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A class period on spacetime-smart 3-vectors with familiar approximates

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Introductory physics students have Newton's laws drilled into their minds, but historically questions related to relativistic motion and accelerated frames have been avoided. With help from the metric equation, these 3-vector laws can be extended into the relativistic regime as long as one sticks with only one reference frame (to define position plus simultaneity), and considers something students are already quite familiar with, namely: motion using map-frame yardsticks as a function of time on clocks of the moving-object. The question here is: How may one class period in an intro-physics class, e.g. as a preview before kinematics or later during a day on relativity-related material, be used to put the material we teach into a spacetime-smart context?

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I. INTRODUCTION

Consider driving a car: When you look at your speedometer, what are you seeing? The reported speed is not relative to some inertial frame with synchronized clocks on the side of the road, since speedometers use the rotation of the wheels¹ (which make static contact with the road) per unit time on the cars on-board clocks. This “proper ratio² of map distance Δx to traveler time $\Delta\tau$ at any speed (e.g. even if lightspeed as for Mr. Tompkins³ was only $\simeq 2.5$ mph) turns out to be proportional to 3-vector momentum \vec{p} , to have no upper limit, and to also be most simply related to kinetic energy and the time available for driver and pedestrian to react after the danger becomes apparent. It reduces to map distance Δx per unit map time Δt only at low speeds.

Another remarkable everyday example of the “traveler point” approach is the fact that your phone accelerometer cannot detect gravity, as shown in Fig. 1. It only detects the normal force which prevents us from falling

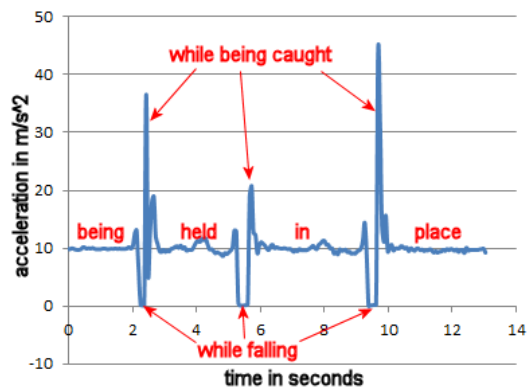


FIG. 1. Accelerometer data from a phone dropped and caught three times. During the free-fall segments, the accelerometer reading drops to zero because gravity is a geometric force (like centrifugal) which acts on every ounce of the phone's structure, and is hence not detected. The positive spikes occur when the fall is arrested as the falling phone is caught (also by hand) before it hit the floor.

through the floor. It also fails to detect inertial forces, like those which push you back into the seat (or to the outside of a curve) when your car accelerates (or follows a curved path). This is good news, coming from general relativity, which says that our accelerometer only detects proper forces but that the “undetected” class of geometric forces (associated with accelerated frames or curved spacetime) can in general be approximated locally as if they are one (or more) proper forces. This honored tradition, of treating geometric forces as proper, was of course started in the 17th century by none other than Issac Newton himself.

II. COLLEGE PHYSICS TABLE

For example, in our college algebra-based basic physics II class, Walker's text⁴ has a chapter near the end on relativity. Our strategy is to introduce the book's tools, along with these traveler-point tools, for dealing with problems of time dilation, unidirectional velocity addition, and rel-

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TABLE I. Newton at any unidirectional speed in (1+1)D, from $(c\Delta\tau)^2 = (c\Delta t)^2 - (\Delta x)^2$.

Quantity↓ Variable→	standard offering	traveler-point version	low-speed version
time dilation $\gamma \equiv \Delta t/\Delta\tau$	$\gamma = 1/\sqrt{1 - (v/c)^2}$	$\gamma = \sqrt{1 + (w/c)^2}$	$\gamma \simeq 1$
relative velocities $v \equiv \Delta x/\Delta t$, $w \equiv \Delta x/\Delta\tau$	$v_{ac} = \frac{(v_{ab}+v_{bc})}{(1+v_{ab}v_{bc}/c^2)}$	$w_{ac} = \gamma_{ac}\gamma_{bc}(v_{ab} + v_{bc})$	$v_{ac} \simeq v_{ab} + v_{bc}$
momentum	$p = m\gamma v$	$p = mw$	$p \simeq mv$
total energy	$E = \gamma mc^2$	$E = \gamma mc^2$	$E \simeq mc^2$
kinetic energy	$K = (\gamma - 1)mc^2$	$K = (\gamma - 1)mc^2$	$K \simeq \frac{1}{2}mv^2$

ativistic energies/momenta. Length contraction is off the table, because it requires two extended frames with synchronized clocks. Proper-velocity $\vec{w} = \gamma\vec{v}$ and rest-mass m is used instead of “relativistic mass” to preserve the standard relationship between momentum and velocity, and students are only being asked to master that subset of problems posed in the book which can be solved with or without these “hybrid kinematic” tools, as shown Table I.

III. INTROPHYSICS IN GENERAL

More generally we suggest initial mention (even if only in passing) of the “traveler-point variables” (chosen because they either have frame-invariant magnitudes or because they don’t require synchronized clocks), namely traveler or proper time τ , proper velocity defined as map distance per unit traveler time $\vec{w} \equiv \Delta\vec{x}/\Delta\tau$, and the net proper force $\Sigma\vec{\xi} = m\vec{a}$ felt by on-board accelerometers. These are approximated at low speeds by the more familiar map time t , coordinate velocity $\vec{v} \equiv \Delta\vec{x}/\Delta t$, and net map-based force $\Sigma\vec{f} \simeq \Delta\vec{p}/\Delta t$. By sticking with displacements $\Delta\vec{x}$ and simultaneity defined by a single bookkeeper or map reference frame (i.e. the metric), as shown in Table II we can simply describe time-dilation $\gamma \equiv \Delta t/\Delta\tau$ and constant unidirectional proper acceleration \vec{a} at any speed, even when there’s no time to explore 3-vector proper velocity/acceleration or multi-frame phenomena like length contraction.

The unidirectional proper-velocity addition equation given in Table II, for example, allows students to see the advantage of colliders over accelerators in more visceral terms, which may even fire up the imagination of NASCAR fans (think of land and relative speed records for particles) as depicted in the relative velocity illustration of Figure 2 (inspired by an XKCD cartoon). Similarly the unidirectional equations of constant proper acceleration given in Table II allow students to easily calculate the map and traveler times elapsed on constant proper-acceleration round trips between stars, as illustrated in Figure 3.

In passing, we should also mention the curious relationship between various energies and the time-dilation or differential-aging factor $\gamma \equiv \Delta t/\Delta\tau$. The basic relationship, given in Table II, allows us to say that kinetic energy of motion with respect to inertial frames in flat spacetime

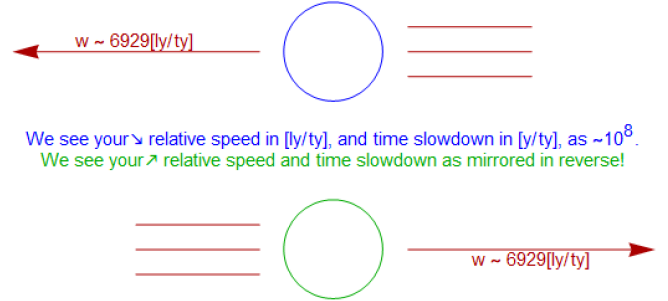


FIG. 2. Two 6.5 TeV LHC protons send messages to each other, while passing at proper velocities of about $\simeq 6929$ lightyears/traveler-year, for a collider energy advantage of $K_{\text{rel}}/K \simeq 13,859$ times the energy of stationary target collision.

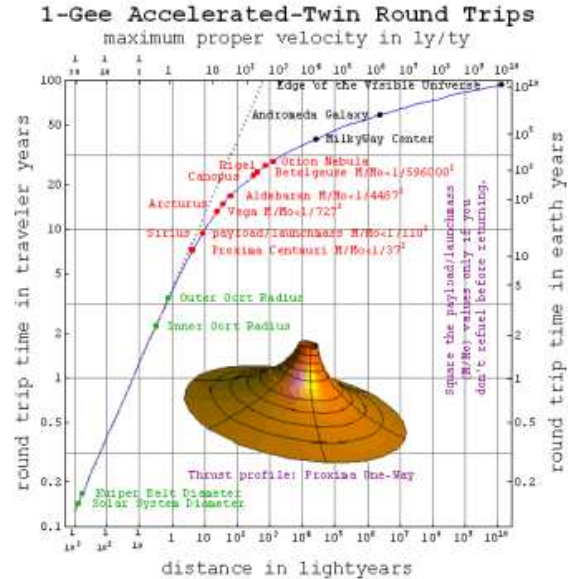


FIG. 3. Round trip times and a sample thrust profile for a spaceship capable of constant 1-gee acceleration and avoiding collision with atoms.

is $K = (\gamma - 1)mc^2$. Remarkably, however, in curved space time and in accelerated frames, relations like this also express potential energies associated with geometric forces. This is easiest to see when standing in an artificial (centrifugal) gravity well, where the rotational kinetic energy from a fixed external point of view looks like a potential

TABLE II. Traveler-point dynamics in flat (1+1)D spacetime: Conserved quantities energy $E = \gamma mc^2$ and momentum $\vec{p} = m\vec{w}$, where differential-aging factor $\gamma \equiv \delta t/\delta\tau$, proper velocity $\vec{w} \equiv \delta\vec{r}/\delta\tau \equiv \gamma\vec{v}$, coordinate acceleration $\vec{a} \equiv \delta\vec{v}/\delta t$. In the first 4 rows, τ is traveler time from “rest” with respect to the map frame, and α is a fixed space-like proper acceleration vector. Asterisk means that the (1+1)D relation also works in (3+1)D.

relation	$w \ll c$	(1+1)D
map time elapsed t	$t \simeq \tau$	$t = \frac{c}{\alpha} \sinh[\frac{\alpha\tau}{c}]$
map displacement \vec{x}	$\vec{x} \simeq \vec{v}_o t + \frac{1}{2}\vec{a}t^2$	$x = \frac{c^2}{\alpha} (\cosh[\frac{\alpha\tau}{c}] - 1)$
aging factor $\gamma \equiv \delta t/\delta\tau$	$\gamma \simeq 1 + \frac{1}{2}(v/c)^2$	$\gamma = \cosh[\frac{\alpha\tau}{c}]$
proper velocity $\vec{w} \equiv \delta\vec{x}/\delta\tau$	$\vec{w} \simeq \vec{v} \simeq \vec{v}_o + \vec{a}t$	$w = c \sinh[\frac{\alpha\tau}{c}]$
*momentum \vec{p}	$\vec{p} \simeq m\vec{v}$	$\vec{p} = m\vec{w} = m(\gamma\vec{v})$
*energy E	$E \simeq mc^2 + \frac{1}{2}mv^2$	$E = \gamma mc^2$
felt ($\vec{\xi}$) \leftrightarrow map-based (\vec{f}) force conversions	$\vec{f} \simeq \vec{\xi}$	$\vec{f} = \vec{\xi}$
*work-energy	$\delta E \simeq \Sigma \vec{f} \cdot \delta \vec{x}$	$\delta E = \Sigma \vec{f} \cdot \delta \vec{x}$
*action-reaction	$\vec{f}_{AB} = -\vec{f}_{BA}$	$\vec{f}_{AB} = -\vec{f}_{BA}$
*map-force:momentum *felt force:acceleration	$\Sigma \vec{f} = \delta \vec{p}/\delta t \simeq m\vec{a}$	$\Sigma \vec{f} = \delta \vec{p}/\delta t ; \Sigma \vec{\xi} = m\vec{a}$

energy well of depth $U \simeq \frac{1}{2}m\omega^2 r^2$ to the rotating inhabitant. However, it also turns out to be true in a spaceship of length L undergoing constant proper acceleration α , where the energy to climb from trailing to leading end is $\Delta U = (\Delta t_{\text{leading}}/\Delta t_{\text{trailing}} - 1)mc^2 \simeq m\alpha L$, and in the gravity of a non-spinning sphere of mass M and radius R , where the escape energy for mass m on the surface (when R is much more than the Schwarzschild radius) is $W_{\text{esc}} = (\Delta t_{\text{far}}/\Delta\tau - 1)mc^2 \simeq GMm/R$, where t_{far} is time elapsed on the clocks of distance observers. This and the kinetic differential-aging factors must, for example, both be considered when calculating your global-positioning-system location.

IV. CAUTIONS

Caution for “traveler-point dynamicists”, especially when considering the vantage point of more than one “map-frame” or bookkeeper metric:

- 1st caution: Specify “which clock” when talking about time elapsed, and which “map frame” when talking about position.
- 2nd caution: Try to stick with a single map frame of yardsticks and bookkeeper or “metric time” clocks. This takes discussion of length contraction and Lorentz transforms (both requiring two extended frames) off the table, but allows 3-vector dynamics to be added.
- 3rd caution: Like rates of energy change at any speed, map-based forces (magnitude & direction) differ from one frame to the next at high speeds, even if the frames are only moving at a constant speed with respect to one another. This frame dependence actually gives rise to a kinetic versus static breakdown of all proper forces, whose

usefulness in the case when there are “oppositely charged” force-carriers is behind the 19th century distinction between magnetic and electrostatic fields.

- 4th caution: The simultaneity of separately located events is also frame dependent, i.e. differently moving observers may disagree on which of two “space-like separated” events came first, just as the filial ordering of non-descendant relatives in a family tree may disagree on an individual’s generation⁵.
- 5th caution: Relative 3-vector proper-velocity addition (e.g. between co-moving reference frames) is possible, but may be complicated by both “clock changes” which affect component magnitudes, and by changes in the reference metric (which affect both component magnitudes and directions).
- 6th caution: If energy is not conserved in an interaction between objects traveling at high speeds, momentum may not be either since differently-moving frames allow trades between energy E and momenta \vec{p} (as well as between motion-through-time $\delta t/\delta\tau$ and motion-through-space $\delta\vec{x}/\delta\tau$) because only a sum of both, e.g. from the flat-space metric $c^2 = c^2(E/mc^2)^2 - (p/m)^2$, is frame invariant.
- 7th caution: Geometric forces like gravity and centrifugal in general only work locally, i.e. in regions within which your reference spacetime metric is “locally flat”. Extensions are possible, e.g. with tidal and Coriolis forces, by combining forces from separate regions.

To provide space for discussing sample problems, and for the development of on-line calculators and simulators to further empower students and introductory teachers with this metric-first⁶ or ‘one-map

two-clock⁷ approach, we've created some space up at sites.google.com/umsl.edu/travelerpointdynamics2 for further discussion.

ACKNOWLEDGMENTS

Thanks to the late Bill Shurcliff (1909-2006) for his counsel on “minimally-variant” approaches.

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Appendix A: possible course notes

The sections to follow might be dropped into the introductory physics schedule as the usual “relativity section”, near the end of the course, or may be rearranged for piecemeal discussion earlier so that the Newtonian models, that students will be working with, are framed as approximations from the start.

1. spacetime version of Pythagoras' theorem

Time is local to a given clock, and simultaneity is determined by your choice of reference frame. Although Maxwell's equations on electromagnetism were “informed” to this reality in the mid 1800's, humans really didn't start to get the picture until the early 1900's. But how might one deal with this quantitatively?

Start with the (1+1)D flat-space **metric equation**, namely $(c\delta\tau)^2 = (c\delta t)^2 - (\delta x)^2$ where x and t are position and time coordinates associated with your reference “map-frame” of yardsticks and synchronized clocks. The quantity τ is the **proper-time** elapsed on the clocks of a traveling observer whose **map-position** x may be written as a function of **map-time** t . As usual c is the spacetime constant (literally the number of meters in a second) which is traditionally referred to as **lightspeed** because it equals the speed of light in a vacuum.

The term on the left in the metric equation is referred to as a **frame-invariant**. Just like a given hypotenuse (cf. Fig. 4) can be expressed in terms of a bunch of different xy coordinate systems, all of which agree on its length, so a given proper-time interval e.g. on a traveling object's clock, can be expressed in terms of many different “bookkeeper” reference-frames, all of which will also

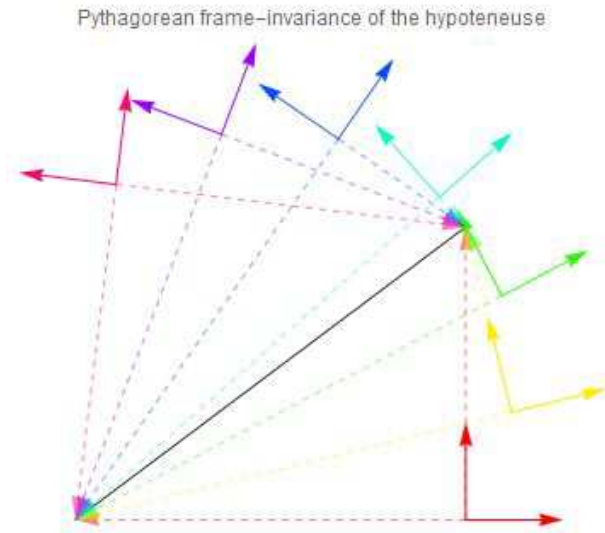


FIG. 4. Pythagorean frame-invariance of the hypotenuse (in black), from the metric equation $(\delta h)^2 = (\delta x)^2 + (\delta y)^2$ for seven different Cartesian coordinate systems, whose unit-vectors are shown in color. Although none of these coordinate systems agrees on the coordinates of the black line's endpoints, all agree that the length of the hypotenuse is 5.

agree on its duration even if they can't agree on which of two spatially-separated events happened first.

This equation is seriously powerful. As Einstein illustrated, if one curves spacetime by tweaking the “unit” coefficients of the terms on the right by only “one part per billion”, we find ourselves in a gravitational field like that on earth where a fall of only a few meters can do you in.

2. traveler-point kinematics

From the foregoing, it is easy to define a **proper-velocity** $\vec{w} \equiv d\vec{x}/d\tau = \gamma\vec{v}$, where $\vec{v} \equiv d\vec{x}/dt$ is **coordinate-velocity** as usual, and **speed of map-time** or “differential-aging factor” $\gamma \equiv dt/d\tau = \sqrt{1 + (w/c)^2} = 1/\sqrt{1 - (v/c)^2} \geq 1$. This last relation tells us that *when simultaneity is defined by a network of synchronized map clocks*, a moving traveler's clock will always run slow. These relationships follow directly from the metric equation itself.

Thus having a new time-variable τ also gives us some new ways to measure rate of travel. Proper-velocity, as we'll see, has no upper limit and is related to conserved-quantity **momentum** by the simple vector relation $\vec{p} = m\vec{w}$ where m is our traveler's frame-invariant rest-mass. The upper limit of $c \geq dx/dt$ on coordinate-velocity results simply from the fact that momentum (and $dx/d\tau$) have to remain finite. Speed of map-time γ , on the other hand, relates to **total energy** by $E = \gamma mc^2$, and to **kinetic energy** by $K = (\gamma - 1)mc^2$.

At this point, a variety of familiar time-dilation and relativistic energy/momentum topics might be covered as examples. If there is added time, one path to take is to introduce length-contraction, velocity addition, and Doppler effect with or without Lorentz transforms, historical notes on lightspeed measurement, etc. In what follows, we instead push the alignment with traveler-point concepts and the Newtonian treatment of kinematics and mechanics a bit further.

It is also conceptually interesting to note that **proper-acceleration**, in turn, is simply the vector-acceleration detected by a cell-phone accelerometer in the traveler's pocket. This quantity, after a couple of proper-time derivatives, pops up on the left side of the metric equation as a frame-invariant as did “hypotenuse” and proper-time.

For unidirectional motion in flat-spacetime (i.e. using map-coordinates in an inertial frame), proper-acceleration $\vec{\alpha}_o = \gamma^3 \vec{a}$, where $\vec{a} \equiv d^2\vec{x}/dt^2$ is the usual **coordinate-acceleration**. These relations also yield a few simple integrals for “constant” proper-acceleration, namely $\alpha = \Delta w/\Delta t = c\Delta\eta/\Delta\tau = c^2\Delta\gamma/\Delta x$.

That 2nd equality involves “hyperbolic velocity-angle” or **rapidity** $\eta = \sinh^{-1}[w/c] = \tanh^{-1}[v/c]$, so that $\gamma = \cosh[\eta]$. The first and third equalities reduce to the familiar conceptual-physics relationships $a = \Delta v/\Delta t = \frac{1}{2}\Delta(v^2)/\Delta x$ at low speed. However they allow beginning students to explore interstellar constant proper-acceleration round-trip problems, almost as easily as they do problems involving projectile trajectories on earth.

3. dynamics in flat spacetime

The net proper-force may in general be written as $\Sigma\vec{F}_o \equiv m\vec{\alpha}_o$. In flat (and unaccelerated) spacetime coordinate-systems, all forces are proper, and proper-acceleration $\vec{\alpha}_o$ equals the **net-acceleration** $\vec{\alpha}$ observed by a traveler. Under a constant net proper-force, we can therefore expect constant net-acceleration.

At high speeds the constant proper-force equations are messier because of that pesky γ in equations like $\Delta w = \Delta(\gamma v) = \alpha_o \Delta t$. The low speed approximation (namely $\Delta v = a\Delta t$) is therefore a bit simpler to deal with, and works fine for speeds well below c .

4. dynamics in curved spacetime

In both curved spacetime and in accelerated frames, Newton's equations still work (at least locally) provided that we recognize the existence of non-proper or geometric forces, like gravity as well as inertial forces that arise in accelerated frames. Newton's law for causes of motion “in the neighborhood of a traveling object” then takes the 3-vector form $\Sigma\vec{F}_o + \Sigma\vec{F}_g = m\vec{\alpha}$, where $\vec{\alpha}$ is the net-acceleration actually observed by the traveler.

The reason that your cell-phone's accelerometer can't see gravity (or centrifugal force), even when gravity is causing a net-acceleration downward, is that these are geometric forces. Geometric forces ($\Sigma\vec{F}_g$), which act on “every ounce” of an object, result from being in an accelerated frame or in curved spacetime. Accelerometers can only detect the result of net proper-forces ($\Sigma\vec{F}_o$), i.e. one's proper-acceleration $\vec{\alpha}_o$.

It's traditional to approximate gravity on the surface of the earth as simply a proper-force that is proportional to mass m , for which $\vec{F} = m\vec{g}$ where \vec{g} is a downward vector with a magnitude of about 9.8 [m/s²]. This works quite well for most applications. However unlike geometric-forces, proper-forces are not associated with positional time-dilation like that which must be figured into GPS system calculations.

Just as kinetic energy in flat spacetime is related via $(dt/d\tau - 1)mc^2$ to the faster passage of map-time (t) with respect to traveler time (τ) when simultaneity is defined by the map-frame, so is the *position-dependent* potential-energy well-depth of some geometric-forces in accelerating frames and curved spacetime. Thus $(dt/d\tau - 1)mc^2$ also describes the potential-energy well-depth for mass- m travelers located: (i) at radius r from the axis in a habitat rotating with angular velocity ω , which in the low-speed limit reduces to the classical value $\frac{1}{2}m\omega^2 r^2$; (ii) a distance L behind the leading edge of a spaceship undergoing constant proper acceleration α , which in the small L limit reduces to the classical value $m\alpha L$; and (iii) on the radius R surface of a mass M planet, which in the $R \gg 2GM/c^2$ limit reduces to the classical value of GMm/R where G is the gravitational constant.