acyclic PARENTHOOD function. The fundamental cycle of an external edge with respect to  $R$  is defined by  $C_{uv} = \text{Ray}(u) \Delta \text{Ray}(v)$ .

Then we can define the external activity for any rooted spanning forests using this extension.

## 1.2 External activity for infinite graphs

Let  $G = (V, E)$  be a infinite simple connected graph and  $R$  be an weakly acyclic PARENTHOOD function. In this setup, rays may be infinite. For each external edge, we associate its fundamental cycle using [Definition 2](#). Let  $uv$  be an external edge. Either  $u$  and  $v$  are in the same connected component of  $R$  then  $C_{uv}$  is finite, or  $\text{Ray}(u)$  and  $\text{Ray}(v)$  are finite then  $C_{uv}$  is finite, or  $\text{Ray}(u)$  and/or  $\text{Ray}(v)$  are infinite then  $C_{uv}$  is infinite. In the finite cases, the activity of  $uv$  is decidable for any order on the edges. In the infinite case, the activity of  $uv$  is not straight forward unless we have some guarantees on the order and  $R$ .

Assuming we can compute the activity of any external edges if rooted spanning forests, we can define a analogue of Tutte polynomial on the distribution of external activity on rooted spanning forests. However, the sum would be infinite and the external activity of a rooted spanning forest might also be infinite.

**Definition 3** (*k-order assumption*). *A pair of an order  $<_E$  and a set  $\mathcal{F}$  of weakly acyclic PARENTHOOD functions verifies the k-order assumption if for any  $f \in \mathcal{F}$  and any vertex  $u \in V$ , the minimal edge of  $\text{Ray}_f(u)$  with respect to  $<_E$  is among the k firsts edges of the ray.*

Under the  $k$ -order assumption, we can compute efficiently the activity of any external edge.

## 1.3 Restriction for summability

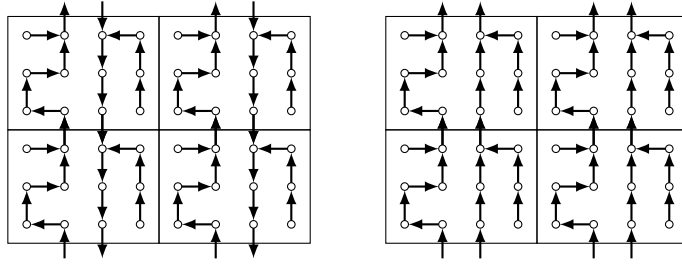
Let  $G = (V, E)$  be a infinite connected graph and  $<_E$  an order on  $E$ . Let  $\mathcal{F}$  be a finite set of weakly acyclic PARENTHOOD function of  $G$  and  $E'$  a finite subset of edges of  $E$ . We assume that the choice of  $<_E$ ,  $\mathcal{F}$  and  $E'$  makes the external and internal activities computable. Then we can define an polynomial

$$T_{\mathcal{F}, E', <_E}(q, t) := \sum_{F \in \mathcal{F}} q^{\text{ext}_{E'}(F)} t^{\text{iny}_{E'}(F)}$$

encoding the distribution of activities on  $\mathcal{F}$  restricted to the edges of  $E'$ .

## 2 Restricted Tutte polynomial on completed square lattice

From this section, we work on the infinite square lattice indexed by  $\mathbb{Z}^2$  where  $(x', y') \in \mathcal{V}(x, y) \Leftrightarrow |x' - x| + |y' - y| = 1$ . Generic restrictions of Tutte polynomials defined in



**Figure 2:** Examples of acyclic periodic PARENTHOOD function of period  $(4, 3)$

previous [Section 1.3](#) allow to consider some well-defined polynomials on this graph. In order to match with Tutte polynomial, we consider only weakly acyclic PARENTHOOD function with at most 1 cycle. First we give two restrictions and an order to explicitly compute the distribution of the external activity on families of endofunctions. Secondly, by observing that the square lattice is self-dual, we define a family of restricted polynomials on the distributions of both activities.

## 2.1 Periodicity

A natural finite subset of PARENTHOOD function of  $\mathbb{Z}^2$  is the set of periodic endofunctions of given fundamental domain. Such PARENTHOOD function has either an infinity of cycle or is acyclic.

**Definition 4.** A PARENTHOOD function  $F$  of  $\mathbb{Z}^2$  is periodic of period  $(W, H) \in \mathbb{N}_{>0}^2$  if for any  $(x, y) \in \mathbb{Z}^2$ , we have  $F(x + W, y) = F(x, y) + (W, 0)$  and  $F(x, y + H) = F(x, y) + (0, H)$ .

An acyclic periodic PARENTHOOD function (see [Figure 3](#)) has periodic infinite branches that share the same slope but can have different orientations. Then each infinite branch  $B$  has a signed slope  $\theta_B \in \mathbb{Z}^2$  with co-prime components. Let  $\vec{\theta} \in \mathbb{Z}^2$  be a non null vector and  $\vec{\theta}^\perp = (-\theta_y, \theta_x)$ . A infinite branch  $B$  is *positive* for  $\vec{\theta}$  if  $\langle \theta, \theta_B \rangle > 0$  or  $\langle \theta, \theta_B \rangle = 0$  and  $\langle \theta^\perp, \theta_B \rangle > 0$ . We said it is *strongly positive* if  $\langle \theta, \theta_B \rangle > 0$ . We denote by *admissible* periodic spanning forests the results of the bijection of [Section 1.1](#) applied to acyclic periodic PARENTHOOD function. Since two PARENTHOOD can share the same spanning forests (see [Figure 2](#)), we map bijectively the admissible periodic spanning forests to acyclic periodic PARENTHOOD function which infinite branches are positive for  $\vec{\theta}$ .

**Definition 5.** An admissible periodic PARENTHOOD for  $\vec{\theta} \neq \vec{0}$  is an acyclic periodic PARENTHOOD function which infinite branches are positive for  $\vec{\theta}$ .

From this point, we only consider sets of admissible periodic PARENTHOOD function for a given  $\vec{\theta}$ . Since the infinite branches are oriented toward the direction  $\vec{\theta}$ , we add

to the graph an additional vertex  $s \notin \mathbb{Z}^2$  that behaves like the crossing point of all the infinite branches at infinity in the direction  $\vec{\theta}$ . This vertex can be seen as a projective root for the PARENTHOOD function.

We denote by  $\mathcal{F}^{W \times H}$  the set of admissible spanning forests of period  $(W, H)$ . The natural finite subset of edges on which we want to compute the activity is the set of edges in a fundamental domain of the forests. For any  $F \in \mathcal{F}^{W \times H}$ , we denote by  $\vec{F}$  the corresponding admissible PARENTHOOD function with respect to  $\vec{\theta}$ . By geometrically identifying each edge by its middle point  $(x_e, y_e)$  in the usual embedding, we denote by  $\mathcal{E}_{W \times H} := \{e_i | \vec{e}_i = (x, y) \text{ where } 0 \leq x \leq W + 1/2 \text{ and } 0 \leq y \leq H + 1/2\}$  this subset of  $2WH$  edges that contains one copy of each edge in the torus  $W \times H$ , in the usual copy of the torus containing the origin and sometimes called the fundamental domain.

In order to respect the  $k$ -order assumption, we need an order that guarantee that (oriented) infinite branches are globally increasing. Since the branches are positive for  $\vec{\theta}$ , we use a total order  $<_{E_{\vec{\theta}}}$  on edges of the square lattice:

$$e_i <_{E_{\vec{\theta}}} e_j \iff (\langle \vec{e}_i | \vec{\theta} \rangle, \langle \vec{e}_i | \vec{\theta}^\perp \rangle) <_{\text{lex}} (\langle \vec{e}_j | \vec{\theta} \rangle, \langle \vec{e}_j | \vec{\theta}^\perp \rangle)$$

where  $<_{\text{lex}}$  is the usual lexicographic order giving priority to the first coordinate. In other word, ignoring ties, the smallest edge is the more distant to the projective root in direction  $\vec{\theta}$ . The pair  $(<_{E_{\vec{\theta}}}, \mathcal{F}^{W \times H})$  verifies the  $k$ -ordered assumption with parameter  $k = 2WH$ .

We aim for

$$T_{\vec{\theta}, W \times H}(q, t) := T_{\mathcal{F}^{W \times H}, \mathcal{E}_{W \times H}, <_{E_{\vec{\theta}}}}(q, t) = \sum_{F \in \mathcal{F}^{W \times H}} q^{\text{ext}_{\mathcal{E}_{W \times H}}(F)} t^{\text{iny}_{\mathcal{E}_{W \times H}}(F)}.$$

Notice that by self-duality of the infinite square lattice and stability by duality of dual pairs these polynomials are symmetric in  $q$  and  $t$ :  $T_{\vec{\theta}, W \times H}(q, t) = T_{\vec{\theta}, W \times H}(t, q)$ . For the partial evaluation  $T_{\vec{\theta}, W \times H}(q, 1)$  at  $t = 1$ , our main result is its independence of  $\vec{\theta}$ .

## 2.2 Result

**Theorem 1.** *For any two rational directions  $\vec{\theta}$  and  $\vec{\theta}'$  we have  $T_{\vec{\theta}, W \times H}(q, 1) = T_{\vec{\theta}', W \times H}(q, 1)$ .*

By symmetry,  $T_{\vec{\theta}, W \times H}(1, t)$  is the same polynomial independent of  $\vec{\theta}$  but the full polynomial  $T_{\vec{\theta}, W \times H}(q, t)$  may depend on  $\vec{\theta}$ , see Section 2.3 for examples. The end of this section gives element of proof for this Theorem 1. This proof follows the same line as in the finite case where a bijection defined for Lemma 1 allows to prove the invariance by an elementary transposition on the order  $<_E$  of edges. The main difficulty is that turning the order  $<_{E_{\vec{\theta}}}$  into the order  $<_{E_{\vec{\theta} + d\vec{\theta}'}}$ , even for small  $d\vec{\theta}'$  requires an infinite number of elementary transpositions on the infinite number of edges so one has to ensure

that updates commute and then directly describe the result of an infinite sequence of elementary updates. The periodicity of forests allows to do so.

Let  $F$  be a periodic admissible forest of  $\mathcal{F}^{W \times H}$ . We show that there exists a finite set of directions  $(\theta_i)_{i \in \mathbb{Z}_k}$  indexed counterclockwise such that the restricted external activity  $\text{ext}_{\mathcal{E}_{W \times H}}(\vec{F})$  is piecewise constant on every  $[\theta_i, \theta_{i+1})$  and that for any  $\theta_i$ , there exists bijectively a forest  $F' \in \mathcal{F}^{W \times H}$  such that  $\text{ext}_{\mathcal{E}_{W \times H}}(\vec{F})$  for  $<_{E_{\theta_i}}$  equals  $\text{ext}_{\mathcal{E}_{W \times H}}(\vec{F}')$  for  $<_{E_{\theta_{i+1}}}$ .

We denote by  $\vec{\pi}(\vec{F}) \in W\mathbb{Z} \times H\mathbb{Z}$  the common period vector of any infinite branch of  $\vec{F}$  with respect to  $W \times H$ . Let  $e$  be an external edge of  $\vec{F}$ , we denote by  $C_{e, \vec{F}}$  the fundamental cycle of  $e$  in  $\vec{F}$  as defined in section 1.3. Let  $H = \mathcal{H}(C_{e, \vec{F}})$  be the convex hull of the middle points of the edges of  $C_{e, \vec{F}}$  (Figure 3). Note that if  $C_{e, \vec{F}}$  is finite, then the corners of  $H$  are finitely many. Otherwise  $C_{e, \vec{F}}$  contains infinite periodic branches then  $H$  is a super set of every  $H + k\vec{\pi}(\vec{F})$  where  $k > 0$ . The convex hull  $H$  has two semi infinite sides of slope  $\vec{\pi}(\vec{F})$  with endpoint in  $C_{e, \vec{F}}$ . Since  $e$  is between the two infinite branches of  $C_{e, \vec{F}}$ , every edges between this branches are inside  $H$ . In particular,  $e + \vec{\pi}(\vec{F})$  is inside  $H$  so it cannot be on the infinite side of  $H$ . Then  $e$  is not an endpoint of this sides, those are internal edges of  $C_{e, \vec{F}}$ . Since the path lengths from the endpoints of  $e$  to infinite branches is bounded by  $2WH$ ,  $H$  has finitely many corners.

**Lemma 2.** *For any  $e$ ,  $\mathcal{H}(C_{e, \vec{F}})$  has finitely many corners.*

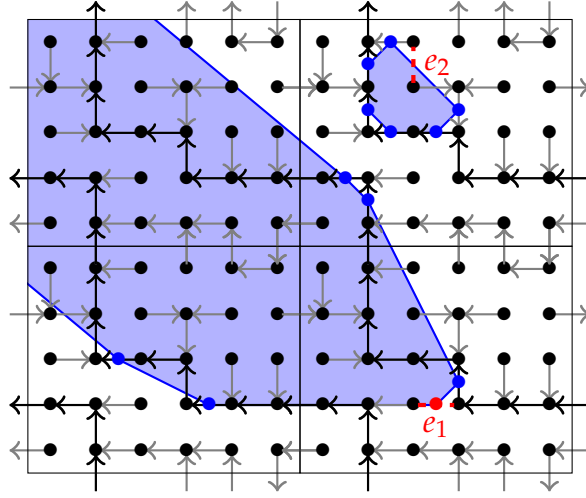
We note  $(h_i)_{1 \leq i \leq k}$  the corners of  $H$  indexed counterclockwise where  $h_1$  and  $h_k$  are the endpoint of the infinite sides when required. Based on the corners of the convex hull we can show the following results.

**Lemma 3.** *Let  $e$  an external edge of  $\vec{F}$ . If an edge is minimal on  $C_{e, \vec{F}}$  for the direction  $\vec{\theta}$ , then it is a corner of  $\mathcal{H}(C_{e, \vec{F}})$ . Reciprocally if an edge is a corner of  $C_{e, \vec{F}}$  that is not adjacent to an infinite side, then there exists a direction for which it is minimal.*

Since the order defined from  $\vec{\theta}$  comes from usual scalar product, the minimality of  $f \in C_{e, \vec{F}}$  for the direction  $\vec{\theta}$  means that  $C_{e, \vec{F}}$  is in the halfplane  $D_{f, \vec{\theta}}$ . This compatibility gives the previous lemma. In particular, for any consecutive corners  $h_i$  and  $h_{i+1}$  of  $H$ ,  $h_{i+1}$  is minimal for the direction  $\overrightarrow{h_i h_{i+1}}^\perp$ . Then an external edge can be active if it is a corner of its fundamental cycle.

When  $C_{e, \vec{F}}$  is infinite,  $e$  cannot be active for the direction  $\vec{\pi}(\vec{F})^\perp$  since it is not on the infinite sides. For this direction, the repetitions of each edge of infinite branches of  $\vec{F}$  are decreasing. We define  $\vec{F}$  the oriented forest obtained by reversing the orientation of the edges of the infinite branches.

**Lemma 4.**  *$\vec{F}$  and  $\vec{F}$  have the same set of active external edges for the direction  $\vec{\pi}(\vec{F})^\perp$ .*



**Figure 3:** Periodic oriented forest  $\vec{F}$ .  $e_1$  is a corner of  $\mathcal{H}(C_{e_1, \vec{F}})$  and  $e_2$  is not a corner of  $\mathcal{H}(C_{e_2, \vec{F}})$ . The corners of the convex hulls are the blue and red dots.

Indeed none of the external edges of fundamental cycles is active and the finite fundamental cycles are preserved in  $\vec{F}$ .

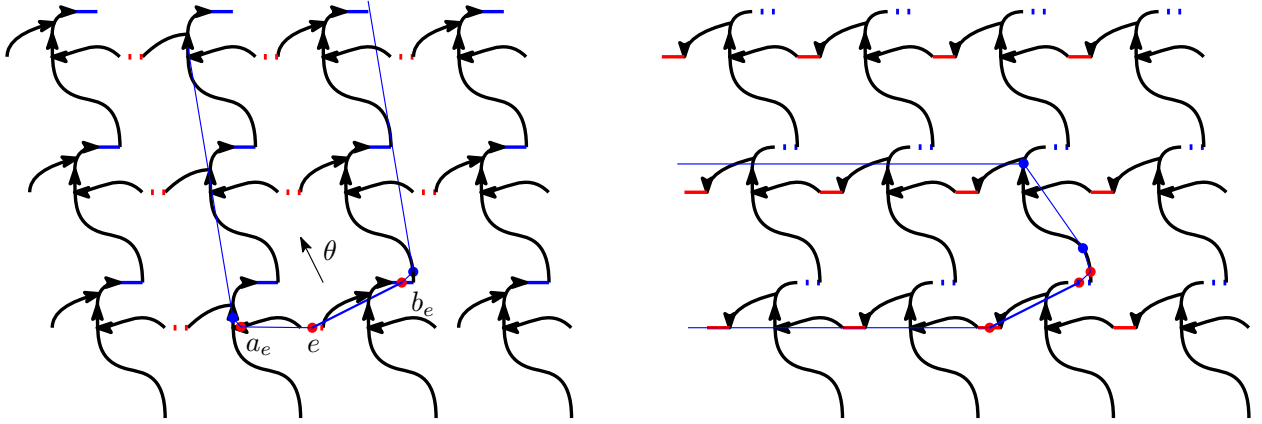
We call a triplet  $(a_e, e, b_e)$  a critical triplet if  $e$  is external and if  $(a_e, e, b_e)$  are consecutive corners of  $\mathcal{H}(C_{e, \vec{F}})$ . Each critical triplet defines two directions  $\vec{a_e e}^\perp$  and  $\vec{e b_e}^\perp$  and  $e$  is active for a direction  $\theta$  if and only if  $\theta \in [\vec{a_e e}^\perp, \vec{e b_e}^\perp)$ . The set of critical triplets is finite up to translations. We denote by  $\Theta_{\vec{F}}$  the finite set of these directions in addition to the direction  $\vec{\pi}(\vec{F})^\perp$ . We select among  $\Theta_{\vec{F}}$  the  $(\theta_i)_{1 \leq i \leq k}$  such that  $(\langle \vec{\pi}(\vec{F}) | \vec{\theta}_i \rangle, \langle \vec{\pi}(\vec{F})^\perp | \vec{\theta}_i \rangle) >_{lex} (0, 0)$ . We index them counterclockwise with  $\theta_k \equiv \vec{\pi}(\vec{F})^\perp$  where  $\vec{u} \equiv \vec{v} \Leftrightarrow \exists \alpha > 0$  s.t.  $\vec{u} = \alpha \vec{v}$ .

**Lemma 5.** *For any  $1 \leq i < k$ , the set of active external edges is invariant for any direction  $\theta \in [\theta_i, \theta_{i+1})$ .*

Let  $\theta_{i+1} \in (\theta_i)_{1 \leq i \leq k}$ . For any critical triplet  $(a_e, e, b_e)$ , the pair  $(a_e, e)$  (resp.  $(e, b_e)$ ) is a critical pair for the direction  $\theta_{i+1}$  if  $\vec{a_e e}^\perp \equiv \theta_{i+1}$  (resp.  $\vec{e b_e}^\perp \equiv \theta_{i+1}$ ). Then, we can show that these critical pairs of  $\vec{F}$  are pairwise disjoint and we note  $Pc_{\vec{F}, \theta_{i+1}}$  the set of the critical pairs.

If  $\theta_{i+1} = \vec{\pi}(\vec{F})^\perp$ , then we replace  $\vec{F}$  by  $\vec{F}$  while preserving the critical pairs thanks to the Lemma 4. We can assume that  $\theta_{i+1} \neq \vec{\pi}(\vec{F})^\perp$ .

We construct step by step  $\vec{F} = F \Delta Pc_{\vec{F}, \theta_{i+1}}$  the forest obtained by swapping the edges of each critical pairs in  $F$ . On each step, we swap periodically one critical pair. Let  $(e, f)$  be a critical pair. We can define  $F' = F \Delta \cup_{v \in W\mathbb{Z} \times H\mathbb{Z}} \{e + v, f + v\}$ . It does not create any finite cycle or finite tree, so  $F'$  and  $\vec{F}$  are admissible periodic forests.



**Figure 4:** Example of critical pair swap that changes the infinite branches

A swap may change the direction of infinite branches of  $F$  (see Figure 4). We skip here the details to only sketch the proof. We can show that the set of critical pairs of  $F'$  for the direction  $\theta_{i+1}$  is the same as  $F$ . And the newly created external edge of  $F'$  has the same activity for  $\theta_{i+1}$  than the previous external edge of  $F$  for the direction  $\theta_i$ . Finally, the other external edges keep their activity through this swap.

Since we preserve the critical pairs and the activity of all external edges that do not appear in critical pairs, the order of the swap is not relevant. We can do all the swaps in parallel. Moreover, since the critical pairs are preserved, this is an involution.

**Corollary 1.** *The external activity of  $\tilde{F}$  for the direction  $\theta_{i+1}$  is the same as the external activity of  $F$  for  $\theta_i$ .*

Let  $\Theta = \bigcup_{\tilde{F}, F \in \mathcal{F}^{W \times H}} \Theta_{\tilde{F}}$ . Since  $\mathcal{F}^{W \times H}$  is finite,  $\Theta$  is finite. We reuse the notation  $\Theta = (\theta_i)_{i \in \mathbb{Z}_{|\Theta|}}$  indexed counterclockwise. For any  $F$  and any  $\theta_{i+1}$ , we map bijectively a  $\tilde{F}$  such that the external activity of  $\tilde{F}$  for the direction  $\theta_{i+1}$  is the same as the external activity of  $F$  for  $\theta_i$ .

So for any rational directions  $\theta$  and  $\theta'$ , we map step by step for each  $F$  a unique  $F_{\theta \rightarrow \theta'}$  preserving the external activity. This ends the proof of Theorem 1.

### 2.3 Restricted Tutte polynomials $T_{\vec{\theta}, W \times H}(q, t)$ for $H = 1$

When  $H = 1$ , the admissible periodic forests different from horizontal cycle on the torus are naturally in bijection with the spanning trees of the wheel graph  $W_{W+1}$ . Each cycle (of length 1) is mapped to an edge toward the center in the wheel graph, and the other edges map to edges on the cycle of the wheel. Thus the  $T_{\vec{\theta}, W \times H}(1, 1) - 1$  counts the number of these trees.

This case gives one of the smallest counter-example against the invariance on  $\theta$  of

$T_{\vec{\theta}, W \times H}(q, t)$ :  $T_{\vec{0}, 3 \times 1}(q, t) = q^3 t^3 + 3q^2 + 3qt + 3t^2 + 3q + 3t + 1$  and  $T_{\vec{\pi}/2, 3 \times 1}(q, t) = q^3 t^3 + 3q^2 t + 3qt^2 + 3q + 3t + 4$ .

We can refine these polynomials taking account of the slope of the infinite branches using  $T_{\vec{\theta}, W \times H}(w, z; q, t)$  where for any  $(i, j)$  coprime and for any  $k > 0$ , the polynomial coefficient  $[w^{ki} z^{kj}] T_{\vec{\theta}, W \times H}(w, z; q, t)$  count the Tutte polynomial restricted to forests with  $k$  cycles on the torus and where the slope of infinite branches is  $\pm(iW, jH)$ . For instance  $T_{\vec{0}, 3 \times 1}(w, z; q, t) = w + 3z(1 + q + t) + 3z^2(q^2 + t^2) + z^3 q^3 t^3$ . The polynomial  $T_{\vec{\theta}, W \times H}(w, z; 1, 1)$  can be extracted from a determinantal formula (see [Ken17, DB18]). Polynomials for some  $(W, H)$  can be found on [https://www.labri.fr/perso/hderycke/biperiodic\\_forests](https://www.labri.fr/perso/hderycke/biperiodic_forests).

### 3 Application for the sandpile model

Sandpile model [Dha90], also called Chip-Firing Game, is a discrete model for diffusion on graphs. Let  $G = (V, E)$  be a simple connected finite graph. A *configuration*  $\eta = (\eta_v)_{v \in V}$  for  $G$  is a function from  $V$  to  $\mathbb{N}$ . A vertex  $v$  is *unstable* in configuration  $\eta$  if  $\eta_v$  is at least the degree of  $v$ . The *toppling* of an unstable vertex  $v$  moves along each edge incident to  $v$  a grain from  $v$  to the opposite endpoint, so  $v$  losses its degree in grains and each neighbour gains one grain. An extra vertex  $s$ , called the sink, is added to the vertices and some edges  $E_s$  connect this sink to some other vertices leading to  $G_s := (V \cup \{s\}, E \cup E_s)$ . A configuration is stable if all vertices are stable, except possibly at the sink  $s$ .

Given a configuration  $\eta$  on  $G_s$ , we topple the unstable vertices except the sink until none remains. This algorithm is called *stabilization* and terminates on a stable configuration noted  $\text{stab}(\eta)$  whatever the order on the toppled vertices is. When  $\eta$  is stable, we denote by  $\text{dhar}(\eta)$  the result of the toppling of the sink  $s$  followed by a stabilization. The application  $\text{dhar}$  is called the Dhar operator. A *recurrent configuration* is a stable configuration that is a fixed point of the Dhar operator. Note that for such configuration, all vertices topple exactly once during this algorithm. This is called the *Dhar criterion*.

Various schedulings of this Dhar criterion provided many bijections from recurrent configurations in  $G_s$  to spanning trees of  $G_s$ , e.g. [MD92, CB03], where one records for each vertex distinct from the sink the edge crossed by the grain allowing toppling.

We consider the generating function  $G_s(q) := \sum_u q^{\text{level}(u)}$  where  $u$  runs over recurrent configurations and  $\text{level}(u) := (\sum_{v \neq s} u_v) - |E|$  is up to a constant the number of grains in configuration  $u$ . This generating function  $G_s(q)$  is also the generating function of spanning trees according to the Tutte external activity [Lóp97] so an evaluation at  $t = 1$  of the bivariate Tutte polynomial  $T_{G_s}(q, t)$ .

In a recent work [DB18], the authors proposed a generalisation of the notion of recur-

rent configuration for the infinite square lattice, usually denoted  $\mathbb{Z}^2$ . Other approaches exist, for example in [GJ14] where recurrent configurations are defined as the image of some spanning trees via an extension of a bijection of the finite case [MD92]. First, we focus on periodic stable configurations of period  $W \times H$ , where  $(W, H) \in \mathbb{N}_+^2$ . Our aim was to define the notion of recurrent configuration for such periodic configurations via an extension of Dhar criterion.

Selecting one vertex, say the origin, as the sink, will break the periodicity. Our intuitive choice was to add the sink as an extra “projective” vertex of the square lattice at infinity in a direction  $\vec{\theta}$  as in Section 2.1. With this choice, the definition of a toppling of the sink now at infinity becomes challenging. We consider the half-plane  $D_{p,\vec{\theta}} := \{v \in \mathbb{Z}^2 \mid \langle v - p, \vec{\theta} \rangle \geq 0\}$  where  $p$  is a vertex and  $\langle \cdot, \cdot \rangle$  the usual scalar product. Intuitively, this half-plane is made of vertices at least as close as  $p$  to the sink at infinity in direction  $\vec{\theta}$ .

**Definition 6** (Weak Dhar Criterion). *A periodic stable configuration is recurrent for the direction  $\vec{\theta}$  if for any  $p$ , after a forced toppling of the half-plane  $D_{p,\vec{\theta}}$ , all other vertices in the complement of  $D_{p,\vec{\theta}}$  topple (once).*

**Proposition 1** ([DB18]). *There exists an algorithm performing weak Dhar criterion on any periodic stable configuration for any rational angle  $\vec{\theta}$  and the result do not depend on the choice of the vertex  $p$  for the half-plane  $D_{p,\vec{\theta}}$ .*

As in the finite case, the algorithm performing the weak Dhar criterion leads to a bijection, extending the one in [CB03] with some admissible PARENTHOOD function for  $\vec{\theta}$ . Let  $\mathcal{F}_+^{W \times H}$  the subset of admissible spanning forests which infinite branches are strongly positive.

**Proposition 2** ([DB18]). *Recurrent configurations of period  $W \times H$  defined by weak Dhar criterion with projective sink in direction  $\vec{\theta}$  are in bijection with admissible spanning forests of  $\mathcal{F}_+^{W \times H}$ , hence excluding those of slope orthogonal to  $\vec{\theta}$ .*

The details of the proof, see [Der18], initially depends on the vertex  $p$  defining the half-plane  $D_{p,\vec{\theta}}$ . But the scheduling of weak Dhar criterion, initially only periodic in the orthogonal direction  $\vec{\theta}^\perp$  becomes also periodic in the direction  $\vec{\theta}$  and then independent of  $p$ . This ultimate periodic behaviour leads via an adaptation of the used finite case bijection, to the expected admissible forests.

We consider a new version of the weak Dhar criterion that will allow to extends the set of recurrent configurations to obtain a bijection with all the admissible spanning forests, hence the number of recurrent configurations of a given period  $W \times H$  will no more depend on  $\vec{\theta}$ .

The bijection from Proposition 2 links the number of grains on recurrent configurations in direction  $\vec{\theta}$  with the activities of the edges on spanning forests with respect to

$\vec{\theta}$ . **Lemma 5** says that the activity per edges is piece-wise constant. Then the bijection is stable for small variation of the direction  $\vec{\theta}$ . Using the same notation and we assume  $\theta \in [\theta_i, \theta_{i+1})$  and define  $\theta^+ = (\theta + \theta_{i+1})/2$ .

**Proposition 3.** *For any direction  $\vec{\theta}$ , the new weak Dhar criterion for  $\vec{\theta}^+$  defines a bijection between recurrent configurations of period  $W \times H$  and all the admissible spanning forests of the same period.*

The polynomials studied in Section 2 satisfies  $T_{\vec{\theta}, W \times H}(q, 1) = \sum_u q^{\text{level}(u)}$  where the left member is the generating function of recurrent configurations of period  $W \times H$  defined for the new weak Dhar criterion  $\vec{\theta}^+$  and the level is the number of grains on  $u^{W \times H}$  on the torus (or one period) minus  $2WH$  that is the number of edges on the torus. This identity relies on the fact that the used bijection, like in the finite case [CB03], turns the level of a configuration into the external activity of the spanning tree for an order of edges related to  $\vec{\theta}$ . It was a priori unclear that this polynomial summing over a strict subset of spanning trees does not depend on  $\theta$  and our main result states that it is.

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