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# DENSITY MODULO 1 OF A SEQUENCE ASSOCIATED TO SOME MULTIPLICATIVE FUNCTIONS EVALUATED AT POLYNOMIAL ARGUMENTS

MOHAMMAD NASIRI-ZARE

ABSTRACT. We study density modulo 1 of the sequence with general term  $\sum_{m \leq n} f(G(m))$  where  $f$  is the strongly multiplicative function of the form  $f(n) = \prod_{p|n} \left(1 - \frac{\nu(p)}{p}\right)$  and  $\nu$  is a multiplicative function for which there exists a real number  $0 < r \leq 1$  such that  $1 \leq |\nu(p)| \leq p^{1-r}$  for all primes  $p$ , and  $G$  is a non constant polynomial with integral coefficients and taking positive values at positive arguments.

## 1. INTRODUCTION

At the Czech-Slovak Number Theory Conference in Smolenice in August 2007, F. Luca asked whether the sequences of arithmetic and geometric means of the first  $n$  values of the Euler totient function are uniformly distributed modulo 1. A. Schinzel modified these questions by asking whether these sequences are dense modulo 1. This question was positively answered in [4], as well as similar questions for different types of mean values of the Euler function taken on the integers. Shortly later, two extensions were introduced. The first one [5] was to consider the mean of the Euler function evaluated at Fibonacci arguments and the second one was to notice that since the sequences have a linear growth with an irrational leading coefficient, it could be uniformly distributed modulo 1; this was shown in [2]. Then the quadratic polynomial values was considered in [1] and [3] for some linear sequences associated to some multiplicative functions. Also, the distribution modulo one of some sequences associated to the arithmetical functions has been investigated in [9] and [10].

Most recently, J.-M. Deshouillers and author proved that the sequence

$$a_n = \sum_{m \leq n} \frac{\varphi(G(m))}{G(m)}$$

is dense modulo 1, where  $G$  is a non constant polynomial with integral coefficients and taking positive values at positive arguments [6]. In this paper an extension of this result for a family of strongly multiplicative functions is given as follows.

**Theorem 1.1.** *Let  $f$  be the strongly multiplicative function defined by*

$$(1) \quad f(n) = \prod_{p|n} \left(1 - \frac{\nu(p)}{p}\right),$$

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where  $\nu$  is a multiplicative function and there exists a real number  $0 < r \leq 1$  such that

$$(2) \quad \forall p : 1 \leq |\nu(p)| \leq p^{1-r},$$

and there exists a family  $\mathcal{P}_G$  of primes such that, for any  $p \in \mathcal{P}_G$  the congruence  $G(x) \equiv 0 \pmod{p}$  has a solution, and

$$(3) \quad \sum_{p \in \mathcal{P}_G} \frac{\nu(p)}{p} = +\infty.$$

Let  $G$  be a non constant polynomial with integral coefficients and taking positive values at positive arguments. Then the sequence with general term

$$(4) \quad a_n = \sum_{m \leq n} f(G(m))$$

is dense modulo 1.

Because of strongly multiplicativity of  $f$ , without loss of generality, we may assume that the polynomial  $G$  has no quadratic irreducible rational factor and that  $\nu(p^k) = 0$  for any prime  $p$  and any integer  $k \geq 2$ . We keep these assumptions throughout the paper, as more as we set the following notations.

For a polynomial  $F$  with integral coefficients, we let

$$(5) \quad \omega(F, p) = \text{Card}\{x \in \mathbb{Z} \cap [0, p) : F(x) \equiv 0 \pmod{p}\},$$

and we say that a prime  $p$  is a fixed divisor of  $F$  if  $\omega(F, p) = p$ .

We denote by  $G$  a non constant polynomial with integral coefficients which takes positive values at positive arguments; we moreover assume that the polynomial  $G$  has no square irreducible rational factor; its degree is denoted by  $g$  and we denote by  $h$  an integer which is at least equal to the minimum of the degree of  $G$  and the maximal fixed prime divisor of  $G$ ; in other words we have

$$(6) \quad h \geq \deg(G) = g \quad \text{and} \quad p > h \Rightarrow \exists x \in \mathbb{Z} : p \nmid G(x).$$

## 2. AUXILIARY LEMMAS

In this section, we state some lemmas we need to prove the main theorem. First we notice that the introduced function has approximately linear growth.

**Lemma 2.1.** *Let  $\nu$  be a multiplicative function and  $f$  be defined by*

$$f(n) = \prod_{p|n} \left(1 - \frac{\nu(p)}{p}\right).$$

and there exists a real number  $0 < r \leq 1$  such that  $\forall p : 1 \leq |\nu(p)| \leq p^{1-r}$ , then there exists  $\alpha \neq 0$  such that

$$(7) \quad \sum_{n \leq x} f(n) = \alpha x + O(x^{1-r} \log x) \quad \text{as} \quad x \rightarrow +\infty.$$

*Proof.* Since  $\nu$  is multiplicative, we have

$$\begin{aligned} \sum_{n \leq x} f(n) &= \sum_{n \leq x} \sum_{d|n} \frac{\mu(d)\nu(d)}{d} = \sum_{d \leq x} \left( \frac{x}{d} + O(1) \right) \frac{\mu(d)\nu(d)}{d} \\ &= x \sum_{d=1}^{\infty} \frac{\mu(d)\nu(d)}{d^2} - x \sum_{d > x} \frac{\mu(d)\nu(d)}{d^2} + O\left( \sum_{d \leq x} \frac{|\nu(d)|}{d} \right) \\ &= x \sum_{d=1}^{\infty} \frac{\mu(d)\nu(d)}{d^2} + O\left( x \sum_{d > x} \frac{|\mu(d)\nu(d)|}{d^2} + \left( \sum_{d \leq x} \frac{|\mu(d)\nu(d)|}{d} \right) \right) \end{aligned}$$

By (2), we have  $|\mu(d)\nu(d)| \leq d^{1-r}$ , and so we have

$$x \sum_{d > x} \frac{|\mu(d)\nu(d)|}{d^2} \leq x \sum_{d > x} \frac{1}{d^{1+r}} = O(x^{1-r})$$

and

$$\sum_{d \leq x} \frac{|\mu(d)\nu(d)|}{d} \leq \sum_{d > x} \frac{1}{d^r} = O(x^{1-r} \log x),$$

which implies

$$\sum_{n \leq x} f(n) = \alpha x + O(x^{1-r} \log x),$$

where

$$\alpha = \sum_{d=1}^{\infty} \frac{\mu(d)\nu(d)}{d^2} = \prod_p \left( 1 - \frac{\nu(p)}{p^2} \right).$$

Notice that the log factor can be dispensed with if  $r < 1$ .  $\square$

**Lemma 2.2.** *Assume that there exists a family  $\mathcal{P}_G$  of primes such that for any  $p$  in  $\mathcal{P}_G$ ,  $\omega(G, p) > 0$  and*

$$(8) \quad \sum_{p \in \mathcal{P}_G} \frac{\nu(p)}{p} = +\infty.$$

*For  $M$  large enough, one can find a finite set of primes  $\mathcal{Q}$  and a positive integer  $n_0$  such that any  $q \in \mathcal{Q}$  is larger than  $2(h+1)M$  and*

$$(9) \quad \forall m \in [1, M] : M^{-1/2} \leq \prod_{\substack{q \in \mathcal{Q} \\ q|G(n_0+m)}} \left( 1 - \frac{\nu(q)}{q} \right) \leq M^{-1/4g}.$$

*Proof.* According to (8), infinitely many primes are available for which  $\nu(p) > 0$  and  $\omega(G, p) \neq 0$  such that over those  $\sum \frac{\nu(p)}{p} = +\infty$  and then the infinite product  $\prod \left( 1 - \frac{\nu(p)}{p} \right)$  tends to zero over such primes. This allows us to repeat the proof of Lemma 1 of [6] to construct  $\mathcal{Q}$  and  $n_0$ .  $\square$

**Lemma 2.3.** *Let  $a, b, c, k$  be positive integers with  $bk > 4a$  and  $u(n, j)$  be real numbers with*

$$(10) \quad 0 \leq u(n, j) \leq 1 \quad \text{and} \quad \forall n \geq 2: \sum_{j \leq bk} u(n, j) \leq \frac{ck}{n}.$$

We have

$$(11) \quad \sum_{a \leq j \leq bk} \prod_{2 \leq n} \left(1 - \frac{1}{n^r}\right)^{u(n, j)} \geq \frac{bk}{4} \exp\left(-\frac{4\eta c}{b}\right).$$

where  $\eta = \frac{1}{r} (\zeta(1+r) - 1)$ .

*Proof.* Let us call  $S$  the sum which appears in (11). We denote as “bad guys” those integers  $j$  for which the product is at most  $\lambda = \exp(-\frac{4\eta c}{b})$  and “good guys” the other ones. A lower bound for  $S$  is  $\lambda$  times the number of good guys, and so it is enough to find an upper bound for the number of bad guys. It will be possible because we have an upper bound for the total number of the  $u(n, j)$ .

It will be always implicit that the integers  $j$  are always limited to  $a \leq j \leq bk$  and the integers  $n$  to  $n \geq 2$ ; let us define  $\mathcal{B} = \{j: \sum_n u(n, j) \log(1 - \frac{1}{n^r}) \leq \log \lambda\}$  and  $\mathcal{G} = [a, bk] \setminus \mathcal{B}$ . We have

$$\sum_j \left( \sum_n u(n, j) \log \left(1 - \frac{1}{n^r}\right) \right) \leq \sum_{j \in \mathcal{B}} \left( \sum_n u(n, j) \log \left(1 - \frac{1}{n^r}\right) \right) \leq \text{Card}(\mathcal{B}) \log \lambda.$$

In the other direction, we have

$$\sum_j \left( \sum_n u(n, j) \log \left(1 - \frac{1}{n^r}\right) \right) = \sum_n \log \left(1 - \frac{1}{n^r}\right) \sum_j u(n, j) \geq ck \sum_n \frac{1}{n} \log \left(1 - \frac{1}{n^r}\right).$$

Using the fact that  $\log \left(1 - \frac{1}{n^r}\right) \geq \frac{-2}{rn^r}$ , we get

$$\frac{4\eta c}{b} \text{Card}(\mathcal{B}) \leq \frac{2ck}{r} \sum_n \frac{1}{n^{1+r}} = 2ck\eta,$$

whence

$$\text{Card}(\mathcal{B}) \leq \frac{bk}{2}$$

and so, using (10)

$$\text{Card}(\mathcal{G}) \geq \frac{bk}{2} - a \geq \frac{bk}{4}$$

which implies

$$S \geq \text{Card}(\mathcal{G}) \lambda \geq \frac{bk}{4} \exp\left(-\frac{4\eta c}{b}\right).$$

□

The working engine of the present paper is the following sieve result which is Proposition 1 in [6].

**Lemma 2.4.** *Let  $M$ ,  $\mathcal{Q}$  and  $n_0$  satisfy Lemma 2.2 and let  $H$  be the polynomial defined by*

$$(12) \quad H(x) = \prod_{m=1}^M G(Qx + n_0 + m), \quad \text{where} \quad Q = \prod_{q \in \mathcal{Q}} q.$$

*There exist infinitely many integers  $x$  such that  $H(x)$  has no other prime factor in the interval  $[2(h+1)M, x^{1/7gM}]$  than those from  $\mathcal{Q}$ .*

### 3. PROOF OF THE MAIN RESULT

In order to prove Theorem 1.1, it is enough to prove that for all  $m \in [1, M]$ ,  $f(G(Qx + n_0 + m))$  tends to zero with  $M$ , and

$$(13) \quad \sum_{m=1}^M f(G(Qx + n_0 + m)) \geq 1.$$

Using the Proposition 2.4 we can find sufficiently large an integer  $x$  such that  $H(x) = \prod_{m=1}^M G(Qx + n_0 + m)$  has no prime factor in the interval  $[2(h+1)M, x^{1/7gM}]$ , except those from  $\mathcal{Q}$ . So we have only three type of prime divisors :

- 1) Belong to  $\mathcal{Q}$ ;
- 2) Smaller than  $2(h+1)M$ ;
- 3) Larger than  $x^{1/7gM}$ .

Since  $G(Qx + n_0 + m) = O(x^g)$ , number of large divisors is at most  $7g^2M$ . Thus for sufficiently large  $x$ ,

$$(14) \quad \prod_{\substack{q|G(Qx+n_0+m) \\ q > x^{1/7gM}}} \left(1 - \frac{\nu(q)}{q}\right) \leq \prod_{\substack{q|G(Qx+n_0+m) \\ q > x^{1/7gM} \\ \nu(q) < 0}} \left(1 + \frac{1}{q^r}\right) \leq (1 + x^{-r/7gM})^{7g^2M} \leq \frac{3}{2},$$

and

$$(15) \quad \prod_{\substack{q|G(Qx+n_0+m) \\ q > x^{1/7gM}}} \left(1 - \frac{\nu(q)}{q}\right) \geq \prod_{\substack{q|G(Qx+n_0+m) \\ q > x^{1/7gM} \\ \nu(q) > 0}} \left(1 - \frac{1}{q^r}\right) \geq (1 - x^{-r/7gM})^{7g^2M} \geq \frac{1}{2}.$$

Considering (9) and (14) we write

$$(16) \quad f(G(Qx + n_0 + m)) \leq \frac{3}{2} M^{-1/4g} \prod_{q \leq 2(h+1)M} \left(1 + \frac{1}{q^r}\right) \ll_r M^{-1/4g} \log M$$

an expression which tends to zero with  $M$ .

Also we have

$$\begin{aligned} \sum_{m=1}^M f(G(Qx + n_0 + m)) &\geq \sum_{m=1}^M \prod_{\substack{q|G(Qx+n_0+m) \\ \nu(q)>0}} \left(1 - \frac{\nu(q)}{q}\right) \\ &\geq \frac{1}{2} M^{-1/2} \sum_{m=1}^M \prod_{\substack{q|G(Qx+n_0+m) \\ q \leq 2(h+1)M}} \left(1 - \frac{1}{q^r}\right). \end{aligned}$$

So its enough to show that

$$\sum_{m=1}^M \prod_{\substack{q|G(Qx+n_0+m) \\ q \leq 2(h+1)M}} \left(1 - \frac{1}{q^r}\right) \geq 2M^{1/2}.$$

We are going to apply Lemma 2.3. We let  $u(n, j) = 1$  when  $n \leq 2(h+1)M$  is a prime which divides  $G(Qx + n_0 + j)$  and 0 otherwise; we thus have

$$\sum_{m=1}^M \prod_{\substack{q|G(Qx+n_0+m) \\ q \leq 2(h+1)M}} \left(1 - \frac{1}{q^r}\right) = \sum_{j=1}^M \prod_{n \geq 2} \left(1 - \frac{1}{n^r}\right)^{u(n,j)}.$$

In order to apply Lemma 2.3, we take  $a = b = 1$ ,  $k = M$  and we have to show that there exists  $c$  such that  $\sum_{j \leq M} u(n, j) \leq \frac{cM}{n}$ . If  $n$  is a fixed divisor of  $G$ , we recall that  $n \leq h$  and we have

$$\sum_{j \leq M} u(n, j) = M \leq \frac{hM}{n}.$$

If  $n$  is not a fixed divisor of  $G$ , for some  $j \in [1, M]$ ,  $n \mid G(Qx + n_0 + j)$ . Indeed the number of such  $js$  is  $\omega(G, n)$  in any interval of length  $n$ , so the total number of those in  $[1, M]$  is  $\left(\frac{M}{n} + 1\right) \omega(G, n)$ . Hence we have

$$\sum_{j \leq M} u(n, j) \leq \left(\frac{M}{n} + 1\right) \omega(G, n) \leq g \left(\frac{M}{n} + 1\right) \leq \frac{(2h+3)gM}{n}.$$

Thus, the second part of (10) is satisfied and we can apply Lemma 2.3 with  $c = g(2h+3)$ , which leads to (with a constant  $C$  depending on  $G$  only)

$$\sum_{m=1}^M \prod_{\substack{q|G(Qx+n_0+m) \\ q \leq 2(h+1)M}} \left(1 - \frac{1}{q^r}\right) \geq CM \geq 2M^{1/2},$$

when  $M$  is large enough. This ends the proof of Theorem 1.1.

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