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Nash embedding, shape operator and Navier-Stokes equation on a Riemannian manifold

Shizan Fang^{1*}

¹I.M.B, Université de Bourgogne, BP 47870, 21078 Dijon, France

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Abstract

What is the suitable Laplace operator on vector fields for the Navier-Stokes equation on a Riemannian manifold? In this note, by considering Nash embedding, we will try to elucidate different aspects of different Laplace operators such as de Rham-Hodge Laplacian as well as Ebin-Marsden's Laplacian. A probabilistic representation formula for Navier-Stokes equations on a general compact Riemannian manifold is obtained when de Rham-Hodge Laplacian is involved.

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1 Introduction

The Navier-Stokes equation on a domain of \mathbb{R}^n or on a torus \mathbb{T}^n ,

$$\begin{cases} \partial_t u + (u \cdot \nabla)u - \nu \Delta u + \nabla p = 0, \\ \nabla \cdot u = 0, \quad u|_{t=0} = u_0, \end{cases} \quad (1.1)$$

describe the evolution of the velocity u of an incompressible viscous fluid with kinematic viscosity $\nu > 0$, as well as the pressure p . Such equation attracts the attention of many researchers, with an enormous quantity of publications in the literature. The model of periodic boundary conditions is introduced to simplify mathematical considerations. There is no doubt on the importance of the Navier-Stokes equation on a Riemannian manifold, which is more suitable for models in aerodynamics, meteorology, and so on.

In a seminal paper [11], the Navier-Stokes equation has been considered on a compact Riemannian manifold M using the framework of the group of diffeomorphisms of M initiated by V. Arnold in [5]; the Laplace operator involved in the text of [11] is de Rham-Hodge Laplacian, however, the authors said in the note added in proof that the convenient Laplace operator comes from deformation tensor. Let's give a brief explanation. Let ∇ be the Levi-Civita connection on M , for a vector field A on M , the deformation tensor $\text{Def}(A)$ is a symmetric tensor of type $(0, 2)$ defined by

$$(\text{Def } A)(X, Y) = \frac{1}{2} \left(\langle \nabla_X A, Y \rangle + \langle \nabla_Y A, X \rangle \right), \quad X, Y \in \mathcal{X}(M), \quad (1.2)$$

*Email: Shizan.Fang@u-bourgogne.fr.

where $\mathcal{X}(M)$ is the space of vector fields on M . Formula (1.2) says that $\text{Def } A$ is the symmetrized part of ∇A . Then $\text{Def} : \text{TM} \rightarrow \text{S}^2\text{T}^*M$ maps a vector field to a symmetric tensor of type $(0, 2)$. Let $\text{Def}^* : \text{S}^2\text{T}^*M \rightarrow \text{TM}$ be the adjoint operator. Following [22] or [26], the Ebin-Marsden's Laplacian $\hat{\square}$ is defined by

$$\hat{\square} = 2\text{Def}^* \text{Def}. \quad (1.3)$$

The following formula holds (see [26] or [3]),

$$\hat{\square}A = -\Delta A - \text{Ric}(A) - \nabla \text{div}(A), \quad A \in \mathcal{X}(M) \quad (1.4)$$

where $\Delta A = \text{Trace}(\nabla^2 A)$. The Ebin-Marsden's Laplacian $\hat{\square}$ has been used in [23] to solve Navier-Stokes equation on a compact Riemannian manifold; recently in [24] to the case of Riemannian manifolds with negative Ricci curvature : it is quite convenient with sign minus in formula (1.4). In [7], the authors gave several arguments from physics in order to convince the relevance of $\hat{\square}$.

On the other hand, the De Rham-Hodge Laplacian $\square = dd^* + d^*d$ was widely used in the field of Stochastic Analysis (see for example [21, 16, 6, 12, 13, 25]). The Navier-Stokes equation with \square on the sphere S^2 was considered by Temam and Wang in [28], the case where Ricci tensor is positive. By Bochner-Weitzenböck formula (see [21]):

$$\square = -\Delta + \text{Ric}, \quad (1.5)$$

examples of positive Ricci tensor are now comfortable with \square . Besides, in [17], S. Kobayashi pleaded for the De Rham-Hodge Laplacian \square in the formalism of the Navier-Stokes equation on manifolds with curvature.

During last decade, a lot of works have been done in order to establish V. Arnold's variational principle [5] for Navier-Stokes on manifolds (see for example [1, 2, 3, 9, 20, 4]). Connections between Navier-Stokes equations and stochastic evolution also have a quite long history: it can be traced back to a work of Chorin [8]. In [15], a representation formula using noisy flow paths for 3-dimensional Navier-Stokes equation was obtained. An achievement has been realized by Constantin and Iyer in [10] by using stochastic flows. We also refer to [10] for a more complete description on the history of the developments. In [3], it was showed that the Ebin-Marsden's Laplacian is naturally involved from the point of view of variational principle; while in [14], the De Rham-Hodge Laplacian was showed to be natural for obtaining probabilistic representations.

2 The case of \mathbb{S}^n

For reader's convenience, we first introduce some elements in Riemannian manifolds. Let M be a compact Riemannian manifold, and $\varphi : M \rightarrow M$ a C^1 -diffeomorphism; the pull-back φ_*A by φ of a vector field A is defined by

$$(\varphi_*A)(x) = d\varphi(\varphi^{-1}(x))A_{\varphi^{-1}(x)}, \quad x \in M;$$

the pull-back $\varphi^*\omega$ by φ of a differential form ω is defined by

$$\langle \varphi^*\omega, A \rangle_x = \langle \omega_{\varphi(x)}, d\varphi(x)A_{\varphi(x)} \rangle \quad x \in M.$$

Let V be a vector field on M , φ_t the group of diffeomorphisms associated to V , then the Lie derivative $\mathcal{L}_V A$ of A with respect to V is defined by

$$\mathcal{L}_V A = \lim_{t \rightarrow 0} \frac{(\varphi_t^{-1})_* A - A}{t},$$

and the Lie derivative $\mathcal{L}_V \omega$ by

$$\mathcal{L}_V \omega = \lim_{t \rightarrow 0} \frac{\varphi_t^* \omega - \omega}{t}.$$

We have $\mathcal{L}_V A = [V, A]$. For a vector field A on M , the divergence $\operatorname{div}(A)$ of A is defined by

$$\int_M \langle \nabla f, A \rangle dx = - \int_M f \operatorname{div}(A) dx,$$

where dx is the Riemannian volume of M . If ∇ denotes the Levi-Civita covariant derivative, then

$$\operatorname{div}(A)(x) = \sum_{i=1}^n \langle \nabla_{v_i} A, v_i \rangle_{T_x M}$$

for any orthonormal basis $\{v_1, \dots, v_n\}$ of $T_x M$. A vector field A on M is said to be of divergence free if $\operatorname{div}(A) = 0$. It is important to emphasize that $\mathcal{L}_V A$ is of divergence free if V and A are (see [14]).

In order to understand the geometry of diffusion processes on a manifold M , Elworthy, Le Jan and Li in [13] embedded M into a higher dimensional Riemannian manifold so that the De Rham-Hodge Laplacian \square on differential forms admits suitable decompositions as sum of squares of Lie derivative of a family of vector fields. More precisely, let $\{A_1, \dots, A_N\}$ be a family of vector fields on M , assume they satisfy

$$|v|_{T_x M}^2 = \sum_{i=1}^N \langle A_i(x), v \rangle^2, \quad v \in T_x M, \quad x \in M, \quad (2.1)$$

$$\sum_{i=1}^N \nabla_{A_i} A_i = 0, \quad (2.2)$$

and

$$\sum_{i=1}^N A_i \wedge \nabla_V A_i = 0, \quad \text{for any vector field } V. \quad (2.3)$$

Then they obtained in [13], for any differential form ω on M ,

$$\square \omega = - \sum_{i=1}^N \mathcal{L}_{A_i}^2 \omega. \quad (2.4)$$

There is a one-to-one correspondence between the space of vector fields and that of differential 1-forms. Given a vector field A (resp. differential 1-form ω), we shall denote by A^\flat (resp. ω^\sharp) the corresponding differential 1-form (resp. vector field). The action of the de Rham-Hodge Laplacian \square on the vector field A is defined as follows:

$$\square A := (\square A^\flat)^\sharp.$$

Surprising enough it was remarked in [14] that the decomposition (2.4) is in general no more valid if ω is replaced by a vector field A . It is why probabilistic representation formulae for Navier-Stokes equations hold only, until now, on symmetric Riemannian manifolds.

Proposition 2.1. Let $\{A_1, \dots, A_N\}$ be a family of vector fields satisfying (2.1)–(2.3). Define

$$\mathcal{T}_1(B) = \sum_{i=1}^N \operatorname{div}(A_i) \mathcal{L}_{A_i} B, \quad B \in \mathcal{X}(M). \quad (2.5)$$

Then \mathcal{T}_1 is a tensor : $\mathcal{X}(M) \rightarrow \mathcal{X}(M)$ such that

$$\sum_{i=1}^N \mathcal{L}_{A_i}^2 B = -\square B - 2\mathcal{T}_1(B). \quad (2.6)$$

Proof. Conditions (2.2) and (2.3) imply the following identity (see [13, 14])

$$\sum_{i=1}^N \operatorname{div}(A_i) A_i = 0. \quad (2.7)$$

By torsion free, $\mathcal{L}_{A_i} B = [A_i, B] = \nabla_{A_i} B - \nabla_B A_i$. Then according to (2.7),

$$\sum_{i=1}^N \operatorname{div}(A_i) \mathcal{L}_{A_i} B = - \sum_{i=1}^N \operatorname{div}(A_i) \nabla_B A_i.$$

It follows that $\mathcal{T}_1(fB) = f\mathcal{T}_1(B)$ for any smooth function f on M . Concerning equality (2.6), it was already calculated in proof of Theorem 3.7 of [14] that

$$\int_M \Delta(\omega(B)) dx = - \int_M (\square\omega)(B) dx - \int_M \omega \left(\sum_{i=1}^N \mathcal{L}_{A_i}^2 B \right) dx - 2 \int_M \omega(\mathcal{T}_1(B)) dx.$$

The left hand side of above equality vanishes and $\int_M (\square\omega)(B) dx = \int_M \omega(\square B) dx$, equality (2.6) follows. \square

The tensor \mathcal{T}_1 has explicit expression on \mathbb{S}^n (see [14]). For the sake of self-contained, we give here a short presentation. We denote by $\langle \cdot, \cdot \rangle$ the canonical inner product of \mathbb{R}^{n+1} . Let $x \in \mathbb{S}^n$, the tangent space $T_x \mathbb{S}^n$ of \mathbb{S}^n at the point x is given by

$$T_x \mathbb{S}^n = \{v \in \mathbb{R}^{n+1}; \langle v, x \rangle = 0\}.$$

Then the orthogonal projection $P_x : \mathbb{R}^{n+1} \rightarrow T_x \mathbb{S}^n$ has the expression:

$$P_x(y) = y - \langle x, y \rangle x.$$

Let $\mathcal{B} = \{e_1, \dots, e_{n+1}\}$ be an orthonormal basis of \mathbb{R}^{n+1} ; then the vector field A_i defined by $A_i(x) = P_x(e_i)$ has the expression: $A_i(x) = e_i - \langle x, e_i \rangle x$ for $i = 1, \dots, n+1$. Let $v \in T_x \mathbb{S}^n$ such that $|v| = 1$, consider

$$\gamma(t) = x \cos t + v \sin t.$$

Then $\{\gamma(t); t \in [0, 1]\}$ is the geodesic on \mathbb{S}^n such that $\gamma(0) = x, \gamma'(0) = v$. We have $A_i(\gamma(t)) = e_i - \langle \gamma(t), e_i \rangle \gamma(t)$. Taking the derivative with respect to t and at $t = 0$, we get

$$(\nabla_v A_i)(x) = P_x(-\langle v, e_i \rangle x - \langle x, e_i \rangle v) = -\langle x, e_i \rangle v. \quad (2.8)$$

It follows that

$$\operatorname{div}(A_i) = -n \langle x, e_i \rangle. \quad (2.9)$$

Hence in this case it is obvious that

$$\sum_{i=1}^{n+1} \operatorname{div}(A_i)A_i = -n \sum_{i=1}^{n+1} (\langle x, e_i \rangle e_i - \langle x, e_i \rangle^2 x) = -n(x - x) = 0. \quad (2.10)$$

Replacing v by A_i in (2.8), we have $\nabla_{A_i} A_i = -\langle x, e_i \rangle e_i + \langle x, e_i \rangle^2 x$; therefore summing over i , we get

$$\sum_{i=1}^{n+1} \nabla_{A_i} A_i = 0. \quad (2.11)$$

Now let $v \in T_x \mathbb{S}^n$ and $a, b \in T_x \mathbb{S}^n$, we have

$$\begin{aligned} \langle A_i \wedge \nabla_v A_i, a \wedge b \rangle &= \langle A_i, a \rangle \langle \nabla_v A_i, b \rangle - \langle A_i, b \rangle \langle \nabla_v A_i, a \rangle \\ &= -\langle a, e_i \rangle \langle x, e_i \rangle \langle v, b \rangle + \langle x, e_i \rangle \langle b, e_i \rangle \langle v, a \rangle. \end{aligned}$$

Summing over i yields

$$\sum_{i=1}^{n+1} \langle A_i \wedge \nabla_v A_i, a \wedge b \rangle = \langle a, x \rangle \langle v, b \rangle - \langle x, b \rangle \langle v, a \rangle = 0. \quad (2.12)$$

By (2.10) and (2.11), we see that the family of vector fields $\{A_1, \dots, A_{n+1}\}$ satisfy above conditions (2.1)–(2.3). Let B be a vector field on \mathbb{S}^n ; by (2.8), $\nabla_B A_i = -\langle x, e_i \rangle B$. Using $\mathcal{L}_{A_i} B = \nabla_{A_i} B - \nabla_B A_i$ and combining with (2.9) and (2.10), we get that

$$\mathcal{T}_1(B) = \sum_{i=1}^m \operatorname{div}(A_i) \mathcal{L}_{A_i} B = -nB. \quad (2.13)$$

In what follows, we will compute $\sum_{i=1}^{n+1} \mathcal{L}_{A_i}^2 B$ directly on \mathbb{S}^n . Again by torsion free and using (2.8), we have

$$\mathcal{L}_{A_i} B = \nabla_{A_i} B - \nabla_B A_i = \nabla_{A_i} B + \langle x, e_i \rangle B,$$

and $\mathcal{L}_{A_i}^2 B = \nabla_{A_i}(\mathcal{L}_{A_i} B) - (\nabla_{\mathcal{L}_{A_i} B} A_i)$. Using (2.8) for last term, taking ∇_{A_i} on the two sides of above equality, we get

$$\mathcal{L}_{A_i}^2 B = \nabla_{A_i} \nabla_{A_i} B + \nabla_{A_i}(\langle x, e_i \rangle B) + \langle x, e_i \rangle \mathcal{L}_{A_i} B.$$

But $\nabla_{A_i}(\langle x, e_i \rangle B) = \langle A_i, e_i \rangle B + \langle x, e_i \rangle \nabla_{A_i} B$; therefore

$$\mathcal{L}_{A_i}^2 B = \nabla_{A_i} \nabla_{A_i} B + \langle A_i, e_i \rangle B + 2\langle x, e_i \rangle \nabla_{A_i} B - \langle x, e_i \rangle \nabla_B A_i.$$

We have, by (2.10)

$$\sum_{i=1}^{n+1} \langle x, e_i \rangle \nabla_{A_i} = -\frac{1}{n} \sum_{i=1}^{n+1} \operatorname{div}(A_i) \nabla_{A_i} B = 0.$$

Besides $\sum_{i=1}^{n+1} \langle A_i, e_i \rangle = \sum_{i=1}^{n+1} (1 - \langle e_i, x \rangle^2) = n$ and $\sum_{i=1}^{n+1} \langle x, e_i \rangle \nabla_B A_i = \sum_{i=1}^{n+1} \langle x, e_i \rangle^2 B = B$.

According to (2.11),

$$\Delta B = \sum_{i=1}^{n+1} \nabla_{A_i} \nabla_{A_i} B.$$

Therefore combining above calculations, we get

$$\sum_{i=1}^{n+1} \mathcal{L}_{A_i}^2 B = \Delta B + (n+1)B. \quad (2.14)$$

Note that $\text{Ric}(B) = (n-1)B$ on \mathbb{S}^n , and by Bochner-Weitzenböck formula $-\square = \Delta - \text{Ric}$, we see that (2.14) is compatible with (2.6) together with (2.13).

Proposition 2.2. *Let $\hat{\Delta}B = \sum_{i=1}^{n+1} \mathcal{L}_{A_i}^2 B$. Then the following dimension free identity holds*

$$-\int_M \langle \hat{\Delta}B, B \rangle dx = 2 \int_M |\text{Def}(B)|^2 dx - 2 \int_M |B|^2 dx, \quad \text{for any } \text{div}(B) = 0.$$

Proof. By (1.4), $\Delta B = -\hat{\square}B + \text{Ric}(B)$ if $\text{div}(B) = 0$. Then $\hat{\Delta}B = -\hat{\square}B + 2B$, due to (2.14). The result follows. \square

3 Nash embedding and sum of squares of Lie derivatives

In what follows, we will find links between above different objects when the compact Riemannian manifold M is isometrically embedded in an Euclidian space \mathbb{R}^N . Let $J : M \rightarrow \mathbb{R}^N$ be a Nash embedding, that is, for any $x \in M$,

$$|dJ(x)v|_{\mathbb{R}^N} = |v|_{T_x M}, \quad v \in T_x M. \quad (3.1)$$

Denote by $(dJ(x))^* : \mathbb{R}^N \rightarrow T_x M$ the adjoint operator of $dJ(x)$:

$$\langle (dJ(x))^* \xi, v \rangle_{T_x M} = \langle \xi, dJ(x)v \rangle_{\mathbb{R}^N}, \quad \xi \in \mathbb{R}^N, v \in T_x M.$$

For each $x \in M$, we denote $E_x = dJ(x)(T_x M)$ and E_x^\perp the orthogonal in \mathbb{R}^N to E_x . Define

$$\Lambda_x = dJ(x)(dJ(x))^*.$$

We have

$$\langle \xi - \Lambda_x \xi, dJ(x)v \rangle = \langle \xi, dJ(x)v \rangle - \langle dJ(x)(dJ(x))^* \xi, dJ(x)v \rangle = 0.$$

Therefore $\Lambda_x : \mathbb{R}^N \rightarrow E_x$ is the orthogonal projection. Set $\Lambda_x^\perp = \text{Id} - \Lambda_x$. By polarization of (3.1), we have

$$(dJ(x))^* dJ(x) = \text{Id}_{T_x M}.$$

The mapping $\Lambda : M \rightarrow L(\mathbb{R}^N, \mathbb{R}^N)$ from M to the space of linear maps of \mathbb{R}^N is smooth. For $\xi \in \mathbb{R}^N$ given, $v \in T_x M$, consider a smooth curve γ on M such that $\gamma(0) = x, \gamma'(0) = v$, we denote

$$d\Lambda_x(v)\xi = \frac{d}{dt}\Big|_{t=0} \left(\Lambda_{\gamma(t)} \xi \right) \in \mathbb{R}^N.$$

Proposition 3.1. *(see [25], ch.5) Let $x \in M$, for any $v \in T_x M$, it holds true:*

$$(i) \quad \Lambda_x \circ d\Lambda_x = d\Lambda_x \circ \Lambda_x^\perp, \quad (ii) \quad \Lambda_x^\perp \circ d\Lambda_x = d\Lambda_x \circ \Lambda_x. \quad (3.2)$$

Proof. We have $\Lambda_x(\Lambda_x\xi) = \Lambda_x\xi$. Taking the derivative with respect to x in direction v on two sides of this equality, we get

$$d\Lambda_x(v)(\Lambda_x\xi) + \Lambda_x(d\Lambda_x(v)\xi) = d\Lambda_x(v)\xi,$$

which yields the first equality in (3.2). For (ii), it suffices to use $\Lambda_x^\perp(\Lambda_x^\perp\xi) = \Lambda_x^\perp\xi$ and $d\Lambda_x^\perp(v) = -d\Lambda_x(v)$. □

Now let $\xi \in E_x$, then by (i) in (3.2), $\Lambda_x(d\Lambda_x(v)\xi) = 0$, which implies that $d\Lambda_x(v)\xi \in E_x^\perp$. This means that $d\Lambda_x(v)$ sends E_x into E_x^\perp . By (ii) in (3.2), $d\Lambda_x(v)$ sends E_x^\perp into E_x .

Now we introduce the Levi-Civita covariant derivative on M : for $B \in \mathcal{X}(M)$ and $v \in T_xM$, set

$$(\nabla_v B)(x) = (dJ(x))^* \left(\partial_v(dJ(\cdot)B)(x) \right), \quad (3.3)$$

where ∂_v denotes the derivative on the manifold M in direction of v . If $B_x = (dJ(x))^*\xi$ for a fix ξ , then

$$\langle \nabla_v B, u \rangle_{T_xM} = \langle d\Lambda_x(v)\xi, dJ(x)u \rangle_{\mathbb{R}^N}, \quad u \in T_xM.$$

The *second fundamental form* α on M is defined as follows: for $u, v \in T_xM$,

$$\alpha_x(u, v) = d\Lambda_x(u) \cdot dJ(x)v. \quad (3.4)$$

By Proposition 3.1, $\alpha_x(u, v) \in E_x^\perp$. For $X, Y \in \mathcal{X}(M)$, we define $\alpha(X, Y)(x) = \alpha_x(X_x, Y_x)$. It is clear that $\alpha(X, fY) = f\alpha(X, Y)$. In what follows, we will see that α is symmetric bilinear application: $\alpha(X, Y) = \alpha(Y, X)$.

Let X be a vector field on M , then $x \rightarrow dJ(x)X_x$ is a \mathbb{R}^N -valued function; therefore there is a function $\bar{X} : \mathbb{R}^N \rightarrow \mathbb{R}^N$ such that $\bar{X}(J(x)) = dJ(x)X_x$ for $x \in M$. Let Y be another vector field on M , and $\bar{Y} : \mathbb{R}^N \rightarrow \mathbb{R}^N$ such that $\bar{Y}(J(x)) = dJ(x)Y_x$, $x \in M$. Taking the derivative of $x \rightarrow \bar{Y}(J(x))$ along X :

$$\partial_X(\bar{Y} \circ J)(x) = \bar{Y}'(J(x)) \cdot dJ(x)X_x = \bar{Y}'(J(x))\bar{X}(J(x)) = (D_{\bar{X}}\bar{Y})(J(x)), \quad (3.5)$$

where \bar{Y}' denotes the differential of \bar{Y} and $D_{\bar{X}}$ the derivative on \mathbb{R}^N with respect to \bar{X} . It follows that

$$[\bar{X}, \bar{Y}](J) = \partial_X(\bar{Y}(J)) - \partial_Y(\bar{X}(J)). \quad (3.6)$$

Since $\bar{Y}(J)(x) \in E_x$, then $\Lambda_x(\bar{Y}(J(x))) = \bar{Y}(J(x))$. Taking the derivative with respect to X and according to definition (3.4), we get

$$\begin{aligned} \partial_X(\bar{Y}(J))(x) &= d\Lambda_x(X_x) \cdot \bar{Y}(J(x)) + \Lambda_x\left(\partial_X(\bar{Y}(J))(x)\right) \\ &= \alpha_x(X_x, Y_x) + \Lambda_x\left(\partial_X(\bar{Y}(J))(x)\right). \end{aligned}$$

Combining this with (3.6), we get

$$[\bar{X}, \bar{Y}](J) = \alpha(Y, X) - \alpha(X, Y) + \Lambda([\bar{X}, \bar{Y}](J)).$$

By property of embedding, $[\bar{X}, \bar{Y}](J(x)) \in E_x$ so that $\Lambda([\bar{X}, \bar{Y}](J)) = [\bar{X}, \bar{Y}]$. Then the symmetry of α follows.

Now according to definition (3.3) and to (3.5), we get $(\nabla_X Y)(x) = (dJ(x))^*(D_{\bar{X}}\bar{Y})(J(x))$. Therefore the following orthogonal decomposition holds

$$(D_{\bar{X}}\bar{Y})(J(x)) = dJ(x)(\nabla_X Y)(x) + \alpha_x(X_x, Y_x). \quad (3.7)$$

Now let $x \rightarrow V_x \in E_x^\perp$ be a field of normal vectors to M . Set, for $X \in \mathcal{X}(M)$,

$$\mathcal{A}(X, V)(x) = -(dJ(x))^*(\partial_X V)(x). \quad (3.8)$$

Since $x \rightarrow \langle dJ(x)Y_x, V_x \rangle_{\mathbb{R}^N} = 0$, taking the derivative with respect to X yields

$$\langle \partial_X(dJ(\cdot)Y)(x), V_x \rangle_{\mathbb{R}^N} + \langle dJ(x)Y_x, (\partial_X V)(x) \rangle_{\mathbb{R}^N} = 0.$$

By (3.7) and (3.8), we get

$$\langle \alpha(X, Y), V \rangle = \langle \mathcal{A}(X, V), Y \rangle. \quad (3.9)$$

From above expression, $\mathcal{A}(X, fV) = f\mathcal{A}(X, V)$. \mathcal{A} is called *shape operator* of M .

Proposition 3.2. *Let $B \in \mathcal{X}(M)$ defined by $B_x = (dJ(x))^*\xi$ for a fixed $\xi \in \mathbb{R}^N$. Then*

$$\nabla_X B = \mathcal{A}(X, \Lambda^\perp \xi), \quad X \in \mathcal{X}(M). \quad (3.10)$$

Proof. By (3.3), $(\nabla_v B)(x) = (dJ(x))^*(\partial_v(\Lambda\xi))$ which is equal to $-(dJ(x))^*(\partial_v(\Lambda^\perp \xi))$ as $\Lambda_x \xi + \Lambda_x^\perp = \xi$. Now definition (3.8) gives the result. \square

For the sake of self-contained, we use (3.10) to check properties (2.2), (2.3) and (2.7). Let $\mathcal{B} = \{e_1, \dots, e_N\}$ be an orthonormal basis of \mathbb{R}^N , we define

$$A_i(x) = (dJ(x))^*e_i, \quad i = 1, \dots, N.$$

Proposition 3.3. *We have*

$$\sum_{i=1}^N \nabla_{A_i} A_i = 0.$$

Proof. Let $u \in T_x M$, then by (3.10) and (3.9) respectively,

$$\langle (\nabla_{A_i} A_i)(x), u \rangle = \langle \mathcal{A}(A_i, \Lambda^\perp \xi_i), u \rangle = \langle \alpha_x((dJ(x))^*e_i, u), e_i \rangle.$$

The sum from $i = 1$ to N of above terms is basis independent. Then taking $\{e_1, \dots, e_n\} \subset E_x$ and $\{e_{n+1}, \dots, e_N\} \subset E_x^\perp$, and remarking $(dJ(x))^*e_j = 0$ for $j > n$, we have

$$\left\langle \sum_{i=1}^N (\nabla_{A_i} A_i)(x), u \right\rangle = \sum_{i=1}^n \langle \alpha_x((dJ(x))^*e_i, u), e_i \rangle = 0,$$

due to the orthogonality. The result follows. \square

Proposition 3.4. *We have*

$$\sum_{i=1}^N A_i \wedge \nabla_V A_i = 0, \quad V \in \mathcal{X}(M).$$

Proof. Let $a, b \in T_x M$, we have

$$\langle A_i \wedge \nabla_V A_i, a \wedge b \rangle = \langle A_i, a \rangle \langle \nabla_V A_i, b \rangle - \langle A_i, b \rangle \langle \nabla_V A_i, a \rangle,$$

which is equal to, by (3.10),

$$\langle A_i, a \rangle \langle \mathcal{A}(V_x, \Lambda_x^\perp e_i), b \rangle - \langle A_i, b \rangle \langle \mathcal{A}(V_x, \Lambda_x^\perp e_i), a \rangle,$$

which is equal to, by (3.9),

$$\langle e_i, dJ(x)a \rangle \langle \alpha(V_x, b), e_i \rangle - \langle e_i, dJ(x)b \rangle \langle \alpha(V_x, a), e_i \rangle.$$

Therefore

$$\sum_{i=1}^N \langle A_i \wedge \nabla_V A_i, a \wedge b \rangle = \langle \alpha(V_x, b), dJ(x)a \rangle - \langle \alpha(V_x, a), dJ(x)b \rangle = 0.$$

□

Proposition 3.5. *Let $B(x) = (dJ(x))^* \xi$ for a fixed $\xi \in \mathbb{R}^N$. Then*

$$\operatorname{div}(B) = \langle \operatorname{Trace}(\alpha), \xi \rangle_{\mathbb{R}^N}. \quad (3.11)$$

Proof. Let $\{v_1, \dots, v_n\}$ be an orthonormal basis of $T_x M$; by (3.10) and (3.9),

$$\operatorname{div}(B) = \sum_{i=1}^n \langle \mathcal{A}(v_i, \Lambda_x^\perp \xi), v_i \rangle = \sum_{i=1}^n \langle \alpha(v_i, v_i), \xi \rangle = \langle \operatorname{Trace}(\alpha), \xi \rangle_{\mathbb{R}^N}.$$

□

Theorem 3.6. *Let $\mathcal{B} = \{e_1, \dots, e_N\}$ be an orthonormal basis of \mathbb{R}^N , and $A_i(x) = (dJ(x))^* e_i$. Define*

$$\mathcal{T}_1(B) = \sum_{i=1}^N \operatorname{div}(A_i) \mathcal{L}_{A_i} B. \quad (3.12)$$

Then

$$\mathcal{T}_1(B) = -\mathcal{A}(B, \operatorname{Trace}(\alpha)). \quad (3.13)$$

Proof. We first see, by (3.11), that

$$\sum_{i=1}^N \operatorname{div}(A_i)(x) A_i(x) = \sum_{i=1}^N \langle \operatorname{Trace}(\alpha_x), e_i \rangle (dJ(x))^* e_i = (dJ(x))^* (\operatorname{Trace}(\alpha_x)) = 0,$$

since $\operatorname{Trace}(\alpha_x) \in E_x^\perp$. Therefore term (3.12) becomes $\mathcal{T}_1(B) = \sum_{i=1}^N \operatorname{div}(A_i) \nabla_B A_i$. Again using (3.10),

$$\mathcal{T}_1(B) = - \sum_{i=1}^N \langle \operatorname{Trace}(\alpha), e_i \rangle \mathcal{A}(B, \Lambda_x^\perp e_i) = -\mathcal{A}(B, \operatorname{Trace}(\alpha)).$$

□

Remark: In the case of \mathbb{S}^n , for $x \in \mathbb{S}^n$, the outer normal unit vector $\mathbf{n}(x) = x$; then for $v \in T_x M$ and $\gamma(t) = x \cos(t) + v \sin(t)$, we see that $\frac{d}{dt}|_{t=0} \mathbf{n}(\gamma(t)) = v \in T_x M$. Therefore

$$\mathcal{A}(v, \mathbf{n}) = v.$$

Using (3.13) gives $\mathcal{T}_1(B) = -nB$.

In what follows, we will give a link between the Ricci tensor and \mathcal{T}_1 . Let $R(B, Y)Z = [\nabla_B, \nabla_Y]Z - \nabla_{[B, Y]}Z$ be the curvature tensor on M . Using second fundamental form, R can be expressed by (see [18]):

$$\langle R(B, Y)Z, W \rangle = \langle \alpha(Y, Z), \alpha(B, W) \rangle - \langle \alpha(B, Z), \alpha(Y, W) \rangle. \quad (3.14)$$

Let A_i be defined as above. Define

$$\psi(B, W) = \sum_{i=1}^N \langle \alpha(B, A_i), \alpha(W, A_i) \rangle,$$

which is basis independent, symmetric bilinear form. Using (3.9), we get

$$\psi(B, W) = \sum_{i=1}^N \langle \mathcal{A}(B, \Lambda^\perp e_i), \mathcal{A}(W, \Lambda^\perp e_i) \rangle.$$

This quantity is independent of the dimension of M , but the co-dimension of M in \mathbb{R}^N . Define the tensor \mathcal{T}_2 by

$$\langle \mathcal{T}_2(B), W \rangle = \psi(B, W), \quad W \in \mathcal{X}(M).$$

The tensor \mathcal{T}_2 is directly related to the manner of the embedding M into \mathbb{R}^N .

Proposition 3.7. *It holds true*

$$\mathcal{T}_1 = -\text{Ric} - \mathcal{T}_2. \quad (3.15)$$

Proof. We have $\langle \text{Ric}(B), W \rangle = \sum_{i=1}^N \langle R(B, A_i)A_i, W \rangle$. The relation (3.14) leads $\langle \text{Ric}(B), W \rangle$ to

$$\begin{aligned} & \sum_{i=1}^N \langle \alpha(A_i, A_i), \alpha(B, W) \rangle - \sum_{i=1}^N \langle \alpha(B, A_i), \alpha(W, A_i) \rangle \\ & = \langle \mathcal{A}(B, \text{Trace}(\alpha)), W \rangle - \psi(B, W) = -\langle \mathcal{T}_1(B), W \rangle - \langle \mathcal{T}_2(B), W \rangle. \end{aligned}$$

□

The result (3.15) follows.

Theorem 3.8. *Let B be a vector field on M of divergence free, then*

$$\sum_{i=1}^N \mathcal{L}_{A_i}^2 B = -\hat{\square} B + 2\mathcal{T}_2(B). \quad (3.16)$$

Proof. Equality (3.16) follows from (2.6) and (3.15). □

4 Probabilistic representation formula for the Navier-Stokes equation

Let's first state Constantin-Iyer's probabilistic representation formula for Navier-Stokes equation (1.1) on \mathbb{T}^n .

Theorem 4.1 (Constantin–Iyer). *Let $\nu > 0$, W be an n -dimensional Wiener process, $k \geq 1$, and $u_0 \in C^{k+1,\alpha}$ a given deterministic divergence-free vector field. Let the pair (X, u) satisfy the stochastic system*

$$\begin{cases} dX_t = \sqrt{2\nu} dW_t + u_t(X_t) dt, \\ u_t = \mathbb{E}\mathbf{P}[(\nabla X_t^{-1})^*(u_0 \circ X_t^{-1})], \end{cases} \quad (4.1)$$

where \mathbf{P} is the Leray–Hodge projection and the star $*$ denotes the transposed matrix. Then u satisfies the incompressible Navier–Stokes equations (1.1).

For a given sufficiently regular initial velocity u_0 , there is a $T > 0$ such that system (4.1) admits a unique solution over $[0, T]$ and X_t is a diffeomorphism of \mathbb{T}^n (see [10]). Using the terminology of pull-back by diffeomorphism of vector fields, the following intrinsic formulation to the second identity in (4.1) was given in [14]:

$$\int_{\mathbb{T}^n} \langle u_t, v \rangle dx = \mathbb{E} \left(\int_{\mathbb{T}^n} \langle u_0, (X_t^{-1})_* v \rangle dx \right), \quad \operatorname{div}(v) = 0, \quad \forall t \geq 0,$$

which means that the evolution of u_t in the direction v is equal to the average of the evolution of v under the inverse flow X_t^{-1} in the initial direction u_0 . The generalization of Theorem 4.1 to a Riemannian manifold gave rise to a problem how to decompose the De Rham–Hodge Laplacian on vector fields as sum of squares of Lie derivatives : this has been achieved in [14] on Riemannian symmetric spaces.

The purpose of this section is to derive a Feymann–Kac type representation for solutions to the following Navier–Stokes equation on M :

$$\begin{cases} \partial_t u + \nabla_u u + \nu \square u + \nabla p = 0, \\ \operatorname{div}(u) = 0, \quad u|_{t=0} = u_0 \end{cases} \quad (4.2)$$

where $\nu > 0$.

Consider a Nash embedding $J : M \rightarrow \mathbb{R}^N$ as in Section 3; let $\mathcal{B} = \{e_1, \dots, e_N\}$ be an orthonormal basis of \mathbb{R}^N and $A_i(x) = (dJ(x))^* e_i, x \in M$. Let $\{u_t, t \in [0, T]\}$ be a time-dependent vector fields in $C^{1,\beta}$ with $\beta > 0$, then the following Stratanovich SDE on M

$$dX_t = \sqrt{2\nu} \sum_{i=1}^N A_i(X_t) \circ dW_t^i + u_t(X_t) dt, \quad X_0 = x \in M \quad (4.3)$$

admits a unique solution $\{X_t, t \in [0, T]\}$ which defines a flow of diffeomorphisms of M (see [6, 16, 12]), where $\{W_t^i; i = 1, \dots, N\}$ is a N -dimensional standard Brownian motion.

The space $\mathcal{X}(M)$ of vector fields on M , equipped with uniform norm $\|B\|_\infty = \sup_{x \in M} |B_x|_{T_x M}$, is a Banach space. We equip $L(\mathcal{X}, \mathcal{X})$, the space of linear map from $\mathcal{X}(M)$ into $\mathcal{X}(M)$, the norm of operator:

$$\|Q\| = \sup_{\|B\|_\infty \leq 1} \|QB\|_\infty.$$

For a diffeomorphism $\varphi : M \rightarrow M$, the pull-back φ_* sends $\mathcal{X}(M)$ into $\mathcal{X}(M)$, which is in the space $L(\mathcal{X}, \mathcal{X})$ with $\|\varphi_*\| \leq \|\varphi\|_{C^1}$. For a tensor \mathcal{T} which sends $\mathcal{X}(M)$ to $\mathcal{X}(M)$, we denote $(\mathcal{T}_*B)(x) = \mathcal{T}(x)B_x$, then $\|\mathcal{T}_*\| \leq \|\mathcal{T}\|_\infty$. Now we consider the following linear differential equation on $L(\mathcal{X}, \mathcal{X})$:

$$\frac{dQ_t}{dt} = \nu Q_t \cdot (X_t^{-1})_* \mathcal{T}_* (X_t)_*, \quad Q_0 = \text{identity} \quad (4.4)$$

where (X_t) is solution to SDE (4.3).

Theorem 4.2. *Let $-\tilde{\square} = \sum_{i=1}^N \mathcal{L}_{A_i}^2 + \mathcal{T}_*$. Then if $\tilde{\square}$ preserves the space of vector fields of divergence free, $u_t \in C^{2,\beta}$ is solution to*

$$\begin{cases} \partial_t u_t + \nabla_{u_t} u_t + \nu \tilde{\square} u_t + \nabla p = 0, \\ \text{div}(u_t) = 0, \quad u|_{t=0} = u_0 \end{cases} \quad (4.5)$$

if and only if, for each $v \in \mathcal{X}(M)$ with $\text{div}(v) = 0$,

$$\int_M \langle u_t, v \rangle dx = \mathbb{E} \left(\int_M \langle u_0, Q_t (X_t^{-1})_* v \rangle dx \right), \quad \forall t \in [0, T]. \quad (4.6)$$

Moreover, u_t has the following more geometric expression

$$u_t = \mathbb{E} \left[\mathbf{P} \left(\rho_t \cdot (X_t^{-1})^* (\tilde{Q}_t u_0)^{\flat} \right)^{\sharp} \right] \quad (4.7)$$

where \tilde{Q}_t is the adjoint of Q_t in the sense that $\int_M \langle \tilde{Q}_t u, v \rangle dx = \int_M \langle u, Q_t v \rangle dx$, ρ_t is the density of $(X_t)_\#(dx)$ with respect to dx and \mathbf{P} is the Leray-Hodge projection on the space of vector fields of divergence free.

Proof. Suppose (4.6) holds. By Itô formula [19, p.265, Theorem 2.1], we have

$$d(X_t^{-1})_*(v) = \sum_{i=1}^N (X_t^{-1})_*(\mathcal{L}_{A_i} v) dW_t^i + \nu \sum_{i=1}^N (X_t^{-1})_*(\mathcal{L}_{A_i}^2 v) dt + (X_t^{-1})_*(\mathcal{L}_{u_t} v) dt.$$

We have, according to (4.4) and above formula,

$$\begin{aligned} d \left[Q_t \cdot (X_t^{-1})_* v \right] &= dQ_t \cdot (X_t^{-1})_* v + Q_t \cdot d(X_t^{-1})_* v \\ &= \nu Q_t \cdot (X_t^{-1})_* \mathcal{T}_* v dt + \sum_{i=1}^N Q_t \cdot (X_t^{-1})_*(\mathcal{L}_{A_i} v) dW_t^i \\ &\quad + \nu \sum_{i=1}^N Q_t \cdot (X_t^{-1})_*(\mathcal{L}_{A_i}^2 v) dt + Q_t \cdot (X_t^{-1})_*(\mathcal{L}_{u_t} v) dt, \end{aligned}$$

since $(X_t)_*(X_t^{-1})_* = \text{identity}$. Now using definition of $\tilde{\square}$, we get

$$\begin{aligned} \int_M \langle u_t, v \rangle dx &= \int_M \langle u_0, v \rangle dx - \nu \int_0^t \mathbb{E} \left(\int_M \langle u_0, Q_s \cdot (X_s^{-1})_* (\tilde{\square} v) \rangle dx \right) ds \\ &\quad + \int_0^t \mathbb{E} \left(\int_M \langle u_0, Q_s \cdot (X_s^{-1})_* (\mathcal{L}_{u_s} v) \rangle dx \right) ds. \end{aligned}$$

Since $\tilde{\square}v$ and $\mathcal{L}_{u_s}v$ are of divergence free, to them we apply (4.6) to get

$$\mathbb{E}\left(\int_M \langle u_0, Q_s \cdot (X_s^{-1})_*(\tilde{\square}v) \rangle dx\right) = \int_M \langle u_s, \tilde{\square}v \rangle dx,$$

and

$$\mathbb{E}\left(\int_M \langle u_0, Q_s \cdot (X_s^{-1})_*(\mathcal{L}_{u_s}v) \rangle dx\right) = \int_M \langle u_s, \mathcal{L}_{u_s}v \rangle dx,$$

Therefore we obtain the following equality:

$$\int_M \langle u_t, v \rangle dx = \int_M \langle u_0, v \rangle dx - \nu \int_0^t \int_M \langle u_s, \tilde{\square}v \rangle dx ds + \int_0^t \int_M \langle u_s, \mathcal{L}_{u_s}v \rangle dx ds.$$

But the last term can be changed

$$\begin{aligned} \int_M \langle u_s, \mathcal{L}_{u_s}v \rangle dx &= \int_M \langle u_s, \nabla_{u_s}v \rangle dx - \int_M \langle u_s, \nabla_v u_s \rangle dx \\ &= \int_M \langle u_s, \nabla_{u_s}v \rangle dx - \frac{1}{2} \int_M v(|u_s|^2) dx = \int_M \langle u_s, \nabla_{u_s}v \rangle dx. \end{aligned}$$

Finally

$$\int_M \langle u_t, v \rangle dx = \int_M \langle u_0, v \rangle dx - \nu \int_0^t \int_M \langle u_s, \tilde{\square}v \rangle dx ds + \int_0^t \int_M \langle u_s, \nabla_{u_s}v \rangle dx ds.$$

It follows that for a.e. $t \geq 0$,

$$\frac{d}{dt} \int_M \langle u_t, v \rangle dx = \nu \int_M \langle u_t, \tilde{\square}v \rangle dx + \int_M \langle u_t, \nabla_{u_t}v \rangle dx.$$

Multiplying both sides by $\gamma \in C_c^1([0, \infty))$ and integrating by parts on $[0, \infty)$, we get

$$\gamma(0) \int_M \langle u_0, v \rangle dx + \int_0^\infty \int_M [\gamma'(t) \langle u_t, v \rangle + \gamma(t) \langle u_t, \nabla_{u_t}v \rangle + \nu \gamma(t) \langle u_t, \tilde{\square}v \rangle] dx dt = 0.$$

The above equation is the weak formulation of the Navier–Stokes (4.5) on the manifold M .

To prove (4.7), we note that

$$\begin{aligned} \int_M \rho_t \langle (X_t^{-1})^*(\tilde{Q}_t u_0)^\flat, v \rangle dx &= \int_M \rho_t \langle (\tilde{Q}_t u_0)^\flat, (X_t^{-1})_* v \rangle_{X_t^{-1}} dx \\ &= \int_M \rho_t(X_t) \tilde{\rho}_t \langle u_0^\flat, Q_t \cdot (X_t^{-1})_* v \rangle dx \\ &= \int_M \langle u_0^\flat, Q_t \cdot (X_t^{-1})_* v \rangle dx = \int_M \langle u_0, Q_t \cdot (X_t^{-1})_* v \rangle_{T_x M} dx \end{aligned}$$

where $\tilde{\rho}_t$ is the density of $(X_t^{-1})_\#(dx)$ with respect to dx . Now by (4.6), we have:

$$\int_M \langle u_t, v \rangle dx = \mathbb{E}\left(\int_M \rho_t \langle (X_t^{-1})^*(\tilde{Q}_t u_0)^\flat, v \rangle dx\right), \quad \operatorname{div}(v) = 0.$$

The formula (4.7) follows.

For proving the converse, we use the idea in [30, Theorem 2.3]. Let $u_t \in C^2$ be a solution to (4.5), then

$$\int_M \langle u_t, v \rangle dx = \int_M \langle u_0, v \rangle dx - \nu \int_0^t \int_M \langle u_s, \tilde{\square}v \rangle dx ds + \int_0^t \int_M \langle u_s, \mathcal{L}_{u_s}v \rangle dx ds.$$

Define

$$\tilde{u}_t = \mathbb{E} \left[\mathbf{P} \left(\rho_t \cdot (X_t^{-1})^* (\tilde{Q}_t u_0)^{\flat} \right)^{\sharp} \right].$$

Then calculations as above lead to

$$\int_M \langle \tilde{u}_t, v \rangle dx = \int_M \langle u_0, v \rangle dx - \nu \int_0^t \int_M \langle \tilde{u}_s, \tilde{\square} v \rangle dx ds + \int_0^t \int_M \langle \tilde{u}_s, \mathcal{L}_{u_s} v \rangle dx ds.$$

Let $z_t = u_t - \tilde{u}_t$; we have

$$\int_M \langle z_t, v \rangle dx = -\nu \int_0^t \int_M \langle z_s, \tilde{\square} v \rangle dx ds + \int_0^t \int_M \langle z_s, \mathcal{L}_{u_s} v \rangle dx ds.$$

It follows that (z_t) solves the following heat equation on M

$$\frac{dz_t}{dt} = -\nu \tilde{\square} z_t - \mathcal{L}_{u_t}^* z_t, \quad z_0 = 0,$$

where $\mathcal{L}_{u_t}^*$ is the adjoint operator. By uniqueness of solutions, we get that $z_t = 0$ for all $t \geq 0$. Thus $u_t = \tilde{u}_t$. Then (4.7) follows. The proof of Theorem 4.2 is complete. \square

Remark 4.3. By Proposition 2.1, $-\square B = \sum_{i=1}^N \mathcal{L}_{A_i}^2 B + 2\mathcal{T}_1(B)$; Taking $\mathcal{T} = 2\mathcal{T}_1$ in Theorem 4.2, we obtain a probabilistic representation formula for Navier-Stokes equation (4.2) on a general compact Riemannian manifold, since \square preserves the space of vector fields of divergence free. According to Theorem 3.8, the Ebin-Marsden Laplacian $\hat{\square}$ has the expression $-\hat{\square} B = \sum_{i=1}^N \mathcal{L}_{A_i}^2 B - 2\mathcal{T}_2(B)$. However, by (1.4) and (1.5),

$$\hat{\square} B = \square B + 2\text{Ric}(B) \quad \text{for } \text{div}(B) = 0;$$

therefore $\hat{\square}$ *does not* preserve the space of vector fields of divergence free, except for the case where $\text{Ric} = k \text{Id}$, that is to say that M is a Einstein manifold.

Remark 4.4. Let g be the Riemannian metric of M . A vector field A on M is said to be a Killing vector field if

$$\mathcal{L}_A g = 0.$$

Let $X, Y \in \mathcal{X}(M)$, we have, using Lie derivatives,

$$\partial_A (g(X, Y)) = (\mathcal{L}_A g)(X, Y) + g(\mathcal{L}_A X, Y) + g(X, \mathcal{L}_A Y), \quad (i)$$

and using covariant derivatives,

$$\partial_A (g(X, Y)) = g(\nabla_A X, Y) + g(X, \nabla_A Y). \quad (ii)$$

Since $\nabla_A X - \mathcal{L}_A X = \nabla_X A$, combining (i) and (ii) yields

$$(\mathcal{L}_A g)(X, Y) = g(\nabla_X A, Y) + g(X, \nabla_Y A) = 2\text{Def}(A)(X, Y).$$

That is to say that A is a Killing vector field if and only if $\text{Def}(A) = 0$, which implies $\hat{\square} A = 0$. Conversely if $\text{div}(A) = 0$ and $\hat{\square} A = 0$, then A is a Killing vector field on M (see [29]). Does it is why Ebin and Marsden said that $\hat{\square}$ is more convenient in [11] ?

Proposition 4.5. *Let ρ_t be given in Theorem 4.2; then the following stochastic partial equation (SPDE) holds*

$$d\rho_t = -\sqrt{2\nu}\langle dJ(x)\nabla\rho_t + \text{Trace}(\alpha)\rho_t, dW_t\rangle + (\nu\Delta\rho_t + \mathcal{L}_{u_t}\rho_t) dt. \quad (4.8)$$

Proof. Let f be a C^2 -function on M , by Itô formula, we have

$$df(X_t) = \sqrt{2\nu} \sum_{i=1}^N (\mathcal{L}_{A_i}f)(X_t) dW_t^i + \nu \sum_{i=1}^N (\mathcal{L}_{A_i}^2f)(X_t) dt + (\mathcal{L}_{u_t}f)(X_t) dt.$$

Since $\Delta f = \sum_{i=1}^N \mathcal{L}_{A_i}^2f$ and

$$\int_M (\mathcal{L}_{A_i}f)(X_t) dx = \int_M \mathcal{L}_{A_i}f \rho_t dx = - \int_M f (\text{div}(A_i)\rho_t + \mathcal{L}_{A_i}\rho_t) dx,$$

$$\int_M (\Delta f)(X_t) dx = \int_M \Delta f \rho_t dx = \int_M f \Delta\rho_t dx,$$

$$\int_M \mathcal{L}_{u_t}(X_t) dx = \int_M \mathcal{L}_{u_t}f \rho_t dx = \int_M f \mathcal{L}_{u_t}\rho_t dx,$$

and according to $\int_M f(X_t) dx = \int_M f \rho_t dx$, we get that ρ_t satisfies the following SPDE

$$d\rho_t = -\sqrt{2\nu} \sum_{i=1}^N (\text{div}(A_i)\rho_t + \mathcal{L}_{A_i}\rho_t) dW_t^i + (\nu\Delta\rho_t + \mathcal{L}_{u_t}\rho_t) dt. \quad (4.9)$$

Using (3.11), we have

$$\sum_{i=1}^N \text{div}(A_i)\rho_t dW_t^i = \langle \text{Trace}(\alpha)\rho_t, dW_t \rangle.$$

Besides, $\sum_{i=1}^N \langle (dJ(x))^*e_i, \nabla\rho_t \rangle dW_t^i = \sum_{i=1}^N \langle e_i, dJ(x)\nabla\rho_t \rangle dW_t^i = \langle dJ(x)\nabla\rho_t, dW_t \rangle$. Therefore (4.8) follows from (4.9). □

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