Nonparametric Estimation of the Trend in Reflected Fractional SDE

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NONPARAMETRIC ESTIMATION OF THE TREND IN REFLECTED FRACTIONAL SDE

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Abstract. This paper deals with the consistency, a rate of convergence and the asymptotic distribution of a nonparametric estimator of the trend in the Skorokhod reflection problem defined by a fractional SDE and a Moreau sweeping process.

Contents

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1. Introduction

Consider $T > 0$ and the Skorokhod reflection problem

$$\left\{ \begin{array}{l} X_\varepsilon(t) = \displaystyle\int_0^t b(X_\varepsilon(s))ds + \varepsilon B(t) + Y_\varepsilon(t) \\ -\dot{Y}_\varepsilon(t) \in \mathcal{N}_{C(t)}(X_\varepsilon(t)) \ |DY_\varepsilon|\text{-a.e. with } Y_\varepsilon(0) = x_0 \end{array} \right. ; t \in [0,T], \tag{1}$$

where $b : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function, $\varepsilon > 0$, B is a fractional Brownian motion of Hurst index $H \in]1/2, 1[$, $\dot{Y}_\varepsilon$ is the Radon-Nikodym derivative of the differential measure DY_ε of Y_ε with respect to its variation measure $|DY_\varepsilon|$, the multifunction $C : [0,T] \rightrightarrows \mathbb{R}$ is Lipschitz continuous for the Hausdorff distance and $x_0 \in C(0)$.

Roughly speaking, if Problem (1) has a solution, it coincides with the solution to $dX_\varepsilon(t) = b(X_\varepsilon(t))dt + \varepsilon dB(t)$, except when X_ε hits the frontier of C. Each time this situation occurs, X_ε is pushed inside of C with a minimal force by Y_ε. The differential inclusion defining the process Y_ε in Problem (1) is a (Moreau) sweeping process. Several authors studied Problem (1) when $H = 1/2$. For instance, the reader can refer to Bernicot and Venel [2], Slominski and Wojciechowski [21] or Castaing et al. [3]. When $H \neq 1/2$, the reader can refer to Falkowski and Slominski [8] or Castaing et al. [4]. In this last paper, the authors proved the existence, uniqueness and the convergence of an approximation scheme of the solution to Problem (1) under a *nonempty interior* condition on C (see Assumption 2.2). In fact, in all these papers, the authors studied the Skorokhod reflection problem defined by a SDE and a sweeping process for a multiplicative and/or multidimensional noise.

Key words and phrases. Nonparametric estimation ; Trend estimation ; Skorokhod reflection problem ; Sweeping process ; Fractional Brownian motion ; Stochastic differential equations.

Let $K : \mathbb{R} \to \mathbb{R}_+$ be a kernel. The paper deals with the consistency, a rate of convergence and the asymptotic distribution of the nonparametric estimator

$$\widehat{\tau}_\varepsilon(t) := \frac{1}{h_\varepsilon} \int_0^t \int_0^T K\left(\frac{s-u}{h_\varepsilon}\right) dX_\varepsilon(s)du \ ; \ t \in [0,T]$$

of the trend

$$\tau := \int_0^\cdot b(x(u))du + y - x_0$$

of Problem (1), where

$$\left\{ \begin{array}{rl} x(t) & = \displaystyle\int_0^t b(x(s))ds + y(t) \\ -\dot{y}(t) & \in \mathcal{N}_{C(t)}(x(t)) \ |Dy|\text{-a.e. with } y(0) = x_0 \end{array} \right. \tag{2}$$

and $h_\varepsilon > 0$ goes to zero when $\varepsilon \to 0$.

Along the last two decades, many authors studied statistical inference in stochastic differential equations driven by the fractional Brownian motion. Most references on the estimation of the trend component in fractional SDE deals with parametric estimators (see Kleptsyna and Le Breton [9], Tudor and Viens [22], Hu and Nualart [11], Chronopoulou and Tindel [7], Neuenkirch and Tindel [19], Mishura and Ralchenko [17], Hu et al. [12], etc.). On the nonparametric estimation of the trend component in fractional SDE, there are only few references. Saussereau [20] and Comte and Marie [6] study the consistency of some Nadaraya-Watson's-type estimators of the drift function in a fractional SDE. In [16], Mishra and Prakasa Rao established the consistency and a rate of convergence of a nonparametric estimator of the whole trend of the solution to a fractional SDE. Our paper generalizes their results to the Skorokhod reflection Problem (1). On the nonparametric estimation in Itô's calculus framework, the reader can refer to Kutoyants [13] and [14]. Up to our knowledge, there is no reference on the nonparametric estimation of the trend in reflected fractional SDE.

Section 2 deals with some preliminaries on the Skorokhod reflection problem defined by a fractional SDE and a sweeping process. Section 3 deals with the consistency, a rate of convergence and the asymptotic distribution of the estimator $\widehat{\tau}_\varepsilon(t)$

Notations and basic properties:

(1) For every $h > 0$, $K_h := 1/hK(\cdot/h)$.
(2) For every closed convex subset K of $\mathbb{R}$ and every $x \in \mathbb{R}$, $\mathcal{N}_K(x)$ is the normal cone of K at x:
$$\mathcal{N}_K(x) := \{y \in \mathbb{R} : \forall z \in K,\ y(z-x) \leqslant 0\}.$$
(3) For every $t \in]0,T]$, $\Delta_t := \{(u,v) \in [0,t]^2 : u < v\}$.
(4) For every function f from $[0,T]$ into $\mathbb{R}$ and $(s,t) \in \Delta_T$, $f(s,t) := f(t) - f(s)$.
(5) Consider $(s,t) \in \Delta_T$. The vector space of continuous functions from $[s,t]$ into $\mathbb{R}$ is denoted by $C^0([s,t],\mathbb{R})$ and equipped with the uniform norm $\|.\|_{\infty,s,t}$ defined by
$$\|f\|_{\infty,s,t} := \sup_{u\in[s,t]} |f(u)| \ ; \forall f \in C^0([s,t],\mathbb{R}),$$
or the semi-norm $\|.\|_{0,s,t}$ defined by
$$\|f\|_{0,s,t} := \sup_{u,v\in[s,t]} |f(v) - f(u)| \ ; \forall f \in C^0([s,t],\mathbb{R}).$$
Moreover, $\|.\|_{\infty,T} := \|.\|_{\infty,0,T}$ and $\|.\|_{0,T} := \|.\|_{0,0,T}$.

(6) Consider $(s,t) \in \Delta_T$. The set of all dissections of $[s,t]$ is denoted by $\mathfrak{D}_{[s,t]}$.
(7) Consider $(s,t) \in \Delta_T$. A function $f : [s,t] \to \mathbb{R}$ is of finite 1-variation if and only if,

$$\|f\|_{1\text{-var},s,t} := \sup\left\{\sum_{k=1}^{n-1} |f(t_k,t_{k+1})| \ ; \ n \in \mathbb{N}^* \text{ and } (t_k)_{k\in[\![1,n]\!]} \in \mathfrak{D}_{[s,t]}\right\} < \infty.$$

Consider the vector space

$$C^{1\text{-var}}([s,t],\mathbb{R}) := \{f \in C^0([s,t],\mathbb{R}) : \|f\|_{1\text{-var},s,t} < \infty\}.$$

The map $\|.\|_{1\text{-var},s,t}$ is a semi-norm on $C^{1\text{-var}}([s,t],\mathbb{R})$. Moreover, $\|.\|_{1\text{-var},T} := \|.\|_{1\text{-var},0,T}$.
(8) The vector space of Lipschitz continuous functions from a closed interval $I \subset \mathbb{R}$ into $\mathbb{R}$ is denoted by $\mathrm{Lip}(I)$ and equipped with the Lipschitz semi-norm $\|.\|_{\mathrm{Lip},I}$ defined by

$$\|f\|_{\mathrm{Lip},I} := \sup\left\{\frac{|f(t)-f(s)|}{|t-s|} \ ; \ s,t \in I \text{ and } s \neq t\right\}$$

for every $f \in \mathrm{Lip}(I)$. Moreover, $\|.\|_{\mathrm{Lip}} := \|.\|_{\mathrm{Lip},\mathbb{R}}$ and $\|.\|_{\mathrm{Lip},T} := \|.\|_{\mathrm{Lip},[0,T]}$.
(9) For every $L > 0$,

$$\Theta_0(L) := \{f \in \mathrm{Lip}(\mathbb{R}) : |f(0)| + \|f\|_{\mathrm{Lip}} \leqslant L\}.$$

2. Preliminaries

This section deals with some preliminaries on the Skorokhod reflection problem defined by a fractional SDE and a sweeping process.

First, the following theorem states a sufficient condition of existence and uniqueness of the solution to the unperturbed sweeping process defined by

$$\begin{cases} -\dot{y}(t) \in \mathcal{N}_{C(t)}(y(t)) \ |Dy|\text{-a.e.} \\ y(0) = y_0 \end{cases} ; t \in [0,T], \tag{3}$$

where $y_0 \in C(0)$.

Theorem 2.1. *Assume that for every $t \in [0,T]$, $C(t)$ is a compact interval of $\mathbb{R}$. Moreover, assume that there exist $r > 0$ and $a \in \mathbb{R}$ such that*

$$[a-r, a+r] \subset \mathrm{int}(C(t)) \ ; \forall t \in [0,T].$$

Then, Problem (3) has a unique continuous solution of finite 1-variation $y : [0,T] \to \mathbb{R}$ such that

$$\|y\|_{1\text{-var},T} \leqslant \max\{0, \|y_0 - a\| - r\}.$$

See Monteiro Marques [18] for a proof.

In the sequel, the multifunction C fulfills the following assumption.

Assumption 2.2. *For every $t \in [0,T]$, $C(t)$ is a compact interval of $\mathbb{R}$. Moreover, there exist $r > 0$ and a continuous selection $\gamma : [0,T] \to \mathbb{R}$ such that*

$$[\gamma(t) - r, \gamma(t) + r] \subset \mathrm{int}(C(t)) \ ; \forall t \in [0,T].$$

Let $\varphi : [0,T] \to \mathbb{R}$ be a continuous function such that $\varphi(0) = 0$ and consider the (generic) Skorokhod reflection problem

$$\begin{cases} v_\varphi(t) = \varphi(t) + w_\varphi(t) \\ -\dot{w}_\varphi(t) \in \mathcal{N}_{C_\varphi(t)}(w_\varphi(t)) \ |Dw_\varphi|\text{-a.e. with } w_\varphi(0) = x_0 \end{cases}, \tag{4}$$

where

$$C_\varphi(t) := \{v - \varphi(t) \ ; \ v \in C(t)\} \ ; \forall t \in [0,T],$$

$v_\varphi : [0,T] \to \mathbb{R}$ is a continuous function and $w_\varphi : [0,T] \to \mathbb{R}$ is a continuous function of finite 1-variation. Under Assumption 2.2, by Theorem 2.1 together with Castaing et al. [5], Lemma 2.2, Problem (4) has a unique solution. Moreover, the following proposition provides a suitable control of $w_\varphi - w_\psi$ for any continuous functions $\varphi, \psi : [0,T] \to \mathbb{R}$ such that $\varphi(0) = \psi(0) = 0$.

Proposition 2.3. *Under Assumption 2.2, for every continuous functions $\varphi, \psi : [0,T] \to \mathbb{R}$ such that $\varphi(0) = \psi(0) = 0$,*

$$\|w_\varphi - w_\psi\|_{\infty,T} \leqslant \|\varphi - \psi\|_{\infty,T}.$$

See Slominski and Wojciechowski [21], Proposition 2.3 for a proof.

Under Assumption 2.2, note that there exist $R > 0$, $N \in \mathbb{N}^*$ and $(t_0, \ldots, t_N) \in \mathfrak{D}_{[0,T]}$ such that

$$[\gamma(t_k) - R, \gamma(t_k) + R] \subset C(t)$$

for every $k \in [\![0, N-1]\!]$ and $t \in [t_k, t_{k+1}]$.

Proposition 2.4. *Consider $(s,t) \in \Delta_T$ and $\rho \in]0, R/2]$. Under Assumption 2.2, if $\|\varphi\|_{0,s,t} \leqslant \rho$, then*

$$\|w_\varphi\|_{1\text{-var},s,t} \leqslant N \sup_{u \in [0,T]} \sup_{v,w \in C(u)} |w - v|.$$

The proof of Proposition 2.4 is the same that the proof of Castaing et al. [4], Proposition 2.5 but with the upper bound for the 1-variation norm of the 1-dimensional unperturbed sweeping process provided in Theorem 2.1 instead of the corresponding upper bound in the multidimensional case provided in Castaing et al. [4], Proposition 2.1.

Since Problem (4) is equivalent to

$$\begin{cases} v_\varphi(t) = \varphi(t) + w_\varphi(t) \\ -\dot{w}_\varphi(t) \in \mathcal{N}_{C(t)}(v_\varphi(t)) \ |Dw_\varphi|\text{-a.e. with } w_\varphi(0) = x_0 \end{cases},$$

one can use the previous results of this section in order to establish the existence and uniqueness of the solution to Problems (1) and (2).

Theorem 2.5. *Under Assumption 2.2,*

(1) *Problem (1) has a unique solution $(X_\varepsilon, Y_\varepsilon)$. Moreover, its paths belong to*

$$C^{p\text{-var}}([0,T], \mathbb{R}) \times C^{1\text{-var}}([0,T], \mathbb{R})$$

for every $p > 1/H$.

(2) *Problem (2) has a unique solution (x,y). Moreover, it is a Lipschitz continuous map from $[0,T]$ into $\mathbb{R}^2$ such that*

$$\|y\|_{\mathrm{Lip},T} \leqslant \|b\|_{\mathrm{Lip}} + \|C\|_{\mathrm{Lip},T}$$

and

$$\|x\|_{\mathrm{Lip},T} \leqslant 2\|b\|_{\mathrm{Lip}} + \|C\|_{\mathrm{Lip},T}.$$

The proof of the existence of solutions to Problem (1) in Theorem 2.5 is the same that the proof of Castaing et al. [4], Theorem 3.1 but with the upper bound for the 1-variation norm of w_φ in Problem (4) provided in Proposition 2.4 instead of the corresponding upper bound in the multidimensional case provided in Castaing et al. [4], Proposition 2.5. Castaing et al. [4], Proposition 4.1 gives the uniqueness of the solution to Problem (1). Castaing et al. [5], Theorem 4.2 gives the existence, uniqueness and the regularity of the solution to Problem (2).

3. Convergence of the trend estimator

This section deals with the consistency, a rate of convergence and the asymptotic distribution of the estimator $\widehat{\tau}_\varepsilon(t)$. First, the following lemma deals with the convergence of X_ε and Y_ε when $\varepsilon \to 0$.

Lemma 3.1. *Under Assumption 2.2, if $b \in \Theta_0(L)$ with $L > 0$, then there exists a deterministic constant $\mathfrak{c}_{H,L,T} > 0$, depending only on T and L, such that*

$$\mathbb{E}(\|X_\varepsilon - x\|^2_{\infty,T}) + \mathbb{E}(\|Y_\varepsilon - y\|^2_{\infty,T}) \leqslant \mathfrak{c}_{H,L,T}\varepsilon^2.$$

Proof. Consider $H_\varepsilon := X_\varepsilon - Y_\varepsilon$ and $h := x - y$. By Proposition 2.3, for any $t \in [0,T]$,

$$\begin{aligned} |X_\varepsilon(t) - x(t)| &\leqslant 2\|H_\varepsilon - h\|_{\infty,t} \\ &\leqslant 2L\int_0^t |X_\varepsilon(s) - x(s)|ds + 2\varepsilon\|B\|_{\infty,t}. \end{aligned}$$

By Gronwall's lemma,

$$|X_\varepsilon(t) - x(t)| \leqslant 2\varepsilon\|B\|_{\infty,T}e^{2LT}.$$

Moreover,

$$\begin{aligned} |Y_\varepsilon(t) - y(t)| &\leqslant |H_\varepsilon(t) - h(t)| + |X_\varepsilon(t) - x(t)| \\ &\leqslant (TL+1)\|X_\varepsilon - x\|_{\infty,T} + \varepsilon\|B\|_{\infty,T} \\ &\leqslant \varepsilon\|B\|_{\infty,T}(2e^{2LT}(TL+1)+1). \end{aligned}$$

This concludes the proof because $\mathbb{E}(\|B\|^2_{\infty,T}) < \infty$. □

In the sequel, the bandwidth h_ε and the kernel K fulfill the following assumptions.

Assumption 3.2. *The bandwidth h_ε satisfies $\varepsilon^2 = o(h_\varepsilon^{1-H})$.*

Assumption 3.3. *The kernel K is bounded and $K^{-1}(\{0\})^c =]A, B[$ with $A < B$.*

Let us now establish the consistency and a rate of convergence for the estimator $\widehat{\tau}_\varepsilon$ of the trend τ of Problem (1).

Theorem 3.4. *Under Assumptions 2.2, 3.2 and 3.3, if $b \in \Theta_0(L)$ with $L > 0$, then there exists a deterministic constant $\mathfrak{c}_{C,H,K,L,T} > 0$, depending only on C, H, K, L and T, such that*

$$\sup_{t\in[0,T]} \mathbb{E}(|\widehat{\tau}_\varepsilon(t) - \tau(t)|^2) \leqslant \mathfrak{c}_{C,H,K,L,T}(\varepsilon^2 + h_\varepsilon^2 + \varepsilon^2 h_\varepsilon^{2H-2}).$$

Proof. First of all, for any $t \in [0,T]$,

$$\begin{aligned} \widehat{\tau}_\varepsilon(t) - \tau(t) &= \int_0^t\int_0^T K_{h_\varepsilon}(s-u)dX_\varepsilon(s)du - \int_0^t b(x(u))du - y(t) + x_0 \\ &= \alpha_\varepsilon(t) + \beta_\varepsilon(t) + \gamma_\varepsilon(t) + \zeta_\varepsilon(t) + \eta_\varepsilon(t), \end{aligned}$$

where

$$\begin{aligned} \alpha_\varepsilon(t) &:= \int_0^t\int_0^T K_{h_\varepsilon}(s-u)(b(X_\varepsilon(s)) - b(x(s)))dsdu, \\ \beta_\varepsilon(t) &:= \int_0^t\int_0^T K_{h_\varepsilon}(s-u)b(x(s))dsdu - \int_0^t b(x(u))du, \\ \gamma_\varepsilon(t) &:= \varepsilon\int_0^t\int_0^T K_{h_\varepsilon}(s-u)dB(s)du, \\ \zeta_\varepsilon(t) &:= \int_0^t\int_0^T K_{h_\varepsilon}(s-u)d(Y_\varepsilon - y)(s)du \text{ and} \\ \eta_\varepsilon(t) &:= \int_0^t\int_0^T K_{h_\varepsilon}(s-u)dy(s)du - y(t) + x_0. \end{aligned}$$

Let us find suitable controls of the supremum on $[0,T]$ of the second order moment of all these components.

- Note that

$$\begin{aligned}|\alpha_\varepsilon(t)| &= \left|\int_0^t \int_{-u/h_\varepsilon}^{(T-u)/h_\varepsilon} K(s)(b(X_\varepsilon(h_\varepsilon s+u)) - b(x(h_\varepsilon s+u)))dsdu\right| \\ &\leqslant \|b\|_{\mathrm{Lip}} \int_0^t \sup_{0\leqslant h_\varepsilon s+u\leqslant T} |X_\varepsilon(h_\varepsilon s+u) - x(h_\varepsilon s+u)|du \leqslant LT\|X_\varepsilon - x\|_{\infty,T}.\end{aligned}$$

  Then, by Lemma 3.1,

$$\sup_{t\in[0,T]} \mathbb{E}(\alpha_\varepsilon(t)^2) \leqslant L^2T^2\mathfrak{c}_{H,L,T}\varepsilon^2.$$

- Since C is a Lipschitz continuous and compact-valued multifunction, x is bounded by a deterministic constant $M>0$ depending only on C (not on b). Then,

$$\begin{aligned}|\beta_\varepsilon(t)| &= \left|\int_0^T b(x(s)) \int_{(s-t)/h_\varepsilon}^{s/h_\varepsilon} K(u)duds - \int_0^t b(x(u))du\right| \\ &= \left|\int_{-\infty}^{\infty} K(u) \int_0^T b(x(s))\mathbf{1}_{[h_\varepsilon u, h_\varepsilon u+t]}(s)dsdu - \int_0^t b(x(s))ds\right| \\ &= \left|\int_A^B K(u)\left(\int_{0\vee(h_\varepsilon u)}^{T\wedge(h_\varepsilon u+t)} b(x(s))ds - \int_0^t b(x(s))ds\right)du\right| \\ &\leqslant 2h_\varepsilon \sup_{z\in[-M,M]} |b(z)| \int_A^B K(u)|u|du.\end{aligned}$$

  Moreover, since $|b(0)| + \|b\|_{\mathrm{Lip}} \leqslant L$,

$$|\beta_\varepsilon(t)| \leqslant 2(|A|\vee|B|)Lh_\varepsilon.$$

- By Memin et al. [15], Theorem 1.1, there exists a deterministic constant $c_1>0$, only depending on H, such that

$$\begin{aligned}\mathbb{E}(\gamma_\varepsilon(t)^2) &\leqslant \varepsilon^2 t \int_0^t \mathbb{E}\left(\left|\int_0^T K_{h_\varepsilon}(s-u)dB(s)\right|^2\right)du \\ &\leqslant c_1\frac{\varepsilon^2 T}{h_\varepsilon^2}\int_0^T \left|\int_0^T K\left(\frac{s-u}{h_\varepsilon}\right)^{1/H} ds\right|^{2H} du \leqslant c_2\varepsilon^2 h_\varepsilon^{2H-2},\end{aligned}$$

  where

$$c_2 := c_1T^2\left|\int_A^B K(s)^{1/H}ds\right|^{2H}.$$

- Since the paths of $Y_\varepsilon - y$ are continuous and of finite 1-variation,

$$\begin{aligned}\zeta_\varepsilon(t) &= \int_0^T \int_0^t K_{h_\varepsilon}(s-u)dud(Y_\varepsilon - y)(s) \\ &= \int_0^T \int_{(s-t)/h_\varepsilon}^{s/h_\varepsilon} K(u)dud(Y_\varepsilon - y)(s) \\ &= \int_{-\infty}^{\infty} K(u) \int_0^T \mathbf{1}_{[h_\varepsilon u, h_\varepsilon u+t]}(s)d(Y_\varepsilon - y)(s)du \\ &= \int_A^B K(u)(Y_\varepsilon - y)(0\vee(h_\varepsilon u), T\wedge(h_\varepsilon u+t))du.\end{aligned}$$

Then, by Lemma 3.1,

$$\sup_{t\in[0,T]} \mathbb{E}(\zeta_\varepsilon(t)^2) \leqslant \mathbb{E}(\|Y_\varepsilon - y\|^2_{\infty,T}) \leqslant \mathfrak{c}_{H,L,T}\varepsilon^2.$$

- Since y is a Lipschitz continuous function (see Theorem 2.5.(2)),

$$\begin{aligned}|\eta_\varepsilon(t)| &= \left|\int_0^T \int_0^t K_{h_\varepsilon}(s-u)dudy(s) - y(t) + x_0\right| \\ &\leqslant \int_A^B K(u)|y(0\vee(h_\varepsilon u), T\wedge(h_\varepsilon u + t)) - y(0,t)|du \leqslant 2(|A|\vee|B|)\|y\|_{\mathrm{Lip},T}h_\varepsilon.\end{aligned}$$

Moreover, since $\|y\|_{\mathrm{Lip},T} \leqslant L + \|C\|_{\mathrm{Lip},T}$,

$$|\eta_\varepsilon(t)| \leqslant 2(|A|\vee|B|)(L + \|C\|_{\mathrm{Lip},T})h_\varepsilon.$$

□

Theorem 3.4 says that the quadratic risk of the estimator $\widehat{\tau}_\varepsilon(t)$ involves a squared bias of order $\varepsilon^2 + h_\varepsilon^2$ and a variance term of order $\varepsilon^2 h_\varepsilon^{2H-2}$. The best possible rate $\varepsilon^{2/(2-H)}$ is reached for a bandwidth choice of order $\varepsilon^{1/(2-H)}$.

Corollary 3.5. *Under Assumptions 2.2 and 3.3, if $h_\varepsilon = \varepsilon^{1/(2-H)}$, then*

$$\lim_{\varepsilon\to 0} \varepsilon^{\alpha-2/(2-H)} \sup_{t\in[0,T]} \mathbb{E}(|\widehat{\tau}_\varepsilon(t) - \tau(t)|^2) = 0 \ ; \ \forall \alpha > 0.$$

Corollary 3.5 is a straightforward consequence of Theorem 3.4 with $v_\varepsilon = \varepsilon^{\alpha-2/(2-H)}$ and $\alpha > 0$.

Finally, the following proposition provides the asymptotic distribution of the estimator $\widehat{\tau}_\varepsilon(t)$.

Proposition 3.6. *Under Assumptions 2.2 and 3.3, if $A \geqslant 0$, $t < T$ and $h_\varepsilon = \varepsilon^{1/(2-H)}$, then*

$$\varepsilon^{-1/(2-H)}(\widehat{\tau}_\varepsilon(t) - \tau(t) - \gamma_\varepsilon(t)) \xrightarrow[\varepsilon\to 0]{\mathbb{L}^2} \mu(t)$$

and

$$\varepsilon^{-1/(2-H)}\dot{\gamma}_\varepsilon(t) \xrightarrow[\varepsilon\to 0]{\Delta} \mathcal{N}(0, \sigma^2_{H,K}),$$

where

$$\mu(t) := (b(x(t)) - b(x(0)) + \dot{y}(t) - \dot{y}(0)) \int_A^B K(u)udu$$

and

$$\sigma^2_{H,K} := H(2H-1)\int_A^B \int_A^B |u-v|^{2H-2}K(u)K(v)dudv.$$

Proof. Since

$$\sup_{t\in[0,T]} \mathbb{E}(\alpha_\varepsilon(t)^2) + \sup_{t\in[0,T]} \mathbb{E}(\zeta_\varepsilon(t)^2) = O(\varepsilon^2)$$

as established in the proof of Theorem 3.4,

$$\varepsilon^{-1/(2-H)}(\alpha_\varepsilon(t) + \zeta_\varepsilon(t)) \xrightarrow[\varepsilon\to 0]{\mathbb{L}^2} 0.$$

Let us study the behaviour of $\varepsilon^{-1/(2-H)}(\beta_\varepsilon(t) + \eta_\varepsilon(t))$ when $\varepsilon \to 0$.

- Since $A \geqslant 0$ and $h_\varepsilon B + t < T$ for ε large enough,

$$\begin{aligned}\beta_\varepsilon(t) &= \int_A^B K(u)\left(\int_t^{h_\varepsilon u+t} b(x(s))ds - \int_0^{h_\varepsilon u} b(x(s))ds\right)du \\ &= h_\varepsilon \int_A^B K(u)u\left(\int_0^1 b(x(sh_\varepsilon u+t))ds - \int_0^1 b(x(sh_\varepsilon u))ds\right)du.\end{aligned}$$

  Therefore, by Lebesgue's theorem,

$$\lim_{\varepsilon\to 0} \varepsilon^{-1/(2-H)}\beta_\varepsilon(t) = (b(x(t)) - b(x(0)))\int_A^B K(u)udu.$$

- Since y is a Lipschitz continuous function, it is absolutely continuous. In other words, for every $(u,v) \in \Delta_T$,

$$y(v) - y(u) = \int_u^v \dot{y}(s)ds.$$

  Then,

$$\begin{aligned}\eta_\varepsilon(t) &= \int_A^B K(u)(y(t, h_\varepsilon u + t) - y(0, h_\varepsilon u))du \\ &= h_\varepsilon \int_A^B K(u)u\left(\int_0^1 \dot{y}(sh_\varepsilon u+t)ds - \int_0^1 \dot{y}(sh_\varepsilon u)ds\right)du.\end{aligned}$$

  Therefore, by Lebesgue's theorem,

$$\lim_{\varepsilon\to 0} \varepsilon^{-1/(2-H)}\eta_\varepsilon(t) = (\dot{y}(t) - \dot{y}(0))\int_A^B K(u)udu.$$

Finally,

$$\dot{\gamma}_\varepsilon(t) = \varepsilon\int_0^T K_{h_\varepsilon}(s-t)dB(s) \rightsquigarrow \mathcal{N}(0, \sigma_\varepsilon(t)^2)$$

where

$$\begin{aligned}\sigma_\varepsilon(t)^2 &:= H(2H-1)\varepsilon^2\int_0^T\int_0^T |s-r|^{2H-2}K_{h_\varepsilon}(r-t)K_{h_\varepsilon}(s-t)drds \\ &= \sigma_{H,K}\varepsilon^2 h_\varepsilon^{2H-2}.\end{aligned}$$

Therefore,

$$\varepsilon^{-1/(2-H)}\dot{\gamma}_\varepsilon(t) \xrightarrow[\varepsilon\to 0]{\Delta} \mathcal{N}(0, \sigma_{H,K}^2).$$

□

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