On the Hilbert Method in the Kinetic Theory of Multicellular Systems: Hyperbolic Limits and Convergence Proof

Mohamed Khaladi, Nisrine Outada, Nicolas Vauchelet

On the Hilbert Method in the Kinetic Theory of Multicellular Systems: Hyperbolic Limits and Convergence Proof*

Mohamed Khaladi, Nisrine Outada† and Nicolas Vauchelet‡

July 26, 2019

Abstract

We consider a system of two kinetic equations modelling a multicellular system : The first equation governs the dynamics of cells, whereas the second kinetic equation governs the dynamics of the chemoattractant. For this system, we first prove the existence of global-in-time solution. The proof of existence relies on a fixed point procedure after establishing some a priori estimates. Then, we investigate the hyperbolic limit after rescaling of the kinetic system. It leads to a macroscopic system of Cattaneo type. The rigorous derivation is established thanks to a compactness method.

Keywords Kinetic systems; Hyperbolic limit; Averaging lemma; hyperbolic limit.

1 Introduction

Our paper deals with derivation of models suitable to describe the behavior of multicellular systems from their description at the microscopic scale delivered by models derived by suitable generalizations of the kinetic theory. This problem can be viewed as a possible generalization of the celebrated sixth Hilbert problem [8] which has been object of several interesting contributions in the classical kinetic theory. The literature in the field is documented in the review papers by Perthame [19] and Saint Raymond [20]. As it is known, the time-space scaling can be referred to the so called parabolic and hyperbolic limits or equivalently low and high field limits. The parabolic limit leads to a drift–diffusion type system (or reaction–diffusion system) in which the diffusion processes dominate the behavior of the solutions. The hyperbolic limit leads to models where the influence of the diffusion terms is of lower (or equal) order of magnitude in comparison with other convective or interaction terms. Accordingly, different macroscopic models are obtained corresponding to different scaling assumptions.

The derivation of macroscopic equations from the kinetic theory description was introduced for dispersed biological entities in the pioneer paper [17] and subsequently developed by various authors as witnessed in the bibliography of the survey [4]. An interesting application has been the derivation of Keller-Segel type models. A broad bibliography has been produced on this challenging topic as reviewed in Sections 5 and 6 of the survey [5]. The rationale of the approach proposed in [17] consists in deriving a kinetic type model corresponding to the transport equation where the collision

*Dedicated to Abdelghani Bellouquid who prematurely passed away on August 2015.

†Université Cadi Ayyad, Faculté des Sciences Semlalia, LMDP, UMMISCO (IRD- UPMC), Marrakech 40000, B.P. 2390, Maroc

‡Université Paris 13, Sorbonne Paris Cité, Laboratoire Analyse Géométrie et Applications, CNRS UMR 7539, 93430 Villetaneuse, France

operator, namely the right hand side term of the kinetic equation, is perturbed by small stochastic term modeling a poisson velocity jump process. The small parameter corresponds to the entity of the perturbation, while an expansion of the dependent variable is developed in terms of powers of the said parameter. Very recent applications have been delivered in [2, 6, 18].

This approach is useful even when it developed at a formal level as it leads to interesting models at the macroscopic scale based on models of the dynamics at the microscopic scale rather than on artificial assumptions to close mass and momentum conservation equations. However, as it is known, most of the literature is developed at a formal level, where *ad hoc* assumptions are needed to prove convergence of the aforementioned power expansions. The derivation of hyperbolic models involves additional problems on the convergence of Hilbert type expansions technically related to loss of regularity. Indeed, this is the main challenge of our paper which is tackled in four sections. In more details, Section 2 presents a kinetic theory model of cross diffusion phenomena, where an hyperbolic scaling is is used to include propagation phenomena with finite speed; a binary mixture is accounted for and the statement of the initial value problem is delivered. Section 3 develops a qualitative analysis of the initial value problem and ends up with a local, in time, existence result and with the extension to arbitrarily large times. Finally, a convergence proof of an Hilbert type expansion is delivered in Section 4, however, due to technical difficulties, we restrict ourself to one dimension.

2 A kinetic model of chemotaxis

In this section we recall briefly the kinetic model presented in [18]. For this aim, let $f(t,x,v)$ and $g(t,x,v)$ denotes, respectively, the density of cells and of the chemoattractant, depending on time t, position $x \in \mathbb{R}^d$ and velocity $v \in V \subseteq \mathbb{R}^d$. Then our kinetic model of chemotaxis reads:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = L(g,f), \\ \partial_t g + v \cdot \nabla_x g = l(g) + G(f,g), \end{cases} \tag{1}$$

where the perturbation turning operators L and l model the dynamics of biological organisms by velocity-jump process, and are integral operators defined by

$$L(f) = \int_V \left(T(v,v') f(t,x,v') - T(v',v) f(t,x,v) \right) dv', \tag{2}$$

$$l(f) = \int_V \left(\mathcal{K}(v,v') f(t,x,v') - \mathcal{K}(v',v) f(t,x,v) \right) dv', \tag{3}$$

while the operator $G(f,g)$, which describe proliferation/destruction interactions, is given by

$$G(f,g) = a \langle f \rangle - b \langle g \rangle , \tag{4}$$

where a, b are real positive constants, and $\langle \cdot \rangle$ stands for the (v)-mean of a function, i.e $\langle h \rangle := \int_V h(t,x,v) dv$ for $h \in L^1(V)$. The turning kernels $T(v,v')$ and $K(v,v')$ describe the reorientation of cells, i.e the random velocity changes from the previous velocity v' to the new v. Moreover, it is assumed that the set of admissible velocities V is a spherically symmetric bounded domain of $\mathbb{R}^d$ with $V \subset B_\nu$ (the ball of radius $\nu > 0$). This corresponds to the assumption that any individual of the population chooses a direction with bounded velocity.

As it is mentioned in the introduction, our contribution in this paper will be the rigorous derivation of a diffusive type model for movement of chemotaxis, obtained as a hydrodynamic limit of the kinetic model (1). In detail, let us assume a hyperbolic scaling for the first population:

$$x \longrightarrow \varepsilon x, \quad t \longrightarrow \varepsilon t, \tag{5}$$

where $\varepsilon > 0$ is a small parameter which will be allowed to tend to zero. In this way we obtain from (1) the following scaled kinetic equation

$$\begin{cases} \partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon = \frac{1}{\varepsilon} L(g_\varepsilon, f_\varepsilon), \\ \partial_t g_\varepsilon + v \cdot \nabla_x g_\varepsilon = l(g_\varepsilon) + G(f_\varepsilon, g_\varepsilon). \end{cases} \tag{6}$$

In addition we assume that the operator L admits the following decomposition:

$$L(g_\varepsilon, f_\varepsilon) = L^0(f_\varepsilon) + \varepsilon L^1(g_\varepsilon, f_\varepsilon), \tag{7}$$

where the perturbation turning operators L^0 and L^1 are linear integral operators with respect to f_ε, and reads:

$$L^0(f_\varepsilon) = \int_V \left(T^0(v, v') f_\varepsilon(t, x, v') - T^0(v', v) f_\varepsilon(t, x, v) \right) dv', \tag{8}$$

$$L^1[g_\varepsilon](f_\varepsilon) = \int_V \left(T^1[g_\varepsilon](v, v') f_\varepsilon(t, x, v') - T^1[g](v', v) f_\varepsilon(t, x, v) \right) dv', \tag{9}$$

while the operator l is still defined by Eq. (3). In this work we consider the following turning kernels T^0, T^1, and $\mathcal{K}$ given by

$$T^0(v, v') = \frac{\mu_0}{|V|}(1 + \gamma^2 v \cdot v'), \qquad \gamma^2 \int_V v \otimes v \, dv = |V| I_d, \tag{10}$$

$$T^1[g](v, v') = \frac{\mu_1}{|V|} - \frac{\mu_2 \gamma^2}{|V|} v' \cdot \alpha(< g_\varepsilon >), \tag{11}$$

$$\mathcal{K}(v, v') = \frac{\sigma}{|V|}, \tag{12}$$

where μ_0, μ_1, μ_2, σ are real positive constants, α is a mapping $\mathbb{R} \longrightarrow \mathbb{R}^d$, and $|V|$ denotes the volume of V. Notice that since V is assumed to be spherically symmetric, the constant γ in (10) is well-defined.

With these considerations and after a straightforward calculation we obtain the following kinetic system, we refer to the paper [18] for more details,

$$\begin{cases} \partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon = \dfrac{\mu_0}{\varepsilon} \left(F_{J_\varepsilon} - f_\varepsilon \right) + \mu_1 \left(\dfrac{n_\varepsilon}{|V|} - f_\varepsilon \right) \\ \qquad\qquad\qquad\qquad -\mu_2 \gamma^2 \left(\dfrac{J_\varepsilon}{|V|} - v f_\varepsilon \right) \cdot \alpha(S_\varepsilon), \\ \partial_t g_\varepsilon + v \cdot \nabla_x g_\varepsilon = \sigma \left(\dfrac{S_\varepsilon}{|V|} - g_\varepsilon \right) + a n_\varepsilon - b S_\varepsilon, \end{cases} \tag{13}$$

where:

- The local densities $n_\varepsilon(t,x)$ and $S_\varepsilon(t,x)$ are defined by

$$n_\varepsilon(t,x) = \int_V f_\varepsilon(t,x,v)dv, \quad \text{and} \quad S_\varepsilon(t,x) = \int_V g_\varepsilon(t,x,v)dv,$$

  while the flux function $J_\varepsilon(t,x)$ fulfills

$$J_\varepsilon(t,x) = \int_V v f_\varepsilon(t,x,v)dv.$$

- The equilibrium function $F_{J_\varepsilon}(t,x,v)$ is assumed to be a linear combination of $1, v_1, \dots, v_d$:

$$F_{J_\varepsilon}(t,x,v) = \frac{1}{|V|}\left(n_\varepsilon(t,x) + \gamma^2 J_\varepsilon(t,x)\cdot v\right). \tag{14}$$

  This equilibrium function is such that (see (10) for the definition of γ)

$$\int_V F_{J_\varepsilon}(t,x,v)\,dv = n_\varepsilon(t,x), \qquad \int_V v F_{J_\varepsilon}(t,x,v)\,dv = J_\varepsilon(t,x).$$

This system is completed with initial condition

$$f_\varepsilon(0,x,v) = f_\varepsilon^0(x,v), \quad \text{and} \quad g_\varepsilon(0,x,v) = g_\varepsilon^0(x,v). \tag{15}$$

3 Existence result

The existence of solutions to kinetic models of chemotaxis coupled to parabolic or elliptic system for the chemoattractant concentration has been studied in several papers (see for instance [7, 10, 13, 24]). However, the study of coupled kinetic systems like Eq. (13) is less common.

The aim of this section is to study the Cauchy problem (13)-(15) for fixed $\varepsilon > 0$. More in detail we will state and prove an existence and uniqueness result for the kinetic model (13)-(15) in Theorem 3.2. The proof is based on a fixed point procedure, after establishing some a priori estimates.

We now introduce some notations which will be used throughout this section: $X_T := L^\infty((0,T)\times\mathbb{R}^d\times V)$ stands for the Lebesgue space of essentially bounded measurable functions, with norm given by

$$\|f\|_{L^\infty_{t,x,v}} = \inf\left\{C \geq 0;\ |f(x)| \leq C \text{ for almost every } (t,x,v)\in(0,T)\times\mathbb{R}^d\times V\right\},$$

and we have analogous definitions for L^∞_x, $L^\infty_{t,x}$ and $L^\infty_{x,v}$. Moreover, we define X_T^+ the subspace of X_T with nonnegative functions.

We assume that α is a bounded and globally Lipschitz continuous function on $\mathbb{R}$: There exists $\alpha_\infty > 0$, $L_\alpha > 0$ such that

$$\forall\, S_1, S_2 \in \mathbb{R}, \qquad \|\alpha(S_1)\| \leq \alpha_\infty, \quad \|\alpha(S_1) - \alpha(S_2)\| \leq L_\alpha |S_1 - S_2|. \tag{16}$$

Definition 3.1 *We say that f is a weak solution of (13)–(15) on X_T for $T > 0$, if $f \in X_T$ and*

satisfies

$$\begin{cases} \displaystyle\int_{(0,T)\times\mathbb{R}^d\times V} (\partial_t\varphi + v\cdot\nabla_x\varphi)\, f\, dxdvdt = -\frac{\mu_0}{\varepsilon}\int_{(0,T)\times\mathbb{R}^d\times V} (F_J - f)\,\varphi\, dxdvdt \\ \qquad\qquad -\mu_1\displaystyle\int_{(0,T)\times\mathbb{R}^d\times V}\left(\frac{n}{|V|} - f\right)\varphi\, dxdvdt - \int_{\mathbb{R}^d\times V} f^0(x,v)\ \varphi(0,x,v)\, dxdv \\ \qquad\qquad +\mu_2\gamma^2\displaystyle\int_{(0,T)\times\mathbb{R}^d\times V}\left(\frac{J}{|V|} - vf\right)\cdot\alpha(S)\,\varphi\, dxdvdt, \\ \displaystyle\int_{(0,T)\times\mathbb{R}^d\times V} (\partial_t\varphi + v\cdot\nabla_x\varphi)\, g\, dxdvdt = -\sigma\int_{(0,T)\times\mathbb{R}^d\times V}\left(\frac{n}{|V|} - g\right)\varphi\, dxdvdt \\ \qquad\qquad +\displaystyle\int_{(0,T)\times\mathbb{R}^d\times V}(an - bS)\,\varphi\, dxdvdt - \int_{\mathbb{R}^d\times V} g^0(x,v)\ \varphi(0,x,v)\, dxdv, \end{cases}$$

for any test function $\varphi \in \mathcal{D}([0,T)\times\mathbb{R}^d\times V)$.

We now state the main result of this section.

Theorem 3.2 (Existence of weak solutions) *Let* $(f^0, g^0) \in L^\infty_{x,v}\times L^\infty_{x,v}$ *be nonnegative and assume that* α *satisfies assumption* (16). *Then the Cauchy problem* (13)-(15) *has a unique global weak solution* (f,g), *with* $(f,g)\in X_T^+\times X_T^+$.

Moreover, if $(f^0,g^0)\in L^1_{x,v}\times L^1_{x,v}$, *then for any* $t\in[0,T]$, $\|f(t,\cdot,\cdot)\|_{L^1_{x,v}} = \|f^0\|_{L^1_{x,v}}$ *and* $\|g(t,\cdot,\cdot)\|_{L^1_{x,v}} = \frac{a}{b}\|f^0\|_{L^1_{x,v}}(1-e^{-b|V|t}) + \|g^0\|_{L^1_{x,v}}e^{-b|V|t}$.

The proof of Theorem 3.2 is divided into several steps. We first establish some a priori estimates thanks to a characteristics method. Then, applying a fixed point procedure, we establish the existence of a local in time solution. This solution can be extended for arbitrary time $T>0$ and therefore we get a global existence result.

3.1 A priori estimates

We start with the following a priori estimates.

Lemma 3.3 (A priori estimates) *Let* $T>0$ *and suppose that* α *satisfies assumption* (16). *Let* (f^0,g^0) *be given in* $L^\infty_{x,v}\times L^\infty_{x,v}$. *Let* (f,g) *be a weak solution of* (13)-(15) *such that* $(f,g)\in X_T^+\times X_T^+$ *and* $(\nabla_x f, \nabla_x g)\in {X_T}^d\times {X_T}^d$. *Then* (f,g) *satisfies the following estimates:*

$$\|n\|_{L^\infty_{t,x}} + \|f\|_{X_T} \le C_1\|f^0\|_{L^\infty_{x,v}}, \tag{17}$$

$$\|S\|_{L^\infty_{t,x}} + \|g\|_{X_T} \le C_2\big(\|f^0\|_{L^\infty_{x,v}} + \|g^0\|_{L^\infty_{x,v}}\big). \tag{18}$$

Furthermore, if the initial data $(f^0,g^0)\in L^1_{x,v}\times L^1_{x,v}$ *then we have,* $\forall\, t\in[0,T]$, $\|f(t,\cdot,\cdot)\|_{L^1_{x,v}} = \|f^0\|_{L^1_{x,v}}$, *and*

$$\|g(t,\cdot,\cdot)\|_{L^1_{x,v}} = \frac{a}{b}\|f^0\|_{L^1_{x,v}}(1-e^{-b|V|t}) + \|g^0\|_{L^1_{x,v}}e^{-b|V|t}.$$

Moreover, if the initial data are given in $W^{1,\infty}_{x,v}\times W^{1,\infty}_{x,v}$ *and assuming that* $\alpha\in C^1(\mathbb{R})$, *then*

$$\|\nabla_x n\|_{(L^\infty_{x,v})^d} + \|\nabla_x f\|_{(X_T)^d} \le C_3\big(\|\nabla_x f^0\|_{(L^\infty_{x,v})^d} + \|\nabla_x g^0\|_{(L^\infty_{x,v})^d}\big), \tag{19}$$

$$\|\nabla_x S\|_{(L^\infty_{x,v})^d} + \|\nabla_x g\|_{(X_T)^d} \le C_4\big(\|\nabla_x f^0\|_{(L^\infty_{x,v})^d} + \|\nabla_x g^0\|_{(L^\infty_{x,v})^d}\big), \tag{20}$$

where the constants C_i, $i=1,2,3,4$, *are independents of time* $T>0$.

Proof. 1. First we begin with the proof of Eq. (17). For this purpose we write the first equation of system (13) in the following way

$$\partial_t f + v \cdot \nabla_x f + Kf = R_1, \tag{21}$$

where the functions K and R_1 are given by

$$K = \frac{\mu_0}{\varepsilon} + \mu_1 - \mu_2 \gamma^2 v \cdot \alpha(S), \text{ and } R_1 = \frac{\mu_0}{\varepsilon} F_J + \frac{\mu_1 n}{|V|} - \mu_2 \gamma^2 J \cdot \alpha(S), \tag{22}$$

where the expression of F_J is given in (14). Integrating (21) along the characteristics, we get

$$\begin{aligned} f(t,x,v) = &\exp\left(\int_t^0 K(\tau, \widetilde{x}_\tau, v) d\tau\right) f^0(x - tv, v) \\ &+ \int_0^t \exp\left(\int_t^s K(\tau, \widetilde{x}_\tau, v) d\tau\right) R_1(x, \widetilde{x}_s, v) ds, \end{aligned} \tag{23}$$

where we set $\widetilde{x}_\tau = x + (\tau - t)v$ (this notation will be used throughout this section). Moreover, using assumption (16), for each $0 \le s \le \tau \le t \le T$ we have

$$|K(\tau, \widetilde{x}_\tau, v)| \le \frac{\mu_0}{\varepsilon} + \mu_1 + \mu_2 \alpha_\infty \gamma^2 \nu. \tag{24}$$

It follows

$$\exp\left(\int_t^s K(\tau, \widetilde{x}_\tau, v) d\tau\right) \le e^{C_1 T} \le C_2. \tag{25}$$

According to Eqs. (23) and (25) we write

$$f(t,x,v) \le C_2 f^0(x - tv, v) + C_2 \int_0^t |R_1(s, \widetilde{x}_s, v)| ds. \tag{26}$$

We estimate the last term of the right hand side of the later inequality as follows:

$$\begin{aligned} \int_0^t |R_1(s, \widetilde{x}_s, v)| ds \le &\left(\frac{\mu_0}{\varepsilon|V|} + \frac{\mu_1}{|V|}\right) \int_0^t n(s, \widetilde{x}_s) ds \\ &+ \left(\frac{\mu_0 \gamma^2 \nu}{\varepsilon |V|} + \mu_2 \alpha_\infty \gamma^2\right) \int_0^t |J(s, \widetilde{x}_s)| ds \\ \le &\left[\frac{\mu_0}{\varepsilon|V|} + \frac{\mu_1}{|V|} + \frac{\mu_0 \gamma^2 \nu^2}{\varepsilon |V|} + \mu_2 \alpha_\infty \gamma^2 \nu\right] \int_0^t n(s, \widetilde{x}_s)\, ds \\ \le &C_3 \int_0^t \|n(s,.)\|_{L^\infty_x}\, ds. \end{aligned}$$

Injecting this last estimate in (26), we obtain

$$f(t,x,v) \le C_2 \|f^0\|_{L^\infty_{x,v}} + C_4 \int_0^t \|n(s,\cdot)\|_{L^\infty_x}\, ds. \tag{27}$$

An integration with respect to v provides

$$\|n(t,\cdot)\|_{L^\infty_x} \le C_2 |V|\, \|f^0\|_{L^\infty_{x,v}} + C_4 |V| \int_0^t \|n(s,\cdot)\|_{L^\infty_x}\, ds. \tag{28}$$

Therefore, applying Gronwall's inequality we get

$$\|n(t,\cdot)\|_{L^\infty_x} \leq C\|f^0\|_{L^\infty_{x,v}}. \tag{29}$$

Using Eq. (27) together with (29), we obtain a similar bound on f in $L^\infty_{x,v}$. This completes the proof of the first assertion (17).

2. The proof of (18) is straightforward and follows the same ideas as of estimate (17). Indeed, we have

$$\partial_t g + v\cdot\nabla_x g + \sigma g = R_2, \quad \text{where } \ R_2 = \left(\frac{\sigma}{|V|} - b\right) S + a\, n. \tag{30}$$

Integrating along the characteristics, we get

$$g(t,x,v) = e^{-\sigma t} g^0(x-tv,v) + \int_0^t e^{(s-t)\sigma} R_2(s,\widetilde{x}_s) ds, \tag{31}$$

and easy computation yields

$$\begin{aligned} g(t,x,v) &\leq g^0(x-tv,v) + \int_0^t |R_2(s,\widetilde{x}_s)| ds \\ &\leq \|g^0\|_{L^\infty_{x,v}} + \left|\frac{\sigma}{|V|} - b\right| \int_0^t |S(s,\widetilde{x}_s)| ds + a\int_0^t |n(s,\widetilde{x}_s)| ds \\ &\leq \|g^0\|_{L^\infty_{x,v}} + \left|\frac{\sigma}{|V|} - b\right| \int_0^t \|S(s,\cdot)\|_{L^\infty_x}\, ds + a\int_0^t \|n(s,\cdot)\|_{L^\infty_x}\, ds. \end{aligned} \tag{32}$$

According to (17) we can write

$$\|n(s,.)\|_{L^\infty_x} \leq C_1 \|f^0\|_{L^\infty_{x,v}},$$

hence, from (32) it follows that

$$g(t,x,v) \leq \|g^0\|_{L^\infty_{x,v}} + C_1\|f^0\|_{L^\infty_{x,v}} + C_2 \int_0^t \|S(s,\cdot)\|_{L^\infty_x}\, ds. \tag{33}$$

Integrating over V, we obtain

$$S(t,x) \leq |V|\, \|g^0\|_{L^\infty_{x,v}} + C_1|V|\, \|f^0\|_{L^\infty_{x,v}} + C_2|V| \int_0^t \|S(s,\cdot)\|_{L^\infty_x}\, ds, \tag{34}$$

and we estimate S thanks to Gronwall's inequality and we conclude the proof of (18) with (33).

3. Assuming the initial data in $L^1_{x,v}$, we have by integration of the first equation in (13): $\|f(t,\cdot,\cdot)\|_{L^1_{x,v}} = \|f^0\|_{L^1_{x,v}}$. Integrating the second equation in (13), we get

$$\frac{d}{dt}\|g(t,\cdot,\cdot)\|_{L^1_{x,v}} = a|V|\|f^0\|_{L^1_{x,v}} - b|V|\|g(t,\cdot,\cdot)\|_{L^1_{x,v}}.$$

We obtain the desired estimate by integrating in time this later identity.

4. We now prove (19) and (20). To begin with, we rewrite (13) in the following way

$$\begin{cases} \partial_t f + v\cdot\nabla_x f + \widetilde{K} f = \widetilde{R}_1, \\ \partial_t g + v\cdot\nabla_x g = R_2, \end{cases} \tag{35}$$

where the functions $\widetilde{K}$ and $\widetilde{R}_1$ are defined by

$$\widetilde{K} = \frac{\mu_0}{\varepsilon} + \mu_1, \ \text{ and } \ \widetilde{R}_1 = \frac{\mu_0}{\varepsilon} F_J + \frac{\mu_1\, n}{|V|} - \frac{\mu_2\gamma^2}{|V|} J\cdot\alpha(S) + \mu_2\gamma^2 v\cdot\alpha(S) f, \tag{36}$$

while R_2 is still given in (30). Therefore, we obtain

$$f(t,x,v) = e^{-t\widetilde{K}} f^0(x-tv,v) + \int_0^t e^{(s-t)\widetilde{K}} \widetilde{R}_1(s,\widetilde{x}_s,v)ds, \tag{37}$$

and

$$g(t,x,v) = e^{-t\sigma} g^0(x-tv,v) + \int_0^t e^{(s-t)\sigma} R_2(s,\widetilde{x}_s)ds. \tag{38}$$

Let $i \in \{1,\dots,d\}$ be arbitrary but fixed index, and for a generic function h we denote by h_i the partial derivate $\partial_{x_i} h$. Hence, from (37) and (38) we get

$$f_i(t,x,v) = e^{-t\widetilde{K}} f_i^0(x-tv,v) + \int_0^t e^{(s-t)\widetilde{K}} \partial_{x_i}\left(\widetilde{R}_1(s,\widetilde{x}_s,v)\right) ds, \tag{39}$$

and

$$g_i(t,x,v) = e^{-t\sigma} g_i^0(x-tv,v) + \int_0^t e^{(s-t)\sigma} \partial_{x_i}\left(R_2(s,\widetilde{x}_s)\right) ds. \tag{40}$$

We now estimate separately f_i and g_i. From (39) it follows that

$$|f_i(t,x,v)| \le \|f_i^0\|_{L^\infty_{x,v}} + \int_0^t \left|\partial_{x_i}\left(\widetilde{R}_1(s,\widetilde{x}_s,v)\right)\right| ds. \tag{41}$$

We have

$$\begin{aligned}
\partial_{x_i}\big(\widetilde{R}_1(s,\widetilde{x}_s,v)\big) &= \frac{\mu_0}{\varepsilon|V|}\left(n_i(s,\widetilde{x}_s) + \gamma^2 J_i(s,\widetilde{x}_s)\cdot v\right) + \frac{\mu_1\, n_i(s,\widetilde{x}_s)}{|V|} \\
&- \frac{\mu_2\gamma^2}{|V|} J_i(s,\widetilde{x}_s)\cdot\alpha\left(S(s,\widetilde{x}_s)\right) - \frac{\mu_2\gamma^2}{|V|} S_i(s,\widetilde{x}_s) J(s,\widetilde{x}_s)\cdot\alpha'\left(S(s,\widetilde{x}_s)\right) \\
&+ \mu_2\gamma^2 v\cdot\alpha\left(S(s,\widetilde{x}_s)\right)\, f_i(s,\widetilde{x}_s,v) + \mu_2\gamma^2 S_i(s,\widetilde{x}_s)\, v\cdot\alpha'\left(S(s,\widetilde{x}_s)\right)\, f(s,\widetilde{x}_s,v) \\
&= \left(\frac{\mu_0}{\varepsilon|V|} + \frac{\mu_1}{|V|}\right) n_i(s,\widetilde{x}_s) + \left(\frac{\mu_0\gamma^2\, v}{\varepsilon|V|} - \frac{\mu_2\gamma^2}{|V|}\alpha\left(S(s,\widetilde{x}_s)\right)\right)\cdot J_i(s,\widetilde{x}_s) \\
&- \frac{\mu_2\gamma^2}{|V|} S_i(s,\widetilde{x}_s) J(s,\widetilde{x}_s)\cdot\alpha'\left(S(s,\widetilde{x}_s)\right) + \mu_2\gamma^2 v\cdot\alpha\left(S(s,\widetilde{x}_s)\right)\, f_i(s,\widetilde{x}_s,v) \\
&+ \mu_2\gamma^2 S_i(s,\widetilde{x}_s)\, v\cdot\alpha'\left(S(s,\widetilde{x}_s)\right)\, f(s,\widetilde{x}_s,v).
\end{aligned} \tag{42}$$

We introduce the following notations

$$\widetilde{n}_i(s) = \int_V \|f_i(s,\cdot,v)\|_{L^\infty_x} dv, \quad \text{and} \quad \widetilde{S}_i(s) = \int_V \|g_i(s,\cdot,v)\|_{L^\infty_x} dv. \tag{43}$$

In this way we have

$$|n_i(s,\widetilde{x}_s)| \le \widetilde{n}_i(s), \quad |J_i(s,\widetilde{x}_s)| \le \nu\widetilde{n}_i(s), \quad \text{and} \quad |S_i(s,\widetilde{x}_s)| \le \widetilde{S}_i(s). \tag{44}$$

Then from (42) we immediately obtain

$$\begin{aligned}
\left|\partial_{x_i}\big(\widetilde{R}_1(s,\widetilde{x}_s,v)\big)\right| &\le C_1\widetilde{n}_i(s) + C_2\widetilde{S}_i(s)\|n(s,\cdot)\|_{L^\infty_x} \\
&+ C_3\|f_i(s,\cdot,v)\|_{L^\infty_x} + C_4\widetilde{S}_i(s)\|f(s,\cdot,\cdot)\|_{L^\infty_{x,v}}.
\end{aligned} \tag{45}$$

According to (17) we have

$$\|n(s,\cdot)\|_{L^\infty_x} \le C_1\|f^0\|_{L^\infty_{x,v}}, \quad \text{and} \quad \|f(s,\cdot,\cdot)\|_{L^\infty_{x,v}} \le C_2\|f^0\|_{L^\infty_{x,v}}. \tag{46}$$

Therefore, using (45) we deduce that

$$\left|\partial_{x_i}\big(\widetilde{R}_1(s,\widetilde{x}_s,v)\big)\right| \leq C_1\widetilde{n}_i(s) + C_2\widetilde{S}_i(s) + C_3\|f_i(s,\cdot,v)\|_{L^\infty_x}. \tag{47}$$

This last estimate together with (41) allow to write

$$\|f_i(t,\cdot,v)\|_{L^\infty_x} \leq \|f_i^0\|_{L^\infty_{x,v}} + C_1\int_0^t \Big(\widetilde{n}_i(s) + \widetilde{S}_i(s)\Big)\,ds + C_3\int_0^t \|f_i(s,\cdot,v)\|_{L^\infty_x}ds. \tag{48}$$

The estimate on g_i can be done similarly to assertion (48). Indeed from Eq. (40) it follows that

$$\|g_i(t,\cdot,v)\|_{L^\infty_x} \leq \|g_i^0\|_{L^\infty_{x,v}} + \int_0^t \left|\partial_{x_i}\left(R_2(s,\widetilde{x}_s)\right)\right| ds, \tag{49}$$

and we compute the first partial derivative of R_2 as follows

$$\partial_{x_i}\left(R_2(s,\widetilde{x}_s)\right) = \left(\frac{\sigma}{|V|} - b\right) S_i + n_i. \tag{50}$$

Hence

$$\partial_{x_i}\left(R_2(s,\widetilde{x}_s)\right) \leq \left|\frac{\sigma}{|V|} - b\right| \widetilde{S}_i(s) + \widetilde{n}_i(s). \tag{51}$$

Taking Eqs. (49) and (51) into account we deduce that

$$\|g_i(t,\cdot,v)\|_{L^\infty_x} \leq \|g_i^0\|_{L^\infty_{x,v}} + \int_0^t \Big(\widetilde{n}_i(s) + \widetilde{S}_i(s)\Big)\,ds. \tag{52}$$

Next integrating, with respect to v. Eqs. (48) and (52) and adding the resulting inequalities, we can write

$$\widetilde{n}_i(t) + \widetilde{S}_i(t) \leq C_1\left(\|f_i^0\|_{L^\infty_{x,v}} + \|g_i^0\|_{L^\infty_{x,v}}\right) + C_2\int_0^t \Big(\widetilde{n}_i(s) + \widetilde{S}_i(s)\Big)\,ds. \tag{53}$$

Therefore, in view of Gronwall's inequality, equation (53) yields

$$|n_i(s,x)| + |S_i(s,x)| \leq \widetilde{n}_i(s) + \widetilde{S}_i(s) \leq C_1\left(\|f_i^0\|_{L^\infty_{x,v}} + \|g_i^0\|_{L^\infty_{x,v}}\right), \tag{54}$$

and a similar estimate is obtained for f_i and g_i using (48), (52) and (54). This complete the a-priori estimates. □

3.2 Proof of Theorem 3.2.

We are now in position to prove the existence result. The idea of the proof follows standard techniques consisting in, first, proving local in time existence by a fixed point procedure, second, iterating this process to obtain global in time existence.

For the local in time existence, let $T > 0$, we introduce the map

$$\mathcal{F} : X_T \longrightarrow X_T, \qquad f \longmapsto \mathcal{F}(f) := \mathcal{F}_2(\mathcal{F}_1(f))$$

where $G = \mathcal{F}_1(f)$ is a weak solution of the following problem:

$$\begin{cases} \partial_t G + v\cdot\nabla_x G = \left(\dfrac{\sigma}{|V|} - b\right)\displaystyle\int_V G dv + an - \sigma G, \\ G(0,x,v) = g^0(x,v) \in L^\infty_{x,v}, \end{cases}$$

with the notation $n(t,x) = \int_V f(t,x,v)dv$, while the functional $\mathcal{F}_2$ is defined by: $F = \mathcal{F}_2(g)$ is a weak solution of

$$\begin{cases} \partial_t F + v\cdot\nabla_x F = \dfrac{\mu_0}{\varepsilon}\left[\dfrac{1}{|V|}\left(\displaystyle\int_V F dv + \gamma^2 \int_V vF dv\cdot v\right) - F\right] \\ \qquad\qquad +\mu_1\left(\dfrac{1}{|V|}\displaystyle\int_V F dv - F\right) - \mu_2\gamma^2\left(\dfrac{1}{|V|}\displaystyle\int_V vF dv - vF\right)\cdot\alpha(S), \\ F(0,x,v) = f^0(x,v) \in L^\infty_{x,v}, \end{cases}$$

with $S(t,x) = \int_V g(t,x,v)dv$. Existence of solutions for these two linear systems is now standard. It is clear, adapting the techniques of Lemma 3.3 that $\mathcal{F}_1$ and $\mathcal{F}_2$ map X_T into itself. Our objective is to show that $\mathcal{F}$ defines a contraction on X_T for T small enough. Let f_1 and f_2 be given in X_T, then we have the following result:

Lemma 3.4 *For $T > 0$ small enough, there exists a constant $C_1(T) < 1$ such that*

$$\|\mathcal{F}_1(f_1) - \mathcal{F}_1(f_2)\|_{X_T} \le C_1(T)\|f_1 - f_2\|_{X_T}. \tag{55}$$

Proof. We set $G_{12} = \mathcal{F}_1(f_1) - \mathcal{F}_1(f_2)$, then we have

$$\partial_t G_{12} + v\cdot\nabla_x G_{12} = \left(\frac{\sigma}{|V|} - b\right)\int_V G_{12}dv - \sigma G_{12} + a(n_1 - n_2), \tag{56}$$

with the notations $n_i(t,x) = \int_V f_i(t,x,v)dv,\ i = 1,2$. Analogously to the proof of Lemma 3.3, we write identity (56) in the following way

$$\partial_t G_{12} + v\cdot\nabla_x G_{12} + \sigma G_{12} = R_1, \tag{57}$$

where

$$R_1 = \left(\frac{\sigma}{|V|} - b\right)\int_V G_{12}dv + a(n_1 - n_2). \tag{58}$$

Moreover, from equation

$$\frac{d}{ds}\left(e^{(s-t)\sigma}G_{12}(s,\widetilde{x}_s,v)\right) = e^{(s-t)\sigma}R_1(s,\widetilde{x}_s), \tag{59}$$

it follows that

$$G_{12}(t,x,v) = \int_0^t e^{(s-t)\sigma}R_1(s,\widetilde{x}_s)ds. \tag{60}$$

$\big($We recall the notation $\widetilde{x}_s = x + (s-t)v\ \big)$. Since $e^{(s-t)\sigma} < 1$, for all $0 \le s \le t \le T$ we deduce from (60) the following estimate

$$|G_{12}(t,x,v)| \le \int_0^t |R_1(s,\widetilde{x}_s)|ds. \tag{61}$$

However, we have

$$\begin{aligned} |R_1(s,\widetilde{x}_s)| &\le \left|\frac{\sigma}{|V|} - b\right| |V|\,\|G_{12}(s,\cdot,\cdot)\|_{L^\infty_{x,v}} + a\,|V|\,\|f_1 - f_2\|_{X_T} \\ &= C_1\,\|G_{12}(s,\cdot,\cdot)\|_{L^\infty_{x,v}} + C_2\,\|f_1 - f_2\|_{X_T}. \end{aligned} \tag{62}$$

Using this last inequality in Eq. (61) we get

$$|G_{12}(t,x,v)| \le \int_0^t C_1\,\|G_{12}(s,\cdot,\cdot)\|_{L^\infty_{x,v}}ds + C_2\,\|f_1 - f_2\|_{X_T}, \tag{63}$$

and the Gronwall lemma gives the desired estimate (55), which finished the proof of Lemma 3.4. □

Now, let us introduce $g_1 = \mathcal{F}_1(f_1)$ and $g_2 = \mathcal{F}_1(f_2)$. Then we claim that:

Lemma 3.5 *For $T > 0$ small enough, there exists a constant $C_2(T) < 1$ such that*

$$\|\mathcal{F}_2(g_1) - \mathcal{F}_2(g_2)\|_{X_T} \leq C_2(T)\|g_1 - g_2\|_{X_T}. \tag{64}$$

Proof. The proof of Lemma 3.5 follows the same techniques as in the proof of Lemma 3.4, but with more technical difficulties. To begin with we set $F_{12} = \mathcal{F}_2(g_1) - \mathcal{F}_2(g_2)$, then we have

$$\begin{aligned}
&\partial_t F_{12} + v \cdot \nabla_x F_{12} = \frac{\mu_0 + \varepsilon\mu_1}{\varepsilon|V|}\int_V F_{12}dv + \frac{\mu_0\gamma^2}{\varepsilon|V|}\int_V vF_{12}dv \cdot v \\
&-\frac{\mu_2\gamma^2}{|V|}\int_V vF_{12}dv \cdot \alpha(S_1) - \frac{\mu_2\gamma^2}{|V|}\int_V \mathcal{F}_2(g_2)dv \cdot (\alpha(S_1) - \alpha(S_2)) \\
&-\left(\frac{\mu_0}{\varepsilon} + \frac{\mu_1}{|V|}\right)F_{12} + \mu_2\gamma^2 vF_{12} \cdot \alpha(S_1) + \mu_2\gamma^2 v\mathcal{F}_2(g_2) \cdot (\alpha(S_1) - \alpha(S_2)),
\end{aligned} \tag{65}$$

with $S_i(t,x) = \int_V g_i(t,x,v)dv,\ i = 1,2$. We introduce the following notations

$$K = \frac{\mu_0}{\varepsilon} + \frac{\mu_1}{|V|} - \mu_2\gamma^2 v \cdot \alpha(S_1),$$

and

$$\begin{aligned}
R_1 \;=\; &\frac{\mu_0 + \varepsilon\mu_1}{\varepsilon|V|}\int_V F_{12}dv + \frac{\mu_0\gamma^2}{\varepsilon|V|}\int_V vF_{12}dv \cdot v - \frac{\mu_2\gamma^2}{|V|}\int_V vF_{12}dv \cdot \alpha(S_1) \\
&-\frac{\mu_2\gamma^2}{|V|}\int_V \mathcal{F}_2(g_2)dv \cdot (\alpha(S_1) - \alpha(S_2)) + \mu_2\gamma^2 v\mathcal{F}_2(g_2) \cdot (\alpha(S_1) - \alpha(S_2)).
\end{aligned}$$

In this way we can write identity (65) as

$$\partial_t F_{12} + v \cdot \nabla_x F_{12} + KF_{12} = R_1. \tag{66}$$

A simple calculation shows that

$$F_{12}(t,x,v) = \int_0^t \left[\exp\left(\int_t^s K(\tau, \widetilde{x}_\tau, v)d\tau\right) R_1(s, \widetilde{x}_s, v)\right] ds, \tag{67}$$

and in view of estimate $\exp\left(\int_t^s K(\tau, \widetilde{x}_\tau, v)d\tau\right) \leq e^{C_1 T}$, we deduce from (67) that

$$|F_{12}(t,x,v)| \leq e^{C_1 T}\int_0^t |R_1(s, \widetilde{x}_s, v)|ds. \tag{68}$$

Moreover, it is easy to see that

$$\begin{aligned}
|R_1(s, \widetilde{x}_s, v)| \leq &C_2 n_{12}(s, \widetilde{x}_s) + C_3 n_2(s, \widetilde{x}_s)\,|\alpha(S_1(s, \widetilde{x}_s)) - \alpha(S_2(s, \widetilde{x}_s))| \\
&+ C_4\mathcal{F}_2(g_2)(s, \widetilde{x}_s, v)\,|\alpha(S_1(s, \widetilde{x}_s)) - \alpha(S_2(s, \widetilde{x}_s))|,
\end{aligned}$$

with the notation $n_{12}(t,x) = \int_V F_{12}(t,x,v)dv$ and $n_2(t,x) = \int_V \mathcal{F}_2(g_2)(t,x,v)dv$. Using Lemma 3.3 together with the assumption (16), we get

$$|R_1(s, \widetilde{x}_s, v)| \;\leq\; C_2|n_{12}(s, \widetilde{x}_x)| + C_5\|f^0\|_{L^\infty_{x,v}} L_\alpha \|S_1 - S_2\|_{L^\infty_{t,x}} \tag{69}$$

We remark that

$$|n_{12}(s, \widetilde{x}_x)| \leq |V|\|F_{12}(s,\cdot,\cdot)\|_{L^\infty_{x,v}}, \tag{70}$$

and

$$\|S_1 - S_2\|_{L^\infty_{t,x}} \leq |V| \|g_1 - g_2\|_{X_T}. \tag{71}$$

Then from (68), (69), (70) and (71) it follows that

$$\|F_{12}(t,\cdot,\cdot)\|_{L^\infty_{x,v}} \leq e^{C_1 T} \int_0^t \|F_{12}(s,\cdot,\cdot)\|_{L^\infty_{x,v}} ds + T C_3 e^{C_1 T} \|g_1 - g_2\|_{X_T}, \tag{72}$$

and we conclude the proof of Lemma 3.5 using Gronwall inequality. □

The local existence in Theorem 3.2 follows from a direct application of the Banach fixed point theorem since $\mathcal{F}$ is a contraction on X_T for T small enough. This gives existence of a unique solution on $[0,T]$ for small enough T. Thanks to a priori estimates in Lemma 3.3 we may iterate this process to extend the solution on $[T,2T]$, then on $[2T,3T]$, ... It concludes the proof of Theorem 3.2. □

4 Hyperbolic limit

Derivation of macroscopic model from the underlaying description at the microscopic scale, provided by the kinetic theory of active particles, is the subject of a growing literature. In [7, 12, 14, 4, 21, 15] it has been proved that the Keller-Segel [5] model can be derived as the limit of a kinetic model by using a moment method. The hyperbolic limit is considered in [11, 3, 16] leading to the same kind of macroscopic model with small diffusion. More recently these results have been extended in [18] dealing with the coupled kinetic system (13). As a consequence a formal derivation of a class of hyperbolic equations of Cattaneo type is obtained. The aim of this section is to purpose a rigorous proof of the formal derivation of the hyperbolic limit performed in [18]. However, due to technical difficulties, we restrict ourself to the one dimensional case, $d = 1$.

The main result can be stated as follows.

Theorem 4.1 *Let $T > 0$, $d = 1$, and V a symmetric bounded domain of $\mathbb{R}$ with $\gamma^2 = |V| \left(\int_V v^2\, dv\right)^{-1}$. Let $(f^0, g^0) \in (L^1_{x,v} \cap L^\infty_{x,v})^2$ be nonnegative and assume that $\alpha \in C^1(\mathbb{R})$ satisfies (16). Let $(f_\varepsilon, g_\varepsilon)$ be the unique nonnegative weak solution of the scaled Cauchy problem (13) on $[0,T]$. Then there exists a subsequence, denoted in the same way, and a couple (f,g) such that*

$$f_\varepsilon \rightharpoonup f, \quad g_\varepsilon \rightharpoonup g \quad \text{in } L^2_{t,x,v}. \tag{73}$$

In addition, the moments

$$n = \int_V f(v)\, dv, \quad S = \int_V g(v)\, dv, \quad J = \int_V v f(v)\, dv, \tag{74}$$

satisfy the following macroscopic system

$$\begin{cases} \partial_t n + \partial_x J = 0 \\ \partial_t J + \frac{1}{\gamma^2} \partial_x n = -\mu_1 J + \mu_2 n\, \alpha(S) \\ \partial_t g + v \partial_x g = \sigma \left(\frac{S}{|V|} - g \right) + an - bS. \end{cases} \tag{75}$$

Moreover, the asymptotic limit f satisfies

$$f = \frac{1}{|V|} \left(n + \gamma^2 J v \right). \tag{76}$$

The first two equations in system (75) form the so-called Cattaneo system for chemosensitive movement [9, 23]. Hence a direct consequence of this Theorem (and Theorem 3.2) is the existence of a solution for the one dimensional Cattaneo system.

Since the last equation has not been rescaled, it cannot be rewritten as a closed system with macroscopic variable. However, we deduce from the last equation in (75) that the moments $S = \langle g \rangle$ and $q = \langle vg \rangle$ verify the (non-closed) system

$$\partial_t S + \partial_x q = an - bS, \qquad \partial_t q + \partial_x Q(g) = -\sigma q,$$

where the second order moment Q is defined by $Q(g) = \int_V v^2 g(v) dv$.

4.1 Uniform a priori estimates

We start with the following a priori estimates uniform with respect to $\varepsilon > 0$:

Lemma 4.2 (A priori estimate in $L^2_{x,v}$) *We suppose that we are in the conditions of theorem 4.1. Then the following estimate*

$$\|f_\varepsilon(t)\|^2_{L^2_{x,v}} + \|g_\varepsilon(t)\|^2_{L^2_{x,v}} \leq C(T) \left(\|f^0\|^2_{L^2_{x,v}} + \|g^0\|^2_{L^2_{x,v}} \right), \tag{77}$$

holds true for a.e $t \in (0,T)$, where the constant $C(T)$ is independent of ε.

Proof. We multiply the first equation of system (13) by f_ε

$$\begin{aligned} \frac{1}{2}\left(\partial_t f_\varepsilon^2 + v\partial_x f_\varepsilon^2\right) &= \frac{\mu_0}{\varepsilon}\left[\frac{1}{|V|}\left(n_\varepsilon f_\varepsilon + J_\varepsilon \gamma^2 v f_\varepsilon\right) - f_\varepsilon^2\right] + \mu_1\left(\frac{n_\varepsilon}{|V|} f_\varepsilon - f_\varepsilon^2\right) \\ &\quad -\mu_2\gamma^2\left(\frac{J_\varepsilon f_\varepsilon}{|V|} - v f_\varepsilon^2\right)\alpha(S_\varepsilon), \end{aligned}$$

and integrate over V to obtain

$$\begin{aligned} \frac{1}{2}\left(\partial_t \int_V f_\varepsilon^2 dv + \partial_x \int_V v f_\varepsilon^2 dv\right) &= \frac{\mu_0}{\varepsilon}\left[\frac{1}{|V|}\left(n_\varepsilon^2 + J_\varepsilon^2\gamma^2\right) - \int_V f_\varepsilon^2 dv\right] \\ &\quad +\mu_1\left(\frac{n_\varepsilon^2}{|V|} - \int_V f_\varepsilon^2 dv\right) - \mu_2\gamma^2\left(\frac{J_\varepsilon n_\varepsilon}{|V|} - \int_V v f_\varepsilon^2 dv\right)\alpha(S_\varepsilon). \end{aligned} \tag{78}$$

Let us introduce the symmetric and the anti-symmetric part of f_ε as follows

$$\begin{aligned} f_\varepsilon^S(v) &= \frac{1}{2}\left(f_\varepsilon(v) + f_\varepsilon(-v)\right), \quad v \in V, \\ f_\varepsilon^A(v) &= \frac{1}{2}\left(f_\varepsilon(v) - f_\varepsilon(-v)\right), \quad v \in V. \end{aligned}$$

Since V is symmetric, it follows that

$$f_\varepsilon = f_\varepsilon^S + f_\varepsilon^A, \quad n_\varepsilon = \int_V f_\varepsilon^S dv, \quad J_\varepsilon = \int_V v f_\varepsilon^A dv, \tag{79}$$

and

$$\int_V f_\varepsilon^2 dv = \int_V \left(f_\varepsilon^S\right)^2 dv + \int_V \left(f_\varepsilon^A\right)^2 dv. \tag{80}$$

Using (78)-(80), we have

$$\frac{1}{2}\left(\partial_t \int_V f_\varepsilon^2 dv + \partial_x \int_V v f_\varepsilon^2 dv\right) = \frac{\mu_0}{\varepsilon}\left[\frac{1}{|V|}\left(\int_V f_\varepsilon^S dv\right)^2 - \int_V \left(f_\varepsilon^S\right)^2 dv \right. \\ \left. + \frac{\gamma^2}{|V|}\left(\int_V v f_\varepsilon^A dv\right)^2 - \int_V \left(f_\varepsilon^A\right)^2 dv\right] \\ + \mu_1\left[\frac{n_\varepsilon^2}{|V|} - \int_V f_\varepsilon^2 dv\right] - \mu_2\gamma^2\left(\frac{J_\varepsilon n_\varepsilon}{|V|} - \int_V v f_\varepsilon^2 dv\right)\alpha(S_\varepsilon), \tag{81}$$

and according to Cauchy-Schwarz inequality we have

$$\left(\int_V f_\varepsilon^S dv\right)^2 \le |V| \int_V \left(f_\varepsilon^S\right)^2 dv, \quad \left(\int_V v f_\varepsilon^A dv\right)^2 \le \frac{|V|}{\gamma^2}\int_V \left(f_\varepsilon^A\right)^2 dv. \tag{82}$$

By combining equations (81) and (82) we get

$$\frac{1}{2}\left(\partial_t \int_V f_\varepsilon^2 dv + \partial_x \int_V v f_\varepsilon^2 dv\right) \le -\mu_2\gamma^2\left(\frac{J_\varepsilon n_\varepsilon}{|V|} - \int_V v f_\varepsilon^2 dv\right)\alpha(S_\varepsilon). \tag{83}$$

Moreover, we have

$$\begin{aligned} -\mu_2\gamma^2\left(\frac{J_\varepsilon n_\varepsilon}{|V|} - \int_V v f_\varepsilon^2 dv\right)\alpha(S_\varepsilon) &= \mu_2\gamma^2\alpha(S_\varepsilon)\left(\int_V v f_\varepsilon^2 dv - \frac{J_\varepsilon n_\varepsilon}{|V|}\right) \\ &\le \mu_2\gamma^2\nu\alpha_\infty\left(\int_V f_\varepsilon^2 dv + \frac{n_\varepsilon^2}{|V|}\right), \end{aligned}$$

and using (82), we obtain

$$-\mu_2\gamma^2\left(\frac{J_\varepsilon n_\varepsilon}{|V|} - \int_V v f_\varepsilon^2 dv\right)\alpha(S_\varepsilon) \le 2\mu_2\gamma^2\nu\alpha_\infty\int_V f_\varepsilon^2 dv. \tag{84}$$

Hence, from (83) and (84) we get

$$\partial_t \int_V f_\varepsilon^2 dv + \partial_x \int_V v f_\varepsilon^2 dv \le C\int_V f_\varepsilon^2 dv,$$

and integration over $x \in \mathbb{R}^d$ yields

$$\frac{d}{dt}\|f_\varepsilon(t)\|_{L^2_{x,v}}^2 \le C\|f_\varepsilon(t)\|_{L^2_{x,v}}^2. \tag{85}$$

To derive a similar estimate for g_ε we multiply the second equation of system (13) by g_ε and we integrate over V to obtain

$$\frac{1}{2}\left(\partial_t \int_V g_\varepsilon^2 dv + \partial_x \int_V v g_\varepsilon^2 dv\right) = \sigma\left(\frac{S_\varepsilon^2}{|V|} - \int_V g_\varepsilon^2 dv\right) + a n_\varepsilon S_\varepsilon - b S_\varepsilon^2.$$

Using the Cauchy-Schwarz inequality we can write

$$\frac{1}{2}\left(\partial_t \int_V g_\varepsilon^2 dv + \partial_x \int_V v g_\varepsilon^2 dv\right) \le \frac{a|V|}{2}\int_V f_\varepsilon^2 dv + \left(\frac{a}{2} + b\right)|V|\int_V g_\varepsilon^2 dv,$$

and integration over the space variable $x \in \mathbb{R}$ gives

$$\frac{d}{dt}\|g_\varepsilon(t)\|^2_{L^2_{x,v}} \leq a|V|\|f_\varepsilon(t)\|^2_{L^2_{x,v}} + (a+2b)|V|\|g_\varepsilon(t)\|^2_{L^2_{x,v}}. \tag{86}$$

Let us now combine equations (85) and (86) to get

$$\frac{d}{dt}\left[\|f_\varepsilon(t)\|^2_{L^2_{x,v}} + \|g_\varepsilon(t)\|^2_{L^2_{x,v}}\right] \leq C\left[\|f_\varepsilon(t)\|^2_{L^2_{x,v}} + \|g_\varepsilon(t)\|^2_{L^2_{x,v}}\right].$$

We conclude the proof thanks to a Gronwall's inequality. □

4.2 Convergence by compactness

According to Lemma 4.2, the sequences f_ε, g_ε are bounded in $L^\infty\left(0,T;L^2_{x,v}\right)$, hence there are bounded in $L^2_{t,x,v}$. Accordingly, it follows that there exist two subsequences, denoted in the same way, and f, $g \in L^2_{t,x,v}$ such that

$$f_\varepsilon \rightharpoonup f, \quad g_\varepsilon \rightharpoonup g \quad \text{in } L^2_{t,x,v}. \tag{87}$$

Moreover, we have

$$\partial_t g_\varepsilon + v\partial_x g_\varepsilon = \sigma\left(\frac{S_\varepsilon}{|V|} - g_\varepsilon\right) + an_\varepsilon - bS_\varepsilon \in L^2_{x,v}. \tag{88}$$

Hence, according to averaging Lemma, see for instance [20] Proposition 3.3.1, we have

$$\int_V g_\varepsilon(v)\,dv = S_\varepsilon \text{ is uniformly bounded in } L^2\left(0,T;H^{\frac{1}{2}}(\mathbb{R})\right). \tag{89}$$

Integrating equation (88) with respect to v, we deduce clearly that $\partial_t S_\varepsilon \in L^2\left(0,T;W^{-1,1}(\mathbb{R})\right)$. Moreover, for each compact $K \subset \mathbb{R}$, we have the embeddings (see e.g. [1])

$$H^{\frac{1}{2}}(K) \xhookrightarrow[compact]{} L^2(K) \hookrightarrow W^{-1,1}(K). \tag{90}$$

From Aubin-Lions compactness Lemma (see [22]), we deduce that the sequence $(S_\varepsilon)_\varepsilon$ is relatively compact in $L^2\left(0,T;L^2(K)\right)$. Hence we can extract a subsequence, still denoted $(S_\varepsilon)_\varepsilon$, which converges strongly towards S in $L^2\left((0,T)\times K\right)$. By uniqueness of the weak limit, we have that $S = \int_V g(v)dv$.

However the convergence is global:

$$S_\varepsilon \to S \quad \text{in } L^2_{t,x}. \tag{91}$$

Indeed, for any compact $[-R,R] \subset \mathbb{R}$ we may extract a subsequence $(S_\varepsilon)_\varepsilon$ such that $S_\varepsilon \to S$ strongly in $L^2([0,T]\times[-R,R])$, and we know that $S_\varepsilon = \int_V g_\varepsilon(v)\,dv$ where

$$\partial_t \int_V (f_\varepsilon^2 + g_\varepsilon^2)dv + \partial_x \int_V v(f_\varepsilon^2 + g_\varepsilon^2)dv \leq C\int_V (f_\varepsilon^2 + g_\varepsilon^2)dv.$$

Multiplying by a function $x \mapsto \phi(x) \in C^1(\mathbb{R})$ with bounded derivative and integrating, we deduce

$$\frac{d}{dt}\int_{\mathbb{R}}\int_V (f_\varepsilon^2 + g_\varepsilon^2)\phi\,dxdv \leq C\int_{\mathbb{R}}\int_V (f_\varepsilon^2 + g_\varepsilon^2)\phi\,dxdv + \int_{\mathbb{R}}\int_V v(f_\varepsilon^2 + g_\varepsilon^2)\phi'\,dxdv. \tag{92}$$

In order to pass from local to global convergence, we need to prove that we have a bound on the tail at infinity. Let us show that $(S_\varepsilon)_\varepsilon$ is a Cauchy sequence in $L^2_{t,x}$. We compute

$$\int_0^T \int_{\mathbb{R}} |S_\varepsilon - S_{\varepsilon'}|^2 \, dxdt = \int_0^T \int_{-R}^{R} |S_\varepsilon - S_{\varepsilon'}|^2 \, dxdt + \int_0^T \int_{\mathbb{R}\setminus[-R,R]} |S_\varepsilon - S_{\varepsilon'}|^2 \, dxdt.$$

From the above result, we know that the first term of the right hand side goes to 0 as $\varepsilon, \varepsilon' \to 0$. For the second term, let us consider $\phi \in C^\infty(\mathbb{R}^d)$ such that $0 \le \phi \le 1$, $\phi(x) = 0$ for $|x| \le 1/2$ and $\phi(x) = 1$ for $|x| \ge 1$. We define $\phi_R(x) = \phi(x/R)$. Then, we have

$$\begin{aligned}\int_0^T \int_{\mathbb{R}\setminus[-R,R]} |S_\varepsilon - S_{\varepsilon'}|^2 \, dxdt &\le \int_0^T \int_{\mathbb{R}\setminus[-R,R]} |S_\varepsilon - S_{\varepsilon'}|^2 \phi_R \, dxdt \\ &\le 2 \int_0^T \int_{\mathbb{R}} (|S_\varepsilon|^2 + |S_{\varepsilon'}|^2) \phi_R \, dxdt.\end{aligned}$$

Let us now use estimate (92) with ϕ_R, since $\phi_R'(x) = \frac{1}{R}\phi'(x/R)$, we have

$$\begin{aligned}\frac{d}{dt} \int_{\mathbb{R}} \int_V (f_\varepsilon^2 + g_\varepsilon^2) \phi_R \, dxdv &\le C \int_{\mathbb{R}} \int_V (f_\varepsilon^2 + g_\varepsilon^2) \phi_R \, dxdv \\ &\quad + \frac{1}{R} \int_{\mathbb{R}} \int_V v (f_\varepsilon^2 + g_\varepsilon^2) \phi'(x/R) \, dxdv.\end{aligned} \tag{93}$$

Applying a Gronwall Lemma, we deduce that

$$\int_{\mathbb{R}} \int_V (f_\varepsilon^2 + g_\varepsilon^2) \phi_R \, dxdv \le e^{CT} \left(\int_{\mathbb{R}} \int_V ((f^0)^2 + (g^0)^2) \phi_R \, dxdv + \frac{C\|\phi'\|}{R} \right).$$

Since the initial data f^0 and g^0 are given in $L^2_{x,v}$ and $\phi_R(x) = 0$ on $B_{R/2}$, we deduce that the left hand side goes to 0 as $R \to +\infty$, uniformly with respect to ε. Thus,

$$\int_0^T \int_{\mathbb{R}} (|S_\varepsilon|^2 + |S_{\varepsilon'}|^2) \phi_R \, dxdt \le |V| \int_0^T \int_{\mathbb{R}} \int_V (f_\varepsilon^2 + f_{\varepsilon'}^2) \phi_R \, dxdvdt$$

goes uniformly to 0 as $R \to +\infty$. We conclude that the sequence $(S_\varepsilon)_\varepsilon$ is a Cauchy sequence in $L^2([0,T] \times \mathbb{R})$. □

4.3 Proof of Theorem 4.1

Multiply the first and second equations of system (13) by 1 and v respectively, and integrate over V to obtain the following system

$$\begin{cases} \partial_t n_\varepsilon + \partial_x J_\varepsilon = 0 \\ \partial_t J_\varepsilon + \partial_x \displaystyle\int_V v^2 f_\varepsilon \, dv = -\mu_1 J_\varepsilon + \mu_2 \gamma^2 \int_V v^2 f_\varepsilon \, dv \, \alpha(S_\varepsilon) \\ \partial_t g_\varepsilon + v \cdot \partial_x g_\varepsilon = \sigma \left(\frac{S_\varepsilon}{|V|} - g_\varepsilon \right) + a n_\varepsilon - b S_\varepsilon. \end{cases} \tag{94}$$

We have

$$f_\varepsilon(t,x,v) \rightharpoonup f(t,x,v) \quad \text{and} \quad g_\varepsilon(t,x,v) \rightharpoonup g(t,x,v) \quad \text{in } L^2_{t,x,v}. \tag{95}$$

Therefore, since the set of velocities V is bounded, we deduce

$$n_\varepsilon(t,x) \rightharpoonup n(t,x), \quad J_\varepsilon(t,x) \rightharpoonup J(t,x) \quad \text{in } L^2_{t,x}, \tag{96}$$

$$S_\varepsilon(t,x) \to S(t,x), \quad q_\varepsilon(t,x) \rightharpoonup q(t,x) \quad \text{in } L^2_{t,x}, \tag{97}$$

$$\int_V v^2 f_\varepsilon(t,x,v)dv \rightharpoonup \int_V v^2 f(t,x,v)dv \quad \text{in } L^2_{t,x}, \tag{98}$$

$$\int_V v^2 g_\varepsilon(t,x,v)dv \rightharpoonup \int_V v^2 g(t,x,v)dv \quad \text{in } L^2_{t,x}, \tag{99}$$

when ε tends to zero. However, according to Section 4.2 we have

$$\alpha(S_\varepsilon(t,x)) \int_V v^2 f_\varepsilon(t,x,v)\,dv \rightharpoonup \alpha(S(t,x)) \int_V v^2 f(t,x,v)\,dv \quad \text{in } L^2_{t,x}. \tag{100}$$

Hence, by passing to limit in (94), in the sense of distributions, and taking into account Eqs. (96)-(100), it follows that

$$\begin{cases} \partial_t n + \partial_x J = 0 \\ \partial_t J + \partial_x \displaystyle\int_V v^2 f dv = -\mu_1 J + \mu_2 \gamma^2 \int_V v^2 f dv\, \alpha(S) \\ \partial_t g + v \cdot \partial_x g = \sigma \left(\dfrac{S}{|V|} - g \right) + an - bS. \end{cases} \tag{101}$$

To identify the term $\int_V v^2 f(t,x,v)dv$, we multiply the first equation of system (13) by ε to get

$$\begin{aligned} &\varepsilon \partial_t f_\varepsilon(t,x,v) + \varepsilon v \cdot \partial_x f_\varepsilon(t,x,v) \\ &= \mu_0 (F_{n_\varepsilon, J_\varepsilon}(t,x,v) - f_\varepsilon(t,x,v)) + \varepsilon \mu_1 \left(\frac{n_\varepsilon(t,x)}{|V|} - f_\varepsilon(t,x,v) \right) \\ &\qquad -\varepsilon \mu_2 \gamma^2 \left(\frac{J_\varepsilon(t,x)}{|V|} - v f_\varepsilon(t,x,v) \right) \alpha\left(S_\varepsilon(t,x)\right). \end{aligned} \tag{102}$$

Then, letting ε go to zero yields

$$f = F_{n,J}(t,x,v) = \frac{1}{|V|} \left(n + \gamma^2 J v \right) \tag{103}$$

and a simple calculations shows that

$$\int_V v^2 f(t,x,v)dv = \int_V v^2 F_{n,J}(t,x,v)dv = \frac{1}{\gamma^2} n(t,x). \tag{104}$$

Using this last equation in system (101) finishes the proof. □

References

[1] H Bahouri, J-Y Chemin, and R Danchin. *Fourier Analysis and Nonlinear Partial Differential Equations*. Springer-Verlag, Berlin, 2011.

[2] N Bellomo, A Bellouquid, and N Chouhad. From a multiscale derivation of nonlinear cross-diffusion models to keller-segel models in a navier-stokes fluid. *Math. Models Methods Appl. Sci.*, 26:2041–2069, 2016.

[3] N Bellomo, A Bellouquid, J Nieto, and J Soler. Multicellular growing systems: Hyperbolic limits towards macroscopic description. *Math. Models Methods Appl. Sci.*, 17:1675–1693, 2007.

[4] N Bellomo, A Bellouquid, J Nieto, and J Soler. On the asymptotic theory from microscopic to macroscopic growing tissue models: an overview with perspectives. *Math. Models Methods Appl. Sci.*, 22, 2012.

[5] N Bellomo, A Bellouquid, Y Tao, and M Winkler. Toward a mathematical theory of keller-segel models of pattern formation in biological tissues. *Math. Models Methods Appl. Sci.*, 25(9):1663–1763, 2015.

[6] D Burini and N Chouhad. An hilbert perturbation method toward a multiscale analysis from kinetic to macroscopic models. *Math. Models Methods Appl. Sci.*, 27(7):1327–1353, 2017.

[7] FACC Chalub, PA Markowich, B Perthame, and C Schmeiser. Kinetic models for chemotaxis and their drift-diffusion limits. *Monatsh. Math.*, 142:123–141, 2004.

[8] Hilbert D. Mathematical problems. *Bull. Amer. Math. Soc.*, 8(10):437–479, 1902.

[9] Y Dolak and T Hillen. Cattaneo models for chemotaxis, numerical solution and pattern formation. *J. Math. Biol.*, 46:461–478, 2003.

[10] R Erban and HJ Hwang. Global existence results for complex hyperbolic models of bacterial chemotaxis. *Disc. Cont. Dyn. Systems-Series B*, 6:1239–1260, 2006.

[11] F Filbet, P Laurençot, and B Perthame. Derivation of hyperbolic models for chemosensitive movement. *J. Math. Biol.*, 50:189–207, 2005.

[12] HJ Hwang, K Kang, and A Stevens. Drift-diffusion limits of kinetic models for chemotaxis: a generalization. *Discrete Contin. Dyn. Syst. Ser. B*, 5(2):319–334, 2005.

[13] HJ Hwang, K Kang, and A Stevens. Global solutions of nonlinear transport equations for chemosensitive movement. *SIAM J. Math. Anal.*, 36:1177–1199, 2005.

[14] HJ Hwang, K Kang, and A Stevens. Global existence of classical solutions for a hyperbolic chemotaxis model and its parabolic limit. *Indiana Univ. Math. J.*, 55:289–316, 2006.

[15] Liao J. Global solution for a kinetic chemotaxis model with internal dynamics and its fast adaptation limit. *J. Differential Equations*, 259:6432–6458, 2015.

[16] J James and N Vauchelet. Chemotaxis: from kinetic equations to aggregate dynamics. *NoDEA Nonlinear Differential Equations Appl.*, 20(1):101–127, 2013.

[17] HG Othmer, SR Dunbar, and W Alt. Models of dispersal in biological systems. *J. Math. Biol.*, 26:263–298, 1988.

[18] N Outada, N Vauchelet, T Akrid, and M Khaladi. From kinetic theory of multicellular systems to hyperbolic tissue equations: Asymptotic limits and computing. *Math. Models Methods Appl. Sci.*, 26:2709–2734, 2016.

[19] B Perthame. Mathematical tools for kinetic equations. *Bull. American Math. Soc.*, 41:205–244, 2004.

[20] L Saint-Raymond. *Hydrodynamic limits of the Boltzmann equation.* Lecture Notes in Mathematics, Springer, Berlin, 2009.

[21] G Si, M Tang, and X Yang. A pathway-based mean-field model for e. coli chemotaxis: Mathematical derivation and its hyperbolic and parabolic limits. *Multiscale Model. Simul.*, 12(2), 2014.

[22] J Simon. Compact sets in the space $l^p(0,t;b)$. *Ann. Mat. Pura Appl.*, 146:65–96, 1987.

[23] Hillen T. Hyperbolic models for chemosensitive movement. *Math. Models Methods Appl. Sci.*, 12:1007–1034, 2002.

[24] N Vauchelet. Numerical simulation of a kinetic model for chemotaxis. *Kinetic and Related Models*, 3(3):501–528, 2010.