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# Mean exit time for the overdamped Langevin process: the case with critical points on the boundary

Boris Nectoux \*

## Abstract

Let  $(X_t)_{t \geq 0}$  be the overdamped Langevin process on  $\mathbb{R}^d$ , i.e. the solution of the stochastic differential equation

$$dX_t = -\nabla f(X_t) dt + \sqrt{h} dB_t.$$

Let  $\Omega \subset \mathbb{R}^d$  be a bounded domain. In this work, when  $X_0 = x \in \Omega$ , we derive new sharp asymptotic equivalents (with optimal error terms) in the limit  $h \rightarrow 0$  of the mean exit time from  $\Omega$  of the process  $(X_t)_{t \geq 0}$  (which is the solution of  $(-\frac{h}{2}\Delta + \nabla f \cdot \nabla)w = 1$  in  $\Omega$  and  $w = 0$  on  $\partial\Omega$ ), when the function  $f : \overline{\Omega} \rightarrow \mathbb{R}$  has critical points on  $\partial\Omega$ . Such a setting is the one considered in many cases in molecular dynamics simulations. This problem has been extensively studied in the literature but such a setting has never been treated. The proof, mainly based on techniques from partial differential equations, uses recent spectral results from [29] and its starting point is a formula from the potential theory. We also provide new sharp leveling results on the mean exit time from  $\Omega$ .

**Key words:** singular perturbation, asymptotic analysis, overdamped Langevin process, mean exit time, metastability, Eyring-Kramers formulas.

**AMS classification (2010):** 31C15, 60J60, 35P15.

## 1 Purpose of this work and main result

### 1.1 Setting and purpose of this work

Let us consider a function  $f : \mathbb{R}^d \rightarrow \mathbb{R}$ . A prototypical process used to model the evolution of a statistical system in a constant temperature environment is the so-called overdamped Langevin process, i.e. the solution  $(X_t)_{t \geq 0}$  of the stochastic differential equation

$$dX_t = -\nabla f(X_t) dt + \sqrt{h} dB_t, \tag{1}$$

where  $h > 0$  is the temperature of the medium and  $(B_t)_{t \geq 0}$  is a  $d$ -dimensional standard Brownian motion. For an open subset  $\Omega$  of  $\mathbb{R}^d$ , one denotes by

$$\tau_{\Omega^c} = \inf\{t \geq 0, X_t \notin \Omega\},$$

the first time the process (1) enters in  $\Omega^c := \mathbb{R}^d \setminus \Omega$  (or equivalently the first exit time from  $\Omega$ ). The aim of this paper is to prove sharp asymptotic estimates when  $h \rightarrow 0$ , with optimal error terms, on the mean exit time  $\mathbb{E}[\tau_{\Omega^c}]$  from a domain  $\Omega \subset \mathbb{R}^d$  of the process (1) when  $f$  has critical point on  $\partial\Omega$ . The resulting equivalents are usually called Eyring-Kramers type formulas, see [19] for a review on this topic. Such a setting is motivated by the fact that in many molecular

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dynamics simulations (where several algorithms have been designed to accelerate the sampling of the exit event  $(\tau_{\Omega^c}, X_{\tau_{\Omega^c}})$ ), the domain  $\Omega$  is defined as a basin of attraction for the  $-\nabla f$  dynamics (see (4) below) of a local minimum point of  $f$  in  $\mathbb{R}^d$  so that  $f$  has critical points on  $\partial\Omega$ . To analyse the mathematical foundations of these algorithms, it is necessary to study the precise asymptotic behavior of the law of  $(\tau_{\Omega^c}, X_{\tau_{\Omega^c}})$  as  $h \rightarrow 0$  (see for instance [12, 30, 33, 47, 50, 51] and references therein). Moreover, our techniques also allow us to consider the case when  $f : \bar{\Omega} \rightarrow \mathbb{R}$  has a degenerate principal barrier in  $\Omega$ , i.e. for some  $m > 1$ ,

$$\lim_{h \rightarrow 0} h \ln \lambda_{1,h} = \lim_{h \rightarrow 0} h \ln \lambda_{2,h} = \dots = \lim_{h \rightarrow 0} h \ln \lambda_{m,h},$$

where  $\lambda_{1,h} < \lambda_{2,h} \leq \dots \leq \lambda_{m,h}$  are the  $m$  smallest eigenvalues of the Dirichlet realization of infinitesimal generator of the diffusion (1) in  $\Omega$  (see the next section for the definition of this operator). Despite the attention this topic has received these past few decades, to the best of our knowledge, such settings have not been treated yet (see Section 1.5 below). Furthermore, dealing with a degenerate principal barrier introduces some subtleties if one wants to study the asymptotic behavior of  $\mathbb{E}[\tau_{\Omega^c}]$  via a spectral approach. This is due to the fact that the  $m$  smallest eigenvalues  $\lambda_{1,h}, \dots, \lambda_{m,h}$  can be asymptotically indistinguishable when  $h \rightarrow 0$ , as explained in [28]. To overcome these difficulties, we introduce local Dirichlet problems (see Section 2.5 below) and use recent spectral results from [29] together with formulas from the potential theory and estimates from the large deviations theory.

## 1.2 Notation and definitions

In all this work,  $\Omega$  is a  $C^\infty$  bounded open and connected subset of  $\mathbb{R}^d$ , and the function  $f : \bar{\Omega} \rightarrow \mathbb{R}$  is  $C^\infty$ . Throughout this work, we will use the following notation: for  $a \in \mathbb{R}$ ,

$$\{f < a\} = \{x \in \bar{\Omega}, f(x) < a\} \text{ and } \{f \leq a\} = \{x \in \bar{\Omega}, f(x) \leq a\}.$$

We will say that a connected component  $M$  of  $\{f \leq a\}$  is isolated if there exists an open set  $V$  of  $\bar{\Omega}$  such that  $M = V \cap \{f \leq a\}^1$ . Moreover,  $B_r(x_*)$  will denote the closed ball centred at  $x_*$  of radius  $r > 0$  in  $\mathbb{R}^d$ . Finally,  $\mathbb{E}_x$  (resp.  $\mathbb{P}_x$ ) will denote the expectation (resp. the probability) when  $X_0 = x \in \mathbb{R}^d$ .

**Dirichlet realization of  $-\frac{h}{2}\Delta + \nabla f \cdot \nabla$  in  $\Omega$ .** Let us denote by  $\mathcal{L}_{f,h} = -\frac{h}{2}\Delta + \nabla f \cdot \nabla$  the infinitesimal generator of the process (1). The operator  $\mathcal{L}_{f,h}$  on  $L^2(\Omega, e^{-\frac{2}{h}f} dx)$  with domain  $\{w \in H^2(\Omega, e^{-\frac{2}{h}f} dx), w = 0 \text{ on } \partial\Omega\}$  is denoted by  $\mathcal{L}_{f,h}^{\text{Dir}}$  (where the notation Dir stands for the fact that we consider  $\mathcal{L}_{f,h}$  with Dirichlet boundary conditions on  $\partial\Omega$ ). The operator  $\mathcal{L}_{f,h}^{\text{Dir}}$  is the Friedrichs extension on  $L^2(\Omega, e^{-\frac{2}{h}f} dx)$  of the closed quadratic form

$$g \in H_0^1(\Omega, e^{-\frac{2}{h}f} dx) \mapsto \frac{2}{h} \int_{\Omega} |\nabla g|^2 e^{-\frac{2}{h}f}. \quad (2)$$

The operator  $\mathcal{L}_{f,h}^{\text{Dir}}$  is a positive self adjoint operator on  $L^2(\Omega, e^{-\frac{2}{h}f} dx)$  with compact resolvent. Its eigenvalues are denoted by

$$\lambda_{1,h} < \lambda_{2,h} \leq \lambda_{3,h} \dots$$

where we recall that by standard results on elliptic operators,  $\lambda_{1,h}$  is simple and any associated eigenfunction  $u_{1,h}$  is  $C^\infty$  on  $\bar{\Omega}$  and has a sign on  $\Omega$ . Finally let us recall that  $\mathcal{L}_{f,h}$  is connected

<sup>1</sup>This is equivalent to say that the connected component  $M$  of  $\{f \leq a\}$  is open in  $\{f \leq a\}$ .

to the Witten Laplacian  $\Delta_{f,h} = h^2 \Delta_H + |\nabla f|^2 - h \Delta f$  through the relation

$$\mathcal{L}_{f,h} = \frac{1}{2h} e^{\frac{1}{h}f} \Delta_{f,h} e^{-\frac{1}{h}f}, \quad (3)$$

which will allow us to transfer the results of [29] on the spectrum of the Dirichlet realization of  $\Delta_{f,h}$  in  $\Omega$  to  $\mathcal{L}_{f,h}^{\text{Dir}}$ .

**Domain of attraction of a subset  $M$  of  $\Omega$  for the  $-\nabla f$  dynamics in  $\Omega$ .** Let  $x \in \Omega$  and  $\varphi_t(x)$  denote the solution to the ordinary differential equation

$$\frac{d}{dt} \varphi_t(x) = -\nabla f(\varphi_t(x)) \text{ with } \varphi_0(x) = x, \quad (4)$$

on the interval  $t \in [0, t_x]$ , where  $t_x = \inf\{t \geq 0, \varphi_t(x) \notin \Omega\} > 0$ . Let  $x \in \Omega$  be such that  $t_x = +\infty$ . The  $\omega$ -limit set of  $x$ , denoted by  $\omega(x)$ , is defined by

$$\omega(x) = \{y \in \overline{\Omega}, \exists (s_n)_{n \in \mathbb{N}} \in (\mathbb{R}_+)^{\mathbb{N}}, \lim_{n \rightarrow \infty} s_n = +\infty, \lim_{n \rightarrow \infty} \varphi_{s_n}(x) = y\}.$$

Let us recall that the  $\omega$ -limit set  $\omega(x)$  is included in the set of the critical points of  $f$  in  $\overline{\Omega}$ . Moreover, if  $f$  has a finite number of critical points in  $\overline{\Omega}$ , there exists  $y \in \overline{\Omega}$  such that  $\omega(x) = \{y\} \subset \{z \in \overline{\Omega}, \nabla f(z) = 0\}$ . The domain of attraction of  $M$  is then defined by

$$\mathcal{A}_\Omega(M) = \{x \in \Omega, t_x = +\infty \text{ and } \omega(x) \subset M\}. \quad (5)$$

**Principal wells of  $f$  in  $\Omega$ .** In all this work, we will assume that the function  $f : \overline{\Omega} \rightarrow \mathbb{R}$  a Morse function, i.e. that all its critical points are non degenerate. In particular,  $f$  has a finite number of critical points in  $\overline{\Omega}$ . Indeed, a critical point of  $f$  is isolated from the other critical points of  $f$  in the compact set  $\overline{\Omega}$  because the Hessian matrix of  $f$  at a critical point of  $f$  is invertible. Moreover, we shall say that  $z \in \overline{\Omega}$  is a saddle point of  $f$  if it is a critical point of index 1 of  $f$ . Let us finally recall the definition of the principal wells of  $f$  in  $\Omega$  which are defined in [29, Section 3.1] when  $f : \overline{\Omega} \rightarrow \mathbb{R}$  is a  $C^\infty$  Morse function (see also [13, Section 2.1]). For all local minimum point  $x$  of  $f$  in  $\Omega$  and  $\lambda > f(x)$ , we denote by  $C(\lambda, x)$  the connected component of  $\{f < \lambda\}$  in  $\overline{\Omega}$  containing  $x$ . Moreover, one defines

$$\lambda(x) := \sup\{\lambda > f(x), C(\lambda, x) \cap \partial\Omega = \emptyset\} \quad \text{and} \quad C(x) := C(\lambda(x), x). \quad (6)$$

For  $x$  a minimum point of  $f$  in  $\Omega$ , we call  $C(x)$  a *principal well* of  $f$  in  $\Omega$ . The set of principal wells of  $f$  in  $\Omega$  is the set

$$\mathcal{C} := \{C(x), x \text{ is a local minimum point of } f \text{ in } \Omega\}. \quad (7)$$

From [29, Proposition 7], each  $C \in \mathcal{C}$  is included in  $\Omega$  and the elements of  $\mathcal{C}$  are two by two disjoint. Notice finally that for each  $C \in \mathcal{C}$ , it holds  $C \subset \mathcal{A}_\Omega(C)$  since  $C$  is stable for the dynamics (4) (because  $f$  is non increasing along the curves defined by (4)).

### 1.3 Assumptions on the function $f$

Let us now introduce the basic assumption of this work.

**Assumption (H1).** *The function  $f : \overline{\Omega} \rightarrow \mathbb{R}$  is a  $C^\infty$  Morse function and the set  $\{f < \min_{\partial\Omega} f\}$  is non empty and contains all the local minima of  $f$  in  $\Omega$ . Furthermore, denoting by  $C_1, \dots, C_N$  the connected components of  $\{f < \min_{\partial\Omega} f\}$ , it holds for all  $j \in \{1, \dots, N\}$ :*

$$\partial C_j \cap \partial\Omega \neq \emptyset \text{ and } \overline{C_j} \text{ is an isolated connected component of } \{f \leq \min_{\partial\Omega} f\}.$$

A schematic representation of  $C_1, \dots, C_N$  is given in Figure 1 when  $N = 2$  (see also Figure 8 below). Let us now comment the assumption **(H1)**. Notice that under **(H1)**, one has:

$$\min_{\overline{\Omega}} f < \min_{\partial\Omega} f \text{ and for all } j \in \{1, \dots, N\}, C_j \subset \Omega.$$

The following lemma will allow us to use the results of [29].

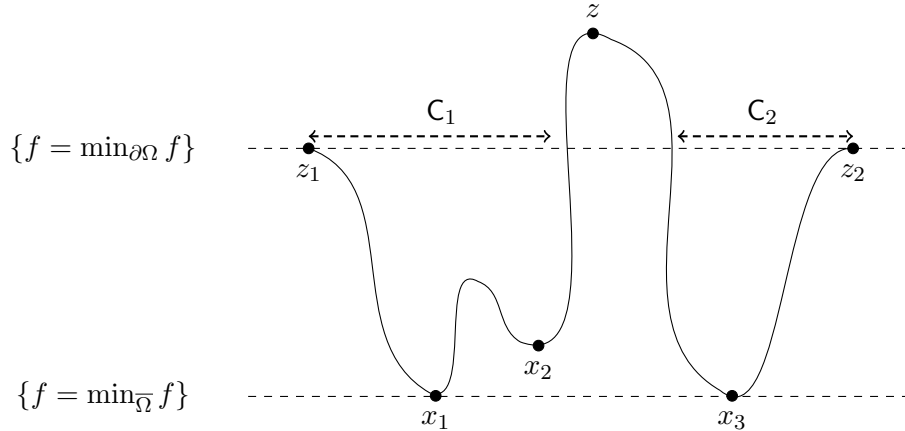
**Lemma 1.** *When **(H1)** holds, according to the terminology introduced in [29] (see also (7)), the function  $f$  has exactly  $N$  principal wells in  $\Omega$  which are  $C_1, \dots, C_N$ .*

*Proof.* Under **(H1)**, it follows that for all local minima  $x$  of  $f$  in  $\Omega$ ,  $\lambda(x) = \min_{\partial\Omega} f$  and  $C(x) = C_j$  where  $j \in \{1, \dots, N\}$  is such that  $x \in C_j$ . Thus, it holds  $\mathcal{C} = \{C_1, \dots, C_N\}$ . ■

From Lemma 1 and [29, Proposition 12], for  $z \in \partial C_j \cap \partial\Omega$  (notice that  $z \in \arg \min_{\partial\Omega} f$ ) one has:

- a. If  $\nabla f(z) = 0$ , then,  $z$  is a saddle point of  $f$ .
- b. If  $\nabla f(z) \neq 0$ , then  $\partial_n f(z) > 0$ , where  $\partial_n f(z) = \nabla f(z) \cdot n_{\Omega}(z)$  is the normal derivative of  $f$  (where  $n_{\Omega}(z)$  is the outward normal to  $\Omega$  at  $z$ ).

For  $j \in \{1, \dots, N\}$ , the assumption that  $\partial C_j \cap \partial\Omega \neq \emptyset$  ensures that the exit event from  $\Omega$  when  $X_0 = x \in \mathcal{A}_{\Omega}(C_j)$  is associated with a point  $z \in \partial C_j \cap \partial\Omega$  when  $h \rightarrow 0$  (see indeed the prefactor appearing in the asymptotic equivalents of  $\mathbb{E}_x[\tau_{\Omega^c}]$  in Theorem 1 below).



**Figure 1:** A one dimensional case when **(H1)** is satisfied with  $N_0 = N = 2$  (notice that **(H2)** is always satisfied in dimension one), see (8). Here, the set of local minima of  $f$  is  $\{x_1, x_2, x_3\}$  with  $\arg \min_{\overline{\Omega}} f = \{x_1, x_3\}$ . The points  $z_1$  and  $z_2$  satisfy  $z_1 \in \partial C_1 \cap \partial\Omega$ ,  $f'(z_1) = 0$ , and  $f'(z_2) = 0$ . Moreover,  $\mathcal{A}_{\Omega}(C_1) = (z_1, z)$ ,  $\mathcal{A}_{\Omega}(C_2) = (z, z_2)$ .

When **(H1)** is satisfied, we denote by  $N_0$  the number of  $C_j$ 's which contain a point in  $\arg \min_{\overline{\Omega}} f$ , i.e.

$$N_0 := \text{Card}\left(j \in \{1, \dots, N\}, \arg \min_{\overline{\Omega}} f \cap C_j \neq \emptyset\right). \quad (8)$$

In all this work, we will assume that  $\{C_1, \dots, C_N\}$  are ordered such that for all  $j \in \{1, \dots, N_0\}$ ,  $\arg \min_{\overline{\Omega}} f \cap C_j \neq \emptyset$ . Let us now introduce the following and last assumption on the function  $f$ .

**Assumption (H2).** The function  $f : \bar{\Omega} \rightarrow \mathbb{R}$  satisfies **(H1)** and for all  $j \in \{1, \dots, N\}$  and  $z \in \partial C_j \cap \partial\Omega$ ,

- a. If  $\nabla f(z) \neq 0$ ,  $z$  is a nondegenerate global minimum point of  $f|_{\partial\Omega}$ .
- b. If  $\nabla f(z) = 0$ , the outward normal  $\mathbf{n}_\Omega(z)$  to  $\Omega$  at  $z$  is an eigenvector associated with the negative eigenvalue of  $\text{Hess } f(z)$ .

Notice that in dimension 1, **(H2)** is always satisfied. When **(H1)** and **(H2)** hold, from [29, Theorem 2], Lemma 1, and (3), there exists  $C > 0$ , for all  $j \in \{1, \dots, N_0\}$ , there exists  $\beta_j \in \{0, -1/2\}$  such that, for  $h$  small enough:

$$C^{-1} h^{\beta_j} e^{-\frac{2}{h}(\min_{\partial\Omega} f - \min_{\bar{\Omega}} f)} \leq \lambda_{j,h} \leq C h^{\beta_j} e^{-\frac{2}{h}(\min_{\partial\Omega} f - \min_{\bar{\Omega}} f)}.$$

Consequently, if  $N_0 \geq 2$ , the principal barrier of  $f$  in  $\Omega$  is thus degenerate and equal  $\min_{\partial\Omega} f - \min_{\bar{\Omega}} f$ . Point  $a$  in **(H2)** is convenient for computations, it can be relaxed using a similar procedure to [42] (see also [4]). When point  $b$  in **(H2)** is satisfied, each  $z \in \partial C_j \cap \partial\Omega$  ( $j \in \{1, \dots, N\}$ ) such that  $\nabla f(z) = 0$  is a nondegenerate global minimum point of  $f|_{\partial\Omega}$  (see [29, Proposition 12]). In particular, under **(H2)**, for all  $j \in \{1, \dots, N\}$ ,  $\partial C_j \cap \partial\Omega$  consists of a finite number of points. Point  $b$  in **(H2)** will be needed to use the results of [29] in Section 2.7 and without this assumption, the prefactors appearing in Theorem 1 below might not be valid anymore, as explained in [29, Section 1.4].

#### 1.4 Main result

When the function  $f$  satisfies the assumptions **(H1)** and **(H2)**, one defines for  $j \in \{1, \dots, N\}$ :

$$\Theta_{j,1} = \frac{\sum_{\substack{z \in \partial C_j \cap \partial\Omega, \\ \nabla f(z) \neq 0}} \frac{\partial_n f(z)}{\pi^{\frac{1}{2}}} (\det \text{Hess } f|_{\partial\Omega}(z))^{-\frac{1}{2}}}{\sum_{y \in \arg \min_{\bar{C}_j} f} (\det \text{Hess } f(y))^{-\frac{1}{2}}} \quad \text{and} \quad \Theta_{j,2} = \frac{\sum_{\substack{z \in \partial C_j \cap \partial\Omega, \\ \nabla f(z) = 0}} \frac{|\mu_z|}{\pi} |\det \text{Hess } f(z)|^{-\frac{1}{2}}}{\sum_{y \in \arg \min_{\bar{C}_j} f} (\det \text{Hess } f(y))^{-\frac{1}{2}}}$$

where  $\mu_z$  is the negative eigenvalue of  $\text{Hess } f(z)$ . By convention, when  $d = 1$ ,  $\det \text{Hess } f|_{\partial\Omega}(z) = 1$ . The main result of this work is the following.

**Theorem 1.** Let  $f : \bar{\Omega} \rightarrow \mathbb{R}$  be such that Assumptions **(H1)** and **(H2)** are satisfied. Then:

1. Let  $\mathbf{K}$  be a compact subset of  $\Omega$ . Then, for any  $\delta > 0$ , it holds for  $h$  small enough:

$$\sup_{x \in \mathbf{K}} \mathbb{E}_x[\tau_{\Omega^c}] \leq e^{\frac{2}{h}(\min_{\partial\Omega} f - \min_{\bar{\Omega}} f + \delta)}.$$

2. Let  $j \in \{1, \dots, N_0\}$  (see (8)) and  $\mathbf{K}$  be a compact subset of  $\mathcal{A}_\Omega(C_j)$  (see (5)). Then, for all  $x \in \mathbf{K}$ , it holds in the limit  $h \rightarrow 0$ :

$$\mathbb{E}_x[\tau_{\Omega^c}] = \frac{e^{\frac{2}{h}(\min_{\partial\Omega} f - \min_{\bar{\Omega}} f)}}{\Theta_{j,1} h^{-\frac{1}{2}} + \Theta_{j,2} + O(\sqrt{h})}, \quad \text{uniformly with respect to } x \in \mathbf{K}.$$

3. Let  $j \in \{\mathbf{N}_0 + 1, \dots, \mathbf{N}\}$  (with the convention that  $\{\mathbf{N}_0 + 1, \dots, \mathbf{N}_0\} = \emptyset$ ) and  $\mathbf{K}$  be a compact subset of  $\mathcal{A}_\Omega(\mathbf{C}_j)$ . Then, it holds for  $h$  small enough:

$$\min_{x \in \mathbf{K}} \mathbb{E}_x[\tau_{\Omega^c}] \geq \frac{e^{\frac{2}{h}(\min_{\partial\Omega} f - \min_{\overline{\mathbf{C}}_j} f)}}{\Theta_{j,1} h^{-\frac{1}{2}} + \Theta_{j,2} + O(\sqrt{h})}.$$

Furthermore, there exists  $\varepsilon > 0$ , such that if  $\min_{\overline{\mathbf{C}}_j} f - \min_{\overline{\Omega}} f < \varepsilon$ , then, when  $h \rightarrow 0$ :

$$\mathbb{E}_x[\tau_{\Omega^c}] = \frac{e^{\frac{2}{h}(\min_{\partial\Omega} f - \min_{\overline{\mathbf{C}}_j} f)}}{\Theta_{j,1} h^{-\frac{1}{2}} + \Theta_{j,2} + O(\sqrt{h})}, \quad \text{uniformly with respect to } x \in \mathbf{K}.$$

As explained above, in many applications where the dynamics (1) is used,  $\Omega$  is a basin of attraction of some local minimum point of  $f$  on  $\mathbb{R}^d$ , and therefore, for all  $j$ ,  $\{z \in \partial\mathbf{C}_j \cap \partial\Omega, \nabla f(z) \neq 0\} = \emptyset$ . Thus only the saddle points of  $f$  appear (through the constants  $\Theta_{j,2}$ ) in the equivalents of  $\mathbb{E}_x[\tau_{\Omega^c}]$  above. This is the main novelty of this work.

The remainder term  $O(\sqrt{h})$  in items 2 and 3 in Theorem 1 is optimal, as shown in Appendix 1 below. The second statement in item 3 in Theorem 1 can be seen as a perturbation result of the completely degenerate case  $\mathbf{N}_0 = \mathbf{N}$  (i.e. when all the  $\mathbf{C}_\ell$ 's have the same depth, namely  $\min_{\overline{\Omega}} f$ ). Though it is assumed that  $\overline{\mathbf{C}}_j$  is an isolated connected component of  $\{f \leq \min_{\partial\Omega} f\}$  (and thus for all  $i \neq j$ ,  $\partial\mathbf{C}_i \cap \partial\mathbf{C}_j = \emptyset$ ), when  $j \in \{\mathbf{N}_0 + 1, \dots, \mathbf{N}\}$ , a tunnelling effect can occur between  $\mathbf{C}_j$  and the others  $\mathbf{C}_\ell$ 's, mixing their properties. This is discussed in Appendix 2 below where we show that when, for some  $j \geq \mathbf{N}_0 + 1$ ,  $\min_{\overline{\mathbf{C}}_j} f$  is much larger than  $\min_{\overline{\Omega}} f$ ,  $\lim_{h \rightarrow 0} h \ln \mathbb{E}_x[\tau_{\Omega^c}] > 2(\min_{\partial\Omega} f - \min_{\overline{\mathbf{C}}_j} f)$ , when  $x \in \mathbf{C}_j$ . Finally, as suggested by computations on one dimensional examples, when  $\overline{\mathbf{C}}_j$  is not an isolated connected component of  $\{f \leq \min_{\partial\Omega} f\}$ , a strong tunnelling effect occurs between  $\mathbf{C}_j$  and the other  $\mathbf{C}_i$ 's which are such that  $\partial\mathbf{C}_i \cap \partial\mathbf{C}_j \neq \emptyset$ , and it is much harder to predict the precise asymptotic equivalents of  $\mathbb{E}_x[\tau_{\Omega^c}]$  when  $x \in \mathbf{C}_j$  due to the various situations that happen.

**Remark 2.** The prefactor  $\Theta_{j,1} h^{-\frac{1}{2}}$  in Theorem 1 differs from the one derived initially in [24]. Indeed, when  $z \in \partial\mathbf{C}_j \cap \partial\Omega$  and  $\nabla f(z) \neq 0$ , it holds  $\partial_n f(z_j) \neq 0^2$  and thus, the Laplace method on which is based the proof of Theorem 1 leads to a dependence on  $h$  in this prefactor. The constant  $\Theta_{j,2}$  in Theorem 1 is the same as the one obtained in [24] up to a multiplication by 2 (see also [6, Theorem 3.2]): this is expected. Indeed, when the process (1) reaches a critical point  $z \in \partial\mathbf{C}_j \cap \partial\Omega$ , because  $z$  is a saddle point of  $f$ , it has, in the limit  $h \rightarrow 0$ , a one-half probability to come back in  $\Omega$  and a one-half probability to go in  $\mathbb{R}^d \setminus \overline{\Omega}$  (this result is not difficult to prove in dimension 1, see [41, Section A.1.2.2], and can be extended to higher dimensions using a suitable set of coordinates around  $z$ ).

According to items 2 and 3 in Theorem 1, for  $x, y \in \mathcal{A}_\Omega(\mathbf{C}_j)$ ,  $\mathbb{E}_x[\tau_{\Omega^c}] = \mathbb{E}_y[\tau_{\Omega^c}](1 + o(1))$  when  $h \rightarrow 0$ , where  $o(1)$  either equal  $O(\sqrt{h})$  or  $O(h)$ . We will actually derive the following more precise leveling result on  $x \in \Omega \mapsto \mathbb{E}_x[\tau_{\Omega^c}]$ .

**Theorem 2.** Let  $f : \overline{\Omega} \rightarrow \mathbb{R}$  which satisfies the assumptions **(H1)** and **(H2)**. Let  $\mathbf{K} \subset \mathcal{A}_\Omega(\mathbf{C}_j)$  for some  $j \in \{1, \dots, \mathbf{N}\}$ . Then, if  $j \in \{1, \dots, \mathbf{N}_0\}$  (see (8)), there exists  $c > 0$  such that for  $h$  small enough and for all  $x, y \in \mathbf{K}$ :

$$\mathbb{E}_x[\tau_{\Omega^c}] = \mathbb{E}_y[\tau_{\Omega^c}](1 + O(e^{-\frac{c}{h}})), \quad \text{uniformly in } x, y \in \mathbf{K}.$$

<sup>2</sup>Such points  $z$  have also been considered by Kramers in [24] to derive formulas for transition rates, as explained in [39].

Moreover, if  $j \geq N_0 + 1$ , there exists  $\varepsilon > 0$ , such that if  $\min_{\overline{C_j}} f - \min_{\overline{\Omega}} f < \varepsilon$ , there exists  $c > 0$ , such that for  $h$  small enough for all  $x, y \in K$ :

$$\mathbb{E}_x[\tau_{\Omega^c}] = \mathbb{E}_y[\tau_{\Omega^c}](1 + O(e^{-\frac{c}{h}})), \quad \text{uniformly in } x, y \in K.$$

Theorem 2 generalizes, as long as the gradient case is concerned, the leveling result [9, Corollary 1] to the case when  $f$  has critical points on  $\partial\Omega$  and several critical points in  $\Omega$ .

To prove Theorem 1, the starting point is a formula from the potential theory for  $\mathbb{E}_{x_*}[\tau_{\Omega^c}]$  when  $x_*$  is a local minimum point of  $f$  in  $\Omega$ . To extract information from this formula, we derive leveling properties on the mean exit time from  $\Omega$  and *committor* functions, and we give sharp equivalents on *capacities*: this is the purpose of Section 2 where more general assumptions than **(H1)** and **(H2)** are considered. Then, in Section 3, we prove Theorems 1 and 2. Let us mention that our analysis is not a straightforward extension of the analysis led in [6], because in our setting, we have to analyse the effect of the boundary  $\partial\Omega$  on the sharp equivalents of  $\mathbb{E}_x[\tau_{\Omega^c}]$  when  $x \in \Omega$  and  $h \rightarrow 0$ . Moreover, we also consider a set of deterministic initial conditions in  $\Omega$  with nonzero Lebesgue measure (here  $\mathcal{A}_{\Omega}(C_j)$ ,  $j \in \{1, \dots, N\}$ ) and not only the case when  $X_0 = x_*$  (which requires a further analysis, see Sections 3.2 and 3.3). This is important for the following reason. In many applications where the dynamics (1) is used, one is interested in having a sharp estimate on the average time this process remains trapped in  $\Omega$  when  $h \ll 1$ . Since the process (1) has a density with respect to the Lebesgue measure in  $\mathbb{R}^d$ , for all  $h > 0$ , the probability that the trajectories of (1) entering in  $\Omega$  pass through each  $x_*$  is zero.

Let us mention that our analysis to prove Theorem 1 will allow us to derive Corollary 10 which states in particular that, when  $\partial C(x) \cap \partial\Omega = \{z\}$  (see (6)) and  $\overline{C(x)}$  is an isolated connected component of  $\{f \leq \lambda(x)\}$ , the process (1) exits  $\Omega$  almost surely through any neighborhood of  $z$  in the limit  $h \rightarrow 0$  when  $X_0 = x \in \mathcal{A}_{\Omega}(C(x))$ . Since  $z$  can be a critical point of  $f$ , to the best of our knowledge, Corollary 10 is new for the dynamics (1) (see indeed [10] or [31, Section 1]).

### 1.5 Link with the previous results on the mean exit time

Let us now recall the main previous contributions on the study of  $\mathbb{E}_x[\tau_{\Omega^c}]$ . The limit of  $h \ln \mathbb{E}[\tau_{\Omega^c}]$  when  $h \rightarrow 0$  and  $x \in \Omega$  has been studied by Freidlin and Wentzell in [16] when  $|\nabla f| \neq 0$  on  $\partial\Omega$  (see also [35]). In [9], it is shown that, for  $x \in \Omega$ ,  $\mathbb{E}_x[\tau_{\Omega^c}] \lambda_{1,h} = 1 + o(1)$  when  $h \rightarrow 0$ , in the case when  $\partial_n f > 0$  on  $\partial\Omega$  and  $f$  has a unique critical point in  $\Omega$  (see also [42]). We also refer to [22, 23] for the study, when  $|\nabla f| \neq 0$  on  $\partial\Omega$ , of the solution of the parabolic equation  $\mathcal{L}_{f,h} u_h = \partial_t u_h$  as  $h \rightarrow 0$ . Let us mention that in [9, 16, 22, 23, 35], the authors also considered non reversible processes. Sharp asymptotic estimates when  $h \rightarrow 0$  of  $\mathbb{E}_x[\tau_{\Omega^c}]$  for some  $x \in \Omega$  have been obtained in [49, Section 4] when  $|\nabla f| \neq 0$  on  $\partial\Omega$  (let us also mention that the assumption that  $\partial\{f < \min_{\partial\Omega} f\} \cap \partial\Omega \neq \emptyset$  is not allowed there according to the assumption (C3) in [49, Section 4]). In [36, Theorem 5], when  $f$  has a non degenerate principal barrier (i.e. in our setting, when  $N_0 = 1$ ), it is shown that  $\lambda_{1,h} \tau_{\Omega^c}$  converges in distribution to an exponential random variable in the limit  $h \rightarrow 0$  when  $X_0 = x$  for some  $x \in \Omega$  (in our setting, for  $x \in C_1$ ). A comprehensive review of the literature on this topic can be found in [1]. We refer to [34, 37–39, 45] where asymptotic formulas for  $\mathbb{E}[\tau_{\Omega^c}]$  when  $h \rightarrow 0$  have been obtained through formal computations in different settings (see also [11, 44]). We finally mention [2, 3, 6, 7, 17, 21, 25, 27, 40] and references therein for asymptotics estimates on eigenvalues, transition times in the boundary less case.



## 2 General results on committor functions and capacities

In all this section, we assume that the function  $f : \bar{\Omega} \rightarrow \mathbb{R}$  is a  $C^\infty$  Morse function. In this section, we derive leveling results on committor functions and we give asymptotic equivalents of capacities in the limit  $h \rightarrow 0$ .

### 2.1 The committor function

For any closed ball  $B \subset \Omega$ , one denotes by

$$\tau_B = \inf\{t \geq 0, X_t \in B\}$$

the first time the process (1) hits  $B$ . Let  $x_*$  be a local minimum point of  $f$  in  $\Omega$ . In all this work, we consider for all  $h > 0$  a constant  $r_h > 0$  such that when  $h \rightarrow 0$ ,

$$r_h = O(e^{-\frac{\delta_*}{h}}), \quad (9)$$

where  $\delta_* > 0$  will be fixed in Lemma 4. In the following,  $h > 0$  is chosen small enough so that  $B_{r_h}(x_*) \subset C(x_*)$ . The committor function (also called *equilibrium potential*) between  $B_{r_h}(x_*)$  and  $\Omega^c$  is defined by:

$$p_{x_*} : x \in \bar{\Omega} \mapsto \mathbb{P}_x[\tau_{B_{r_h}(x_*)} < \tau_{\Omega^c}]. \quad (10)$$

A schematic representation of  $C(x_*)$  and  $B_{r_h}(x_*)$  is given in Figure 2 when  $\partial C(x_*) \cap \partial\Omega \neq \emptyset$ .

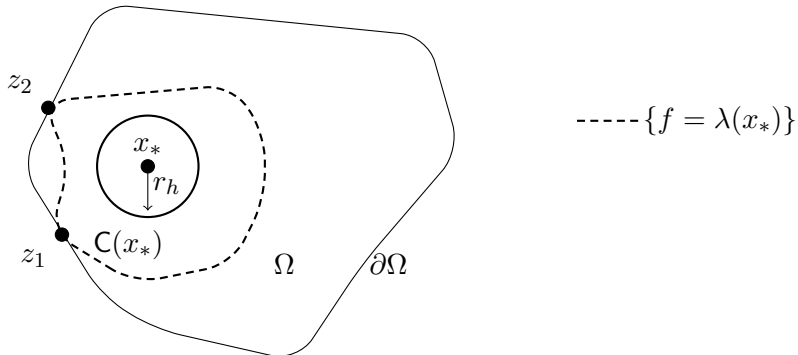


Figure 2: Schematic representation of  $C(x_*)$  (see (6)) and  $B_{r_h}(x_*)$ . On the figure  $\partial C(x_*) \cap \partial\Omega$  is non empty and equals  $\{z_1, z_2\}$ .

### 2.2 A formula from the potential theory

In this section, we recall a formula from the potential theory which is used in this work (see (11)), which can be found for instance in [5]. Let us denote by  $G_\Omega$  be the Green function of  $\mathcal{L}_{f,h}$  associated with homogeneous Dirichlet boundary conditions on  $\partial\Omega$ . The *equilibrium measure*  $e_{B_{r_h}(x_*), \Omega^c}$  associated with the couple  $(B_{r_h}(x_*), \Omega^c)$  (see [6, Section 2] and more precisely (2.10) there) is defined as the unique measure on  $\partial B_{r_h}(x_*)$  such that

$$p_{x_*}(x) = \int_{\partial B_{r_h}(x_*)} G_\Omega(x, y) e_{B_{r_h}(x_*), \Omega^c}(dy).$$

From [6, Section 2] (see (2.27) there), one has the following relation:

$$\int_{\partial\mathbb{B}_{r_h}(x_*)} \mathbb{E}_z[\tau_{\Omega^c}] e^{-\frac{2}{h}f(z)} e_{\mathbb{B}_{r_h}(x_*), \Omega^c}(dz) = \int_{\Omega} e^{-\frac{2}{h}f(x)} \mathbf{p}_{x_*}(x) dx, \quad (11)$$

where  $\mathbf{p}_{x_*}$  is defined by (10). Let us now define, as in [6, Section 2] (see (2.13) there), the capacity associated with  $(\mathbb{B}_{r_h}(x_*), \Omega^c)$ :

$$\text{cap}_{\mathbb{B}_{r_h}(x_*)}(\Omega^c) = \int_{\partial\mathbb{B}_{r_h}(x_*)} e^{-\frac{2}{h}f(z)} e_{\mathbb{B}_{r_h}(x_*), \Omega^c}(dz). \quad (12)$$

Finally, let us recall that from [6, Section 2], one has the following variational principle:

$$\text{cap}_{\mathbb{B}_{r_h}(x_*)}(\Omega^c) = \frac{h}{2} \int_{\Omega \setminus \mathbb{B}_{r_h}(x_*)} |\nabla \mathbf{p}_{x_*}|^2 e^{-\frac{2}{h}f} = \inf_{\mathbf{p} \in \mathcal{H}_{\mathbb{B}_{r_h}(x_*), \Omega^c}} \frac{h}{2} \int_{\Omega \setminus \mathbb{B}_{r_h}(x_*)} |\nabla \mathbf{p}|^2 e^{-\frac{2}{h}f}, \quad (13)$$

where we recall  $\mathbf{p}_{x_*}$  is defined by (10) and

$$\mathcal{H}_{\mathbb{B}_{r_h}(x_*), \Omega^c} = \{\mathbf{p} \in H^1(\Omega \setminus \mathbb{B}_{r_h}(x_*), e^{-\frac{2}{h}f} dx), \mathbf{p} = 1 \text{ on } \partial\mathbb{B}_{r_h}(x_*), \mathbf{p} = 0 \text{ on } \partial\Omega\}.$$

### 2.3 Leveling results on the mean exit time

In order to extract  $\mathbb{E}_{x_*}[\tau_{\Omega^c}]$  from the left hand side of (11), we need a leveling result on the function  $z \mapsto \mathbb{E}_z[\tau_{\Omega^c}]$  in  $\mathbb{B}_{r_h}(x_*)$ . This is the purpose of the next lemma.

**Lemma 3.** *Let  $x_*$  be a local minimum point of  $f$  in  $\Omega$  and  $r_h > 0$  such that (9) holds for some  $\delta_* > 0$ . Then, there exist  $h_0 > 0$  and  $c > 0$  such that for all  $h \in (0, h_0)$  it holds*

$$\max_{z \in \mathbb{B}_{r_h}(x_*)} |\mathbb{E}_z[\tau_{\Omega^c}] - \mathbb{E}_{x_*}[\tau_{\Omega^c}]| \leq e^{-\frac{c}{h}} \mathbb{E}_{x_*}[\tau_{\Omega^c}].$$

*Proof.* Let us recall that from the Dynkin's formula, the function  $w : x \mapsto \mathbb{E}_x[\tau_{\Omega^c}]$  is the unique weak solution to

$$\mathcal{L}_{f,h} w = 1 \text{ in } \Omega \text{ and } w = 0 \text{ on } \partial\Omega, \quad (14)$$

and  $w \in C^\infty(\bar{\Omega}, \mathbb{R})$ . Set  $b := \nabla f$ . For  $z \in \mathbb{B}_{2\sqrt{h}}(x_*)$ , set  $y = (z - x_*)/\sqrt{h} + x_* \in \mathbb{B}_2(x_*)$  and  $v(y) = w((y - x_*)\sqrt{h} + x_*)$  which satisfies for  $y \in \mathbb{B}_2(x_*)$ ,

$$-\frac{1}{2}\Delta v(y) + \frac{b((y - x_*)\sqrt{h} + x_*)}{\sqrt{h}} \cdot \nabla v(y) = 1.$$

Then, since  $b(x_*) = 0$  (and thus  $y \mapsto b((y - x_*)\sqrt{h} + x_*)/\sqrt{h}$  on  $\mathbb{B}_2(x_*)$  is bounded uniformly with respect to  $h$ , as well as its gradient), using the Schauder estimate [18, Corollary 6.3] on  $y \mapsto v(y)$  on  $\mathbb{B}_2(x_*)$ , there exists  $C > 0$  independent of  $h$  such that,

$$\max_{y \in \mathbb{B}_1(x_*)} |\nabla v(y)| \leq C \left(1 + \max_{y \in \mathbb{B}_2(x_*)} v(y)\right).$$

Thus, coming back to the variable  $z$ , it holds:

$$\max_{z \in \mathbb{B}_{\sqrt{h}}(x_*)} |\nabla \mathbb{E}_z[\tau_{\Omega^c}]| \leq \frac{C}{\sqrt{h}} \left(1 + \max_{z \in \mathbb{B}_{2\sqrt{h}}(x_*)} \mathbb{E}_z[\tau_{\Omega^c}]\right).$$

Moreover, there exist  $M > 0$  and  $c > 0$  such that for  $h$  small enough and for all  $z \in \mathbb{B}_{2\sqrt{h}}(x_*)$ ,  $\mathbb{E}_z[\tau_{\Omega^c}] \geq M e^{\frac{c}{h}}$ . Indeed, this follows from [15, Lemma 1] together with the fact that for all

$z \in \mathbf{B}_{2\sqrt{h}}(x_*)$ ,  $\mathbb{E}_z[\tau_{\Omega^c}] \geq \mathbb{E}_z[\tau_{\mathbf{U}^c}]$  where  $\mathbf{U}$  is a  $C^\infty$  subdomain of  $\Omega$  such that  $\partial_n f > 0$  on  $\partial\mathbf{U}$  and  $\{z \in \mathbf{U}, \nabla f(z) = 0\} = \{x_*\}$  (such a  $\mathbf{U}$  exists since  $x_*$  is a nondegenerate local minimum point of  $f$  in  $\Omega$ ). Hence, one has, for  $h$  small enough,

$$\max_{z \in \mathbf{B}_{\sqrt{h}}(x_*)} |\nabla \mathbb{E}_z[\tau_{\Omega^c}]| \leq \frac{C}{\sqrt{h}} \max_{z \in \mathbf{B}_{2\sqrt{h}}(x_*)} \mathbb{E}_z[\tau_{\Omega^c}].$$

Then, for all  $z \in \mathbf{B}_{\sqrt{h}}(x_*)$ ,

$$|\mathbb{E}_z[\tau_{\Omega^c}] - \mathbb{E}_{x_*}[\tau_{\Omega^c}]| \leq \frac{C}{\sqrt{h}} |z - x_*| \max_{z \in \mathbf{B}_{2\sqrt{h}}(x_*)} \mathbb{E}_z[\tau_{\Omega^c}],$$

and thus, for all  $z \in \mathbf{B}_{r_h}(x_*)$ ,

$$|\mathbb{E}_z[\tau_{\Omega^c}] - \mathbb{E}_{x_*}[\tau_{\Omega^c}]| \leq C \frac{e^{-\frac{\delta_*}{h}}}{\sqrt{h}} \max_{z \in \mathbf{B}_{2\sqrt{h}}(x_*)} \mathbb{E}_z[\tau_{\Omega^c}].$$

Finally, it follows from [18, Theorem 8.20] (applied to the function  $v$ ):

$$\max_{z \in \mathbf{B}_{2\sqrt{h}}(x_*)} \mathbb{E}_z[\tau_{\Omega^c}] \leq C \min_{z \in \mathbf{B}_{2\sqrt{h}}(x_*)} \mathbb{E}_z[\tau_{\Omega^c}],$$

for some  $C > 0$ . In conclusion, one has for  $h$  small enough,  $\max_{z \in \mathbf{B}_{r_h}(x_*)} |\mathbb{E}_z[\tau_{\Omega^c}] - \mathbb{E}_{x_*}[\tau_{\Omega^c}]| \leq C e^{-\frac{\delta_*}{2h}} \mathbb{E}_{x_*}[\tau_{\Omega^c}]$ . This concludes proof of Lemma 3.  $\blacksquare$

Using Lemma 3 together with (11) and (12), one obtains, for all local minima  $x_*$  of  $f$  in  $\Omega$ , in the limit  $h \rightarrow 0$ :

$$\mathbb{E}_{x_*}[\tau_{\Omega^c}] = \frac{\int_{\Omega} e^{-\frac{2}{h}f(x)} \mathbf{p}_{x_*}(x) dx}{\text{cap}_{\mathbf{B}_{r_h}(x_*)}(\Omega^c)} (1 + O(e^{-\frac{c}{h}})), \text{ for some } c > 0 \text{ independent of } h. \quad (15)$$

To compute a sharp asymptotic equivalent of  $\mathbb{E}_{x_*}[\tau_{\Omega^c}]$  when  $h \rightarrow 0$ , we study the asymptotic behaviour of  $\mathbf{p}_{x_*}(x)$  and  $\text{cap}_{\mathbf{B}_{r_h}(x_*)}(\Omega^c)$  in the limit  $h \rightarrow 0$ . This is the purpose of the next sections.

## 2.4 Leveling results on the committor function $\mathbf{p}_{x_*}$ in $\mathbf{C}(x_*)$

Let us prove the following leveling property on the committor function  $\mathbf{p}_{x_*}$  (see (10)) in a  $h$ -independent neighborhood  $\mathbf{U}$  of  $x_*$  in  $\mathbf{C}(x_*)$  (see (6)).

**Lemma 4.** *Let  $x_*$  be a local minimum point of  $f$  in  $\Omega$ . Let  $\mathbf{U}$  be a  $C^\infty$  subdomain of  $\mathbf{C}(x_*)$  (see (6)) such that  $\partial_n f > 0$  on  $\partial\mathbf{U}$  and  $\{z \in \mathbf{U}, \nabla f(z) = 0\} = \{x_*\}$ . Let  $\mathbf{K}$  be a  $C^\infty$  compact subset of  $\mathbf{U}$ . Then, for some  $\delta_* > 0$ , there exist  $h_0 > 0$  and  $c > 0$ , for all  $h \in (0, h_0)$ :*

$$\max_{y \in \mathbf{K}} |\mathbf{p}_{x_*}(y) - 1| \leq e^{-\frac{c}{h}}.$$

From now on, the constant  $\delta_* > 0$  in (9) is specified as the constant in Lemma 4.

*Proof.* Since the trajectories of the process (1) are continuous, one has for all  $y \in \mathbf{K}$ ,  $\{\tau_{\Omega^c} < \tau_{\mathbf{B}_{r_h}(x_*)}\} \subset \{\tau_{\mathbf{U}^c} < \tau_{\mathbf{B}_{r_h}(x_*)}\}$  when  $X_0 = y$ . Using [8, Theorem 2], there exist  $\delta_* > 0$  (see (9)),  $h_0 > 0$  and  $c > 0$  such that for all  $h \in (0, h_0)$  and  $y \in \mathbf{K}$ ,  $\mathbb{P}_y[\tau_{\mathbf{U}^c} \geq \tau_{\mathbf{B}_{r_h}(x_*)}] \leq e^{-\frac{c}{h}}$ . This concludes the proof of Lemma 4.  $\blacksquare$

With Lemma 4, we can extend the leveling property on  $\mathbf{p}_{x_*}$  to  $\mathcal{A}_\Omega(\mathbf{C}(x_*))$ .

**Proposition 5.** *Let  $x_*$  be a local minimum point of  $f$  in  $\Omega$ . Let  $\mathbf{K}$  be a compact subset of  $\mathcal{A}_\Omega(\mathbf{C}(x_*))$  (see (6) and (5)). Then, there exist  $h_0 > 0$  and  $c > 0$  such that for all  $h \in (0, h_0)$ , one has:*

$$\max_{y \in \mathbf{K}} |\mathbf{p}_{x_*}(y) - 1| \leq e^{-\frac{c}{h}}.$$

This implies in particular that, using the Laplace method, for all  $k \in \{1, 2\}$ , in the limit  $h \rightarrow 0$ :

$$\int_{\mathbf{C}(x_*)} \mathbf{p}_{x_*}^k e^{-\frac{2}{h}f} = (h\pi)^{\frac{d}{2}} e^{-\frac{2}{h} \min_{\overline{\mathbf{C}(x_*)}} f} \sum_{y \in \arg \min_{\overline{\mathbf{C}(x_*)}} f} (\det \text{Hess} f(y))^{-\frac{1}{2}} (1 + O(h)).$$

*Proof.* Let us introduce the function  $\mathbf{q}_{x_*} : y \mapsto 1 - \mathbf{p}_{x_*}(y)$  which is the  $\mathcal{C}^\infty(\overline{\Omega \setminus \mathbf{B}_{r_h}(x_*)}, \mathbb{R})$  solution to

$$-\frac{h}{2} \Delta \mathbf{q}_{x_*} = -\nabla f \cdot \nabla \mathbf{q}_{x_*} \text{ in } \Omega \setminus \mathbf{B}_{r_h}(x_*), \mathbf{q}_{x_*} = 1 \text{ on } \partial\Omega \text{ and } \mathbf{q}_{x_*} = 0 \text{ on } \partial\mathbf{B}_{r_h}(x_*). \quad (16)$$

Working with  $\mathbf{q}_{x_*}$  instead of  $\mathbf{p}_{x_*}$  will be convenient to derive the key estimate (25) below. The proof of Proposition 5 is divided into several steps. The first two steps are inspired by [11] but we have to take care here with the regularity of  $\mathbf{q}_{x_*}$  which does not belong to  $H^2(\mathbf{C}(x_*))$ . For that reason, before starting the proof of Proposition 5, we introduce the following sets. Let  $r_* > 0$  and  $\mathbf{U}$  be a smooth subdomain of  $\mathbf{C}(x_*)$  as in Lemma 4 with moreover  $\overline{\mathbf{U}} \subset \mathbf{C}(x_*)$ , such that:

$$\mathbf{B}_{r_*}(x_*) \subset \mathbf{U}. \quad (17)$$

For  $m \in \mathbb{N}$  with  $m \geq 1$ , we consider a sequence

$$(r_k)_{k=1, \dots, m} \in (\mathbb{R})^m \text{ such that } 0 < r_1 < r_2 < \dots < r_m < r_{m+1} := r_*. \quad (18)$$

It then follows that  $\mathbf{B}_{r_k}(x_*) \subset \text{int}(\mathbf{B}_{r_{k+1}}(x_*))$ , for  $k = 1, \dots, m$ . In the following,  $h > 0$  is chosen small enough so that

$$\mathbf{B}_{r_h}(x_*) \subset \text{int}(\mathbf{B}_{r_1}(x_*)). \quad (19)$$

In the following,  $C > 0$  is a constant which can change from one occurrence to another and which does not depend on  $h$ . We are now in position to prove Proposition 5.

**Step 1:** Let us prove that there exist  $h_0 > 0$ ,  $\gamma > 0$  and  $C > 0$  such that for all  $h \in (0, h_0)$ ,

$$\|\nabla \mathbf{q}_{x_*}\|_{L^\infty(\Omega \setminus \mathbf{B}_{r_*}(x_*))} \leq Ch^{-\gamma}. \quad (20)$$

To this end, we apply recursively  $H^2$ -interior elliptic estimates outside a neighborhood of  $\mathbf{B}_{r_h}(x_*)$ . This will allow us to use Sobolev injections with constants independent of  $h$ . Using (16) and combining  $H^2$ -interior elliptic estimates and  $H^2$ -boundary elliptic estimates near  $\partial\Omega$  (see [14, Theorem 1 and Theorem 4 in Section 6.3]), there exists  $C > 0$  such that:

$$\|\mathbf{q}_{x_*}\|_{H^2(\Omega \setminus \mathbf{B}_{r_2}(x_*))} \leq C(1 + h^{-1} \|\nabla \mathbf{q}_{x_*}\|_{L^2(\Omega \setminus \mathbf{B}_{r_1}(x_*))}). \quad (21)$$

Moreover, using (16), the trace theorem and (21) together with the fact that  $\mathbf{q}_{x_*} = 0$  on  $\partial\mathbf{B}_{r_h}(x_*)$  and  $\mathbf{q}_{x_*} = 1$  on  $\partial\Omega$ , there exists  $C > 0$  such that for any  $\varepsilon > 0$  and  $\varepsilon' > 0$ ,

$$\begin{aligned} h \|\nabla \mathbf{q}_{x_*}\|_{L^2(\Omega \setminus \mathbf{B}_{r_h}(x_*))}^2 &\leq C \left( h \int_{\partial\Omega} |\partial_n \mathbf{q}_{x_*}| d\sigma + \int_{\Omega \setminus \mathbf{B}_{r_h}(x_*)} |\nabla f \cdot \nabla \mathbf{q}_{x_*}| \right) \\ &\leq C \left( \frac{h}{\varepsilon} + h\varepsilon \|\mathbf{q}_{x_*}\|_{H^2(\Omega \setminus \mathbf{B}_{r_2}(x_*))}^2 + \frac{1}{\varepsilon'} + \varepsilon' \|\nabla \mathbf{q}_{x_*}\|_{L^2(\Omega \setminus \mathbf{B}_{r_h}(x_*))}^2 \right) \\ &\leq C \left( \frac{h}{\varepsilon} + h^{-1}\varepsilon \left( \|\nabla \mathbf{q}_{x_*}\|_{L^2(\Omega \setminus \mathbf{B}_{r_1}(x_*))}^2 + 1 \right) + \frac{1}{\varepsilon'} + \varepsilon' \|\nabla \mathbf{q}_{x_*}\|_{L^2(\Omega \setminus \mathbf{B}_{r_h}(x_*))}^2 \right). \end{aligned}$$

Choosing  $\varepsilon' = \frac{h}{4(C+1)}$ , one obtains

$$\frac{3}{4}h \|\nabla \mathbf{q}_{x_*}\|_{L^2(\Omega \setminus \mathbf{B}_{r_h}(x_*))}^2 \leq C \left( \frac{h}{\varepsilon} + h^{-1}\varepsilon \left( \|\nabla \mathbf{q}_{x_*}\|_{L^2(\Omega \setminus \mathbf{B}_{r_1}(x_*))}^2 + 1 \right) + h^{-1} \right).$$

Thus, since  $\Omega \setminus \mathbf{B}_{r_1}(x_*) \subset \Omega \setminus \mathbf{B}_{r_h}(x_*)$  (see (19)),

$$\frac{3h}{4} \|\nabla \mathbf{q}_{x_*}\|_{L^2(\Omega \setminus \mathbf{B}_{r_1}(x_*))}^2 \leq C \left( \frac{h}{\varepsilon} + h^{-1}\varepsilon \left( \|\nabla \mathbf{q}_{x_*}\|_{L^2(\Omega \setminus \mathbf{B}_{r_1}(x_*))}^2 + 1 \right) + h^{-1} \right).$$

Then, one chooses  $\varepsilon = \frac{h^2}{4(C+1)}$  which implies that  $\|\nabla \mathbf{q}_{x_*}\|_{L^2(\Omega \setminus \mathbf{B}_{r_1}(x_*))} \leq Ch^{-1}$ . Thus, from (21), there exist  $C > 0$  and  $h_0 > 0$  such that for all  $h \in (0, h_0)$ :

$$\|\mathbf{q}_{x_*}\|_{H^2(\Omega \setminus \mathbf{B}_{r_2}(x_*))} \leq Ch^{-2}.$$

Then, for  $k \in \{3, \dots, m\}$ , using  $H^k$ -interior elliptic estimates and  $H^k$ -boundary elliptic estimates [14, Theorems 4 and 5 in Section 6.3.2], one obtains recursively that there exists  $C > 0$  such that for  $h$  small enough:

$$\|\mathbf{q}_{x_*}\|_{H^k(\Omega \setminus \mathbf{B}_{r_k}(x_*))} \leq Ch^{-1} (1 + \|\nabla \mathbf{q}_{x_*}\|_{H^{k-1}(\Omega \setminus \mathbf{B}_{r_k}(x_*))}) \leq Ch^{-k}.$$

Then, choosing  $m$  (see (18)) such that  $m > \frac{d}{2}$  yields (20) using the Sobolev injection  $H^m(\Omega \setminus \mathbf{B}_{r_m}(x_*))$  is continuously embedded in  $L^\infty(\Omega \setminus \mathbf{B}_{r_m}(x_*))$ .

**Step 2:** Let us define for  $\eta > 0$  the set (see (6))

$$\mathbf{C}_\eta(x_*) = \mathbf{C}(x_*) \cap \{f < \lambda(x_*) - \eta\} \subset \Omega, \quad (22)$$

Let

$$c_{*,M} := \max\{f(y), y \in \mathbf{C}(x_*) \text{ such that } \nabla f(y) = 0\}. \quad (23)$$

Since  $f$  is a Morse function on  $\bar{\Omega}$ ,  $f$  has a finite number of critical points in  $\bar{\Omega}$ , and thus:  $c_{*,M} < \lambda(x_*)$ . Let us consider

$$\eta_* \in (0, \lambda(x_*) - c_{*,M}) \text{ such that } \bar{\mathbf{U}} \subset \mathbf{C}_{\eta_*}(x_*) \text{ (see (17)–(19)) and } \eta \in (0, \eta_*]. \quad (24)$$

According to [29, Proposition 10],  $\mathbf{C}_\eta(x_*)$  is a connected component of  $\{f < \lambda(x_*) - \eta\}$ . Moreover,  $\mathbf{C}_\eta(x_*)$  is  $\mathcal{C}^\infty$  because its boundary does not contain critical points of  $f$  by choice of  $\eta$  and  $\mathbf{C}_\eta(x_*) \subset \Omega$ . Let us now prove that for all  $\eta_0 \in (0, \eta_*]$ , there exists  $\alpha_0 > 0$  such that for  $h$  small enough:

$$\|\nabla \mathbf{q}_{x_*}\|_{L^\infty(\mathbf{C}_{\eta_0}(x_*) \setminus \mathbf{B}_{r_*}(x_*))} \leq e^{-\frac{\alpha_0}{h}}. \quad (25)$$

Let  $\eta$  such that  $2^m \eta = \eta_0$  (see (18)) where  $m$  will be fixed later (notice that  $\eta \leq \eta_*$ ). Notice that (16) can be rewritten as  $\operatorname{div}(e^{-\frac{2}{h}f} \nabla \mathbf{q}_{x_*}) = 0$  on  $\Omega \setminus \mathbf{B}_{r_h}(x_*)$ . Therefore, using (20), the Green formula,  $\mathbf{q}_{x_*} \leq 1$ , and  $\mathbf{q}_{x_*} = 0$  on  $\partial \mathbf{B}_{r_h}(x_*)$ , there exist  $C > 0$  and  $\gamma > 0$  such that,

$$\int_{\mathbf{C}_{\eta/2}(x_*) \setminus \mathbf{B}_{r_h}(x_*)} |\nabla \mathbf{q}_{x_*}|^2 e^{-\frac{2}{h}f} = \int_{\partial \mathbf{C}_{\eta/2}(x_*) \setminus \mathbf{B}_{r_h}(x_*)} \mathbf{q}_{x_*} e^{-\frac{2}{h}f} \partial_n \mathbf{q}_{x_*} \leq \frac{C}{h^\gamma} e^{-\frac{2}{h}(\lambda(x_*) - \frac{\eta}{2})},$$

since  $\partial \mathbf{C}_{\eta/2}(x_*) \subset \{f = \lambda(x_*) - \frac{\eta}{2}\}$ . In addition, since  $\mathbf{C}_\eta(x_*) \setminus \mathbf{B}_{r_h}(x_*) \subset \mathbf{C}_{\eta/2}(x_*) \setminus \mathbf{B}_{r_h}(x_*)$  it holds,

$$e^{-\frac{2}{h}(\lambda(x_*) - \eta)} \int_{\mathbf{C}_\eta(x_*) \setminus \mathbf{B}_{r_h}(x_*)} |\nabla \mathbf{q}_{x_*}|^2 \leq \int_{\mathbf{C}_{\eta/2}(x_*) \setminus \mathbf{B}_{r_h}(x_*)} |\nabla \mathbf{q}_{x_*}|^2 e^{-\frac{2}{h}f} \leq \frac{C}{h^\gamma} e^{-\frac{2}{h}(\lambda(x_*) - \frac{\eta}{2})}.$$

Therefore, using in addition (19), there exists  $\beta > 0$  such that for  $h$  small enough,

$$\int_{\mathbf{C}_\eta(x_*) \setminus \mathbf{B}_{r_1}(x_*)} |\nabla \mathbf{q}_{x_*}|^2 \leq \int_{\mathbf{C}_\eta(x_*) \setminus \mathbf{B}_{r_h}(x_*)} |\nabla \mathbf{q}_{x_*}|^2 \leq \frac{C}{h^\gamma} e^{-\frac{\eta}{h}} \leq C e^{-\frac{\beta}{h}}.$$

Thus, from (16), we deduce that  $h$  small enough,  $\|\Delta \mathbf{q}_{x_*}\|_{L^2(\mathbf{C}_\eta(x_*) \setminus \mathbf{B}_{r_1}(x_*))} \leq C e^{-\frac{\beta}{2h}}$ . In the following,  $\beta > 0$  is a constant which may change from one occurrence to another and does not depend on  $h$ . Let  $\chi_1 \in \mathcal{C}_c^\infty(\mathbf{C}_\eta(x_*) \setminus \mathbf{B}_{r_1}(x_*), [0, 1])$  be such that  $\chi_1 \equiv 1$  on  $\mathbf{C}_{*, 2\eta} \setminus \mathbf{B}_{r_2}(x_*)$ . Since  $\Delta(\chi_1 \mathbf{q}_{x_*}) = \chi_1 \Delta \mathbf{q}_{x_*} + \mathbf{q}_{x_*} \Delta \chi_1 + 2\nabla \chi_1 \cdot \nabla \mathbf{q}_{x_*}$ , there exists  $C$ , such that  $\|\Delta(\chi_1 \mathbf{q}_{x_*})\|_{L^2(\mathbf{C}_\eta(x_*) \setminus \mathbf{B}_{r_1}(x_*))} \leq C$  for  $h$  small enough. From the  $H^2$ -elliptic regularity estimate on  $\mathbf{C}_\eta(x_*) \setminus \mathbf{B}_{r_1}(x_*)$ , one has:

$$\|\mathbf{q}_{x_*}\|_{H^2(\mathbf{C}_{2\eta}(x_*) \setminus \mathbf{B}_{r_2}(x_*))} \leq C.$$

Let  $\alpha \in (0, 1)$  be an irrational number such that  $p_1 = \frac{2d}{d-2\alpha} > 0$ . From the Gagliardo-Nirenberg interpolation inequality (see [43, Lecture II]), the following inequality holds:

$$\begin{aligned} C \|\nabla \mathbf{q}_{x_*}\|_{L^{p_1}(\mathbf{C}_{2\eta}(x_*) \setminus \mathbf{B}_{r_2}(x_*))} &\leq \|\mathbf{q}_{x_*}\|_{H^2(\mathbf{C}_{2\eta}(x_*) \setminus \mathbf{B}_{r_2}(x_*))}^\alpha \|\nabla \mathbf{q}_{x_*}\|_{L^2(\mathbf{C}_{2\eta}(x_*) \setminus \mathbf{B}_{r_2}(x_*))}^{1-\alpha} \\ &+ \|\nabla \mathbf{q}_{x_*}\|_{L^2(\mathbf{C}_{2\eta}(x_*) \setminus \mathbf{B}_{r_2}(x_*))} \leq e^{-\frac{\beta}{h}}. \end{aligned}$$

Then, from (16), one deduces that  $\|\Delta \mathbf{q}_{x_*}\|_{L^{p_1}(\mathbf{C}_{2\eta}(x_*) \setminus \mathbf{B}_{r_2}(x_*))} \leq C e^{-\frac{\beta}{h}}$ . Using a cutoff function  $\chi_2 \in \mathcal{C}_c^\infty(\mathbf{C}_{2\eta}(x_*) \setminus \mathbf{B}_{r_2}(x_*), [0, 1])$  such that  $\chi_2 \equiv 1$  on  $\mathbf{C}_{4\eta}(x_*) \setminus \mathbf{B}_{r_3}(x_*)$ , we get from the  $W^{2, p_1}$ -elliptic regularity estimate, that  $\|\mathbf{q}_{x_*}\|_{W^{2, p_1}(\mathbf{C}_{4\eta}(x_*) \setminus \mathbf{B}_{r_3}(x_*))} \leq C$ . Let  $p_2 = \frac{2d}{d-4\alpha}$ , from the Gagliardo-Nirenberg interpolation inequality (see [43, Lecture II]), one obtains:

$$\begin{aligned} C \|\nabla \mathbf{q}_{x_*}\|_{L^{p_2}(\mathbf{C}_{4\eta}(x_*) \setminus \mathbf{B}_{r_3}(x_*))} &\leq \|\mathbf{q}_{x_*}\|_{W^{2, p_1}(\mathbf{C}_{4\eta}(x_*) \setminus \mathbf{B}_{r_3}(x_*))}^\alpha \|\nabla \mathbf{q}_{x_*}\|_{L^{p_1}(\mathbf{C}_{4\eta}(x_*) \setminus \mathbf{B}_{r_3}(x_*))}^{1-\alpha} \\ &+ \|\nabla \mathbf{q}_{x_*}\|_{L^{p_1}(\mathbf{C}_{4\eta}(x_*) \setminus \mathbf{B}_{r_3}(x_*))} \leq e^{-\frac{\beta}{h}}. \end{aligned}$$

We repeat this procedure  $m - 2$  times such that  $d - 2m\alpha \leq 0$ , the Gagliardo-Nirenberg interpolation inequality implies that  $\|\nabla \mathbf{q}_{x_*}\|_{L^\infty(\mathbf{C}_{2^m \eta}(x_*) \setminus \mathbf{B}_{r_{m+1}}(x_*))} \leq C e^{-\frac{\beta}{h}}$  which ends the proof of (25) since  $\eta_0 = 2^m \eta$  and  $r_{m+1} = r_*$  (see (18)).

**Step 3:** Let  $r_{**} > r_*$  such that  $\mathbf{B}_{r_{**}}(x_*) \subset \mathbf{U}$  (see (17)). From Lemma 4, for  $h$  small enough, it holds:

$$\max_{y \in \mathbf{B}_{r_{**}}(x_*)} |\mathbf{p}_{x_*}(y) - 1| = O(e^{-\frac{c}{h}}). \quad (26)$$

When the dimension  $d \geq 2$ , the open set  $\mathbf{C}_\eta(x_*) \setminus \mathbf{B}_{r_*}(x_*)$  is path-connected. Indeed  $\mathbf{B}_{r_*}(x_*)$  is a closed ball included in the open connected set  $\mathbf{C}_\eta(x_*)$  (see (17) and (24)) and thus, when  $d \geq 2$ ,  $\mathbf{C}_\eta(x_*) \setminus \mathbf{B}_{r_*}(x_*)$  is connected. Since it is moreover locally path-connected,  $\mathbf{C}_\eta(x_*) \setminus \mathbf{B}_{r_*}(x_*)$  is path-connected. When  $d = 1$ ,  $\mathbf{C}_\eta(x_*) \setminus \mathbf{B}_{r_*}(x_*)$  is the disjoint union of two open intervals  $\mathbf{I}_1$  and  $\mathbf{I}_2$ . Let  $z_{**} \in \partial \mathbf{B}_{r_{**}}(x_*)$ . Then, when  $d \geq 2$  for  $\eta > 0$  small enough and for all  $y \in \mathbf{C}_\eta(x_*) \setminus \mathbf{B}_{r_*}(x_*)$ , considering a smooth curve  $\gamma : [0, 1] \rightarrow \mathbf{C}_\eta(x_*) \setminus \mathbf{B}_{r_*}(x_*)$  such that  $\gamma(0) = y$  and  $\gamma(1) = z$  (because  $\mathbf{C}_\eta(x_*) \setminus \mathbf{B}_{r_*}(x_*)$  is open, path-connected, and locally smooth), it follows from (25) that there exists  $\beta > 0$  such that for  $h$  small enough:

$$\sup_{y \in \mathbf{C}_\eta(x_*) \setminus \mathbf{B}_{r_*}(x_*)} |\mathbf{p}_{x_*}(y) - \mathbf{p}_{x_*}(z_{**})| \leq e^{-\frac{\beta}{h}}.$$

Let us mention that when  $d = 1$ , the previous estimate also holds by choosing  $z_{**} \in \mathbf{I}_k$  if  $y \in \mathbf{I}_k$  ( $k \in \{1, 2\}$ ). Using in addition (26), for  $\eta > 0$  small enough, there exists  $c > 0$ , for  $h$  small enough, one has:

$$\mathbf{p}_{x_*}(y) = 1 + O(e^{-\frac{c}{h}}), \text{ uniformly on } y \in \mathbf{C}_\eta(x_*). \quad (27)$$

This concludes the proof of Proposition 5 when  $K \subset C(x_*)$  (indeed in this case there exists  $\eta > 0$  such that  $K \subset C_\eta(x_*)$ ).

**Step 4:** Let us consider  $K \subset \mathcal{A}_\Omega(C(x_*))$ . Since  $C(x_*) \subset \mathcal{A}_\Omega(C(x_*))$  and  $C(x_*)$  is open, there exists  $T_K \geq 0$  such that for all  $x \in K$ ,

$$\varphi_{T_K}(x) \in C(x_*), \quad (28)$$

where we recall  $\varphi_t(x)$  is defined by (4). Moreover, because  $K \subset \mathcal{A}_\Omega(C(x_*))$ , for all  $x \in K$ ,  $t_x = +\infty$  (i.e.  $\varphi_t(x) \in \Omega$  for all  $t \geq 0$ ) and thus  $\{\varphi_t(x), t \in [0, T_K] \text{ and } x \in K\} \subset \Omega$ . Since  $K$  is compact,  $\{\varphi_t(x), t \in [0, T_K] \text{ and } x \in K\}$  is therefore a compact subset of  $\Omega$ . Therefore, there exists  $\delta > 0$  such that all continuous curves  $\gamma : [0, T_K] \rightarrow \bar{\Omega}$  such that

$$\exists x \in K, \quad \sup_{t \in [0, T_K]} |\gamma(t) - \varphi_t(x)| \leq \delta,$$

satisfy:

$$\forall t \in [0, T_K], \quad \gamma(t) \in \Omega. \quad (29)$$

Moreover, up to choosing  $\delta > 0$  smaller, there exists  $\alpha_K > 0$  such that

$$\{\varphi_{T_K}(x) + z, \quad x \in K \text{ and } |z| \leq \delta\} \subset C_{\alpha_K}(x_*). \quad (30)$$

Let us notice that when  $X_0 = x \in K$  and  $\sup_{t \in [0, T_K]} |X_t - \varphi_t(x)| \leq \delta$ , it holds from (29) and (30):

$$\tau_\Omega > T_K \text{ and } X_{T_K} \in C_{\alpha_K}(x_*). \quad (31)$$

From the proof of [9, Lemma 1] and its note, there exists  $\eta_K > 0$  such that for  $h$  small enough, it holds:

$$\sup_{x \in K} \mathbb{P}_x \left[ \sup_{t \in [0, T_K]} |X_t - \varphi_t(x)| \geq \delta \right] \leq e^{-\frac{\eta_K}{h}}. \quad (32)$$

Let us now concludes the proof of Proposition 5. For  $x \in K$ , one writes  $\mathbf{p}_{x_*}(x) = a(x) + b(x)$ , where

$$a(x) = \mathbb{P}_x \left[ \left\{ \tau_{B_{r_h}(x_*)} < \tau_{\Omega^c} \right\} \cap \left\{ \sup_{t \in [0, T_K]} |X_t - \varphi_t(x)| < \delta \right\} \right],$$

and

$$b(x) = \mathbb{P}_x \left[ \left\{ \tau_{B_{r_h}(x_*)} < \tau_{\Omega^c} \right\} \cap \left\{ \sup_{t \in [0, T_K]} |X_t - \varphi_t(x)| \geq \delta \right\} \right].$$

Using (32), it holds for  $h$  small enough:  $\max_{x \in K} b(x) \leq e^{-\frac{\eta_K}{h}}$ . Using (31), (27) with  $C_{\alpha_K}(x_*)$ , (32), and the Markov property of the process (1), there exists  $c > 0$  such that for all  $x \in K$ , one has when  $h \rightarrow 0$ :

$$\begin{aligned} a(x) &= \mathbb{E}_x \left[ \mathbb{E}_{X_{T_K}} \left[ \mathbf{1}_{\{\tau_{B_{r_h}(x_*)} < \tau_{\Omega^c}\}} \mathbf{1}_{\sup_{t \in [0, T_K]} |X_t - \varphi_t(x)| < \delta} \right] \right] \\ &= (1 + O(e^{-\frac{c}{h}})) \times \mathbb{P}_x \left[ \sup_{t \in [0, T_K]} |X_t - \varphi_t(x)| \leq \delta \right] = 1 + O(e^{-\frac{c}{h}}), \text{ uniformly in } x \in K. \end{aligned}$$

Hence, there exists  $c > 0$  such that for all  $x \in K$ , one has when  $h \rightarrow 0$ :  $\mathbf{p}_{x_*}(x) = 1 + O(e^{-\frac{c}{h}})$ , uniformly in  $x \in K$ . This concludes the proof of Proposition 5.  $\blacksquare$

We will also need the slightly different version of Proposition 5.

**Proposition 6.** *Let  $x_*$  be a local minimum point of  $f$  in  $\Omega$ . Consider a compact subset  $\mathbf{K}$  of  $\mathcal{A}_\Omega(\mathbf{C}(x_*))$ . Let  $T_{\mathbf{K}}$  be such that (28) holds. Then, there exist  $\delta_{\mathbf{K}} > 0$ ,  $\alpha_{\mathbf{K}} > 0$ , and a smooth domain  $\Omega_{\mathbf{K}}$  containing  $\mathbf{K}$  with  $\overline{\Omega_{\mathbf{K}}} \subset \Omega$ , such that all continuous curves  $\gamma : [0, T_{\mathbf{K}}] \rightarrow \overline{\Omega}$  such that*

$$\exists x \in \mathbf{K}, \quad \sup_{t \in [0, T_{\mathbf{K}}]} |\gamma(t) - \varphi_t(x)| \leq \delta_{\mathbf{K}} \quad (\text{see (4)}),$$

*satisfy:*

1.  $\forall t \in [0, T_{\mathbf{K}}], \gamma(t) \in \Omega_{\mathbf{K}}$ .
2.  $\{\varphi_{T_{\mathbf{K}}}(x) + z, x \in \mathbf{K} \text{ and } |z| \leq \delta_{\mathbf{K}}\} \subset \mathbf{C}_{\alpha_{\mathbf{K}}}(x_*)$  (see (22)), and  $\overline{\mathbf{C}_{\alpha_{\mathbf{K}}}(x_*)} \subset \Omega_{\mathbf{K}}$ .

*Moreover, there exist  $h_0 > 0$  and  $c > 0$  such that for all  $h \in (0, h_0)$ , it holds:*

$$\max_{y \in \mathbf{K}} |\mathbb{P}_y[\tau_{\mathbf{B}_{r_h}(x_*)} < \tau_{\Omega_{\mathbf{K}}}^c] - 1| \leq e^{-\frac{c}{h}}. \quad (33)$$

*Proof.* Let us first prove items 1 and 2 in Proposition 6. Let  $\mathbf{K} \subset \mathcal{A}_\Omega(\mathbf{C}(x_*))$  and  $T_{\mathbf{K}}$  be such that (28) holds. The set  $\{\varphi_t(x), t \in [0, T_{\mathbf{K}}] \text{ and } x \in \mathbf{K}\}$  is a compact subset of  $\Omega$  and the set  $\{\varphi_{T_{\mathbf{K}}}(x) + z, x \in \mathbf{K} \text{ and } |z| \leq \delta\}$  is a compact subset of  $\mathbf{C}(x_*)$ . Thus, for  $\delta > 0$  small enough, the compact set  $\mathbf{M} := \{y \in \overline{\Omega}, \exists x \in \mathbf{K} \text{ and } t \in [0, T_{\mathbf{K}}] \text{ such that } |y - \varphi_t(x)| \leq \delta\}$  is included in  $\Omega$ , and  $\{\varphi_{T_{\mathbf{K}}}(x) + z, x \in \mathbf{K} \text{ and } |z| \leq \delta\}$  is included in  $\mathbf{C}_{\alpha_{\mathbf{K}}}(x_*)$  for some  $\alpha_{\mathbf{K}} > 0$  small enough. Because  $\mathbf{K}$ ,  $\mathbf{M}$ , and  $\overline{\mathbf{C}_{\alpha_{\mathbf{K}}}(x_*)}$  are compact subsets of  $\Omega$ , there exists a smooth domain  $\Omega_{\mathbf{K}}$  such that  $\mathbf{K} \cup \mathbf{M} \cup \overline{\mathbf{C}_{\alpha_{\mathbf{K}}}(x_*)} \subset \Omega_{\mathbf{K}}$  and  $\overline{\Omega_{\mathbf{K}}} \subset \Omega$ . This proves items 1 and 2 in Proposition 6. Up to choosing  $\Omega_{\mathbf{K}}$  larger, let us assume that  $\alpha_{\mathbf{K}} \in (0, \lambda(x_*) - c_{*,M})$ , see (23). Let us now prove (33). The proof of (33) is the same as the one made to prove Proposition 5. For the sake of completeness, we recall the main arguments. Let us denote by  $\tilde{\mathbf{p}}_{x_*} : y \in \overline{\Omega_{\mathbf{K}}} \mapsto \mathbb{P}_y[\tau_{\mathbf{B}_{r_h}(x_*)} < \tau_{\Omega_{\mathbf{K}}}^c]$ . Since  $\alpha_{\mathbf{K}} \in (0, \lambda(x_*) - c_{*,M})$ , there exists  $\varepsilon_0 > 0$  such that for all  $\varepsilon \in (0, \varepsilon_0)$ ,  $\mathbf{C}_{\alpha_{\mathbf{K}} + \varepsilon}$  is a  $\mathcal{C}^\infty$  open connected component of  $\{f < \lambda(x_*) - \alpha_{\mathbf{K}} - \varepsilon\}$  which is included in  $\Omega_{\mathbf{K}}$  (see the lines after (24)). Thus, with the same arguments as those used to prove (27) above, one obtains that for all  $\varepsilon \in (0, \varepsilon_0)$ , there exists  $c > 0$  such that for  $h$  small enough:

$$\tilde{\mathbf{p}}_{x_*}(y) = 1 + O(e^{-\frac{c}{h}}), \quad \text{uniformly on } y \in \mathbf{C}_{\alpha_{\mathbf{K}} + \varepsilon}(x_*). \quad (34)$$

Since  $\{\varphi_{T_{\mathbf{K}}}(x) + z, x \in \mathbf{K} \text{ and } |z| \leq \delta\}$  is a compact subset of the open set  $\mathbf{C}_{\alpha_{\mathbf{K}}}(x_*)$ , for  $\varepsilon \in (0, \varepsilon_0)$  small enough, it holds

$$\{\varphi_{T_{\mathbf{K}}}(x) + z, x \in \mathbf{K} \text{ and } |z| \leq \delta\} \subset \mathbf{C}_{\alpha_{\mathbf{K}} + \varepsilon}(x_*). \quad (35)$$

From item 1 in Proposition 6, when  $X_0 = x \in \mathbf{K}$  and  $\sup_{t \in [0, T_{\mathbf{K}}]} |X_t - \varphi_t(x)| \leq \delta$ , it holds:

$$\tau_{\Omega_{\mathbf{K}}}^c > T_{\mathbf{K}}, \quad \text{and from (35), } X_{T_{\mathbf{K}}} \in \mathbf{C}_{\alpha_{\mathbf{K}} + \varepsilon}(x_*). \quad (36)$$

One then writes for all  $x \in \mathbf{K}$ ,  $\tilde{\mathbf{p}}_{x_*}(x) = c(x) + d(x)$ , where  $c(x) = \mathbb{P}_x[\{\tau_{\mathbf{B}_{r_h}(x_*)} < \tau_{\Omega_{\mathbf{K}}}^c\} \cap \{\sup_{t \in [0, T_{\mathbf{K}}]} |X_t - \varphi_t(x)| < \delta\}]$  and  $d(x) = \mathbb{P}_x[\{\tau_{\mathbf{B}_{r_h}(x_*)} < \tau_{\Omega_{\mathbf{K}}}^c\} \cap \{\sup_{t \in [0, T_{\mathbf{K}}]} |X_t - \varphi_t(x)| \geq \delta\}]$ . Using (32), it holds for  $h$  small enough:  $\sup_{x \in \mathbf{K}} d(x) \leq e^{-\frac{c}{h}}$ , for some  $c > 0$  independent of  $h$ . The Markov property together with (34) and (36), imply that there exists  $c > 0$  such that when  $h \rightarrow 0$ ,  $c(x) = 1 + O(e^{-\frac{c}{h}})$ , uniformly in  $x \in \mathbf{K}$ . This proves (33) and concludes the proof of Proposition 6.  $\blacksquare$

We will also need a leveling result of  $\mathbf{p}_{x_*}$  in  $\mathbf{C} \subset \mathcal{C}$  (see (7)) when  $\mathbf{C} \neq \mathbf{C}(x)$ . This leveling result is the purpose of Proposition 8 below which is based on an estimate on the most probable places of exit from a bounded domain when  $\mathbf{N} = 1$ , see the next section.



## 2.5 The case when $N = 1$

In this section, we prove the following result.

**Proposition 7.** *Assume that **(H1)** is satisfied with  $N = 1$  (i.e.  $\{f < \min_{\partial\Omega} f\}$  is connected). Let  $K$  be compact subset of  $\mathcal{A}_\Omega(\{f < \min_{\partial\Omega} f\})$  (see (5)) and  $\Gamma$  be an open subset of  $\partial\Omega$  such that  $\min_{\bar{\Gamma}} f > \min_{\partial\Omega} f$ . Then,*

$$\text{there exists } c > 0 \text{ s.t. for } h \text{ small enough: } \max_{x \in K} \mathbb{P}_x [X_{\tau_{\Omega^c}} \in \Gamma] \leq e^{-\frac{c}{h}}. \quad (37)$$

In Corollary 10 below, we will generalize (37) to more general settings (e.g. when  $N > 1$ ).

Before going through the proof of Proposition 7, we recall the notion of separating saddle points of  $f$  in  $\Omega$ . A point  $z \in \Omega$  is a *separating saddle point* of  $f$  in  $\Omega$  if it is a saddle point of  $f$  and for  $r > 0$  small enough,  $B_r(z) \cap \{f < f(z)\}$  has two connected components which are included in two different connected components of  $\{f < f(z)\}$ . The notion of separating saddle points was first introduced in [21] in the boundaryless case and then adapted to the boundary case in [13, 29].

*Proof.* Let  $f : \bar{\Omega} \rightarrow \mathbb{R}$  be a  $C^\infty$  function. Before starting the proof of Proposition 7, we introduce some preliminary results we will need. Let us choose an eigenfunction  $u_{1,h}$  associated with the principal eigenvalue  $\lambda_{1,h}$  of  $\mathcal{L}_{f,h}^{\text{Dir}}$  such that

$$u_{1,h} > 0 \text{ on } \Omega \text{ and } \int_{\Omega} u_{1,h}^2 e^{-\frac{2}{h}f} = 1. \quad (38)$$

For ease of notation, we write  $L_w^2(\Omega)$  (resp.  $L^2(\Omega)$ ) for  $L^2(\Omega, e^{-\frac{2}{h}f} dx)$  (resp. for  $L^2(\Omega, dx)$ ). The subscript  $w$  stands for the fact that we consider the weight function  $e^{-\frac{2}{h}f}$  in the scalar product. The proof of Proposition 7 is based on the use of the quasi-stationary distribution of the process (1) in  $\Omega$ : we will indeed first prove (37) when  $X_0$  is distributed according to this distribution, and then, with the help of a leveling result, we will derive it when  $X_0 = x \in K$ . A quasi-stationary distribution for the process (1) in  $\Omega$  is a probability measure  $\mu$  supported in  $\Omega$  such that for all measurable sets  $A \subset \Omega$  and for all  $t \geq 0$ ,

$$\mu(A) = \frac{\int_{\Omega} \mathbb{P}_x [X_t \in A, t < \tau_{\Omega}] \mu(dx)}{\int_{\Omega} \mathbb{P}_x [t < \tau_{\Omega^c}] \mu(dx)}.$$

Let us recall that (see for example [26]) the unique quasi-stationary distribution  $\nu_h$  of the dynamics (1) in  $\Omega$  is the measure

$$\nu_h(dx) = \frac{u_{1,h}(x) e^{-\frac{2}{h}f(x)}}{\int_{\Omega} u_{1,h} e^{-\frac{2}{h}f}} dx. \quad (39)$$

Moreover, according to [26], when  $X_0 \sim \nu_h$ ,  $X_{\tau_{\Omega^c}}$  has a density with respect to the Lebesgue measure on  $\partial\Omega$  which is given by:

$$z \in \partial\Omega \mapsto -\frac{h}{2\lambda_{1,h}} \frac{\partial_n u_{1,h}(z) e^{-\frac{2}{h}f(z)}}{\int_{\Omega} u_{1,h} e^{-\frac{2}{h}f}}. \quad (40)$$

Let us now assume that  $f : \bar{\Omega} \rightarrow \mathbb{R}$  is a Morse function. From the analysis led in the proof of [29, Theorem 1] together with (3), for any  $L_w^2(\Omega)$ -normalized eigenfunction  $w_h$  of  $\mathcal{L}_{f,h}^{\text{Dir}}$  associated with

an eigenvalue of order  $o(1)$  when  $h \rightarrow 0$ , it holds, for every neighborhood  $W$  of the local minima of  $f$  in  $\Omega$ :

$$\lim_{h \rightarrow 0} \|w_h \mathbf{1}_W\|_{L_w^2(\Omega)} = 1. \quad (41)$$

Let us now assume that **(H1)** is satisfied with  $N = 1$ . Let us define  $c_M = \max\{f(x), x \in \{f < \min_{\partial\Omega} f\} \text{ such that } \nabla f(x) = 0\} < \min_{\partial\Omega} f$ . In the following, for  $\alpha > 0$ ,  $C_\alpha$  denotes the open set  $\{f < \min_{\partial\Omega} f - \alpha\}$ . Let us recall that for  $\alpha < \min_{\partial\Omega} f - c_M$ ,  $C_\alpha$  is a smooth connected open subset of  $\Omega$  (see indeed the lines after (22)). Moreover, for any  $c_0 > 0$ , we denote by  $V_0$  a smooth open connected subset of  $\Omega$  such that  $\{f < \min_{\partial\Omega} f\} \subset \overline{V_0}$  and  $f \leq \min_{\partial\Omega} f + c_0$  on  $V_0$ . Since  $\partial\{f < \min_{\partial\Omega} f\} \cap \partial\Omega \neq \emptyset$ , we choose  $V_0$  such that  $\partial V_0 \cap \partial\Omega$  has a non zero Lebesgue measure (in  $\partial\Omega$ ). We are now ready to prove Proposition 7. The proof is divided into several steps.

**Step 1:** Let us prove that:

$$\lim_{h \rightarrow 0} h \log \lambda_{1,h} = -2\left(\min_{\partial\Omega} f - \min_{\overline{\Omega}} f\right) \text{ and } \liminf_{h \rightarrow 0} h \log \lambda_{2,h} \geq -2\left(c_M - \min_{\overline{\Omega}} f\right). \quad (42)$$

When **(H2)** holds, (42) is a consequence of [29, Theorem 2] and (3). Moreover, under **(H1)**, the set  $\partial\{f < \min_{\partial\Omega} f\} \cap \Omega$  does not contain separating saddle point of  $f$  in  $\Omega$ . Thus, if **(H2)** holds, according to (3) and [29, Theorem 3] (with  $m^* = 1$  and  $C_j(x_1) = \{f < \min_{\partial\Omega} f\}$  there), one has more precisely when  $h \rightarrow 0$ :

$$\lambda_{1,h} = \left(\Theta_{1,1} h^{-\frac{1}{2}}(1 + O(h)) + \Theta_{1,2}(1 + O(\sqrt{h}))\right) e^{-\frac{2}{h}(\min_{\partial\Omega} f - \min_{\overline{\Omega}} f)}, \quad (43)$$

where  $\Theta_{1,1}$  and  $\Theta_{1,2}$  are defined at the beginning of Section 1.4 (recall that  $C_1 = \{f < \min_{\partial\Omega} f\}$  because we assume here that  $N = 1$ ). Actually, we will prove (42) and (37) without assuming **(H2)**. Let us first prove the first limit in (42). Let  $\chi \in C_c^\infty(\{f < \min_{\partial\Omega} f\})$  be such that  $\chi = 1$  on  $C_\alpha$ , for  $\alpha > 0$ . Then, it follows from the Laplace method that for all  $\alpha > 0$  small enough such that  $\arg \min_{\overline{\Omega}} f \subset C_\alpha$ , there exists  $C > 0$  such that for  $h$  small enough:

$$\lambda_{1,h} \leq \left\langle \frac{\mathcal{L}_{f,h}\chi}{\|\chi\|_{L_w^2(\Omega)}}, \frac{\chi}{\|\chi\|_{L_w^2(\Omega)}} \right\rangle_{L_w^2(\Omega)} = \frac{h \|\nabla \chi\|_{L_w^2(\Omega)}^2}{2 \|\chi\|_{L_w^2(\Omega)}^2} \leq C h^{-\frac{d}{2}+1} e^{-\frac{2}{h}(\min_{\partial\Omega} f - \min_{\overline{\Omega}} f - \alpha)}. \quad (44)$$

Because  $\alpha > 0$  is arbitrary, this proves that  $\lim_{h \rightarrow 0} h \log \lambda_{1,h} \leq -2(\min_{\partial\Omega} f - \min_{\overline{\Omega}} f)$ . To prove the reverse inequality, inspired by [32, Lemma 5.9], let us argue by contradiction: assume that there exists  $c > 0$  and a sequence  $(h_n)_{n \in \mathbb{N}}$  such that  $\lim_{n \rightarrow +\infty} h_n = 0$  and for all  $n \geq 0$ ,  $\lambda_{1,h_n} \leq e^{-\frac{2}{h_n}(\min_{\partial\Omega} f - \min_{\overline{\Omega}} f + c)}$ . Let  $c_0 < c$ . From (38) and since  $f \leq \min_{\partial\Omega} f + c_0$  on  $V_0$ , it holds:

$$\frac{h_n e^{-\frac{2}{h_n}(\min_{\partial\Omega} f + c_0)} \int_{V_0} |\nabla u_{1,h_n}|^2}{2} \leq \frac{h_n}{2} \int_{V_0} |\nabla u_{1,h_n}|^2 e^{-\frac{2}{h_n} f} \leq e^{-\frac{2}{h_n}(\min_{\partial\Omega} f - \min_{\overline{\Omega}} f + c)}, \quad (45)$$

which implies that:  $\|\nabla(u_{1,h_n} e^{-\frac{1}{h_n} \min_{\overline{\Omega}} f})\|_{L^2(V_0)}^2 \leq \frac{2}{h_n} e^{-\frac{c-c_0}{h_n}} \rightarrow 0$  as  $n \rightarrow \infty$ . Defining  $c_n = \int_{V_0} u_n e^{-\frac{1}{h_n} \min_{\overline{\Omega}} f} (\int_{V_0} 1)^{-1}$ , the Poincaré-Wirtinger inequality (valid because  $V_0$  is a smooth bounded domain) then implies:

$$\lim_{n \rightarrow +\infty} \|u_n e^{-\frac{1}{h_n} \min_{\overline{\Omega}} f} - c_n\|_{L^2(V_0)} = 0 \text{ and thus, } \lim_{n \rightarrow +\infty} \|u_n e^{-\frac{1}{h_n} \min_{\overline{\Omega}} f} - c_n\|_{H^1(V_0)} = 0.$$

Because  $u_n|_{\partial V_0 \cap \partial\Omega} = 0$ , it follows from the trace theorem, that  $\lim_{n \rightarrow +\infty} c_n = 0$ . Therefore, it holds:  $\lim_{n \rightarrow +\infty} \|u_n e^{-\frac{1}{h_n} \min_{\overline{\Omega}} f}\|_{L^2(V_0)} = 0$  and thus,  $\lim_{n \rightarrow +\infty} \|u_n e^{-\frac{1}{h_n} f}\|_{L^2(V_0)} = 0$  which

contradicts (41). This concludes the proof of the first limit in (42). Let us now prove the second limit in (42). For that purpose, we argue again by contradiction: assume that there exists a constant  $c \in (0, \min_{\partial\Omega} f - c_M)$  and a decreasing sequence  $(h_n)_{n \in \mathbb{N}}$  such that  $\lim_{n \rightarrow +\infty} h_n = 0^+$  and for all  $n \in \mathbb{N}$ ,  $\lambda_{2, h_n} \leq e^{-\frac{2}{h_n}(c_M - \min_{\overline{\Omega}} f + c)}$ . Then, for  $\alpha > 0$  small enough (such that  $\mathbf{1}_{C_\alpha} \neq 0$ ), using the Max-Min principle (see for example [20, Theorem 11.7]), for all  $n \in \mathbb{N}$ , there exists a  $L_w^2(\Omega)$ -normalized function  $\varphi_n \in D(\mathcal{L}_{f, h}^{\text{Dir}})$  such that

$$\langle \varphi_n, 1 \rangle_{L_w^2(C_\alpha)} = 0 \quad \text{and} \quad \langle \mathcal{L}_{f, h_n} \varphi_n, \varphi_n \rangle_{L_w^2(\Omega)} = \frac{h_n}{2} \|\nabla \varphi_n\|_{L_w^2(\Omega)}^2 \leq e^{-\frac{2}{h_n}(c_M - \min_{\overline{\Omega}} f + c)}.$$

Choosing  $\alpha = \min_{\partial\Omega} f - c_M - \frac{c}{2}$  and reasoning as in (45), one deduces that:  $\|\nabla(\varphi_n e^{-\frac{1}{h_n} \min_{\overline{\Omega}} f})\|_{L^2(C_\alpha)}^2 \leq \frac{2}{h_n} e^{-\frac{c-c/2}{h_n}} \rightarrow 0$  as  $n \rightarrow \infty$ . The Poincaré-Wirtinger inequality (valid because  $C_\alpha$  is a smooth bounded domain since  $\alpha < \min_{\partial\Omega} f - c_M$ ), there exists  $c_n \in \mathbb{R}$  such that  $\lim_{n \rightarrow +\infty} \|\varphi_n e^{-\frac{1}{h_n} \min_{\overline{\Omega}} f} - c_n\|_{L^2(C_\alpha)} = 0$ . Therefore,

$$\lim_{n \rightarrow +\infty} \|\varphi_n e^{-\frac{1}{h_n} f} - c_n e^{-\frac{1}{h_n}(f - \min_{\overline{\Omega}} f)}\|_{L^2(C_\alpha)} = 0.$$

Since  $\langle \varphi_n, 1 \rangle_{L_w^2(C_\alpha)} = 0$ , it holds:

$$\left\| \varphi_n e^{-\frac{1}{h_n} f} - c_n e^{-\frac{1}{h_n}(f - \min_{\overline{\Omega}} f)} \right\|_{L^2(C_\alpha)}^2 = \|\varphi_n e^{-\frac{1}{h_n} f}\|_{L^2(C_\alpha)}^2 + c_n^2 \|e^{-\frac{1}{h_n}(f - \min_{\overline{\Omega}} f)}\|_{L^2(C_\alpha)}^2.$$

Thus,  $\lim_{n \rightarrow +\infty} \|\varphi_n e^{-\frac{1}{h_n} f}\|_{L^2(C_\alpha)} = 0$  which contradicts (41). This concludes the proof (42).

**Step 2:** In this step, one proves that in the limit  $h \rightarrow 0$ :

$$\int_{\Omega} u_{1, h} e^{-\frac{2}{h} f} = (\pi h)^{\frac{d}{4}} \sqrt{\sum_{y \in \arg \min_{\overline{\Omega}} f} (\det \text{Hess} f(y))^{-\frac{1}{2}} e^{-\frac{1}{h} \min_{\overline{\Omega}} f}}. \quad (46)$$

By (42), there exists  $c > 0$  such that for  $h$  small enough,  $\lambda_{1, h} = O(e^{-\frac{c}{h}}) \lambda_{2, h}$ . Therefore, there exists  $\beta_0 > 0$  such that for all  $\beta \in (0, \beta_0)$ , there exists  $h_0 > 0$  such that for all  $h \in (0, h_0)$ , the  $L_w^2$ -orthogonal projector  $\pi_h := \pi_{[0, e^{-\frac{2}{h}(\min_{\partial\Omega} f - \min_{\overline{\Omega}} f - \beta)}]}(\mathcal{L}_{f, h}^{\text{Dir}})$  has rank 1. From the following spectral estimate

$$\left\| (1 - \pi_h) \frac{\chi}{\|\chi\|_{L_w^2(\Omega)}} \right\|_{L_w^2(\Omega)}^2 \leq C e^{\frac{2}{h}(\min_{\partial\Omega} f - \min_{\overline{\Omega}} f - \beta)} \langle \mathcal{L}_{f, h} \frac{\chi}{\|\chi\|_{L_w^2(\Omega)}}, \frac{\chi}{\|\chi\|_{L_w^2(\Omega)}} \rangle_{L_w^2(\Omega)},$$

and in view of (44), it holds for some  $C > 0$ ,

$$\left\| (1 - \pi_h) \frac{\chi}{\|\chi\|_{L_w^2(\Omega)}} \right\|_{L_w^2(\Omega)}^2 \leq C h^{\frac{d}{2} + 1} e^{-\frac{2}{h}(\beta - \alpha)}.$$

Then, choosing  $\alpha < \beta$ , there exists  $c > 0$  such that for  $h$  small enough

$$\left\| (1 - \pi_h) \frac{\chi}{\|\chi\|_{L_w^2(\Omega)}} \right\|_{L_w^2(\Omega)} = O(e^{-\frac{c}{h}}).$$

Thus, there exists  $c > 0$  such that (using in addition the fact that  $u_{1, h} > 0$  and  $\chi \geq 0$  on  $\Omega$ ),  $u_{1, h} = \frac{\chi}{\|\chi\|_{L_w^2(\Omega)}} + O(e^{-\frac{c}{h}})$  in  $L_w^2$ . This implies that there exists  $c > 0$  such that for  $h$  small enough:

$$\int_{\Omega} u_{1, h} e^{-\frac{2}{h} f} = \int_{\Omega} \frac{\chi e^{-\frac{2}{h} f}}{\|\chi\|_{L_w^2(\Omega)}} + O(e^{-\frac{c}{h}}) \sqrt{\int_{\Omega} e^{-\frac{2}{h} f}} = \int_{\Omega} \frac{\chi}{\|\chi\|_{L_w^2(\Omega)}} e^{-\frac{2}{h} f} + O\left(e^{-\frac{1}{h}(\min_{\overline{\Omega}} f + c)}\right).$$

The lower bound in (46) is then obtained with the Laplace method for the terms  $\int_{\Omega} \chi e^{-\frac{2}{h}f}$  and  $\int_{\Omega} \chi^2 e^{-\frac{2}{h}f}$ .

**Step 3:** Let us now prove that for all measurable subset  $\Sigma$  of  $\partial\Omega$ , one has:

$$\limsup_{h \rightarrow 0} h \log \mathbb{P}_{\nu_h} [X_{\tau_{\Omega^c}} \in \Sigma] \leq - \left( \min_{\Sigma} f - \min_{\partial\Omega} f \right), \quad (47)$$

where we recall that  $\nu_h$  is the quasi stationary distribution of the process (1) in  $\Omega$  (see (39)). Notice that (47) implies (37) when  $X_0 \sim \nu_h$ . According to (40) and (46), using the Trace theorem, for any  $\delta > 0$  there exists  $h_0 > 0$  such that for all  $h \in (0, h_0)$ ,

$$\mathbb{P}_{\nu_h} [X_{\tau_{\Omega^c}} \in \Sigma] \leq \frac{h}{2\lambda_{1,h}} \|e^{-\frac{1}{h}f} u_h\|_{H^1(\Omega)} e^{-\frac{1}{h} \min_{\Sigma} f} e^{\frac{1}{h} \min_{\partial\Omega} f} h^{\frac{d}{2}}. \quad (48)$$

Let us now prove that there exists  $C > 0$  such that for  $h$  small enough,

$$\|e^{-\frac{1}{h}f} u_h\|_{H^1(\Omega)} \leq Ch^{-\frac{3}{2}} \sqrt{\lambda_{1,h}}. \quad (49)$$

One first has from (38),  $\frac{h}{2} \|e^{-\frac{1}{h}f} \nabla u_{1,h}\|_{L^2}^2 = \lambda_{1,h}$ . Moreover, since  $u_{1,h} = 0$  on  $\partial\Omega$ , it holds  $\nabla_T(e^{-\frac{1}{h}f} u_{1,h}) = 0$  on  $\partial\Omega$  (where  $\nabla_T$  is the tangential gradient on  $\partial\Omega$ ), the Gaffney inequality (see [46, Corollary 2.1.6]) implies that there exist  $C > 0$  independent of  $h$  such that,

$$\begin{aligned} C \|e^{-\frac{1}{h}f} \nabla u_{1,h}\|_{H^1(\Omega)} &\leq \|e^{-\frac{1}{h}f} \nabla u_{1,h}\|_{L^2(\Omega)} + \|\operatorname{div}(e^{-\frac{1}{h}f} \nabla u_{1,h})\|_{L^2(\Omega)} \\ &+ \sum_{i,j=1}^d \|\partial_i(e^{-\frac{1}{h}f} \partial_j u_{1,h}) - \partial_j(e^{-\frac{1}{h}f} \partial_i u_{1,h})\|_{L^2(\Omega)}. \end{aligned}$$

Therefore, there exist  $C' > 0$  such that for all  $h > 0$ ,

$$\begin{aligned} \|e^{-\frac{1}{h}f} \nabla u_{1,h}\|_{H^1(\Omega)} &\leq \left( C' + \frac{C'}{h} \right) \|e^{-\frac{1}{h}f} \nabla u_{1,h}\|_{L^2(\Omega)} + C' \|e^{-\frac{1}{h}f} \Delta u_{1,h}\|_{L^2(\Omega)} \\ &\leq \frac{C' \sqrt{\lambda_{1,h}}}{h^{3/2}} + C' \|e^{-\frac{1}{h}f} \Delta u_{1,h}\|_{L^2(\Omega)}. \end{aligned}$$

It remains to estimate the term  $\|e^{-\frac{1}{h}f} \Delta u_{1,h}\|_{L^2}$ . Multiplying  $\mathcal{L}_{f,h} u_{1,h} = \lambda_{1,h} u_{1,h}$  by  $e^{-\frac{1}{h}f}$  and using (38), one obtains that there exists  $C > 0$  such that for  $h$  small enough,

$$\|e^{-\frac{1}{h}f} \Delta u_{1,h}\|_{L^2(\Omega)} \leq \frac{2}{h} \left( \|\nabla f\|_{L^2(\Omega)} \|e^{-\frac{1}{h}f} \nabla u_{1,h}\|_{L^2(\Omega)} + \lambda_{1,h} \right) \leq \frac{C}{h} \left( \frac{\sqrt{\lambda_{1,h}}}{\sqrt{h}} + \lambda_{1,h} \right) \leq C \frac{\sqrt{\lambda_{1,h}}}{h^{\frac{3}{2}}},$$

where we used the fact that  $\lim_{h \rightarrow 0} \lambda_{1,h} = 0$  (which follows from (42)). This concludes the proof of (49). Using (48) together with (49) and (42), one gets (47).

**Step 4:** Let  $\mathbf{K}$  be compact subset of  $\{f < \min_{\partial\Omega} f\}$ . To extend (47) to the case when  $x \in \mathbf{K}$ , let us prove that for any  $F \in C^\infty(\partial\Omega, \mathbb{R})$ , there exists  $c > 0$  such that for  $h$  small enough,

$$\max_{(x,y) \in \mathbf{K} \times \mathbf{K}} \left| \mathbb{E}_x [F(X_{\tau_{\Omega^c}})] - \mathbb{E}_y [F(X_{\tau_{\Omega^c}})] \right| \leq e^{-\frac{c}{h}}. \quad (50)$$

The inequality (50) is a leveling result on  $x \in \Omega \mapsto \mathbb{E}_x [F(X_{\tau_{\Omega^c}})]$ . Let us denote by  $v_h \in H^1(\Omega)$  the unique weak solution to the elliptic boundary value problem

$$\frac{h}{2} \Delta v_h - \nabla f \cdot \nabla v_h = 0 \text{ in } \Omega \quad \text{and} \quad v_h = F \text{ on } \partial\Omega,$$

which actually belongs to  $C^\infty(\overline{\Omega}, \mathbb{R})$ . Moreover, the Dynkin's formula implies that  $\forall x \in \overline{\Omega}$ ,  $v_h(x) = \mathbb{E}_x [F(X_{\tau_{\Omega^c}})]$ . Notice that for all  $h > 0$ ,  $\|v_h\|_{L^\infty(\Omega)} \leq \|F\|_{L^\infty(\partial\Omega)}$ . With the same arguments as those used to prove (25), for  $\alpha > 0$  small enough there exist  $h_0 > 0$  and  $c > 0$  such that for all  $h \in (0, h_0)$ ,

$$\|\nabla v_h\|_{L^\infty(C_\alpha)} \leq e^{-\frac{c}{h}}.$$

Then, since for  $\alpha > 0$  small enough, the open set  $C_\alpha$  is path-connected (because it is locally path-connected and connected), this proves (50).

**Step 5:** Proof of (37).

**Step 5.1:** Let  $K$  be a compact subset of  $\{f < \min_{\partial\Omega} f\}$ . Let us prove that for any  $F \in C^\infty(\partial\Omega, \mathbb{R})$ , there exists  $c > 0$  such that for  $h$  small enough,

$$\max_{y \in K} |\mathbb{E}_y [F(X_{\tau_{\Omega^c}})] - \mathbb{E}_{\nu_h} [F(X_{\tau_{\Omega^c}})]| \leq e^{-\frac{c}{h}}. \quad (51)$$

From (39), it holds for  $\alpha > 0$  small enough:

$$\mathbb{E}_{\nu_h} [F(X_{\tau_{\Omega^c}})] = \frac{\int_{\Omega} v_h u_{1,h} e^{-\frac{2}{h}f}}{\int_{\Omega} u_h e^{-\frac{2}{h}f}} = \frac{1}{Z_h} \int_{C_\alpha} v_h u_{1,h} e^{-\frac{2}{h}f} + \frac{1}{Z_h} \int_{\Omega \setminus \overline{C_\alpha}} v_h u_{1,h} e^{-\frac{2}{h}f}, \quad (52)$$

where  $Z_h := \int_{\Omega} u_{1,h} e^{-\frac{2}{h}f}$ . Let us first deal with the second term in (52). From (46), there exists  $C > 0$  such that for  $h$  small enough:  $Z_h^{-1} \leq Ch^{-\frac{d}{2}} e^{\frac{1}{h} \min_{\overline{\Omega}} f}$ . For all  $\alpha > 0$  small enough, one has  $(\overline{\Omega} \setminus C_\alpha) \cap \arg \min_{\overline{\Omega}} f = \emptyset$ . Therefore, using in addition (38), for all  $\alpha > 0$  small enough, there exists  $c > 0$  such that when  $h \rightarrow 0$ :

$$\int_{\Omega \setminus \overline{C_\alpha}} u_{1,h} e^{-\frac{2}{h}f} \leq \sqrt{\int_{\Omega \setminus \overline{C_\alpha}} e^{-\frac{2}{h}f}} \leq e^{-\frac{1}{h}(\min_{\overline{\Omega}} f + c)}.$$

Then, since  $\|v_h\|_{L^\infty(\Omega)} \leq \|F\|_{L^\infty(\partial\Omega)}$ , one obtains that

$$\frac{1}{Z_h} \int_{\Omega \setminus \overline{C_\alpha}} v_h u_{1,h} e^{-\frac{2}{h}f} = O(e^{-\frac{c}{h}}). \quad (53)$$

Let us now deal with the first term in (52). Let  $\alpha > 0$  be small enough. Then, from (50), there exists  $\delta_\alpha > 0$  such that for all  $y \in \overline{C_\alpha}$ ,

$$\frac{1}{Z_h} \int_{\overline{C_\alpha}} v_h u_{1,h} e^{-\frac{2}{h}f} = \frac{v_h(y)}{Z_h} \int_{\overline{C_\alpha}} u_{1,h} e^{-\frac{2}{h}f} + \frac{O(e^{-\frac{\delta_\alpha}{h}})}{Z_h} \int_{\overline{C_\alpha}} u_{1,h} e^{-\frac{2}{h}f} \quad (54)$$

in the limit  $h \rightarrow 0$  and uniformly with respect to  $y \in \overline{C_\alpha}$ . Moreover, for all  $\alpha > 0$  small enough, there exists  $c > 0$  such that in the limit  $h \rightarrow 0$ :

$$\frac{1}{Z_h} \int_{\overline{C_\alpha}} u_{1,h} e^{-\frac{2}{h}f} = 1 + O(e^{-\frac{c}{h}}). \quad (55)$$

This indeed follows from the fact that

$$\frac{1}{Z_h} \int_{\overline{C_\alpha}} u_{1,h} e^{-\frac{2}{h}f} = 1 - \frac{1}{Z_h} \int_{\Omega \setminus \overline{C_\alpha}} u_{1,h} e^{-\frac{2}{h}f},$$

together with (53). Let us now fix  $\alpha > 0$  sufficiently small. Then, using (54) and (55), there exist  $c > 0$ ,  $\delta_\alpha > 0$  such that for all  $y \in \overline{C_\alpha}$ :

$$\frac{1}{Z_h} \int_{\overline{C_\alpha}} v_h u_{1,h} e^{-\frac{2}{h}f} = v_h(y)(1 + O(e^{-\frac{c}{h}})) + O(e^{-\frac{\delta_\alpha}{h}}) \quad (56)$$

in the limit  $h \rightarrow 0$  and uniformly with respect to  $y \in \overline{C_\alpha}$ . Therefore, using (52), (53) and (56), there exists  $\alpha_0 > 0$ , such that for all  $\alpha \in (0, \alpha_0)$ , there exists  $c > 0$  such that for  $h$  small enough:

$$\mathbb{E}_{\nu_h}[F(X_{\tau_{\Omega^c}})] = \mathbb{E}_y[F(X_{\tau_{\Omega^c}})] + O(e^{-\frac{c}{h}}), \text{ uniformly with respect to } y \in \overline{C_\alpha}.$$

This proves (51) since any compact  $K$  of  $\{f < \min_{\partial\Omega} f\}$  is included in  $C_\alpha$  for some  $\alpha > 0$ .

**Step 5.2:** End of the proof of (37). To this end, we will use (47) and (51). Let us now consider  $\Gamma$  be an open subset of  $\partial\Omega$  such that  $\min_{\overline{\Gamma}} f > \min_{\partial\Omega} f$ . Let us mention that (51) holds for smooth functions  $F$  on  $\partial\Omega$  and thus we cannot directly consider  $F = 1_\Gamma$ . Let us consider an open set  $\Sigma \subset \partial\Omega$  such that  $\overline{\Gamma} \subset \Sigma$  and  $\min_{\overline{\Sigma}} f > \min_{\partial\Omega} f$ . Let  $F$  be a  $C^\infty$  function  $F$  on  $\partial\Omega$  with values in  $[0, 1]$  such that  $F = 1$  on  $\Gamma$  and  $F = 0$  on  $\partial\Omega \setminus \Sigma$ . For all  $y \in K$  (where  $K$  is a compact subset of  $\{f < \min_{\partial\Omega} f\}$ ), it holds from (51):

$$\mathbb{P}_y[X_{\tau_{\Omega^c}} \in \Gamma] \leq \mathbb{E}_y[F(X_{\tau_{\Omega^c}})] = \mathbb{E}_{\nu_h}[F(X_{\tau_{\Omega^c}})] + O(e^{-\frac{c}{h}}).$$

Finally, since  $\mathbb{E}_{\nu_h}[F(X_{\tau_{\Omega^c}})] \leq \mathbb{P}_{\nu_h}[X_{\tau_{\Omega^c}} \in \Sigma]$ , one deduces from (47), that there exists  $c > 0$  such that for  $h$  small enough:

$$\max_{y \in K} \mathbb{P}_y[X_{\tau_{\Omega^c}} \in \Gamma] \leq e^{-\frac{c}{h}}.$$

This proves (37) when  $K \subset \{f < \min_{\partial\Omega} f\}$ . The case when  $K$  is a compact subset of  $\mathcal{A}_\Omega(\{f < \min_{\partial\Omega} f\})$  is treated with the same arguments as those used in the fourth step in the proof of Proposition 5 (with the large deviations estimate (32)). This concludes the proof of (37) and the proof of Proposition 7.  $\blacksquare$

## 2.6 Leveling result of $p_{x_*}$ outside $C(x_*)$

In this section we prove the following leveling property on  $p_{x_*}$  in another sublevel set of  $f$  in  $\Omega$ .

**Proposition 8.** *Let  $x_{*,1}$  and  $x_{*,2}$  be two local minima of  $f$  in  $\Omega$  such that  $C(x_{*,1}) \neq C(x_{*,2})$  (see (6)). Assume moreover that  $\overline{C(x_{*,2})}$  is an isolated connected component of  $\{f \leq \lambda(x_{*,2})\}$  which intersects  $\partial\Omega$ . Let  $K$  be a compact set of  $\mathcal{A}_\Omega(C(x_{*,2}))$ . Then,*

$$\text{there exist } h_0 > 0 \text{ and } c > 0 \text{ s.t. for all } h \in (0, h_0): \max_{y \in K} p_{x_{*,1}}(y) \leq e^{-\frac{c}{h}}.$$

**Remark 9.** *Let us mention that Proposition 8 is not a consequence of [6, Corollary 4.8] for the following two reasons. On the one hand, [6, Corollary 4.8] does not directly apply in our setting because the saddle points connecting  $B_{r_h}(x_{*,1})$  and  $\Omega^c$  belongs to  $\Omega^c$ . On the other hand, even if the inequality in [6, Corollary 4.8] holds, it is possible that the height of the saddle points connecting  $x \in C(x_{*,2})$  and  $B_{r_h}(x_{*,1})$  in  $\mathbb{R}^d$  is equal to the height of the those connecting  $x$  and  $\Omega^c$ .*

*Proof.* To prove Proposition 8, the strategy consists in using Proposition 7 with a suitable domain  $\Omega_{x_{*,2}} \subset \Omega$  such that:

- (i)  $\Omega_{x_{*,2}}$  is a  $C^\infty$  connected open subset of  $\Omega$  containing  $C(x_{*,2})$ .
- (ii)  $\partial\Omega_{x_{*,2}} \cap \partial\Omega$  is a neighborhood of  $\partial C(x_{*,2}) \cap \partial\Omega$  in  $\partial\Omega$ .
- (iii)  $\operatorname{argmin}_{\partial\Omega_{x_{*,2}}} f = \partial C(x_{*,2}) \cap \partial\Omega$ .

(iv)  $\{f < \min_{\partial\Omega_{x_*,2}} f\} \cap \overline{\Omega_{x_*,2}} = \mathbf{C}(x_{*,2})$  and  $\mathbf{C}(x_{*,2})$  contains all the local minima  $x$  of  $f$  in  $\Omega_{x_*,2}$ .

(v)  $\{f \leq \min_{\partial\Omega_{x_*,2}} f\} \cap \overline{\Omega_{x_*,2}} = \overline{\mathbf{C}(x_{*,2})}$ .

Notice that item (iii) above implies that

$$\min_{\partial\Omega_{x_*,2}} f = \lambda(x_{*,2}) \text{ (see (6)).}$$

For  $\delta > 0$ , let  $\mathbf{V}_\delta(x_{*,2}) = \{x \in \overline{\Omega}, d(x, \overline{\mathbf{C}(x_{*,2})}) < \delta\}^3$  be the  $\delta$ -neighborhood of  $\overline{\mathbf{C}(x_{*,2})}$  in  $\overline{\Omega}$ . The set  $\mathbf{V}_\delta(x_{*,2})$  is an open neighborhood of  $\overline{\mathbf{C}(x_{*,2})}$  in  $\overline{\Omega}$ . Because  $\overline{\mathbf{C}(x_{*,2})}$  is an isolated connected component of  $\{f \leq \lambda(x_{*,2})\}$  (by assumption), for  $\delta > 0$  small enough,

$$\{f \leq \lambda(x_{*,2})\} \cap \mathbf{V}_\delta(x_{*,2}) = \overline{\mathbf{C}(x_{*,2})}. \quad (57)$$

Notice that this implies in particular that

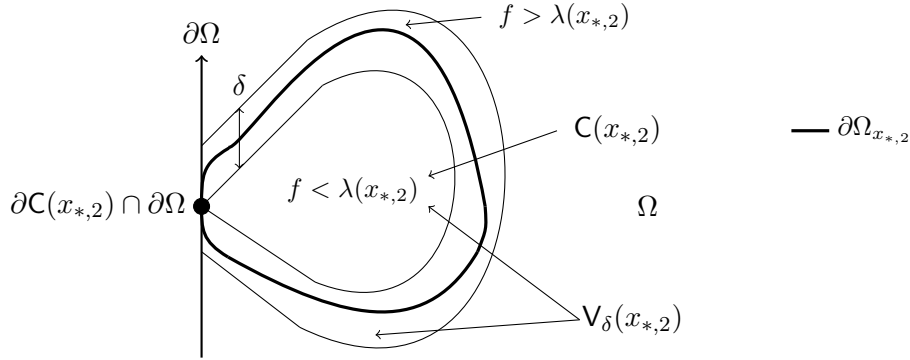


Figure 3: The sets  $\mathbf{V}_\delta(x_{*,2})$  and  $\Omega_{x_{*,2}}$ . On the figure  $\partial\mathbf{C}(x_{*,2}) \cap \partial\Omega$  is reduced to a single point.

$$\{f < \lambda(x_{*,2})\} \cap \mathbf{V}_\delta(x_{*,2}) = \mathbf{C}(x_{*,2}). \quad (58)$$

Moreover, since  $\mathbf{V}_\delta(x_{*,2}) \cap \partial\Omega$  is an open subset of  $\partial\Omega$  (because  $\mathbf{V}_\delta(x_{*,2})$  is an open subset of  $\overline{\Omega}$ ) and since  $\partial\mathbf{C}(x_{*,2}) \cap \partial\Omega \subset \mathbf{V}_\delta(x_{*,2}) \cap \partial\Omega$  (because  $\overline{\mathbf{C}(x_{*,2})} \subset \mathbf{V}_\delta(x_{*,2})$ ), it holds:

$$\mathbf{V}_\delta(x_{*,2}) \cap \partial\Omega \text{ is a neighborhood of } \partial\mathbf{C}(x_{*,2}) \cap \partial\Omega \text{ in } \partial\Omega. \quad (59)$$

Recall that  $f$  has a finite number of critical points in  $\overline{\Omega}$ . In addition,  $\partial\mathbf{C}(x_{*,2})$  does not contain local minimum point of  $f$  in  $\Omega$ . Indeed, let  $z$  be a local minimum point of  $f$  in  $\Omega$ . Then  $\nabla f(z) = 0$  and  $\text{Hess } f(z) > 0$ . Consequently, for some  $r > 0$ ,  $f(x) > f(z)$  for  $x \in \mathbf{B}_r(z) \setminus \{z\}$ , and thus,  $z \notin \overline{\{f < f(z)\}}$ . Therefore, one can assume, up to choosing  $\delta > 0$  smaller, that

$$\overline{\mathbf{V}_\delta(x_{*,2})} \setminus \overline{\mathbf{C}(x_{*,2})} \text{ does not contain local minimum point of } f \text{ in } \Omega.$$

<sup>3</sup>Where  $d(x, \overline{\mathbf{C}(x_{*,2})}) = \inf\{|y - x|, y \in \overline{\mathbf{C}(x_{*,2})}\}$  is the distance of  $x \in \overline{\Omega}$  to  $\overline{\mathbf{C}(x_{*,2})}$ .

See Figure 3 for a schematic representation of  $V_\delta(x_{*,2})$ . At this stage, a good candidate to be  $\Omega_{x_{*,2}}$  is intuitively  $\Omega \cap V_\delta(x_{*,2})$  (up to considering  $\delta > 0$  smaller such that (57) holds with  $\overline{V_\delta(x_{*,2})}$  instead of  $V_\delta(x_{*,2})$ ), but this set is not smooth. However, since  $V_\delta(x_{*,2})$  is a neighborhood of  $\overline{C(x_{*,2})}$  in  $\overline{\Omega}$  and in view of (59), by increasing  $\overline{C(x_{*,2})}$  in  $V_\delta(x_{*,2})$ , one can construct a  $C^\infty$  connected open set  $\Omega_{x_{*,2}}$  of  $\Omega$  such that

- a.  $C(x_{*,2}) \subset \Omega_{x_{*,2}}, \overline{\Omega_{x_{*,2}}} \subset V_\delta(x_{*,2})$ ,
- b.  $\partial\Omega_{x_{*,2}} \cap \partial\Omega$  is a neighborhood of  $\partial C(x_{*,2}) \cap \partial\Omega$  in  $\partial\Omega$ ,
- c.  $\overline{\partial\Omega_{x_{*,2}} \cap \Omega} \subset V_\delta(x_{*,2}) \setminus \overline{C(x_{*,2})}$ .

A schematic representation of  $\Omega_{x_{*,2}}$  is given in Figure 3. Let us now give some direct properties of  $\Omega_{x_{*,2}}$  which will imply that it satisfies (i)-(v) above. First of all, it holds  $f \geq \lambda(x_{*,2})$  on  $\partial\Omega_{x_{*,2}} \cap \partial\Omega$ . Indeed,  $\{f < \lambda(x_{*,2})\} \cap \partial\Omega_{x_{*,2}} \cap \partial\Omega \subset \{f < \lambda(x_{*,2})\} \cap V_\delta(x_{*,2}) \cap \partial\Omega = C(x_{*,2}) \cap \partial\Omega = \emptyset$ , where we have used (58) and the fact that  $C(x_{*,2}) \subset \Omega$  (see the line just after (7)). Similarly  $\{f \leq \lambda(x_{*,2})\} \cap \partial\Omega_{x_{*,2}} \cap \partial\Omega \subset \{f \leq \lambda(x_{*,2})\} \cap V_\delta(x_{*,2}) \cap \partial\Omega = \overline{C(x_{*,2})} \cap \partial\Omega = \partial C(x_{*,2}) \cap \partial\Omega$ . Thus, one has:

$$\arg \min_{\partial\Omega_{x_{*,2}} \cap \partial\Omega} f = \partial C(x_{*,2}) \cap \partial\Omega \subset \{f = \lambda(x_{*,2})\}.$$

In addition, by (57), it holds:

$$f > \lambda(x_{*,2}) \text{ on the tube } V_\delta(x_{*,2}) \setminus \overline{C(x_{*,2})} \text{ of width } \delta \text{ around } \overline{C(x_{*,2})} \text{ in } \overline{\Omega}.$$

Using also that  $\overline{\partial\Omega_{x_{*,2}} \cap \Omega} \subset V_\delta(x_{*,2}) \setminus \overline{C(x_{*,2})}$ , it then holds:

$$\text{for some } \beta > 0, \quad f \geq \lambda(x_{*,2}) + \beta \text{ on } \overline{\partial\Omega_{x_{*,2}} \cap \Omega}.$$

It is then easy to check that  $\Omega_{x_{*,2}}$  satisfy items (i)-(v) above. See Figure 4 for a schematic representation of  $\Omega$  with  $\Omega_{x_{*,2}}, x_{*,2}, x_{*,1}, C(x_{*,2})$ , and  $C(x_{*,1})$ .

Let us now end the proof of Proposition 8. Since for  $h$  small enough,  $B_{r_h}(x_{*,1}) \cap \overline{\Omega_{x_{*,2}}} = \emptyset$ , using the continuity of the trajectories of the process (1), it holds for all  $x \in C(x_{*,2})$  and  $h$  small enough:

$$\{\tau_{B_{r_h}(x_{*,1})} < \tau_{\Omega^c}\} \subset \{X_{\tau_{\Omega_{x_{*,2}}^c}} \in \partial\Omega_{x_{*,2}} \setminus \partial\Omega\}.$$

The domain  $\Omega_{x_{*,2}}$  and  $f : \overline{\Omega_{x_{*,2}}} \rightarrow \mathbb{R}$  satisfies all the required assumptions of Proposition 7. Moreover, since  $\inf_{\partial\Omega_{x_{*,2}} \setminus \partial\Omega} f > \min_{\Omega_{x_{*,2}}} f = \lambda(x_{*,2})$  (see items (ii) and (iii) above), from Proposition 7, for all compact subset  $K$  of  $C(x_{*,2}) = \{v \in \overline{\Omega_{x_{*,2}}}, f(v) < \lambda(x_{*,2})\}$  (see item (iv) above), there exists  $c > 0$  such that for  $h$  small enough (see (37)):

$$\max_{y \in K} \mathbb{P}_y[X_{\tau_{\Omega_{x_{*,2}}^c}} \in \partial\Omega_{x_{*,2}} \setminus \partial\Omega] \leq e^{-\frac{c}{h}}. \quad (60)$$

Recall that by definition (see (10))  $\mathbf{p}_{x_{*,1}}(y) = \mathbb{P}_{x_{*,1}}[\tau_{B_{r_h}(x_{*,1})} < \tau_{\Omega^c}]$ , and thus, there exists  $c > 0$  such that for  $h$  small enough:  $\max_{y \in K} \mathbf{p}_{x_{*,1}}(y) \leq e^{-\frac{c}{h}}$ . The proof of Proposition 8 is complete when  $K \subset C(x_{*,2})$ . The proof of Proposition 8 for  $K \subset \mathcal{A}_\Omega(C(x_{*,2}))$  is obtained using the same procedure as the one used in the fourth step in the proof of Proposition 5.  $\blacksquare$

A direct consequence of the proof of Proposition 8 is the following generalization of Proposition 7.



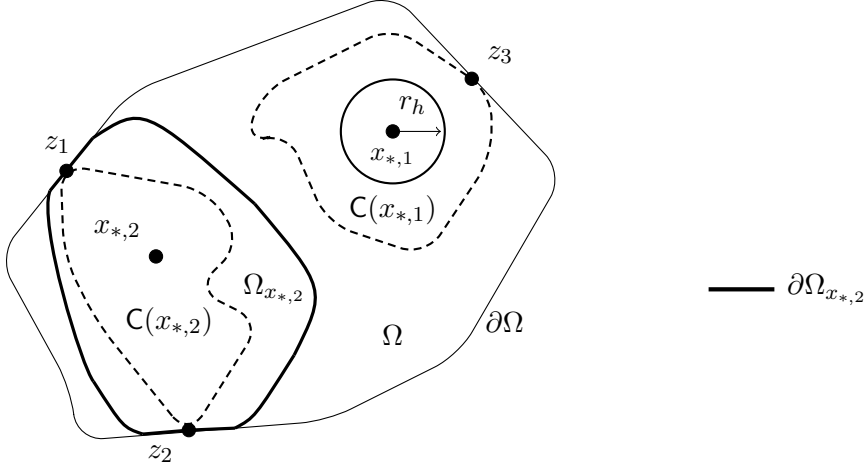


Figure 4: Schematic representation of  $\Omega_{x_{*,2}}$  satisfying items (i)-(v). On the figure,  $\partial\mathbb{C}(x_{*,2}) \cap \partial\Omega = \{z_1, z_2\}$  (see (7)).

**Corollary 10.** *Let  $x_*$  be a local minimum point of  $f$  in  $\Omega$ . Assume that  $\overline{\mathbb{C}(x_*)}$  (see (7)) is an isolated connected component of  $\{f \leq \lambda(x_*)\}$  which intersects  $\partial\Omega$ . Let  $\mathbb{K}$  be compact subset of  $\mathcal{A}_\Omega(\mathbb{C}(x_*))$  (see (5)) and  $\Gamma$  be an open subset of  $\partial\Omega$ . Then, if  $\overline{\Gamma} \cap (\partial\mathbb{C}(x_*) \cap \partial\Omega) = \emptyset$ ,*

*there exist  $h_0 > 0$  and  $c > 0$  s.t. for all  $h \in (0, h_0)$ :  $\max_{x \in \mathbb{K}} \mathbb{P}_x [X_{\tau_{\Omega^c}} \in \Gamma] \leq e^{-\frac{c}{h}}$ .*

*Proof.* Set  $x_* = x_{*,2}$  so that we can use the notation of the proof of Proposition 8. Since  $\overline{\Gamma} \cap (\partial\mathbb{C}(x_{*,2}) \cap \partial\Omega) = \emptyset$ , one can assume, up to choosing  $V(x_{*,2})$  smaller, that  $\overline{\Gamma} \cap V(x_{*,2}) = \emptyset$  (and therefore,  $\overline{\Gamma} \cap \overline{\Omega_{x_{*,2}}} = \emptyset$ ). Thus, it holds for all  $y \in \mathbb{C}(x_{*,2})$ , when  $X_0 = y$ ,  $\{X_{\tau_{\Omega^c}} \in \Gamma\} \subset \{X_{\tau_{\Omega_{x_{*,2}}^c}} \in \partial\Omega_{x_{*,2}} \setminus \partial\Omega\}$ . Together with (60), this concludes the proof of Corollary 10 when  $y \in \mathbb{K} \subset \mathbb{C}(x_{*,2})$ . To get it for  $\mathbb{K} \subset \mathcal{A}_\Omega(\mathbb{C}(x_{*,2}))$ , one uses the procedure of the the fourth step in the proof of Proposition 5.  $\blacksquare$

Notice that assumptions of Corollary 10 are satisfied when **(H1)** holds (see indeed the proof of Lemma 1). Corollary 10 implies that the process (1) exits  $\Omega$  almost surely through any neighborhood of  $\partial\mathbb{C}(x_*) \cap \partial\Omega$  in the limit  $h \rightarrow 0$  when  $X_0 = x \in \mathcal{A}_\Omega(\mathbb{C}(x))$ .

## 2.7 Sharp asymptotic estimate of the capacities

In this section, one proves an asymptotic equivalent of  $\text{cap}_{\mathbb{B}_{r_h}(x_*)}(\Omega^c)$  in the limit  $h \rightarrow 0$ . We start with the following lemma.

**Lemma 11.** *Consider a local minimum point  $x_*$  of  $f$  in  $\Omega$ . Let us assume that  $\partial\mathbb{C}(x_*) \cap \partial\Omega \neq \emptyset$  (see (7)),  $\overline{\mathbb{C}(x_*)}$  is an isolated connected component of  $\{f \leq \lambda(x_*)\}$ , and that all  $z \in \partial\mathbb{C}(x_*) \cap \partial\Omega$  satisfy item a or item b in **(H2)**. Then, there exists a smooth function  $\phi_{x_*} \in H_0^1(\Omega, e^{-\frac{2}{h}f} dx)$*

with values in  $[0, 1]$  which equals 1 in a neighborhood of  $x_*$  in  $\Omega$ , such that in the limit  $h \rightarrow 0$ :

$$\begin{aligned} \frac{h}{2} \int_{\Omega} |\nabla \phi_{x_*}|^2 e^{-\frac{2}{h}f} &= \sum_{z \in \partial\mathbf{C}(x_*) \cap \partial\Omega, \nabla f(z) \neq 0} \frac{(h\pi)^{\frac{d-1}{2}} \partial_n f(z)}{(\det \text{Hess} f|_{\partial\Omega}(z))^{\frac{1}{2}}} e^{-\frac{2}{h}\lambda(x_*)} (1 + O(h)) \\ &+ \sum_{z \in \partial\mathbf{C}(x_*) \cap \partial\Omega, \nabla f(z) = 0} \frac{(h\pi)^{\frac{d}{2}} |\mu_z|}{\pi |\det \text{Hess} f(z)|^{\frac{1}{2}}} e^{-\frac{2}{h}\lambda(x_*)} (1 + O(\sqrt{h})). \end{aligned}$$

*Proof.* Because  $\mathbf{C}(x_*) \in \mathcal{C}$  and all the  $z \in \partial\mathbf{C}(x_*) \cap \partial\Omega$  satisfy item *a* or item *b* in **(H2)**, we can use the results of [29]. First of all, consider the function  $\phi_{x_*} \in H_0^1(\Omega, e^{-\frac{2}{h}f} dx)$  with values in  $[0, 1]$  introduced in [29, Definition 22] which is associated to  $x_*$ . Then set (see [29, Definition 23])

$$\psi_{x_*} = \phi_{x_*} e^{-\frac{f}{h}} / Z_x \text{ where } Z_x^2 := \int_{\Omega} \phi_{x_*}^2 e^{-\frac{2}{h}f}.$$

As  $h \rightarrow 0$ , it holds (see at the beginning of the proof of [29, Proposition 24] and the fact that  $\mathbf{C}_{\mathbf{j}}(x_*) = \mathbf{C}(x_*)$ ):

$$Z_x^2 = (\pi h)^{d/2} e^{-\frac{2}{h}f(x_*)} \sum_{q \in \text{argmin}_{\mathbf{C}(x_*)} f} (\det \text{Hess } f(q))^{-1/2} (1 + O(h)).$$

Define  $\mathbf{S}_* = \{z \in \partial\mathbf{C}(x_*), z \text{ is a separating saddle point of } f \text{ in } \Omega\}$ . Let us recall that by the construction of the map  $\mathbf{j}$  in [29, Section 3.3],

$$\mathbf{j}(x_*) = \mathbf{S}_* \cup (\partial\mathbf{C}(x_*) \cap \partial\Omega) \text{ and } \mathbf{S}_* = \mathbf{j}(x_*) \cap \Omega \subset \{x \in \Omega, \nabla f(x) = 0\}.$$

By item 1 in [29, Proposition 26] and since  $f(\mathbf{j}(x_*)) = \lambda(x_*)$  there,

$$\int_{\Omega} |d_{f,h} \psi_{x_*}|^2 = [\sqrt{h}(\mathbf{K}_{1,x_*}(1 + O(h)) + h(\mathbf{K}_{2,x_*}(1 + O(h)))] e^{-\frac{2}{h}(\lambda(x_*) - f(x_*))},$$

as  $h \rightarrow 0$  and where

$$\mathbf{K}_{1,x_*} = \sum_{z \in \mathbf{j}(x_*), \nabla f(z) \neq 0} \mathbf{c}_{x_*,z} \text{ and } \mathbf{K}_{2,x_*} = \sum_{z \in \mathbf{j}(x_*), \nabla f(z) = 0} \mathbf{c}_{x_*,z},$$

see [29, Proposition 24] for the definition of the  $\mathbf{c}_{x_*,z}$ 's. Since  $d_{f,h} = h e^{-\frac{1}{h}f} \nabla e^{\frac{1}{h}f}$ , one has:

$$\int_{\Omega} |d_{f,h} \psi_{x_*}|^2 = \frac{h^2}{Z_x^2} \int_{\Omega} |\nabla \phi_{x_*}|^2 e^{-\frac{2}{h}f}.$$

Moreover, because  $\overline{\mathbf{C}(x_*)}$  is a connected component of  $\{f \leq \lambda(x_*)\}$ ,  $\mathbf{S}_* = \emptyset$ . One then deduces that:

$$\mathbf{K}_{2,x_*} = \sum_{z \in \mathbf{j}(x_*) \cap \partial\Omega, \nabla f(z) = 0} \mathbf{c}_{x_*,z}.$$

Using the previous asymptotic formulas and the definition of the  $\mathbf{c}_{x_*,z}$ 's (see [29, item 1 in Proposition 24]), one deduces the asymptotic equivalent on  $\int_{\Omega} |\nabla \phi_{x_*}|^2 e^{-\frac{2}{h}f}$  given in Lemma 11.  $\blacksquare$

The following proposition provides the asymptotic equivalent of  $\text{cap}_{\mathbf{B}_{r_h}(x_*)}(\Omega^c)$  when  $h \rightarrow 0$ .

**Proposition 12.** Consider  $x_*$  a local minimum point of  $f$  in  $\Omega$ . Assume that the assumptions on  $C(x_*)$  stated in Lemma 11 are satisfied. Then, it holds, in the limit  $h \rightarrow 0$ :

$$\begin{aligned} \text{cap}_{\mathbb{B}_{r_h}(x_*)}(\Omega^c) &= \sum_{z \in \partial C(x_*) \cap \partial \Omega, \nabla f(z) \neq 0} \frac{(h\pi)^{\frac{d-1}{2}} \partial_n f(z)}{(\det \text{Hess} f|_{\partial \Omega}(z))^{\frac{1}{2}}} e^{-\frac{2}{h}\lambda(x_*)} (1 + O(h)) \\ &+ \sum_{z \in \partial C(x_*) \cap \partial \Omega, \nabla f(z) = 0} \frac{(h\pi)^{\frac{d}{2}} |\mu_z|}{\pi |\det \text{Hess} f(z)|^{\frac{1}{2}}} e^{-\frac{2}{h}\lambda(x_*)} (1 + O(\sqrt{h})). \end{aligned} \quad (61)$$

*Proof.* From (13), it holds:  $\text{cap}_{\mathbb{B}_{r_h}(x_*)}(\Omega^c) \leq \frac{h}{2} \int_{\Omega \setminus \mathbb{B}_{r_h}(x_*)} |\nabla \phi_{x_*}|^2 e^{-\frac{2}{h}f}$ . Using Lemma 11, this gives the required asymptotic upper bound on  $\text{cap}_{\mathbb{B}_{r_h}(x_*)}(\Omega^c)$ . Let us now obtain a lower bound on  $\text{cap}_{\mathbb{B}_{r_h}(x_*)}(\Omega^c)$ . To this end, we will use (43). Let us consider two smooth subdomains  $\Omega_{x_*}^1$  and  $\Omega_{x_*}^2$  of  $\Omega$ , both satisfying items (i)-(v) listed at the beginning of the proof of Proposition 8, such that  $C(x_*) \subset \Omega_{x_*}^1 \subset \Omega_{x_*}^2$  and  $\overline{\Omega_{x_*}^2}$  is a neighborhood of  $\overline{\Omega_{x_*}^1}$  in  $\overline{\Omega}$ . This implies in particular that

$$\text{there exists } r > 0 \text{ s.t. for all } x \in \Omega_{x_*}^2 \setminus \overline{\Omega_{x_*}^1}, \quad f(x) \geq \lambda(x_*) + r. \quad (62)$$

Let  $\chi \in C^\infty(\overline{\Omega}, [0, 1])$  be such that  $\chi = 1$  on  $\overline{\Omega_{x_*}^1}$  and  $\chi = 0$  on  $\overline{\Omega} \setminus \overline{\Omega_{x_*}^2}$ . From (13), it holds:

$$\text{cap}_{\mathbb{B}_{r_h}(x_*)}(\Omega^c) = \frac{h}{2} \int_{\Omega \setminus \mathbb{B}_{r_h}(x_*)} |\nabla \mathbf{p}_{x_*}|^2 e^{-\frac{2}{h}f} \geq \frac{h}{2} \int_{\Omega_{x_*}^1 \setminus \mathbb{B}_{r_h}(x_*)} |\nabla \mathbf{p}_{x_*}|^2 e^{-\frac{2}{h}f}.$$

Let us now obtain an asymptotic lower bound on  $\frac{h}{2} \int_{\Omega_{x_*}^1 \setminus \mathbb{B}_{r_h}(x_*)} |\nabla \mathbf{p}_{x_*}|^2 e^{-\frac{2}{h}f}$  in the limit  $h \rightarrow 0$ . To this end, let us write:

$$\begin{aligned} \int_{\Omega_{x_*}^2 \setminus \mathbb{B}_{r_h}(x_*)} |\nabla(\chi \mathbf{p}_{x_*})|^2 e^{-\frac{2}{h}f} &= \int_{\Omega_{x_*}^2 \setminus \mathbb{B}_{r_h}(x_*)} (\chi^2 |\nabla \mathbf{p}_{x_*}|^2 + 2\mathbf{p}_{x_*} \chi \nabla \chi \cdot \nabla \mathbf{p}_{x_*} + \mathbf{p}_{x_*}^2 |\nabla \chi|^2) e^{-\frac{2}{h}f} \\ &= \int_{\Omega_{x_*}^1 \setminus \mathbb{B}_{r_h}(x_*)} |\nabla \mathbf{p}_{x_*}|^2 e^{-\frac{2}{h}f} + R_*, \end{aligned}$$

where

$$R_* = \int_{\Omega_{x_*}^2 \setminus \Omega_{x_*}^1} \chi^2 |\nabla \mathbf{p}_{x_*}|^2 e^{-\frac{2}{h}f} + \int_{\Omega_{x_*}^2 \setminus \Omega_{x_*}^1} (2\mathbf{p}_{x_*} \chi \nabla \chi \cdot \nabla \mathbf{p}_{x_*} + \mathbf{p}_{x_*}^2 |\nabla \chi|^2) e^{-\frac{2}{h}f},$$

where we used the fact that  $\nabla \chi = 0$  on  $\overline{\Omega_{x_*}^1}$  and  $\chi = 1$  on  $\overline{\Omega_{x_*}^1}$ . Moreover, from (20), (62), and since  $\mathbf{p}_{x_*} \leq 1$ , there exists  $c > 0$ , such that for  $h$  small enough:  $R_* = O(e^{-\frac{2}{h}(\lambda(x_*)+c)})$ . Now since  $\chi \mathbf{p}_{x_*} \in H_0^1(\Omega_{x_*}^2, e^{-\frac{2}{h}f} dx)$  (because  $\mathbf{p}_{x_*} = 0$  on  $\partial \Omega$  and  $\chi = 0$  on  $\partial \Omega_{x_*}^2 \cap \Omega$ ), using the min-max principle (see (2)):

$$\frac{h}{2} \int_{\Omega_{x_*}^2 \setminus \mathbb{B}_{r_h}(x_*)} |\nabla(\chi \mathbf{p}_{x_*})|^2 e^{-\frac{2}{h}f} = \frac{h}{2} \int_{\Omega_{x_*}^2} |\nabla(\chi \mathbf{p}_{x_*})|^2 e^{-\frac{2}{h}f} \geq \lambda_{1,h} \int_{\Omega_{x_*}^2} |\chi \mathbf{p}_{x_*}|^2 e^{-\frac{2}{h}f},$$

where  $\lambda_{1,h}$  denotes here the principal eigenvalue of the Dirichlet realization of  $\mathcal{L}_{f,h}$  in  $\Omega_{x_*}^2$ . In conclusion, for  $h$  small enough:

$$\text{cap}_{\mathbb{B}_{r_h}(x_*)}(\Omega^c) \geq \lambda_{1,h} \int_{\Omega_{x_*}^2} |\chi \mathbf{p}_{x_*}|^2 e^{-\frac{2}{h}f} + O(e^{-\frac{2}{h}(\lambda(x_*)+c)}). \quad (63)$$

Since  $\chi = 1$  on  $C(x_*)$ , from Proposition 5 and since  $\arg \min_{\overline{C(x_*)}} f = \arg \min_{\overline{\Omega_{x_*}^2}} f$  (see item (iv) at the beginning of the proof of Proposition 8), the Laplace method provides when  $h \rightarrow 0$ :

$$\int_{\Omega_{x_*}^2} |\chi \mathbf{p}_{x_*}|^2 e^{-\frac{2}{h}f} = (h\pi)^{\frac{d}{2}} e^{-\frac{2}{h} \min_{\overline{C(x_*)}} f} \sum_{y \in \arg \min_{\overline{C(x_*)}} f} (\det \text{Hess} f(y))^{-\frac{1}{2}} (1 + O(h)). \quad (64)$$

Furthermore, the domain  $\Omega_{x_*}^2$  and  $f : \overline{\Omega_{x_*}^2} \rightarrow \mathbb{R}$  satisfy all the required assumptions of Proposition 7 (see indeed items (i)-(iv) at the beginning of the proof of Proposition 8) with:

$$\min_{\partial\Omega_{x_*}^2} f = \lambda(x_*) \text{ and } \{v \in \overline{\Omega_{x_*}^2}, f(v) < \min_{\partial\Omega_{x_*}^2} f\} = \mathbf{C}(x_*).$$

Since **(H2)** holds for  $f : \overline{\Omega_{x_*}^2} \rightarrow \mathbb{R}$ , the asymptotic equivalent of  $\lambda_{1,h}$  when  $h \rightarrow 0$  given in (43) (with  $\Omega = \Omega_{x_*}^2$  there) holds. Together with (63) and (64), this provides the expected lower bounds on  $\text{cap}_{\mathbb{B}_{r_h}(x_*)}(\Omega^c)$ . The proof of Proposition 12 is complete.  $\blacksquare$

### 3 Proof of Theorems 1 and 2

In all this section, we assume that the function  $f : \overline{\Omega} \rightarrow \mathbb{R}$  satisfies **(H1)**. Then, the results of the previous section can be used with  $\mathbf{C}(x_*) = \mathbf{C}_j$ , where  $x_*$  is a local minimum point of  $f$  in  $\Omega$  such that  $x_* \in \mathbf{C}_j$ ,  $j \in \{1, \dots, \mathbf{N}\}$  (in this case  $\lambda(x_*) = \min_{\partial\Omega} f$ ), see Lemma 1 and its proof.

#### 3.1 Mean value of the committor functions

The following proposition gives sharp asymptotic equivalents of  $\int_{\Omega} \mathbf{p}_{x_j} e^{-\frac{2}{h}f}$  for  $j \in \{1, \dots, \mathbf{N}\}$ , where  $x_j$  is a local minimum point of  $f$  in  $\mathbf{C}_j$ .

**Proposition 13.** *Let  $f : \overline{\Omega} \rightarrow \mathbb{R}$  be such that the assumption **(H1)** holds. Let  $x_j$  be a local minimum point of  $f$  in  $\mathbf{C}_j$ , for some  $j \in \{1, \dots, \mathbf{N}\}$ , and  $k \in \{1, 2\}$ . If  $j \in \{1, \dots, \mathbf{N}_0\}$ , it holds when  $h \rightarrow 0$ :*

$$\int_{\Omega} \mathbf{p}_{x_j}^k e^{-\frac{2}{h}f} = (h\pi)^{\frac{d}{2}} e^{-\frac{2}{h} \min_{\overline{\Omega}} f} \sum_{y \in \arg \min_{\overline{\mathbf{C}_j}} f} (\det \text{Hess}f(y))^{-\frac{1}{2}} (1 + O(h)),$$

where we recall that  $\mathbf{p}_{x_j}$  is defined by (10) with  $x_* = x_j$  there. Finally, if  $j \geq \mathbf{N}_0 + 1$ , there exists  $\varepsilon > 0$ , such that if  $\min_{\overline{\mathbf{C}_j}} f - \min_{\overline{\Omega}} f < \varepsilon$ , in the limit  $h \rightarrow 0$ , it holds:

$$\int_{\Omega} \mathbf{p}_{x_j}^k e^{-\frac{2}{h}f} = (h\pi)^{\frac{d}{2}} e^{-\frac{2}{h} \min_{\overline{\mathbf{C}_j}} f} \sum_{y \in \arg \min_{\overline{\mathbf{C}_j}} f} (\det \text{Hess}f(y))^{-\frac{1}{2}} (1 + O(h)).$$

*Proof.* Let  $j \in \{1, \dots, \mathbf{N}\}$  and writes:

$$\int_{\Omega} \mathbf{p}_{x_j}^k e^{-\frac{2}{h}f} = \int_{\mathbf{C}_j} \mathbf{p}_{x_j}^k e^{-\frac{2}{h}f} + \int_{\Omega \setminus \mathbf{C}_j} \mathbf{p}_{x_j}^k e^{-\frac{2}{h}f}.$$

To prove Proposition 13, in view of Proposition 5 (applied with  $x_* = x_j$  and  $\mathbf{C}(x_*) = \mathbf{C}_j$ ), it remains to show that  $\int_{\Omega \setminus \mathbf{C}_j} \mathbf{p}_{x_j}^k e^{-\frac{2}{h}f}$  is an error term for  $h$  small enough. Let us consider a neighborhood  $\mathbf{V} \subset \Omega$  of the points belonging to  $\arg \min_{\overline{\Omega}} f \setminus \mathbf{C}_j$ . Let us moreover choose  $\mathbf{V}$  such that  $\overline{\mathbf{V}} \subset \cup_{\ell \neq j, \ell=1, \dots, \mathbf{N}_0} \mathbf{C}_\ell$  (this possible by the choice of  $\mathbf{N}_0$ ). Since there exists  $\alpha_1 > 0$  such that  $f(x) \geq \min_{\overline{\Omega}} f + \alpha_1$  for all  $x \in \overline{\Omega} \setminus (\mathbf{C}_j \cup \mathbf{V})$ , it holds, using in addition the fact that  $\mathbf{p}_{x_j} \leq 1$ ,

$$\int_{\Omega \setminus (\mathbf{C}_j \cup \mathbf{V})} \mathbf{p}_{x_j}^k e^{-\frac{2}{h}f} \leq C e^{-\frac{2}{h}(\min_{\overline{\Omega}} f + \alpha_1)}, \text{ for some } C > 0 \text{ independent of } h.$$

Since  $\bar{V} \subset \cup_{\ell \neq j, \ell=1, \dots, N_0} C_\ell$ , from Proposition 8 (applied with  $x_{1,*} = x_j$ ,  $x_{2,*} = x_\ell$ , and  $K = \bar{V} \cap C_\ell$ , for  $\ell \in \{1, \dots, N_0\}$ ,  $\ell \neq j$ ), one has:

$$\int_V \mathbf{p}_{x_j}^k e^{-\frac{2}{h}f} \leq C e^{-\frac{2}{h}(\min_{\bar{\Omega}} f + \alpha_2)}, \text{ for some } \alpha_2 > 0 \text{ and } h \text{ small enough.}$$

Recall that  $\min_{\bar{C}_j} f \geq \min_{\bar{\Omega}} f$  for all  $j$ , and that  $\min_{\bar{C}_j} f > \min_{\bar{\Omega}} f$  iff  $j > N_0$ . Therefore, when  $j \leq N_0$ , one deduces the second statement in Proposition 13. When  $j \geq N_0 + 1$ , one sees that if  $\min_{\bar{C}_j} f - \min_{\bar{\Omega}} f < \min(\alpha_1, \alpha_2)$ , ones obtains the last statement in Proposition 13. This concludes the proof of Proposition 13.  $\blacksquare$

Let us assume that **(H1)** and **(H2)** hold. Let  $x_j$  be a local minimum point of  $f$  in  $C_j$ , for some  $j \in \{1, \dots, N\}$ . Using (15) and Proposition 12 together with the first statement in Proposition 13, one deduces that in the limit  $h \rightarrow 0$ , when  $j \in \{1, \dots, N_0\}$ :

$$\mathbb{E}_{x_j}[\tau_{\Omega^c}] = \frac{e^{\frac{2}{h}(\min_{\partial\Omega} f - \min_{\bar{\Omega}} f)}}{\Theta_{j,1} h^{-\frac{1}{2}} + \Theta_{j,2} + O(\sqrt{h})}. \quad (65)$$

Using (15), Proposition 12, and the second statement in Proposition 5 together with the fact that  $\int_{\Omega} \mathbf{p}_{x_j} e^{-\frac{2}{h}f} \geq \int_{C_j} \mathbf{p}_{x_j} e^{-\frac{2}{h}f}$ , one deduces that when  $j \in \{N_0 + 1, \dots, N\}$ , it holds for  $h$  small enough:

$$\mathbb{E}_{x_j}[\tau_{\Omega^c}] \geq \frac{e^{\frac{2}{h}(\min_{\partial\Omega} f - \min_{\bar{C}_j} f)}}{\Theta_{j,1} h^{-\frac{1}{2}} + \Theta_{j,2} + O(\sqrt{h})}. \quad (66)$$

Moreover, according to the last statement in Proposition 13, (15), and Proposition 12, there exists  $\varepsilon > 0$  such that if  $\min_{\bar{C}_j} f - \min_{\bar{\Omega}} f < \varepsilon$  then (66) is actually an equality when  $h \rightarrow 0$ . This proves items 2 and 3 in Theorem 1 when  $X_0 = x_j$ . To prove Theorem 1, it thus remains to deal with the case when  $X_0 = x \in \mathcal{A}_{\Omega}(C_j)$ . For that purpose, we need an upper bound on  $x \in \Omega \mapsto \mathbb{E}_x[\tau_{\Omega^c}]$ . This is the goal of the next section.

### 3.2 Upper bound on the mean exit time

**Proposition 14.** *Let  $f : \bar{\Omega} \rightarrow \mathbb{R}$  be such that the hypotheses **(H1)** and **(H2)** hold. Let  $K$  be a compact subset of  $\Omega$ . Then, for any  $\delta > 0$ , there exists  $h_0 > 0$  such that for all  $h \in (0, h_0)$ :*

$$\max_{x \in K} \mathbb{E}_x[\tau_{\Omega^c}] \leq e^{\frac{2}{h}(\min_{\partial\Omega} f - \min_{\bar{\Omega}} f + \delta)}. \quad (67)$$

Notice that Proposition 14 is item 1 in Theorem 1.

*Proof.* To prove (67), we will use [48, Theorem 1]. To this end, we construct a  $\mathcal{C}^\infty$  bounded subdomain  $D_\Omega$  of  $\mathbb{R}^d$  containing  $\Omega$  such that  $|\nabla f| \neq 0$  on  $\partial D_\Omega$ : this is indeed required to use [48, Theorem 1].

**Step 1:** Construction of the domain  $D_\Omega$  containing  $\Omega$ . The domain  $D_\Omega$  is constructed by slightly "extending"  $\Omega$  near the critical points of  $f$  on  $\partial\Omega$ . This is made as follows. Since  $f : \bar{\Omega} \rightarrow \mathbb{R}$  is a smooth function, it is by definition, the restriction of a smooth function defined on an open neighborhood  $\tilde{\Omega}$  of  $\bar{\Omega}$  in  $\mathbb{R}^d$ . This extension is still denoted by  $f$  in the following. In addition, since  $f : \bar{\Omega} \rightarrow \mathbb{R}$  is a Morse function, the critical points of  $f$  in  $\bar{\Omega}$  are isolated in  $\bar{\Omega}$ . Thus, there exists  $\varepsilon > 0$ , for all  $z \in \partial\Omega$  such that  $\nabla f(z) = 0$ , it holds:

$$|\nabla f| \neq 0 \text{ on } B_\varepsilon(z) \setminus \{z\}.$$

Up to choosing  $\varepsilon > 0$  smaller, the following properties are satisfied: the balls  $B_\varepsilon(z)$  are two by two disjoint,

$$\text{for all } \ell \in \{1, \dots, N\} \text{ and } y \in \partial C_\ell \cap \partial\Omega, \text{ if } \nabla f(y) \neq 0 \text{ then } y \notin B_\varepsilon(z), \quad (68)$$

and there exists a smooth coordinate system  $\Psi : y \in B_\varepsilon(z) \mapsto (x', x_d) \in \mathbb{R}^d$  such that  $\Psi(z) = 0$ ,

$$\Omega \cap B_\varepsilon(z) = \{y \in B_\varepsilon(z), x_d(y) < 0\} \text{ and } \partial\Omega \cap B_\varepsilon(z) = \{y \in B_\varepsilon(z), x_d(y) = 0\}.$$

Notice that (68) can be indeed satisfied because  $f$  has finite number of critical points in  $\partial\Omega$  and for all  $k \in \{1, \dots, N\}$ ,  $\partial C_k \cap \partial\Omega$  contains a finite number of points when **(H1)** and **(H2)** hold. For  $\alpha > 0$ , let us consider a  $C^\infty$  function  $x' \in \mathbb{R}^{d-1} \mapsto \chi(x') \in [0, \alpha]$ , such that:

$$\chi(x') = 0 \text{ for all } x' \in \mathbb{R}^{d-1} \setminus \Psi(\partial\Omega \cap B_{\varepsilon/2}(z)) \text{ and } \chi(x') = \alpha \text{ for all } x' \in \Psi(\partial\Omega \cap B_{\varepsilon/4}(z)).$$

Moreover, one chooses  $\alpha > 0$  small enough such that

$$\{(x', \chi(x')), x' \in \Psi(\Omega \cap B_\varepsilon(z))\} \subset \Psi(B_\varepsilon(z)). \quad (69)$$

A schematic representation of  $\{(x', x_d) \in \Psi(B_\varepsilon(z)), \text{ such that } x_d < \chi(x')\}$  is given in Figure 5. Let us define,

$$O_z := \Psi^{-1}\left(\{(x', x_d) \in \Psi(B_\varepsilon(z)), \text{ such that } x_d < \chi(x')\}\right).$$

For ease of notation, we omitted to write explicitly the dependence on  $z$  for  $\Psi$ ,  $\chi$ , and  $\alpha$ . The bounded subset  $D_\Omega$  of  $\mathbb{R}^d$  is then defined by:

$$D_\Omega := \Omega \cup \bigcup_{z \in \partial\Omega, \nabla f(z)=0} O_z.$$

From the definition of  $\chi$  it holds:

$$\partial D_\Omega \setminus \bigcup_{z \in \partial\Omega, \nabla f(z)=0} B_{\varepsilon/2}(z) = \partial\Omega \setminus \bigcup_{z \in \partial\Omega, \nabla f(z)=0} B_{\varepsilon/2}(z). \quad (70)$$

From (69), for all critical point  $z$  of  $f$  on  $\partial\Omega$  (see Figure 5), it holds:

$$\partial D_\Omega \cap B_\varepsilon(z) = \Psi^{-1}\left(\{(x', x_d) \in \Psi(B_\varepsilon(z)), \text{ such that } x_d = \chi(x')\}\right). \quad (71)$$

The set  $D_\Omega$  is thus a smooth open and connected subset of  $\mathbb{R}^d$ . Finally, it holds:

$$|\nabla f| \neq 0 \text{ on } \partial D_\Omega,$$

and all the critical points of  $f$  in  $D_\Omega$  belong to  $\bar{\Omega}$ . In the following, we will need other properties on  $D_\Omega$  which might require to reduce the parameter  $\varepsilon > 0$  one again. Let us state these properties. First, up to choosing  $\varepsilon > 0$  smaller, it holds for all local minima  $x \in \partial\Omega$  of  $f$  in  $\mathbb{R}^d$ ,

$$\max_{t \in [-\varepsilon, \varepsilon]} f(x + t\mathbf{n}_\Omega(x)) - f(x) < \min_{\partial\Omega} f - \min_{\bar{\Omega}} f, \quad (72)$$

where we recall that  $\mathbf{n}_\Omega(x)$  is the unit outward normal to  $\Omega$  at  $x$ . Let us also recall that, for all  $j \in \{1, \dots, N\}$  and  $z \in \partial C_j \cap \partial\Omega$  up to choosing  $\varepsilon > 0$  smaller, it holds:

$$B_\varepsilon(z) \cap \{f < \min_{\partial\Omega} f\} = C_j \cap B_\varepsilon(z), \quad (73)$$

which follows from the fact that the  $\overline{C}_\ell$ 's are two by two disjoint (see **(H1)**). Let us now consider, for some  $j \in \{1, \dots, \mathbf{N}\}$ ,  $z_* \in \partial C_j \cap \partial \Omega$  such that  $z_*$  is a critical point of  $f$  in  $\mathbb{R}^d$  (recall that  $f(z_*) = \min_{\partial \Omega} f$  and  $z_*$  is a saddle point of  $f$  in  $\mathbb{R}^d$ ). By assumption (see point  $b$  in **(H2)**), up to choosing  $\varepsilon > 0$  smaller,

$$z_* \text{ is the unique maximum point of } f : t \in [-\varepsilon, \varepsilon] \mapsto f(z_* + t n_\Omega(z_*)), \quad (74)$$

with  $z_* + t n_\Omega(z_*) \in \Omega$  when  $t \in [-\varepsilon, 0)$ , and,  $z_* + t n_\Omega(z_*) \in \mathbb{R}^d \setminus \overline{\Omega}$  when  $t \in (0, \varepsilon]$ . Then, one defines:

$$\gamma_* : t \in [-\varepsilon, \varepsilon] \mapsto z_* + t n_\Omega(z_*). \quad (75)$$

Hence, since for  $t \in [-\varepsilon, 0)$ ,  $z_* + t n_\Omega(z_*) \in \Omega$  and  $f(z_* + t n_\Omega(z_*)) < f(z_*) = \min_{\partial \Omega} f$ , from (73), one deduces that

$$\text{for all } t \in [-\varepsilon, 0), \gamma_*(z_* + t n_\Omega(z_*)) \in C_j, \quad (76)$$

Let us define

$$s_* := \inf\{t \in (0, \varepsilon], \gamma_*(t) \in \partial D_\Omega\}. \quad (77)$$

Since  $z_* \notin \partial D_\Omega$  (see (71) and  $\chi(0) = \alpha > 0$ ), it holds:  $s_* > 0$ . A schematic representation of  $\gamma_*$  is given in Figure 6. The parameter  $\varepsilon > 0$  defining  $D_\Omega$  is now fixed.

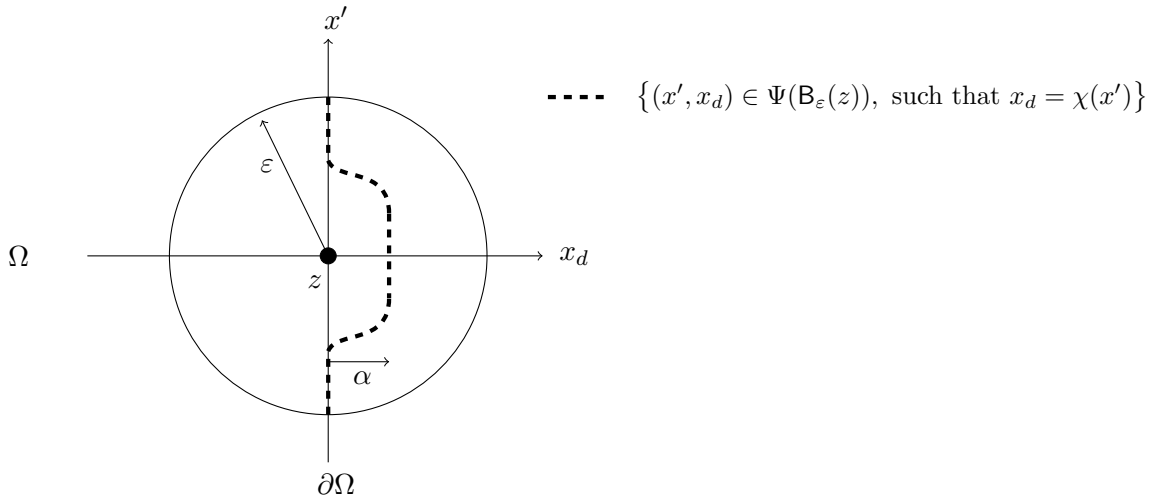


Figure 5: Schematic representation of  $\{(x', x_d) \in \Psi(B_\varepsilon(z)), \text{ such that } x_d = \chi(x')\}$  in the  $(x', x_d)$  coordinates (see the dashed line).

**Step 2:** Proof of (67). Let us introduce  $\tau_{D_\Omega^c} = \inf\{t \geq 0, X_t \notin D_\Omega\}$  the first exit time of the process (1) from  $D_\Omega$ . Notice that when  $X_0 = x \in \Omega$  it holds  $\tau_{\Omega^c} \leq \tau_{D_\Omega^c}$  (since the trajectories of the process (1) are continuous). Consequently,

$$\text{for all } x \in \Omega, \mathbb{E}_x[\tau_{\Omega^c}] \leq \mathbb{E}_x[\tau_{D_\Omega^c}]. \quad (78)$$

For all local minima  $x \in D_\Omega$  of  $f$ , one defines

$$H(x) := \inf \left\{ \max_{t \in [0, 1]} f(\gamma(t)), \gamma \in C^0([0, 1], \overline{D_\Omega}), \gamma(0) = x, \gamma(1) \in \partial D_\Omega \right\} \quad (79)$$

where  $C^0([0, 1], \overline{D_\Omega})$  is the set of continuous paths from  $[0, 1]$  to  $\overline{D_\Omega}$ , and,

$$H_{\max} := \max \{H(x) - f(x), x \in D_\Omega \text{ is a local minimum point of } f \text{ in } D_\Omega\}.$$

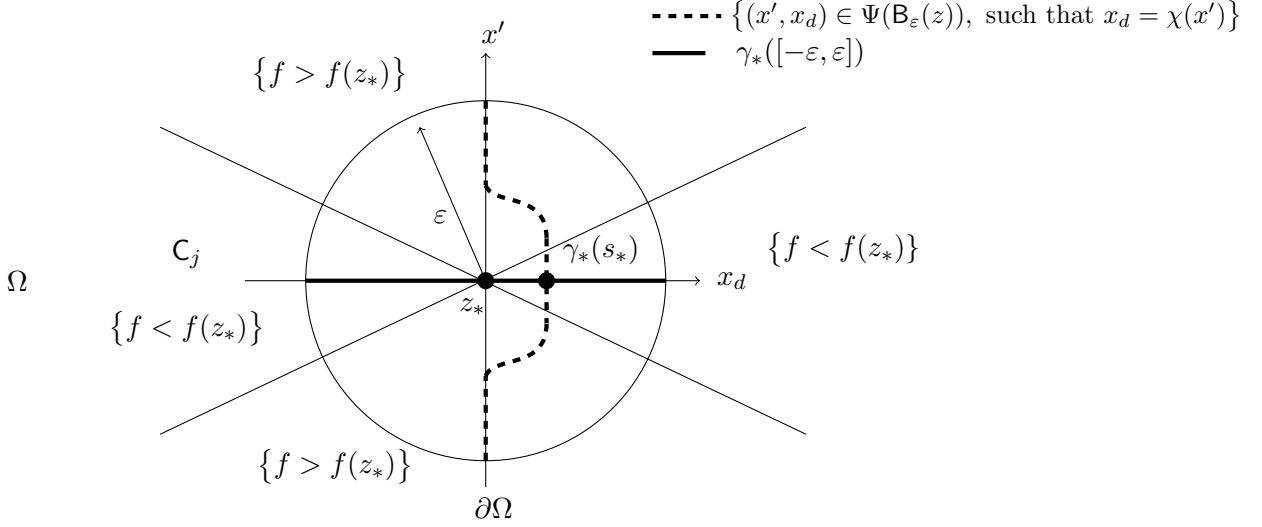


Figure 6: Schematic representation of  $\gamma_*$  for  $z_* \in \partial C_j \cap \partial \Omega$  in the  $(x', x_d)$  coordinates.

From [48, Theorem 1], it holds, for all  $K \subset \Omega$  and for all  $\delta > 0$ , there exists  $h_0 > 0$ , for all  $h \in (0, h_0)$ ,

$$\max_{x \in K} \mathbb{E}_x[\tau_{D_\Omega^c}] \leq e^{\frac{2}{h}(H_{\max} + \delta)}, \quad (80)$$

Let us mention that [48, Theorem 1] is not stated uniformly in compact subset  $D_\Omega$  but this is actually clear from [48, Theorems 3.2 and 4.1]. To prove (67) and in view of (78), let us prove that

$$H_{\max} = \min_{\partial \Omega} f - \min_{\bar{\Omega}} f.$$

Let  $x \in D_\Omega$  be a local minimum point of  $f$  in  $D_\Omega$ . Then, by construction of  $D_\Omega$ ,  $x \in \bar{\Omega}$ . Hence from **(H1)**,  $x \in \partial \Omega$  or  $x \in C_j$  for some  $j \in \{1, \dots, N\}$ . If  $x \in \partial \Omega$ , then, from (79), (72) and (71), it holds:

$$H(x) - f(x) \leq \max_{t \in [0, 1]} f(x + \varepsilon t n_\Omega(x)) - f(x) < \min_{\partial \Omega} f - \min_{\bar{\Omega}} f. \quad (81)$$

Let us now consider the case when

$$x \in C_j \text{ for some } j \in \{1, \dots, N\}.$$

Let  $\gamma \in C^0([0, 1], \bar{D}_\Omega)$  such that  $\gamma(0) = x$  and  $\gamma(1) \in \partial D_\Omega$ . Because  $\gamma$  is continuous and  $C_j \subset D_\Omega$ , there exists  $t_* \in [0, 1]$  such that  $\gamma(t_*) \in \partial C_j$ . Hence, one has:

$$\max_{t \in [0, 1]} f(\gamma(t)) \geq f(\gamma(t_*)) = \min_{\partial \Omega} f. \quad (82)$$

Let us now prove that there exists  $T_1 < T_2$ , and  $\gamma \in C^0([T_1, T_2], \bar{D}_\Omega)$  such that  $\gamma(T_1) = x$ ,  $\gamma(T_2) \in \partial D_\Omega$ , and

$$\max_{t \in [T_1, T_2]} f(\gamma(t)) = \min_{\partial \Omega} f. \quad (83)$$

**Case 1:**  $\partial C_j \cap \partial \Omega$  contains a critical point  $z_*$  of  $f$ .

Let  $z_* \in \partial C_j \cap \partial \Omega$  be a critical point of  $f$ . Since  $\gamma_*(-\varepsilon) \in C_j$  (see (75) and (76)) and  $C_j$  is path-connected (because it is open, connected and locally path-connected), there exists a continuous



curve  $\gamma_0 : [-2\varepsilon, -\varepsilon] \rightarrow \mathbf{C}_j$  such that  $\gamma_0(-2\varepsilon) = x$  and  $\gamma_0(-\varepsilon) = \gamma_*(-\varepsilon)$ . Let us finally define the continuous curve  $\gamma : [-2\varepsilon, s_*] \rightarrow \overline{\mathbf{D}}_\Omega$  by:

$$\text{for all } t \in [-2\varepsilon, -\varepsilon], \gamma(t) = \gamma_0(t), \text{ and, for all } t \in (-\varepsilon, s_*], \gamma(t) = \gamma_*(t),$$

where  $s_* > 0$  is defined by (77). It then follows from (74) together with the fact that  $\gamma_0([-2\varepsilon, -\varepsilon]) \subset \mathbf{C}_j \subset \{f < \min_{\partial\Omega} f\}$  that  $\max_{t \in [-2\varepsilon, s_*]} f(\gamma(t)) = f(z_*) = \min_{\partial\Omega} f$ . This proves (83) in this case.

**Case 2:**  $\partial\mathbf{C}_j \cap \partial\Omega$  does not contain critical point of  $f$ .

Because, by assumption (see **(H1)**),  $\partial\mathbf{C}_j \cap \partial\Omega$  is non empty, there exists  $z_* \in \partial\mathbf{C}_j \cap \partial\Omega$ . Notice that from (68) and (70),  $z_* \in \partial\mathbf{D}_\Omega$ . In addition, since  $\partial_n f(z_*) > 0$ , for  $t > 0$  small enough,  $\varphi_t(z_*) \in \Omega$  (see (4)). Furthermore, because there is no critical point of  $f$  in a neighborhood of  $z_*$  in  $\mathbb{R}^d$ , it holds for all  $t > 0$  sufficiently small,

$$f(\varphi_t(z_*)) - f(z_*) = \int_0^t \frac{d}{ds} f(\varphi_s(z_*)) ds = - \int_0^t |\nabla f|^2(\varphi_s(z_*)) ds < 0.$$

Thus, from (73), for all  $t > 0$  sufficiently small,  $\varphi_t(z_*) \in \mathbf{C}_j$  (and since  $\mathbf{C}_j$  is stable for the dynamics (4),  $\varphi_t(z_*) \in \mathbf{C}_j$  for all  $t > 0$ ). Let us consider  $t_* > 0$  and a curve  $\gamma_0 : [-2t_*, -t_*] \rightarrow \mathbf{C}_j$  such that  $\gamma_0(-2t_*) = x$  and  $\gamma_0(-t_*) = \varphi_{t_*}(z_*)$ . Then, one defines  $\gamma : [-2t_*, 0] \rightarrow \overline{\mathbf{D}}_\Omega$  by:

$$\text{for all } t \in [-2t_*, -t_*], \gamma(t) = \gamma_0(t), \text{ and, for all } t \in (-t_*, 0], \gamma(t) = \varphi_{-t}(z_*).$$

By construction, it holds,  $\max_{t \in [-2t_*, 0]} f(\gamma(t)) = f(z_*) = \min_{\partial\Omega} f$ . This concludes the proof of (83).

From (79), (82), and (83), one deduces that for all local minimum point  $x$  of  $f$  in  $\Omega$ ,  $\mathbf{H}(x) = \min_{\partial\Omega} f$ . Together with (81) and by definition of  $\mathbf{H}_{\max}$ , it holds

$$\mathbf{H}_{\max} = \min_{\partial\Omega} f - \min_{\overline{\Omega}} f.$$

Then, (78) and (80) conclude the proof of Proposition 14. ■

### 3.3 Proof of Theorem 2 and end of the proof of Theorem 1

*End of the proof of Theorems 1 and 2.* Let  $f : \overline{\Omega} \rightarrow \mathbb{R}$  which satisfies the assumptions **(H1)** and **(H2)**. Let  $j \in \{1, \dots, \mathbf{N}\}$ ,  $\mathbf{K}$  be compact subset of  $\mathcal{A}_\Omega(\mathbf{C}_j)$  and  $x \in \mathbf{K}$ . Let  $x_j$  be a local minimum point of  $f$  in  $\mathbf{C}_j$ . Let us write

$$\mathbb{E}_x[\tau_{\Omega^c}] = \mathbb{E}_x \left[ \tau_{\Omega^c} \mathbf{1}_{\tau_{\mathbf{B}_{r_h}(x_j)} < \tau_{\Omega^c}} \right] + \mathbb{E}_x \left[ \tau_{\Omega^c} \mathbf{1}_{\tau_{\mathbf{B}_{r_h}(x_j)} \geq \tau_{\Omega^c}} \right] \quad (84)$$

By the strong Markov Property,

$$\mathbb{E}_x \left[ \tau_{\Omega^c} \mathbf{1}_{\tau_{\mathbf{B}_{r_h}(x_j)} < \tau_{\Omega^c}} \right] = \mathbb{E}_x \left[ \mathbf{1}_{\tau_{\mathbf{B}_{r_h}(x_j)} < \tau_{\Omega^c}} \mathbb{E}_{X_{\tau_{\mathbf{B}_{r_h}(x_j)}}} [\tau_{\Omega^c}] \right].$$

Using Lemma 3 and Proposition 5, there exists  $c > 0$  such that for all  $h$  small enough and  $x \in \mathbf{K}$ , it holds:

$$\mathbb{E}_x \left[ \tau_{\Omega^c} \mathbf{1}_{\tau_{\mathbf{B}_{r_h}(x_j)} < \tau_{\Omega^c}} \right] = (1 + O(e^{-\frac{c}{h}})) \times \mathbb{E}_{x_j}[\tau_{\Omega^c}], \text{ uniformly in } x \in \mathbf{K}. \quad (85)$$

Let us now deal with the last term in (84). Let  $\Omega_K$  be as introduced in Proposition 6 (applied with  $C(x_*) = C_j$ ). Since  $\Omega_K \subset \Omega$ ,  $\mathbb{B}_{r_h}(x_j) \subset \Omega_K$  (for  $h$  small enough) and by continuity of the trajectories of the process (1), it holds for  $h$  small enough and for all  $x \in K$ ,

$$\{\tau_{\mathbb{B}_{r_h}(x_j)} \geq \tau_{\Omega^c}\} \subset \{\tau_{\mathbb{B}_{r_h}(x_j)} \geq \tau_{\Omega_K^c}\}, \quad \text{when } X_0 = x.$$

Consequently, and using the strong Markov property, for all  $x \in K$ , one has

$$\mathbb{E}_x \left[ \tau_{\Omega^c} \mathbf{1}_{\tau_{\mathbb{B}_{r_h}(x_j)} \geq \tau_{\Omega^c}} \right] \leq \mathbb{E}_x \left[ \tau_{\Omega^c} \mathbf{1}_{\tau_{\mathbb{B}_{r_h}(x_j)} \geq \tau_{\Omega_K^c}} \right] = \mathbb{E}_x \left[ \mathbf{1}_{\tau_{\mathbb{B}_{r_h}(x_j)} \geq \tau_{\Omega_K^c}} \mathbb{E}_{X_{\tau_{\Omega_K^c}}} [\tau_{\Omega^c}] \right].$$

Then, using Proposition 6 (with  $x_* = x_j$  and thus,  $C(x_*) = C_j$ ) and Proposition 14 (with the compact set  $\overline{\Omega_K}$  there), there exists  $c > 0$  such that for all  $h$  small enough and all  $x \in K$ , it holds:

$$\mathbb{E}_x \left[ \tau_{\Omega^c} \mathbf{1}_{\tau_{\mathbb{B}_{r_h}(x_j)} \geq \tau_{\Omega^c}} \right] \leq \max_{y \in K} \mathbb{P}_y [\tau_{\mathbb{B}_{r_h}(x_j)} \geq \tau_{\Omega_K^c}] \times \max_{y \in \Omega_K} \mathbb{E}_y [\tau_{\Omega^c}] \leq e^{\frac{2}{h}(\min_{\partial\Omega} f - \min_{\overline{\Omega}} f)} e^{-\frac{c}{h}}, \quad (86)$$

uniformly in  $x \in K$ . Thus, (85), (86), (65), and (84) imply that for all  $h$  small enough and  $x \in K$ , it holds when  $j \in \{1, \dots, N_0\}$ :

$$\mathbb{E}_x [\tau_{\Omega^c}] = \mathbb{E}_{x_j} [\tau_{\Omega^c}] (1 + O(e^{-\frac{c}{h}})) = \frac{e^{\frac{2}{h}(\min_{\partial\Omega} f - \min_{\overline{\Omega}} f)}}{\Theta_{j,1} h^{-\frac{1}{2}} + \Theta_{j,2} + O(\sqrt{h})},$$

uniformly in  $x \in K$ . When  $j \in \{N_0 + 1, \dots, N\}$ , from (84), (85), and (66), it holds when  $h \rightarrow 0$ :

$$\mathbb{E}_x [\tau_{\Omega^c}] \geq \frac{e^{\frac{2}{h}(\min_{\partial\Omega} f - \min_{\overline{C_j}} f)}}{\Theta_{j,1} h^{-\frac{1}{2}} + \Theta_{j,2} + O(\sqrt{h})}, \quad \text{uniformly in } x \in K.$$

Finally, from (84), (85), (86), and the lines just after (66), there exists  $\varepsilon > 0$ , such that if  $\min_{\overline{C_j}} f - \min_{\overline{\Omega}} f < \varepsilon$ , the previous inequality is an equality which holds uniformly in  $x \in K$ . This concludes the proof of Theorem 2, and the proofs of items 2 and 3 in Theorem 1. The proof of Theorem 1 is thus complete.  $\blacksquare$

## 4 Appendix

**Appendix 1.** In this section, we show the optimality of the error term  $O(\sqrt{h})$  in items 2 and 3 in Theorem 1. For that purpose, let us consider a function  $f \in \mathcal{C}^\infty([z_1, z_2], \mathbb{R})$  such that  $\{z \in [z_1, z_2], f'(z) = 0\} = \{z_1, x_1, z\}$ ,  $f''(z_1) < 0$ ,  $f''(z) < 0$ ,  $f''(x_1) > 0$ ,  $f'(z_2) < 0$ ,  $z_1 < x_1 < z < z_2$ , and  $f(x_1) < f(z_1) < f(z_2) < f(z)$  (see Figure 7). Assumption **(H1)** is thus satisfied with  $N = 1$ . In addition, one has  $\min_{\partial\Omega} f = f(z_1)$  and

$$C_1 = \{f < f(z_1)\} = (z_1, z_*),$$

where  $z_*$  is the unique point  $y$  in  $(z_1, z_2]$  such that  $f(y) = f(z_1)$  (see Figure 7). The following formula holds for  $x \in [z_1, z_2]$ :

$$\mathbb{E}_x [\tau_{(z_1, z_2)^c}] = \frac{2}{h} \int_{z_1}^{z_2} \int_{z_1}^y e^{\frac{2}{h}(f(y)-f(t))} dt dy \frac{\int_{z_1}^x e^{\frac{2}{h}f}}{\int_{z_1}^{z_2} e^{\frac{2}{h}f}} - \frac{2}{h} \int_{z_1}^x \int_{z_1}^y e^{\frac{2}{h}(f(y)-f(t))} dt dy. \quad (87)$$

This formula is a consequence of the Dynkin's formula: the function  $x \in [z_1, z_2] \mapsto \mathbb{E}_x[\tau_{(z_1, z_2)^c}]$  is the  $\mathcal{C}^\infty([z_1, z_2], \mathbb{R})$  solution to

$$\frac{h}{2} e^{\frac{2}{h}f} \frac{d}{dx} \left( e^{-\frac{2}{h}f} \frac{d}{dx} v \right) = -1 \text{ in } \Omega, \text{ with } v(z_1) = 0 \text{ and } v(z_2) = 0.$$

Consider now  $x \in \mathbf{C}_1 = (z_1, z_*)$ . For all  $t$  and  $y$  such that  $z_1 \leq t \leq y \leq x$ , it holds  $f(y) - f(t) \leq f(x) - f(x_1) < f(z_1) - f(x_1)$ . Thus, for some  $c > 0$  it holds for all  $h > 0$ :

$$\int_{z_1}^x \int_{z_1}^y e^{\frac{2}{h}(f(y)-f(t))} dt dy \leq e^{\frac{2}{h}(f(z_1)-f(x_1))} e^{-\frac{c}{h}}.$$

Using the Laplace method, since the maximum value of  $t < y \in [z_1, z_2] \mapsto f(y) - f(t)$  attains uniquely is maximum at the point  $(x_1, z) \in \{t < y, (t, y) \in (z_1, z_2)^2\}$ , it holds when  $h \rightarrow 0$ :

$$\int_{z_1}^{z_2} \int_{z_1}^y e^{\frac{2}{h}(f(y)-f(t))} dt dy = \frac{\pi h}{\sqrt{f''(x_1)|f''(z)|}} e^{\frac{2}{h}(f(z)-f(x_1))} (1 + O(h)).$$

Moreover, since the maximum value of  $y \in [z_1, z_2] \mapsto f(y)$  is uniquely attained when  $y = z \in (z_1, z_2)$ , one has using again the Laplace method, in the limit  $h \rightarrow 0$ :

$$\int_{z_1}^{z_2} e^{\frac{2}{h}f} = \frac{\sqrt{\pi h}}{\sqrt{|f''(z)|}} e^{\frac{2}{h}f(z)} (1 + O(h)).$$

Finally, since the maximum value of  $y \in [z_1, x] \mapsto f(y)$  is uniquely attained when  $y = z_1 \in \partial(z_1, x_1)$ ,  $f'(z_1) = 0$ , and  $f''(z_1) < 0$ , using Watson's lemma, it holds when  $h \rightarrow 0$ :

$$\int_{z_1}^x e^{\frac{2}{h}f} = \frac{1}{2} \frac{\sqrt{\pi h}}{\sqrt{|f''(z_1)|}} e^{\frac{2}{h}f(z_1)} (1 + c\sqrt{h} + O(h)),$$

where  $c \in \mathbb{R}$  can be expressed in terms of the gamma function and the derivatives of  $f$  at  $z_1$ . In conclusion, it holds for  $x \in \mathbf{C}_1$ , in the limit  $h \rightarrow 0$ :

$$\mathbb{E}_x[\tau_{(z_1, z_2)^c}] = \frac{\pi e^{\frac{2}{h}(f(z_1)-f(x_1))}}{\sqrt{f''(x_1)|f''(z_1)|}} (1 + c\sqrt{h} + O(h)) = \frac{(1 + c\sqrt{h} + O(h))}{\Theta_{1,2}} e^{\frac{2}{h}(f(z_1)-f(x_1))}.$$

If we now assume that  $f'(z_1) < 0$ , we obtain that, using the Laplace method, when  $h \rightarrow 0$ :

$$\int_{z_1}^x e^{\frac{2}{h}f} = \frac{h}{2|f'(z_1)|} e^{\frac{2}{h}f(z_1)} (1 + O(h)), \text{ for some } c \in \mathbb{R}.$$

and thus, in the limit  $h \rightarrow 0$ :

$$\mathbb{E}_x[\tau_{(z_1, z_2)^c}] = \frac{\sqrt{h\pi} e^{\frac{2}{h}(f(z_1)-f(x_1))}}{|f'(z_1)| \sqrt{f''(x_1)}} (1 + O(h)) = \sqrt{h} \frac{(1 + O(h))}{\Theta_{1,1}} e^{\frac{2}{h}(f(z_1)-f(x_1))},$$

where, by the Laplace method,  $O(h)$  equals  $O(h) = \sum_{k \geq 1} a_k h^k$ , and where the  $a_k$ 's can be explicitly computed with the derivatives of  $f$  at  $z$ ,  $x_1$ , and  $z_1$ . In general,  $a_1 \neq 0$ , and this  $O(h)$  is thus not a  $o(h)$ . These computations show that the error term  $O(\sqrt{h})$  in Theorem 1 is optimal.

**Appendix 2.** In this section, we discuss the strong tunnelling effect which occurs when a well  $\mathbf{C}_i$  of  $f$  in  $\Omega$  is much deeper than the other wells  $\mathbf{C}_j$ ,  $j \neq i$ . To this end, we consider a function  $f \in \mathcal{C}^\infty([z_2, z_1], \mathbb{R})$  such that  $\{z \in [z_2, z_1], f'(z) = 0\} = \{x_2, z, x_1\}$ ,  $z_2 < x_2 < z < x_1 < z_1$ ,

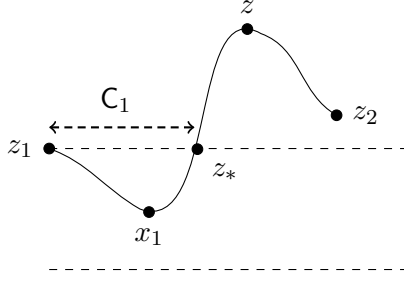


Figure 7: A one dimensional case **(H1)** is satisfied with  $N = 1$  ( $\partial C_1 \cap \partial \Omega = \{z_1\}$ ).

$f''(z) < 0$ ,  $f''(x_2) > 0$ ,  $f''(x_1) > 0$ ,  $f'(z_2) < 0$ ,  $f'(z_1) > 0$ , and  $f(x_1) < f(x_2) < f(z_1) = f(z_2) < f(z)$ . Such a function  $f$  is represented in Figure 8. It thus holds  $f(x_2) = \min_{\overline{C_2}} f > \min_{\overline{\Omega}} f = f(x_1)$ , and  $\{f < \min_{\partial \Omega} f\}$  has two connected components:  $C_1$  containing  $x_1$  and  $C_2$  containing  $x_2$ . Moreover, **(H1)** holds with  $N_0 = 1 < N = 2$ . When  $f(z) - f(x_2) > f(z_1) - f(x_1)$ , one has, using the Laplace method in (87) (replaced  $z_1$  by  $z_2$ , and  $z_2$  by  $z_1$  there):

$$\lim_{h \rightarrow 0} h \ln \mathbb{E}_{x_2}[\tau_{(z_2, z_1)^c}] = 2(f(z_2) - f(x_2)),$$

whereas, when  $f(z) - f(x_2) < f(z_1) - f(x_1)$ , it holds:

$$\lim_{h \rightarrow 0} h \ln \mathbb{E}_{x_2}[\tau_{(z_2, z_1)^c}] = 2(f(z_1) - f(x_1) - (f(z) - f(z_2))).$$

We thus observe that when  $f(z) - f(x_2) < f(z_1) - f(x_1)$ , even if  $\overline{C_1} \cap \overline{C_2} = \emptyset$ , a strong tunnelling effect between the wells  $C_1$  and  $C_2$  occurs. Let us mention that the conclusion remains unchanged if  $f'(z_1) = 0$  or  $f'(z_2) = 0$ .

**Remark 15.** In the example depicted in Figure 8, it has been proved in [29, Theorem 2], that  $\lim_{h \rightarrow 0} h \ln \lambda_{1,h} = -2(f(z_1) - f(x_1))$  and  $\lim_{h \rightarrow 0} h \ln \lambda_{2,h} = -2(f(z_2) - f(x_2))$ . Thus, when  $f(z) - f(x_2) < f(z_1) - f(x_1)$ , for all  $i \in \{1, 2\}$ ,  $\lim_{h \rightarrow 0} h \ln \mathbb{E}_{x_2}[\tau_{(z_1, z_2)^c}] \neq \lim_{h \rightarrow 0} h \ln \lambda_i^{-1}$ .

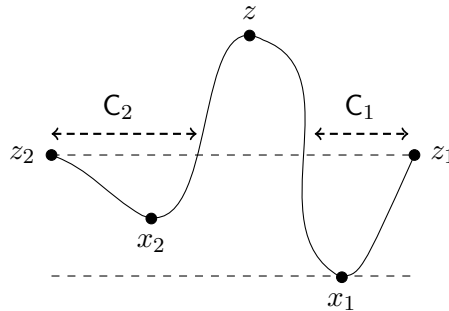


Figure 8: A one dimensional case when **(H1)** holds with  $N_0 = 1$  and  $N = 2$ .

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