# A new Direct Connected Component Labeling and Analysis Algorithm for GPUs 

Arthur Hennequin ${ }^{1,2}$, Lionel Lacassagne ${ }^{1}$

LIP6, Sorbonne University, CNRS, France ${ }^{1}$ LHCb experiment, CERN, Switzerland ${ }^{2}$

GTC 2019 March $21^{\text {st }}$

## What are Connected Component Labeling and Analysis ?

Connected Components Labeling (CCL) consists in assigning a unique number (label) to each connected component of a binary image to cluster pixels Connected Components Analysis (CCA) consists in computing some features associated to each connected component like the bounding box $\left[x_{\min }, x_{\max }\right] \times$ [ $y_{\min }, y_{\max }$ ], the sum of pixels $S$, the sums of $x$ and $y$ coordinates $S x, S y$

gray level image

binary level image (segmentation by (motion detection)

connected component labeling

connected component analysis

- seems easy for a human being who has a global view of the image
- ill-posed problem: the computer has only a local view around a pixel (neighborhood)
- important in computer vision for pattern recognition, motion detection ...


## Two classes of CCL algorithms

- multi-pass iterative algorithms
- compute the local positive min over a $3 \times 3$ neighborhood
- until stabilization : the number of iterations depends on the data
- not predictable, nor suited for embedded systems
- two-pass direct algorithms
- first pass $=$ temporary label creation and equivalence building
- need an equivalence table to memorize the connectivity between labels
- then compute transitive closure of the tree associated to the equivalence table
- second pass $=$ image relabeling (apply table T to the image)
- what are the existing algorithms on CPU and GPU ?
- on CPU, scalar algorithms are all direct and can be parallelized
- on SIMD CPU, until 2019, all SIMD algorithms are iterative, except 1
- on GPU, until 2018, all algorithms are iterative, except 3
- Why so few direct algorithms on GPU and SIMD ?
$\Rightarrow$ because extremely complex to design (not suited for SIMD nor GPU)


## Direct algorithms are based on Union-Find structure

## Algorithm 1: Rosenfeld labeling algorithm

$$
\text { for } i=0: h-1 \text { do }
$$

$$
\text { for } j=0: w-1 \text { do }
$$

if $I[i][j] \neq 0$ then
$e_{1} \leftarrow E[i-1][j]$
$e_{2} \leftarrow E[i][j-1]$
if $\left(e_{1}=e_{2}=0\right)$ then
$n e \leftarrow n e+1$
$e_{x} \leftarrow n e$
else
$r_{1} \leftarrow \operatorname{Find}\left(e_{1}, T\right)$
$r_{2} \leftarrow \operatorname{Find}\left(e_{2}, T\right)$
$e_{x} \leftarrow \min ^{+}\left(r_{1}, r_{2}\right)$
if $\left(r_{1} \neq 0\right.$ and $\left.r_{1} \neq e_{x}\right)$ then $T\left[r_{1}\right] \leftarrow e_{x}$
if $\left(r_{2} \neq 0\right.$ and $\left.r_{2} \neq e_{x}\right)$ then $T\left[r_{2}\right] \leftarrow e_{x}$

| Algorithm 2: Find $(e, T)$ |
| :--- |
| while $T[e] \neq e$ do |
| $L e T / e]$ |
| return $e / /$ the root of the tree |

Algorithm 3: Union $\left(e_{1}, e_{2}, T\right)$

```
r
r
if (r
    T[r2]}\leftarrow\mp@subsup{r}{1}{
else
    T[r, ]}\leftarrow\mp@subsup{r}{2}{
```

Algorithm 4: Transitive Closure

```
for i=0: ne do
    T[e]}\leftarrowT[T[e]
```

Parallel algorithms have to do:

- sparse addressing $\Rightarrow$ scatter/gather SIMD instructions (AVX512/SVE)
- concurrent $\min$ computation $\Rightarrow$ recursive atomic min (CUDA)


## Classic direct algorithm: Rosenfeld

Rosenfeld algorithm is the first 2-pass algorithm with an equivalence table

- when two labels belong to the same component, an equivalence is created and stored into the equivalence table T
- for example, there is an equivalence between 2 and 3 (stair pattern) and between 4 and 2 (concavity pattern)
- stair and concavity are the only two two patterns generating equivalence
- here, background in gray and foreground in white, 4-connectivity algorithm

| 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 1 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| 1 | 0 | 1 | 0 | 1 | 1 | 1 | 1 |
| 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| binary image of piA |  |  |  |  |  |  |  |



|  |  |  |  |  |  |
| :---: | :--- | :--- | :--- | :--- | :--- |
| e | 0 | 1 | 2 | 3 | 4 |
| $\mathrm{~T}[\mathrm{e}]$ | 0 | 1 | 2 | 2 | 2 |

equivalence table

image of labels

| 1 | 1 | 1 | 0 | 0 | 0 | 2 | 2 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 1 | 0 | 0 | 0 | 2 | 2 | 2 | 2 |
| 1 | 0 | 2 | 0 | 2 | 2 | 2 | 2 |
| 1 | 0 | 2 | 2 | 2 | 2 | 2 | 2 |
| image of labels after relabeling |  |  |  |  |  |  |  |


equivalence trees

patterns generator of equivalence

## Back to iterative Labeling algorithms

The number of iterations depends on data structure:

| 1 | 2 | 3 | 4 | 5 |
| :---: | :---: | :---: | :---: | :---: |
| 6 | 7 | 8 | 9 | 10 |
| 11 | 12 | 13 | 14 | 15 |
| 16 | 17 | 18 | 19 | 20 |
| 21 | 22 | 23 | 24 | 25 |



9 iterations for a $5 \times 5$ square

but 16 iterations for a $5 \times 5$ zig-zag or a spirale
the number of iterations is equal to the longest path aka the max geodesic distance
... and the max geodesic distance for a $n \times n$ image is $\simeq n^{2} / 2$

## Parallel State-of-the-art

- Parallel Light Speed Labeling[1](L. Cabaret, L. Lacassagne, D. Etiemble) (2018)
- parallel algorithm for CPU
- based on RLE (Run Length Encoding) to speed up processing and save memory accesses
- current fastest CCA algorithm on CPU
- Distanceless Label Propagation[2](L. Cabaret, L. Lacassagne, D. Etiemble) (2018)
- direct CCL algorithm for GPU
- Playne-Equivalence[3](D. P. Playne, K.A. Hawick) (2018)
- direct CCL algorithm for GPU (2D and 3D versions)
- based on the analysis of local pixels configuration to avoid unnecessary and costly atomic operations to save memory accesses.
$\Rightarrow$ no CCA for GPU, right now ...


## Equivalence merge function \& concurrency issue

The direct CCL algorithms rely on Union-Find to manage equivalences A parallel merge operation can lead to concurrency issues:


- $1^{\text {st }}$ example (top-left): no concurrency, $\mathrm{T}[3] \leftarrow 1, \mathrm{~T}[4] \leftarrow 1$
- $2^{\text {nd }}$ example (top-right): no concurrency, $\mathrm{T}[3] \leftarrow 1, \mathrm{~T}[4] \leftarrow 2$
- $3^{r d}$ example (bottom-left): non-problematic concurrency, $\mathrm{T}[4] \leftarrow 1, \mathrm{~T}[4] \leftarrow 1$
- $4^{\text {th }}$ example (bottom-right): concurrency issue, $\mathrm{T}[4] \leftarrow 1, \mathrm{~T}[4] \leftarrow 2$
- 4 can't be equal to 1 and 2
- $\Rightarrow 4$ has to point to 1 and 2 has to point to 1 too...


## Equivalence merge function (aka recursive Union)

The merge function, introduced by Playne and Hawick, solves the concurrency issues by iteratively merging labels using atomic operations

```
Algorithm 5: merge( \(\mathrm{L}, \mathrm{e}_{1}, \mathrm{e}_{2}\) )
while \(\mathrm{e}_{1} \neq \mathrm{e}_{2}\) and \(\mathrm{e}_{1} \neq \mathrm{L}\left[\mathrm{e}_{1}\right]\) do
    \(\mathrm{e}_{1} \leftarrow \mathrm{~L}\left[\mathrm{e}_{1}\right] \quad / /\) root of \(e_{1}\)
while \(\mathrm{e}_{1} \neq \mathrm{e}_{2}\) and \(\mathrm{e}_{2} \neq \mathrm{L}\left[\mathrm{e}_{2}\right]\) do
    \(\mathrm{e}_{2} \leftarrow \mathrm{~L}\left[\mathrm{e}_{2}\right] \quad / /\) root of \(e_{2}\)
while \(e_{1} \neq e_{2}\) do
    if \(e_{1}<e_{2}\) then \(\operatorname{swap}\left(e_{1}, e_{2}\right)\)
        \(\mathrm{e}_{3} \leftarrow\) atomicMin \(\left(\mathrm{L}\left[\mathrm{e}_{1}\right], \mathrm{e}_{2}\right) \quad / /\) recursive min
        if \(e_{3}=e_{1}\) then \(e_{1} \leftarrow e_{2}\)
        else \(e_{1} \leftarrow e_{3}\)
```

By definition, $e_{3} \leq \mathrm{L}\left[e_{1}\right]$, so:

- if $e_{3}=e_{1}$ : no concurrent write, update of $L$ is successful, terminates the loop
- if $e_{3}<e_{1}$ : concurrent write, $L$ was updated by another thread, need to merge $e_{3}$ and $e_{2}$


## Hardware Accelerated algorithm : HA4

Analysis of state-of-the-art weaknesses:

- vertical borders (non-coalescent memory accesses)
- expensive atomic operations

Analysis of state-of-the-art strengths:

- equivalence table embedded in the image (Cabaret, Playne)
- merge function (Komura [4] + Playne)
- segments labeling (Light Speed Labeling)
- necessary condition to merge two equivalence trees (Playne)


Figure 1: All possible 4 pixels configurations. Only (f) needs to merge labels. (Playne)

## Hardware Accelerated: HA4

The algorithm is divided into 3 kernels:

- strip labeling: the image is split into horizontal strips of 4 rows. Each strip is processed by a block of $32 \times 4$ threads (one warp per row). Only the head of segment is labeled
- border merging: to merge the labels on the horizontal borders between strips
- relabeling / features computation: to propagate the label of each segment to the pixels or to compute the features associated to the connected components



## Example - Strip labeling initialization (Step \#0)

The $8 \times 8$ image is divided into 2 strips of $8 \times 4$ pixels, warp size $=8$

Initial strip labeling:

- only the head of each segment (start node) is labeled with an unique label
- equal to its linear address: $L[k]=k$ with $k \triangleq y \times$ width $+x$
- warning: label numbering starts at 0 , not 1

|  | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 0 |  |  |  |  |  | 6 |  |
| 1 | 8 |  |  |  | 12 |  |  |  |
| 2 | 16 |  | 18 |  | 20 |  |  |  |
| 3 | 24 |  | 26 |  |  |  |  |  |
| 0 | 32 |  | 34 |  |  |  |  |  |
| 1 | 40 |  |  | 43 |  |  |  | 47 |
| 2 | 48 |  |  |  |  |  | 54 |  |
| 3 | 56 |  |  |  |  |  | 62 |  |

(a) Initialization

## Example - Strip labeling (Step \#1)

After initialization:

- detection of merging nodes using necessary conditions in each thread
- update of start nodes only

Strips' segments are now labeled

(b) Strip labeling

(c) Strip labeled


Here, a CC spanning over several strips is represented by 3 disjoint trees of labels

## Example - Border merging (Step \#2)

Same merging operations on border nodes only All the segments are correctly labeled. A CC spanning to several strips is represented by 1 tree.


| 0 | 32 | 32 |  |  |  |  |  |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
|  | 32 |  |  | 34 |  |  |  | 34 |
|  | 32 | 40 |  |  |  |  |  | 47 |
|  | 48 |  |  |  |  |  | 54 |  |
|  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |

(d) Border merging


## Example - Re-Labeling / Analysis (Step \#3)

In the final step only, each start node (blue) flattens its equivalence tree

- to Label the image: broadcast the label to the whole segment
- to Analyse the image: accumulate features into global memory using atomics example of features associated to segment $\left[x_{0}, x_{1}\right]$ at line $y$ :
- $S=x_{1}-x_{0}, \quad S_{y}=S \times y_{0}, \quad S_{x}=\frac{1}{2}\left[x_{1}\left(x_{1}-1\right)-\left(x_{0}\left(x_{0}-1\right)\right]\right.$

|  | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 0 |  |  |  |  |  | 0 |  |
| 1 | 0 |  |  |  | 0 |  |  |  |
| 2 | 0 |  | 0 |  | 0 |  |  |  |
| 3 | 0 |  | 0 |  |  |  |  |  |
| 0 | 0 |  | 0 |  |  |  |  |  |
| 1 | 0 |  |  | 0 |  |  |  | 0 |
| 2 | 0 |  |  |  |  |  | 0 |  |
| 3 | 0 |  |  |  |  |  | 0 |  |

FindRoot


## Implementation details: Grid-stride loop

- first weakness of previous GPU algorithms is the vertical border merging: the non-coalescent memory accesses are slower
- we used the grid-stride loop [5] design pattern to divide the image in strips instead of tiles

```
kernel Classic(width)
    x}\leftarrow\mathrm{ blockDim.x x blockldx.x + threadldx.x
    if }x<<\mathrm{ width then
        L // do stuff..
kernel Grid_stride_loop(width)
    for }x\leftarrow\mathrm{ threadldx.x to width by blockDim. }x\mathrm{ do
        L // do stuff..
```

Benefits:

- thread reuse: less thread creation. Helps to amortize the cost of thread creation/destruction
- thread context is preserved: the loop ensures that pixels are processed in a specific order and allows to reuse previously computed values


## Implementation details: horizontal data exchange

All threads working on the same row are from the same warp, CUDA Warp-Level Primitives [6] can be used to directly exchange data from threads registers

- _-ballot_sync primitive returns a 32 -bit bitmask based on the value of a boolean within each thread (1 bit per thread)
- __shfl_sync primitive exchanges a 32-bit value between any pair of threads in a warp. Each thread specifies a thread ID to read and a value to share


## Implementation details: segments

- each thread needs to find its distance to the segment's start node
- distance to the end is also needed for features computation
- bitwise operations can accelerate the computation of these distances ( $\mathrm{tx}=$ thread number)

|  | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| pixels | 0 | 1 | 1 | 1 | 0 | 0 | 1 | 0 |
| start_distance |  | 0 | 1 | 2 |  |  | 0 |  |
| end_distance |  | 3 | 2 | 1 |  |  | 1 |  |

operator start_distance (pixels, $t x$ )
return __clz $(\sim($ pixels << $(32-t x))) / / ~ c l z=$ Count Leading Zeros
operator end_distance (pixels, $t x$ )
return _ffs $(\sim($ pixels $\gg(t x+1))) / / f f s=$ Find First Set

## Implementation details: vertical data exchange

- classic way of optimizing memory accesses: copying data from global to shared memory
- shared memory is divided in 32 banks: same bank memory accesses at different addresses get serialized [7]



## Implementation details: vertical data exchange

- for each row, we store the bitmasks of the 32 neighbor pixels in different banks
- store: no serialization, load: broadcast



## One final optimization...

- two pixels directly next to each other either belong to the same segment or have a different color
- we can assign a thread two pixels instead of one.
- 32-bit $\rightarrow$ 64-bit bitmask: modified distance operators.
- new version: HA464


```
operator start_distance64(pixels, tx)
    b}\leftarrow\mathrm{ get bit tx of ~pixels
    txb}\leftarrowtx+
    return __clzll(~(pixels << (64-txb)))
operator end_distance64(pixe/s, tx)
    b}\leftarrow\mathrm{ get bit tx of ~pixels
    txb}\leftarrow\textrm{tx}+\textrm{b
    return __ffsll(~(pixels >> (txb+1)))
```


## Benchmark of CCL and CCA algorithms

- random $2048 \times 2048$ (2k) images of varying density ( $0 \%-100 \%$ ), granularity ( $1-16$, granularity $=4$ close to natural image complexity)
- percolation threshold: transition from many smalls CCs to few larges CCs
- 8C: density $=45 \%$
- 4C: density $=64 \%$




## Comparison of CCL algorithms on Jetson TX2

Comparison with 2 state-of-the-art algorithms [Playne, Cabaret]

- Cabaret and Playne lose time updating all the temporary labels
- thanks to the use of segments, HA4's processing time decreases after the percolation threshold $\mathrm{d}=64 \%$
- HA4 ${ }_{64}$ is $2 \times$ faster in average than Playne and Cabaret
- CCL throughput: $1.2 \mathrm{Gpx} / \mathrm{s}$ ( $\mathrm{HA} 4_{64}, 2 \mathrm{k}, \mathrm{g}=4$ )

(a) Playne

(c) $\mathrm{HA}_{32}(\mathrm{ccl})$

(b) Cabaret

(d) $\mathrm{HA}_{64}(\mathrm{ccl})$


## Comparison of CCA algorithms on Jetson TX2

- HA4 ${ }_{64}$ CCA: labeling kernel is replaced by on-the-fly analysis kernel
- other algorithms: features computation kernel after relabeling kernel
- 7 features: $\mathrm{S}, \mathrm{Sx}, \mathrm{Sy}, \mathrm{x}_{\min }, \mathrm{y}_{\min }, \mathrm{x}_{\max }, \mathrm{y}_{\max } \rightarrow 1.1 \mathrm{Gpx} / \mathrm{s}\left(\mathrm{HA} 4_{64}, 2 \mathrm{k}, \mathrm{g}=4\right)$

(a) Playne

(a) $\mathrm{HA}_{32}$

(b) Cabaret

(b) $\mathrm{HA}_{64}$


## Performance of CCL on Jetson AGX \& V100

## Latest results on Volta architecture:

- AGX: $4.6 \mathrm{Gpx} / \mathrm{s}\left(\mathrm{HA}_{64}, 2 \mathrm{k}, \mathrm{g}=4\right)$
- V100: $27.0 \mathrm{Gpx} / \mathrm{s}\left(\mathrm{HA}_{64}, 2 \mathrm{k}, \mathrm{g}=4\right)$

(a) $\mathrm{HA4}_{32}$ Jetson AGX

(c) $\mathrm{HA}_{32} \mathrm{~V} 100$

(b) $\mathrm{HA4}_{64}$ Jetson AGX

(d) $\mathrm{HA}_{64} \mathrm{~V} 100$


## Performance of CCA on Jetson AGX \& V100

## Latest results on Volta architecture:

- AGX: 3.4 Gpx/s ( $\mathrm{HA}_{64}, 2 \mathrm{k},\left(\mathrm{S}, \mathrm{Sx}, \mathrm{Sy}, \mathrm{x}_{\text {min }}, \mathrm{y}_{\text {min }}, \mathrm{x}_{\text {max }}, \mathrm{y}_{\max }\right), \mathrm{g}=4$ )
- V100: $14.9 \mathrm{Gpx} / \mathrm{s}\left(\mathrm{HA}_{64}, 2 \mathrm{k},\left(\mathrm{S}, \mathrm{Sx}, \mathrm{Sy}, \mathrm{Sx}^{2}, \mathrm{Sy}^{2}\right), \mathrm{g}=4\right)$

(a) $\mathrm{HA}_{32}$ Jetson AGX

(c) $\mathrm{HA}_{32} \mathrm{~V} 100$

(b) $\mathrm{HA}_{64}$ Jetson AGX

(d) HA4 ${ }_{64}$ V100


## Observations for Jetson AGX \& V100

- strong scalability for CCL
- weak scalability for CCA (concurrent accesses in atomic operations)
- some features are faster to compute than others: the first statistical moments, computed with atomic addition, are faster than the bounding boxes computed with atomic min and max

(a) $\mathrm{HA}_{64}(\mathrm{cca}) \mathrm{V} 100\left(\mathrm{~S}, \mathrm{Sx}, \mathrm{Sy}, \mathrm{x}_{\text {min }}, \mathrm{y}_{\text {min }}, \mathrm{x}_{\text {max }}, \mathrm{y}_{\text {max }}\right)$

(b) $\mathrm{HA}_{64}$ (cca) V100 (S, Sx, Sy, Sx $\left.{ }^{2}, \mathrm{Sy}^{2}\right)$


## Conclusion

- two new algorithms for 4-connectivity connected component processing on GPU:
- CCL $2 \times$ faster than State-of-the-Art
- CCA new on GPU
- introduced a new way to efficiently reduce the number of global memory accesses using segments, combined with low-level intrinsics
- $\mathrm{HA}_{64}$ ready for realtime embedded processing.
- CCL throughput: 4.6 Gpix/s on AGX (1920×1080: 2208 fps ) or
- CCA throughput: 3.4 Gpix/s on AGX (1920x1080: 1615 fps$)$
- future works:
- Design 8-connectivity versions on GPUs
- Improve CCA by implementing different merging strategies
- algorithm and benchmarks were published at DASIP 2018 [8]

Thank you!

## References I

T
L. Cabaret, L. Lacassagne, and D. Etiemble, "Parallel Light Speed Labeling for connected component analysis on multi-core processors," Journal of Real Time Image Processing, no. 15,1, pp. 173-196, 2018.
L. Cabaret, L. Lacassagne, and D. Etiemble, "Distanceless label propagation: an efficient direct connected component labeling algorithm for GPUs," in IEEE International Conference on Image Processing Theory, Tools and Applications (IPTA), pp. 1-8, 2017.
D. P. Playne and K. Hawick, "A new algorithm for parallel connected-component labelling on GPUs," IEEE Transactions on Parallel and Distributed Systems, 2018.
Y. Komura, "Gpu-based cluster-labeling algorithm without the use of conventional iteration: application to swendsen-wang multi-cluster spin flip algorithm," Computer Physics Communications, pp. 54-58, 2015.
M. Harris,
"https://devblogs.nvidia.com/cuda-pro-tip-write-flexible-kernels-grid-stride-loops/," 2013.
Y. Lin and V. Grover, "https://devblogs.nvidia.com/using-cuda-warp-level-primitives/," 2018.
M. Harris, "https://devblogs.nvidia.com/using-shared-memory-cuda-cc/," 2013.
A. Hennequin, L. Lacassagne, L. Cabaret, and Q. Meunier, "A new Direct Connected Component Labeling and Analysis Algorithms for GPUs," in DASIP, (Porto, Portugal), Oct. 2018.

## Backup: average throughput with $\mathrm{g}=16$

- TX2 2k:
- CCL: $1.37 \mathrm{Gpx} / \mathrm{s}$
- CCA: $1.36 \mathrm{Gpx} / \mathrm{s}$
- AGX 2k:
- CCL: $5.75 \mathrm{Gpx} / \mathrm{s}$
- CCA: $5.61 \mathrm{Gpx} / \mathrm{s}$
- V100 2k:
- CCL: $32.02 \mathrm{Gpx} / \mathrm{s}$
- CCA: $24.42 \mathrm{Gpx} / \mathrm{s}$
- V100 4k:
- CCL: $42.92 \mathrm{Gpx} / \mathrm{s}$
- CCA: $30.35 \mathrm{Gpx} / \mathrm{s}$


## Backup: full post-features analysis (TX2)



## Direct algorithms are based on Union-Find structure

What are the issues (for parallel architectures) ?

```
Algorithm 6: Find \((e, T)\)
while \(T[e] \neq e\) do
    \(\llcorner e \leftarrow T[e]\)
return \(e / /\) the root of the tree
```

Algorithm 7: Union $\left(e_{1}, e_{2}, T\right)$
$r_{1} \leftarrow \operatorname{Find}\left(e_{1}, T\right)$
$r_{2} \leftarrow \operatorname{Find}\left(e_{2}, T\right)$
if $r_{1}<r_{2}$ then $T\left[r_{2}\right] \leftarrow r_{1}$
else $T\left[r_{1}\right] \leftarrow r_{2}$

- SIMD CPU \& sparse addressing
- requires scatter/gather instructions (AVX512/SVE)
- CPU pyramidal/parallel merge:
- pyramidal merge requires disjoint-sets
- parallel merge requires recursive atomic instructions
- SIMD pyramidal merge needs emulated atomic instructions within registers (conflict detection)
- GPU parallel merge
- requires recursive atomic instructions
but capability is *not* efficiency

