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Logarithmic Space Verifiers on NP-complete

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Abstract

P versus NP is considered as one of the most important open problems in computer science. This consists in knowing the answer of the following question: Is P equal to NP? A precise statement of the P versus NP problem was introduced independently by Stephen Cook and Leonid Levin. Since that date, all efforts to find a proof for this problem have failed. NP is the complexity class of languages defined by polynomial time verifiers M such that when the input is an element of the language with its certificate, then M outputs a string which belongs to a single language in P. Another major complexity classes are L and NL. The certificate-based definition of NL is based on logarithmic space Turing machine with an additional special read-once input tape: This is called a logarithmic space verifier. NL is the complexity class of languages defined by logarithmic space verifiers M such that when the input is an element of the language with its certificate, then M outputs 1. To attack the P versus NP problem, the NP-completeness is a useful concept. We demonstrate there is an NP-complete language defined by a logarithmic space verifier M such that when the input is an element of the language with its certificate, then M outputs a string which belongs to a single language in L.

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1 Preliminaries

In 1936, Turing developed his theoretical computational model [12]. The deterministic and nondeterministic Turing machines have become in two of the most important definitions related to this theoretical model for computation [12]. A deterministic Turing machine has only one next action for each step defined in its program or transition function [12]. A nondeterministic Turing machine could contain more than one action defined for each step of its program, where this one is no longer a function, but a relation [12].

Let Σ be a finite alphabet with at least two elements, and let Σ^* be the set of finite strings over Σ [3]. A Turing machine M has an associated input alphabet Σ [3]. For each string w in Σ^* there is a computation associated with M on input w [3]. We say that M accepts w if this computation terminates in the accepting state, that is $M(w) = 1$ (when M outputs 1 on the input w) [3]. Note that M fails to accept w either if this computation ends in the rejecting state, that is $M(w) = 0$, or if the computation fails to terminate, or the computation ends in the halting state with some output, that is $M(w) = y$ (when M outputs the string y on the input w) [3].

The language accepted by a Turing machine M , denoted $L(M)$, has an associated alphabet Σ and is defined by:

$$L(M) = \{w \in \Sigma^* : M(w) = 1\}.$$

We denote by $t_M(w)$ the number of steps in the computation of M on input w [3]. For $n \in \mathbb{N}$ we denote by $T_M(n)$ the worst case run time of M ; that is:

$$T_M(n) = \max\{t_M(w) : w \in \Sigma^n\}$$

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where Σ^n is the set of all strings over Σ of length n [3]. We say that M runs in polynomial time if there is a constant k such that for all n , $T_M(n) \leq n^k + k$ [3]. In other words, this means the language $L(M)$ can be decided by the Turing machine M in polynomial time. Therefore, P is the complexity class of languages that can be decided by deterministic Turing machines in polynomial time [5]. A verifier for a language L_1 is a deterministic Turing machine M , where:

$$L_1 = \{w : M(w, c) = 1 \text{ for some string } c\}.$$

We measure the time of a verifier only in terms of the length of w , so a polynomial time verifier runs in polynomial time in the length of w [3]. A verifier uses additional information, represented by the symbol c , to verify that a string w is a member of L_1 . This information is called certificate. NP is the complexity class of languages defined by polynomial time verifiers [10].

► **Lemma 1.** *Given a language $L_1 \in P$, a language L_2 is in NP if there is a deterministic Turing machine M , where:*

$$L_2 = \{w : M(w, c) = y \text{ for some string } c \text{ such that } y \in L_1\}$$

and M runs in polynomial time in the length of w . In this way, NP is the complexity class of languages defined by polynomial time verifiers M such that when the input is an element of the language with its certificate, then M outputs a string which belongs to a single language in P .

Proof. If L_1 can be decided by the Turing machine M' in polynomial time, then the deterministic Turing machine $M''(w, c) = M'(M(w, c))$ will output 1 when $w \in L_2$. Consequently, M'' is a polynomial time verifier of L_2 and thus, L_2 is in NP . ◀

2 Hypothesis

A function $f : \Sigma^* \rightarrow \Sigma^*$ is a polynomial time computable function if some deterministic Turing machine M , on every input w , halts in polynomial time with just $f(w)$ on its tape [12]. Let $\{0, 1\}^*$ be the infinite set of binary strings, we say that a language $L_1 \subseteq \{0, 1\}^*$ is polynomial time reducible to a language $L_2 \subseteq \{0, 1\}^*$, written $L_1 \leq_p L_2$, if there is a polynomial time computable function $f : \{0, 1\}^* \rightarrow \{0, 1\}^*$ such that for all $x \in \{0, 1\}^*$:

$$x \in L_1 \text{ if and only if } f(x) \in L_2.$$

An important complexity class is NP -complete [8]. A language $L_1 \subseteq \{0, 1\}^*$ is NP -complete if:

- $L_1 \in NP$, and
- $L' \leq_p L_1$ for every $L' \in NP$.

If L_1 is a language such that $L' \leq_p L_1$ for some $L' \in NP$ -complete, then L_1 is NP -hard [5]. Moreover, if $L_1 \in NP$, then $L_1 \in NP$ -complete [5]. A principal NP -complete problem is SAT [6]. An instance of SAT is a Boolean formula ϕ which is composed of:

1. Boolean variables: x_1, x_2, \dots, x_n ;
2. Boolean connectives: Any Boolean function with one or two inputs and one output, such as \wedge (AND), \vee (OR), \neg (NOT), \Rightarrow (implication), \Leftrightarrow (if and only if);

3. and parentheses.

A truth assignment for a Boolean formula ϕ is a set of values for the variables in ϕ . A satisfying truth assignment is a truth assignment that causes ϕ to be evaluated as true. A formula with a satisfying truth assignment is a satisfiable formula. The problem *SAT* asks whether a given Boolean formula is satisfiable [6]. We define a *CNF* Boolean formula using the following terms:

A literal in a Boolean formula is an occurrence of a variable or its negation [5]. A Boolean formula is in conjunctive normal form, or *CNF*, if it is expressed as an AND of clauses, each of which is the OR of one or more literals [5]. A Boolean formula is in 3-conjunctive normal form or *3CNF*, if each clause has exactly three distinct literals [5].

For example, the Boolean formula:

$$(x_1 \vee \neg x_1 \vee \neg x_2) \wedge (x_3 \vee x_2 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee \neg x_4)$$

is in *3CNF*. The first of its three clauses is $(x_1 \vee \neg x_1 \vee \neg x_2)$, which contains the three literals x_1 , $\neg x_1$, and $\neg x_2$. Another relevant *NP-complete* language is *3CNF* satisfiability, or *3SAT* [5]. In *3SAT*, it is asked whether a given Boolean formula ϕ in *3CNF* is satisfiable.

A logarithmic space Turing machine has a read-only input tape, a write-only output tape, and read/write work tapes [12]. The work tapes may contain at most $O(\log n)$ symbols [12]. In computational complexity theory, *L* is the complexity class containing those decision problems that can be decided by a deterministic logarithmic space Turing machine [10]. *NL* is the complexity class containing the decision problems that can be decided by a nondeterministic logarithmic space Turing machine [10].

A logarithmic space transducer is a Turing machine with a read-only input tape, a write-only output tape, and read/write work tapes [12]. The work tapes must contain at most $O(\log n)$ symbols [12]. A logarithmic space transducer M computes a function $f : \Sigma^* \rightarrow \Sigma^*$, where $f(w)$ is the string remaining on the output tape after M halts when it is started with w on its input tape [12]. We call f a logarithmic space computable function [12]. We say that a language $L_1 \subseteq \{0, 1\}^*$ is logarithmic space reducible to a language $L_2 \subseteq \{0, 1\}^*$, written $L_1 \leq_l L_2$, if there exists a logarithmic space computable function $f : \{0, 1\}^* \rightarrow \{0, 1\}^*$ such that for all $x \in \{0, 1\}^*$,

$$x \in L_1 \text{ if and only if } f(x) \in L_2.$$

The logarithmic space reduction is frequently used for *L* and *NL* [10]. A Boolean formula is in 2-conjunctive normal form, or *2CNF*, if it is in *CNF* and each clause has exactly two distinct literals. There is a problem called *2SAT*, where we asked whether a given Boolean formula ϕ in *2CNF* is satisfiable. *2SAT* is complete for *NL* [10]. Another special case is the class of problems where each clause contains *XOR* (i.e. exclusive or) rather than (plain) *OR* operators. This is in *P*, since an *XOR SAT* formula can also be viewed as a system of linear equations mod 2, and can be solved in cubic time by Gaussian elimination [9]. We denote the *XOR* function as \oplus . The *XOR 2SAT* problem will be equivalent to *XOR SAT*, but the clauses in the formula have exactly two distinct literals. *XOR 2SAT* is in *L* [2], [11].

We can give a certificate-based definition for *NL* [3]. The certificate-based definition of *NL* assumes that a logarithmic space Turing machine has another separated read-only tape [3]. On each step of the machine the machine's head on that tape can either stay in place or move to the right [3]. In particular, it cannot reread any bit to the left of where the head currently is [3]. For that reason this kind of special tape is called "read once" [3].

► **Definition 2.** A language L_1 is in NL if there exists a deterministic logarithmic space Turing machine M with an additional special read-once input tape polynomial $p : \mathbb{N} \rightarrow \mathbb{N}$ such that for every $x \in \{0, 1\}^*$,

$$x \in L_1 \Leftrightarrow \exists u \in \{0, 1\}^{p(|x|)} \text{ such that } M(x, u) = 1$$

where by $M(x, u)$ we denote the computation of M where x is placed on its input tape and u is placed on its special read-once tape, and M uses at most $O(\log |x|)$ space on its read/write tapes for every input x where $|\dots|$ is the bit-length function [3]. M is called a logarithmic space verifier [3].

We state the following Hypothesis:

▷ **Hypothesis 3.** Given a language $L_1 \in L$, there is a language L_2 in NP -complete with a deterministic Turing machine M , where:

$$L_2 = \{w : M(w, u) = y \text{ for some string } u \text{ such that } y \in L_1\}$$

when M runs in logarithmic space in the length of w , u is placed on the special read-once tape of M , and u is polynomially bounded by w . In this way, there is an NP -complete language defined by a logarithmic space verifier M such that when the input is an element of the language with its certificate, then M outputs a string which belongs to a single language in L .

3 Motivation

The P versus NP problem is a major unsolved problem in computer science [4]. This is considered by many to be the most important open problem in the field [4]. It is one of the seven Millennium Prize Problems selected by the Clay Mathematics Institute to carry a US\$1,000,000 prize for the first correct solution [4]. It was essentially mentioned in 1955 from a letter written by John Nash to the United States National Security Agency [1]. However, the precise statement of the $P = NP$ problem was introduced in 1971 by Stephen Cook in a seminal paper [4]. In 2012, a poll of 151 researchers showed that 126 (83%) believed the answer to be no, 12 (9%) believed the answer is yes, 5 (3%) believed the question may be independent of the currently accepted axioms and therefore impossible to prove or disprove, 8 (5%) said either do not know or do not care or don't want the answer to be yes nor the problem to be resolved [7]. It is fully expected that $P \neq NP$ [10]. Indeed, if $P = NP$ then there are stunning practical consequences [10]. For that reason, $P = NP$ is considered as a very unlikely event [10]. Certainly, P versus NP is one of the greatest open problems in science and a correct solution for this incognita will have a great impact not only in computer science, but for many other fields as well [1]. Whether $P = NP$ or not is still a controversial and unsolved problem [1]. In this work, we show some results that might help us to solve one of the most important open problems in computer science.

4 Problems

These are the problems that we are going to discuss:

► **Definition 4. MONOTONE NAE 3SAT**

INSTANCE: A Boolean formula ϕ in 3CNF such that each clause has no negation variables.

QUESTION: Is there a truth assignment for ϕ such that each clause has at least one true literal and at least one false literal?

REMARKS: This is equivalent to the special case of the NP-complete problem known as SET SPLITTING when the sets in the input have exactly three elements and therefore, MONOTONE NAE 3SAT \in NP-complete [6].

► **Definition 5. MINIMUM EXCLUSIVE-OR 2-SATISFIABILITY**

INSTANCE: A positive integer K and a Boolean formula ϕ that is an instance of XOR 2SAT such that each clause has no negation variables.

QUESTION: Is there a truth assignment in ϕ such that at most K clauses are unsatisfiable?

REMARKS: We denote this problem as $MIN \oplus 2SAT$.

► **Definition 6. K EXACT COVER-2**

INSTANCE: A positive integer K , a “universe” set U of natural numbers and a family of n sets $S_i \subseteq U$ with the property that every element in U appears at most twice in the list S_1, \dots, S_n .

QUESTION: Is it the case there is a subfamily S'_1, \dots, S'_m with $m \leq n$ after removing K different numbers in U from the whole list S_1, \dots, S_n , such that $S'_i \cap S'_j = \emptyset$ for $1 \leq i \neq j \leq m$ and $S'_1 \cup \dots \cup S'_m = U'$ where U' is equal to the set U without the removed K different numbers?

REMARKS: We denote this problem as $KEC-2$.

► **Definition 7. EXACT SEPARATE COVER-2**

INSTANCE: A positive integer m , a family of sets U_1, \dots, U_k from a “universe” set $U_1 \cup \dots \cup U_k = U$ and a collection of n pairs $T_i = (x, S_i)$ such that x is a positive integer and $S_i \subseteq U$ is a set with the property that every element in U appears at most twice in the list S_1, \dots, S_n from the pairs T_1, \dots, T_n . For every pair $T_j = (x, S_j)$, the positive integer x appears exactly once in a single pair $T_i = (x, S_i)$ for $i \leq m$. We assume the elements in the set S_i of each pair T_i appear sorted in the input with ascending order. Moreover, a set S_i from a pair T_i could be equal to the set \emptyset . Furthermore, if we have two pairs $T_i = (x, S_i)$ and $T_j = (y, S_j)$ such that $x = y$, $i < j$, $S_i \neq \emptyset$ and $S_j \neq \emptyset$, then the minimum element of S_j is greater than the maximum element of S_i .

QUESTION: Is it the case there is a family of sets $S'_1, \dots, S'_{n'}$ with $n' \leq n$, such that $S'_i \cap S'_j = \emptyset$ for $1 \leq i \neq j \leq n'$ where S'_i is equal to the union of sets $\bigcup_j S_j$ for all $T_j = (y, S_j)$ when $x = y$ for a single value x and $S'_1 \cup \dots \cup S'_{n'} = U$?

REMARKS: We denote this problem as $ESC-2$.

► **Definition 8. EXACT COVER-2**

INSTANCE: A “universe” set U and a family of n sets $S_i \subseteq U$ with the property that every element in U appears at most twice in the list S_1, \dots, S_n .

QUESTION: Is it the case there is a subfamily S'_1, \dots, S'_m with $m \leq n$, such that $S'_i \cap S'_j = \emptyset$ for $1 \leq i \neq j \leq m$ and $S'_1 \cup \dots \cup S'_m = U$?

REMARKS: We denote this problem as $EC-2$. $EC-2 \in L$ [2], [11].

5 Results

► **Theorem 9.** $MIN \oplus 2SAT \in NP$ -complete.

Proof. It is trivial to see $MIN \oplus 2SAT \in NP$ [10]. Given a Boolean formula ϕ in 3CNF with n variables and m clauses such that each clause has no negation variables, we create

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three new variables a_{c_i} , b_{c_i} and d_{c_i} for each clause $c_i = (x \vee y \vee z)$ in ϕ , where x , y and z are positive literals, in the following formula:

$$P_i = (a_{c_i} \oplus b_{c_i}) \wedge (b_{c_i} \oplus d_{c_i}) \wedge (a_{c_i} \oplus d_{c_i}) \wedge (x \oplus a_{c_i}) \wedge (y \oplus b_{c_i}) \wedge (z \oplus d_{c_i}).$$

We can see P_i has at most one unsatisfiable clause for some truth assignment if and only if at least one member of $\{x, y, z\}$ is true and at least one member of $\{x, y, z\}$ is false for the same truth assignment. Hence, we can create the Boolean formula ψ as the conjunction of the P_i formulas for every clause c_i in ϕ , such that $\psi = P_1 \wedge \dots \wedge P_m$. Finally, we obtain that

$$\phi \in \text{MONOTONE NAE 3SAT} \text{ if and only if } (\psi, m) \in \text{MIN} \oplus \text{2SAT}.$$

Consequently, we prove $\text{MONOTONE NAE 3SAT} \leq_p \text{MIN} \oplus \text{2SAT}$ where we already know $\text{MONOTONE NAE 3SAT} \in \text{NP-complete}$ [6]. To sum up, we show $\text{MIN} \oplus \text{2SAT} \in \text{NP-hard}$ and $\text{MIN} \oplus \text{2SAT} \in \text{NP}$ and thus, $\text{MIN} \oplus \text{2SAT} \in \text{NP-complete}$. ◀

► **Theorem 10.** $\text{KEC-2} \in \text{NP-complete}$.

Proof. It is trivial to see $\text{KEC-2} \in \text{NP}$ [10]. Given a Boolean formula ϕ that is an instance of XOR 2SAT with n variables and m clauses such that each clause has no negation variables, we create a new set S_x for each variable x in ϕ and we iterate for each clause $c_i = (x \oplus y)$ in ϕ from $i = 1$ to m , where x and y are positive literals, and modify the following sets: $S_x = S_x \cup \{i\}$ and $S_y = S_y \cup \{i\}$. We create the “universe” set U as $\{1, \dots, m\}$. In this way, we obtain a “universe” set U of natural numbers and a family of n sets $S_j \subseteq U$ with the property that every element in U appears at most twice in the list S_1, \dots, S_n .

We can see if we have a subfamily $S'_1, \dots, S'_{n'}$ with $n' \leq n$ after removing K different numbers in U from the whole list S_1, \dots, S_n , such that $S'_i \cap S'_j = \emptyset$ for $1 \leq i \neq j \leq n'$ and $S'_1 \cup \dots \cup S'_{n'} = U'$ where U' is equal to the set U without the removed K different numbers, then we obtain exactly $m - K$ satisfiable clauses in ϕ for a truth assignment where the variable x is true if and only if S_x belongs to the subfamily $S'_1, \dots, S'_{n'}$. However, this would mean if there are exactly $m - K$ satisfiable clauses in ϕ for a truth assignment, then there are at most K unsatisfiable clauses in ϕ for the same truth assignment. Finally, we obtain that

$$(\phi, K) \in \text{MIN} \oplus \text{2SAT} \text{ if and only if } (K, U, S_1, \dots, S_n) \in \text{KEC-2}.$$

Consequently, we prove $\text{MIN} \oplus \text{2SAT} \leq_p \text{KEC-2}$ where we already know $\text{MIN} \oplus \text{2SAT} \in \text{NP-complete}$ by Theorem 9. To sum up, we show $\text{KEC-2} \in \text{NP-hard}$ and $\text{KEC-2} \in \text{NP}$ and thus, $\text{KEC-2} \in \text{NP-complete}$. ◀

► **Theorem 11.** $\text{ESC-2} \in \text{L}$.

Proof. Given a valid instance $m, U_1, \dots, U_k, T_1, \dots, T_n$ for ESC-2 , we can create a family of sets $S_1, \dots, S_{m'}$ with $m' \leq n$ where S_i is equal to the union of sets $\bigcup_j S_j$ for all $T_j = (y, S_j)$ when $x = y$ for a single value x . This family of sets $S_1, \dots, S_{m'}$ with the “universe” set $U = U_1 \cup \dots \cup U_k$ of the instance of ESC-2 is actually a valid instance for EC-2 since every element in U appears at most twice in the list $S_1, \dots, S_{m'}$. Moreover, due to the properties of the acceptance instances of ESC-2 , we obtain that:

$$(m, U_1, \dots, U_k, T_1, \dots, T_n) \in \text{ESC-2} \text{ if and only if } (U, S_1, \dots, S_{m'}) \in \text{EC-2}.$$

Furthermore, we can make this reduction in logarithmic space since for every pair $T_j = (x, S_j)$, the positive integer x appears exactly once in a single pair $T_i = (x, S_i)$ for $i \leq m$. Hence, we

only need to iterate from $i = 1$ to m on the pairs $T_i = (x, S_i)$ and join the sets $S_i \cup (\bigcup_j S_j)$ for all $T_j = (y, S_j)$ when $x = y$ for a single value x and $m < j \leq n$. This logarithmic space reduction will be the Algorithm 1. Certainly, the variables of the Algorithm 1 use at most logarithmic space in relation to the length of the input. If some problem L_1 is logarithmic space reduced to another problem in L , then $L_1 \in L$. Consequently, $ESC-2 \in L$ because $EC-2 \in L$ [2], [11]. ◀

Algorithm 1 Logarithmic space reduction

```

1: /*A valid instance for ESC-2*/
2: procedure REDUCTION( $m, U_1, \dots, U_k, T_1, \dots, T_n$ )
3:   /*Output the "universe" set  $U^*$ */
4:   output  $U_1 \cup \dots \cup U_k$ 
5:   /*Iterate for the first  $m$  pairs*/
6:   for  $i \leftarrow 1$  to  $m$  do
7:     /*Initialize the set output*/
8:     output "{"
9:      $(x, S_i) \leftarrow T_i$ 
10:    /*Output the elements of  $S_i^*$ */
11:    for all  $e_i \in S_i$  do
12:      /*The first element has not been printed in the output*/
13:      output ",  $e_i$ "
14:    end for
15:    /*Iterate for the next elements*/
16:    for  $j \leftarrow m + 1$  to  $n$  do
17:       $(y, S_j) \leftarrow T_j$ 
18:      /*If  $x$  is equal to  $y^*$ */
19:      if  $x = y$  then
20:        /*Output the elements of  $S_j^*$ */
21:        for all  $e_j \in S_j$  do
22:          output ",  $e_j$ "
23:        end for
24:      end if
25:    end for
26:    /*Finalize the set output*/
27:    output "}, "
28:  end for
29: end procedure

```

► **Theorem 12.** *There is a deterministic Turing machine M , where:*

$$KEC-2 = \{w : M(w, u) = y \text{ for some string } u \text{ such that } y \in ESC-2\}$$

when M runs in logarithmic space in the length of w , u is placed on the special read-once tape of M , and u is polynomially bounded by w .

Proof. Given a valid instance K, U, S_1, \dots, S_m for $KEC-2$, we can create a certificate array A which contains the K different natural numbers in U sorted in ascending order that we are going to remove from the instance. We read at once the elements of the array A and we reject whether this is not a valid certificate: That is when the numbers are not

sorted in ascending order, or the array A does not contain exactly K elements, or the array A contains a number that is not in U . While we read the elements of the array A , we remove them from the instance K, U, S_1, \dots, S_m for $KEC-2$ just creating another instance $m, U_1, \dots, U_{k'}, T_1, \dots, T_{n'}$ for $ESC-2$ where the “universe” set $U' = U_1 \cup \dots \cup U_{k'}$ is equal to U without the K different numbers in A and m is the number of different sets from the list S_1, \dots, S_m . The final pairs $T_i = (x, S_i)$ will not contain inside of S_i any of the K different natural numbers in U . Therefore, we obtain that:

$$(K, U, S_1, \dots, S_m) \in KEC-2 \text{ if and only if } (m, U_1, \dots, U_{k'}, T_1, \dots, T_{n'}) \in ESC-2.$$

Furthermore, we can make this verification in logarithmic space such that the array A is placed on the special read-once tape, because we read at once the elements in the array A and we remove the K different natural numbers from the output. Hence, we only need to iterate from the elements of the array A to verify whether the array is a valid certificate and also remove all the K different natural numbers from the sets S_i in the pairs $T_i = (x, S_i)$. This logarithmic space verification will be the Algorithm 2. In this Algorithm we guarantee:

1. Every element in U' appears at most twice in the list $S_1, \dots, S_{n'}$ from the pairs $T_1, \dots, T_{n'}$.
2. For every pair $T_j = (x, S_j)$, the positive integer x appears exactly once in a single pair $T_i = (x, S_i)$ for $i \leq m$.
3. The elements in the set S_i of each pair T_i appear sorted in the input with ascending order.
4. A set S_i from a pair T_i could be equal to the set \emptyset .
5. If we have two pairs $T_i = (x, S_i)$ and $T_j = (y, S_j)$ such that $x = y$, $i < j$, $S_i \neq \emptyset$ and $S_j \neq \emptyset$, then the minimum element of S_j is greater than the maximum element of S_i .

Note, in the loop e_j from min to $max - 1$, we do nothing when $max - 1 < min$: Indeed, in that iteration the output pairs $T_i = (x, S_i)$ will comply with $S_i = \emptyset$. Certainly, the variables of the Algorithm 2 use at most logarithmic space in relation to the length of the input. ◀

► **Theorem 13.** *The Hypothesis 3 is true.*

Proof. This is a consequence of Theorems 10, 11 and 12. ◀

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Algorithm 2 Logarithmic space verifier

```

1: /*A valid instance for KEC-2 with its certificate*/
2: procedure VERIFIER((K, U, S1, . . . , Sm), A)
3:   /*Output the value of m*/
4:   output m
5:   /*Initialize minimum and maximum values*/
6:   min ← 1
7:   max ← 0
8:   /*Iterate for the elements of the certificate array A*/
9:   for i ← 1 to K + 1 do
10:    if i = K + 1 then
11:      /*There exists a K + 1 element in the array*/
12:      if A[i] ≠ undefined then
13:        /*Reject the certificate*/
14:        return 0
15:      end if
16:      /*maximum(U) is equal to the maximum number in U*/
17:      max ← maximum(U) + 1
18:      else if A[i] = undefined ∨ A[i] ≤ max ∨ A[i] ∉ U then
19:        /*Reject the certificate*/
20:        return 0
21:      else
22:        max ← A[i]
23:      end if
24:      /*Initialize the set output Ur*/
25:      output "{"
26:      for er ← min to max − 1 do
27:        if er ∈ U then
28:          output ", er"
29:        end if
30:      end for
31:      /*Finalize the set output*/
32:      output "}, "
33:      /*Iterate for the sets Sj*/
34:      for j ← 1 to m do
35:        /*Initialize the pair output*/
36:        output "(j, {"
37:        for ej ← min to max − 1 do
38:          if ej ∈ Sj then
39:            output ", ej"
40:          end if
41:        end for
42:        /*Finalize the pair output*/
43:        output "}), "
44:      end for
45:      min ← max + 1
46:    end for
47: end procedure

```

10 Logarithmic Space Verifiers on NP-complete

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