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Julien Bensmail, Foivos Fioravantes. On BMRN*-colouring of planar digraphs. 2019. hal-02195028v3

HAL Id: hal-02195028

https://hal.science/hal-02195028v3

Preprint submitted on 2 Sep 2019 (v3), last revised 15 Feb 2021 (v6)

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On BMRN*-colouring of planar digraphs

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Abstract

In a recent work, Bensmail, Blanc, Cohen, Havet and Rocha, motivated by applications for TDMA scheduling problems, have introduced the notion of BMRN*-colouring of digraphs, which is a type of arc-colouring with specific colouring constraints. In particular, they gave a special focus to planar digraphs. They notably proved that every planar digraph can be 8-BMRN*-coloured, while there exist planar digraphs for which 7 colours are needed in a BMRN*-colouring. They also proved that the problem of deciding whether a planar digraph can be 3-BMRN*-coloured is NP-hard.

In this work, we pursue these investigations on planar digraphs, in particular by answering some of the questions left open by the authors in that seminal work. We exhibit planar digraphs needing 8 colours to be BMRN*-coloured, thus showing that the upper bound of Bensmail *et al.* cannot be decreased in general. We also generalize their complexity result by showing that the problem of deciding whether a planar digraph can be k-BMRN*-coloured is NP-hard for every $k \in \{3, ..., 6\}$. Finally, we investigate the connection between the girth of a planar digraph and the least number of colours in its BMRN*-colourings.

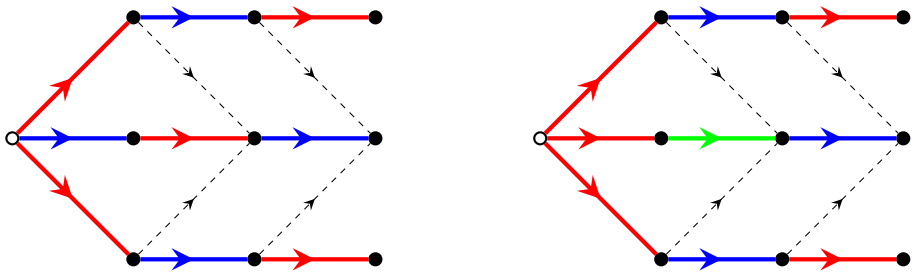
Keywords: BMRN*-colouring; planar digraphs; TDMA scheduling.

1. Introduction

The colouring concepts studied in this work are motivated by the following real-life problem. Suppose we have a network of entities communicating via radio waves. One particular of our entities, the *center of command*, continuously, through time, emits messages that must be propagated to all entities of the network. Because all entities might not be in the range of the center of command, these transmissions are performed in a multi-hop fashion. That is, for each entity (different from the center of command), a *preferred entity* is defined, from which the messages will be received. Entities and their respective preferred entity thus define a set of *links*, which is commonly referred to as the network's *backbone*, along which the messages of the center of command will be broadcasted.

In an ideal world, an entity receiving a message from its preferred entity via its link would just forward the message right away to some close entities. In practice, however, this cannot be done that simply because of particular types of physical rules. For instance, since messages are forwarded via radio waves, an entity a will not only receive messages from its preferred entity, but also from any other entity that has a in its range. In other words, *interferences* might occur, resulting in unexpected and unwanted alterations of messages.

In the TDMA method, this issue is overcome by dividing time into several successive units of time, called *time slots*, during each of which only some of the backbone links become active. The goal is then to design a *scheduling* of the links which will prevent interferences during transmissions. A downside of this method, however, is that links now become active during a short period of time only, i.e., during the time slot they are active



(a) A 2-BMRN-colouring. (b) A 3-BMRN*-colouring.

Figure 1: Examples of BMRN-arc-colouring and BMRN*-arc-colouring for a given spanned digraph. In each picture, the white vertex is the root of the backbone. Thick solid arcs are backbone arcs. Thin dashed arcs are interference arcs.

in. The main objective is thus to find a scheduling of the links with minimum number of slots, so that the waiting time of links is minimized.

The problem above can be studied under a graph colouring formalism. The network is modelled by a digraph D whose vertices are the entities, while the presence of an arc (u, v) indicates that v is in the emission range of u. The backbone is modelled by a subdigraph B of D being a spanning subdigraph in which all connected components are out-trees, i.e., oriented trees with all arcs oriented from a root r (modelling a local center of command) towards the leaves. We call the pair (D, B) a *backboned digraph*, the arcs of B being the *backbone arcs* and the arcs in $A(D) \setminus A(B)$ being the *interference arcs*. In case B is connected, i.e., its underlying undirected graph $\mathrm{und}(B)$ has only one connected component, we call (D, B) a *spanned digraph*. Since, in the problem above, we are assuming that the center of command is unique, the notion of spanned digraph is the precise one modelling our network context; however, the notion of backboned digraph is more general and will be of great use in most of our investigations in this paper.

A scheduling of the backbone links is modelled by an arc-colouring of the arcs of B verifying certain properties. In [3], Bensmail, Blanc, Cohen, Havet and Rocha introduced the notion of *BMRN-colourings* of backboned digraphs, which are arc-colourings ϕ without the following two configurations occuring:

- **Direct conflict**: two "consecutive" backbone arcs $(u, v), (v, w)$ with $\phi((u, v)) = \phi((v, w))$.
- **Indirect conflict**: an interference arc (u_1, v_2) such that $(u_1, v_1), (u_2, v_2)$ are two backbone arcs with $\phi((u_1, v_1)) = \phi((u_2, v_2))$.

Direct conflicts model the fact that, regarding the TDMA scheduling problem above, an entity v should not both receive and emit messages during a same time slot. This is because entities can be assumed to be very simple devices, that are not able to perform both actions at the same time. Indirect conflicts model interferences due to some entities being in the range of other emitting entities: during a given time slot, an entity v_2 should not receive a message from both its preferred entity v_1 and another entity u_1 also sending a message.

For a backboned digraph (D, B), the least number of colours in a BMRN-colouring is called the *BMRN-index* of (D, B), denoted by $\mathrm{BMRN}(D, B)$. The authors of [3] also considered the following slightly modified variation: a *BMRN*-colouring* of (D, B) is a BMRN-colouring where it holds that, for every vertex v of D with $\delta_B^+(v) \geq 1$, all backbone arcs of B out-going from v are assigned the same colour. The *BMRN*-index* of (D, B) is then the least number of colours (denoted $\mathrm{BMRN}^*(D, B)$) in a BMRN*-colouring of (D, B).

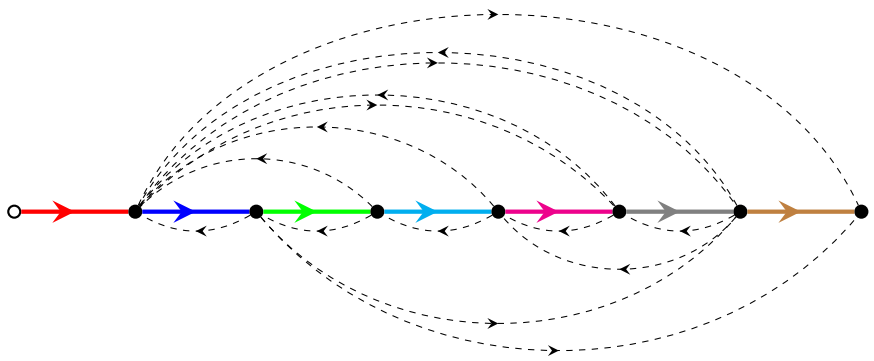


Figure 2: A planar spanned digraph (D, T) with $\mathrm{BMRN}^*(D, T) = 7$. The white vertex is the root of the backbone. Thick solid arcs are backbone arcs. Thin dashed arcs are interference arcs.

In the concrete network problem above, finding a BMRN*-colouring would be equivalent to finding a link scheduling where all entities send their messages during a unique time slot.

Refer to Figure 1 for an illustration of these concepts; in particular, note that the colouring on picture (a) is not a BMRN*-colouring, as the white vertex does not have all of its incident out-going backbone arcs assigned a same colour.

In [3], the authors gave a number of general results on both the BMRN-index and the BMRN*-index of digraphs. Note that $\mathrm{BMRN}(D, B) \leq \mathrm{BMRN}^*(D, B)$ always holds. Although the authors proved that the BMRN*-index of a given spanned digraph can be arbitrarily larger than its BMRN-index, they observed that, at least for particular classes of digraphs, considering BMRN*-colourings can be a way to deduce optimal BMRN-colourings, which is a reason why this modified notion is useful. They also considered algorithmic aspects related to the problem of determining the BMRN-index or the BMRN*-index of a given spanned digraph, which in general is NP-hard, even if one is allowed to construct the backbone from a given root. Also, they gave a number of more specific results for particular classes of digraphs, such as bounded-degree digraphs, outerplanar digraphs, and more generally planar digraphs.

An interesting aspect of BMRN*-colouring is its connection with the notion of *distance-2 colourings* of undirected graphs, which are vertex-colourings where no two vertices at distance at most 2 are assigned the same colour. For an undirected graph G, we denote by $\chi_2(G)$ the least number of colours in a distance-2 colouring of G. It was noticed in [3] that, for any backboned digraph (D, B), a distance-2 colouring ϕ of $\mathrm{und}(D)$ yields a BMRN*-colouring of (D, B) (by assigning, for every vertex v, colour $\phi(v)$ to all its incident out-going backbone arcs); this is because having no two adjacent vertices receiving the same colour by ϕ takes care of direct conflicts, while having no two vertices at distance 2 receiving the same colour takes care of indirect conflicts. Then, $\mathrm{BMRN}^*(D, B) \leq \chi_2(\mathrm{und}(D))$. Although these two chromatic parameters seem related, there are actually cases where they differ a lot. A good illustration is the case of planar graphs: while $\chi_2(G)$ is bounded below by $\Delta(G) + 1$ for every planar graph G, on the other hand we have $\mathrm{BMRN}^*(D, B) \leq 8$ for every backboned digraph (D, B) (see below). This phenomenon is quite intriguing, and one of our aims in this work is thus to understand BMRN*-colourings further.

The current paper is devoted to investigating further the case of planar digraphs, more

precisely the behaviour of the BMRN*-index for these digraphs, as many questions raised in [3] remain open to date. Focusing further on this class of digraphs is motivated by the central role they play in colouring problems, and by the fact that, regarding the practical problem introduced earlier, planar digraphs stand as a rather realistic class to consider. Making use of the Four-Colour Theorem [1, 2], the authors of [3] proved that for every planar backboned digraph (D, B), we have $\mathrm{BMRN}^*(D, B) \leq 8$. They also noticed that there exist planar spanned digraphs (D, T) with $\mathrm{BMRN}^*(D, T) = 7$, such as that one depicted in Figure 2. An interesting property of this example is that its backbone is a directed path; on the one hand, this implies that even $\mathrm{BMRN}(D, T) = 7$ holds, while, on the second hand, it illustrates the fact that, even in a planar backboned digraph, having a simple backbone topology does not prevent the BMRN-index and BMRN*-index from being relatively large. Yet, the authors left open the question of whether their upper bound is tight or not.

Question 1.1 ([3]). *Do we have $BMRN(D, T) \leq 7$ for every planar spanned digraph (D, T)? Similarly, do we always have $BMRN^*(D, T) \leq 7$?*

As a first result in this paper, we answer negatively to Question 1.1 by exhibiting, in Section 2, a planar spanned digraph (D, T) verifying $\mathrm{BMRN}(D, T), \mathrm{BMRN}^*(D, T) = 8$. This shows that the upper bound above is tight.

We then consider algorithmic aspects in Section 3. In [3], the authors proved that deciding if $\mathrm{BMRN}^*(D, B) \leq 3$ (and similarly $\mathrm{BMRN}(D, B) \leq 3$) holds for a given planar backboned digraph (D, B) is NP-hard, even when restricted to planar spanned digraphs. As mentioned earlier, we know that the BMRN-index and BMRN*-index of a planar backboned digraph can be as large as 8, and it thus makes sense investigating, for such a planar backboned digraph (D, B), the complexity of deciding whether $\mathrm{BMRN}^*(D, B) \leq k$ (and similarly $\mathrm{BMRN}(D, B) \leq k$) for $k \in \{4, .., 7\}$.

Question 1.2 ([3]). *For every $k \in \{4, 5, 6, 7\}$, what is the complexity of the Planar k-BMRN*-Colouring problem? What is that of the Planar k-BMRN-Colouring problem?*

Towards Question 1.2, we prove that deciding whether $\mathrm{BMRN}^*(D, B) \leq k$ holds for a given planar backboned digraph (D, B) is NP-hard for every $k \in \{4, ..., 6\}$, even when restricted to planar spanned digraphs.

For a planar backboned digraph, the BMRN*-index can be as large as 8. In all our extremal examples (such as those in Figure 2 and upcoming Figure 3) we note that small cycles are one of the main reasons why the number of needed colours is that high. Thus, it seems judicious to investigate the behaviour of the BMRN*-index of planar backboned digraphs when small cycles are excluded, which is a classical aspect in graph colouring theory. In that line, Section 4 is devoted to investigating the effects of a large girth on the BMRN*-index of a planar backboned digraph.

Conclusions and perspectives are gathered in Section 5.

Definitions, notation and terminology

For a given digraph D, we denote by $V(D)$ and $A(D)$ its vertex and arc sets, respectively. The *in-degree* (resp. *out-degree*) $d_D^-(v)$ (resp. $d_D^+(v)$) of a vertex v of D is the number of in-coming (resp. out-going) arcs incident to v. The subscript in this notation will be omitted whenever no ambiguity is possible. The *minimum in-degree* (resp. *minimum out-degree*) $\delta^-(D)$ (resp. $\delta^+(D)$) of D is the minimum in-degree (resp. out-degree) over the vertices of D. Conversely, the *maximum in-degree* (resp. *maximum out-degree*) $\Delta^-(D)$ (resp. $\Delta^+(D)$) of D is the maximum in-degree (resp. out-degree) over the vertices of D.

Abusing notions and notations, we voluntarily employ some terms or notations usually defined for undirected graph in the context of digraphs. Whenever we do so for a digraph D, it should be understood that we are referring to $\mathrm{und}(D)$, the undirected graph underlying D. In particular, we consider that D is *connected* as soon as $\mathrm{und}(D)$ is. The *degree* of a vertex v of D is its degree in $\mathrm{und}(D)$. The *maximum degree* $\Delta(D)$ of D is the maximum degree of $\mathrm{und}(D)$. We say D is *planar* if $\mathrm{und}(D)$ itself is planar. Given a planar embedding of D in the plane, we denote by $F(D)$ the set of faces. The *degree* $d(f)$ of a face f is the length of a walk enclosing f (in particular, if f is incident to a pendant arc, then that arc is counted twice). The *girth* $g(D)$ of D is the girth of $\mathrm{und}(D)$, which is the length of its smallest cycles.

In turn, whenever referring to a digraph notion or notation for a backboned digraph (D, B), we implicitly refer to the corresponding notion or notation for D.

2. Planar spanned digraphs with BMRN-index 8

Answering Question 1.1 negatively, we point out that there exist planar spanned digraphs with BMRN-index (and BMRN*-index) 8. Thus, according to the upper bound exhibited in [3], the maximum value of $\mathrm{BMRN}(D, T)$ (and $\mathrm{BMRN}^*(D, T)$) over all planar spanned digraphs (D, T) is 8. Our result is built upon the following straight observation:

Observation 2.1. *Let (D, B) be a backboned digraph. For every backboned surdigraph (D', B') (i.e., $V(D') = V(D)$ and $A(D) \subseteq A(D')$), we have $BMRN(D, B) \leq BMRN(D', B')$ and $BMRN^*(D, B) \leq BMRN^*(D', B')$.*

The proof of Observation 2.1 is obvious, as adding backbone arcs or interference arcs to a backboned digraph can only make its BMRN-index and BMRN*-index increase.

Now consider the planar backboned digraph depicted in Figure 3.

Observation 2.2. *The backboned digraph depicted in Figure 3 is a planar backboned digraph with BMRN-index 8.*

Proof. As shown via the embedding depicted in the figure, this backboned digraph (D, B) is indeed planar. The backbone B has exactly eight arcs, and it can be checked that no two of them can be assigned the same colour by a BMRN-colouring, either because of a backbone arc or because of an interference arc. In particular, for every backbone arc (u, v), we have precisely $d^-_B(u) + d^+_{D-A(B)}(u) + d^+_B(v) + d^-_{D-A(B)}(v) = 7$. Thus, $\mathrm{BMRN}(D, B) = 8$. □

Theorem 2.3. *There exist planar spanned digraphs with BMRN-index 8.*

Proof. Consider the planar backboned digraph in Figure 3 (which has BMRN-index 8 by Observation 2.2). It can easily be turned into a planar spanned digraph by adding, for instance, a backbone arc from v_2 to v_6, and a backbone arc from v_2 to v_9. Since, in the depicted embedding, v_2 and v_6 belong to a common face, and similarly v_2 and v_9 belong to a common face, adding these two arcs does not break planarity. Then the backbone becomes an out-tree with root v_1. By Observation 2.1 the resulting planar spanned digraph retains BMRN-index 8. □

3. On the complexity of Planar k-BMRN*-Colouring

Throughout this section, for any $k \geq 1$ the k-BMRN*-Colouring problem is that where a backboned digraph (D, B) is given, and the task is to determine whether $\mathrm{BMRN}^*(D, B) \leq$

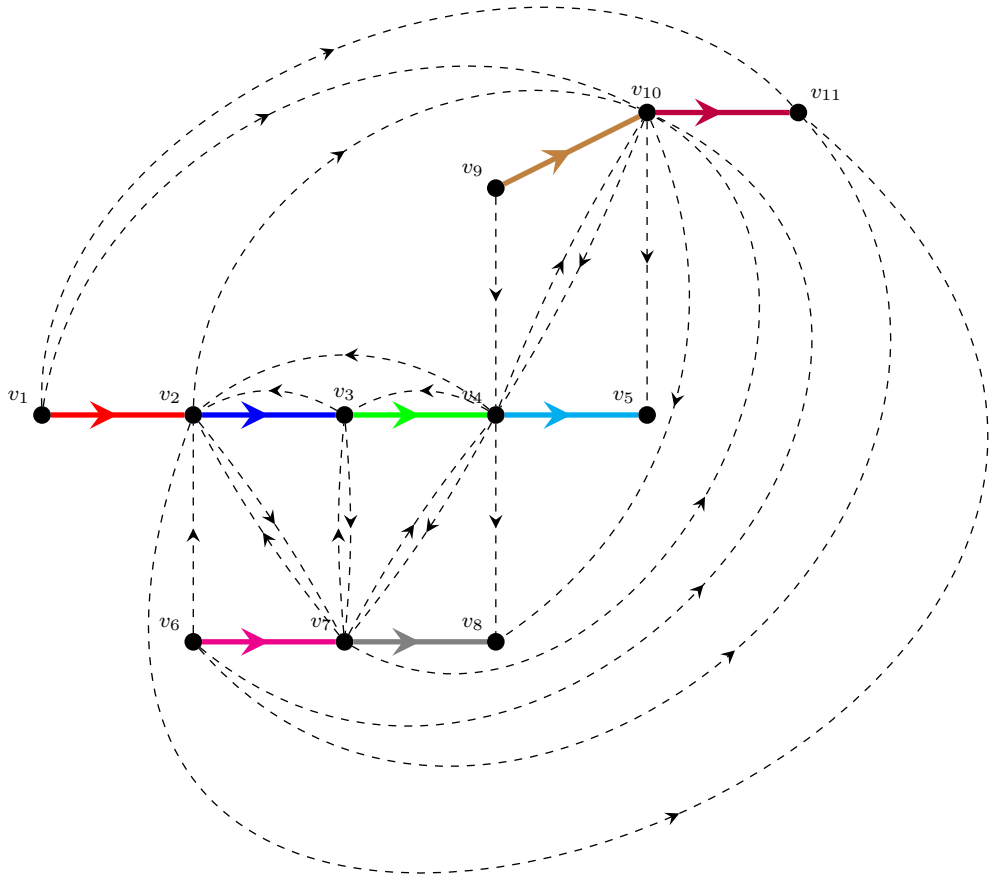


Figure 3: A planar backboned digraph (D, B) with $\mathrm{BMRN}(D, B) = 8$. Thick solid arcs are backbone arcs. Thin dashed arcs are interference arcs.

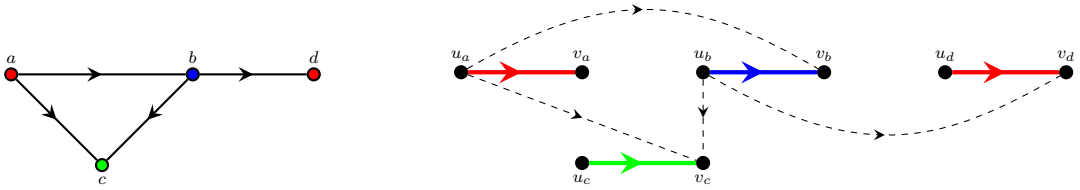
k. The Planar k-BMRN*-Colouring problem is the restriction of k-BMRN*-Colouring to planar backboned digraphs.

In [3], the authors noticed that k-BMRN*-Colouring is equivalent to the usual k-Colouring problem (where one aims at deciding, for a given undirected graph G, whether $\chi(G) \leq k$, i.e., whether G admits proper k-vertex-colourings), and is thus polynomial-time solvable for $k = 1, 2$, and NP-hard for every $k \geq 3$ even when restricted to spanned digraphs. Regarding Planar k-BMRN*-Colouring, they proved that Planar 3-BMRN*-Colouring is also NP-hard, even when restricted to planar spanned digraphs. As seen in previous Section 2, planar spanned digraphs can have BMRN*-index as large as 8, and it thus makes sense wondering about those digraphs with BMRN*-index exactly k for every $k \in \{4, ..., 7\}$ (recall Question 1.2).

3.1. Auxiliary tools, and main result

The complexity results above were established using the following construction (illustrated in Figure 4). Given an oriented graph $\vec{G}$, let $(D_{\vec{G}}, M_{\vec{G}})$ be the backboned digraph obtained from $\vec{G}$ as follows:

- For every vertex a of $\vec{G}$, add a backbone arc (u_a, v_a) to $M_{\vec{G}}$, where u_a, v_a are two new vertices.



(a) An oriented graph $\vec{G}$.

(b) The matched digraph $(D_{\vec{G}}, M_{\vec{G}})$.

Figure 4: An oriented graph $\vec{G}$ and the corresponding matched digraph $(D_{\vec{G}}, M_{\vec{G}})$. In figure (b), thick solid arcs are backbone arcs. Thin dashed arcs are interference arcs. The colours show a proper 3-vertex-colouring of $\text{und}(\vec{G})$ (a) and the equivalent 3-BMRN*-colouring of $(D_{\vec{G}}, M_{\vec{G}})$ (b).

- For every arc (a, b) of $\vec{G}$, add the interference arc (u_a, v_b) to $A(D_{\vec{G}}) \setminus A(M_{\vec{G}})$.

We call the resulting backboned digraph $(D_{\vec{G}}, M_{\vec{G}})$ the *matched digraph* (associated to $\vec{G}$). In that backboned digraph, to every vertex of $\vec{G}$ is associated a backbone arc, while to every arc of $\vec{G}$ is associated an interference arc. There is then a straight equivalence between finding a proper k-vertex-colouring of $\text{und}(\vec{G})$ and a k-BMRN*-colouring of $(D_{\vec{G}}, B_{\vec{G}})$.

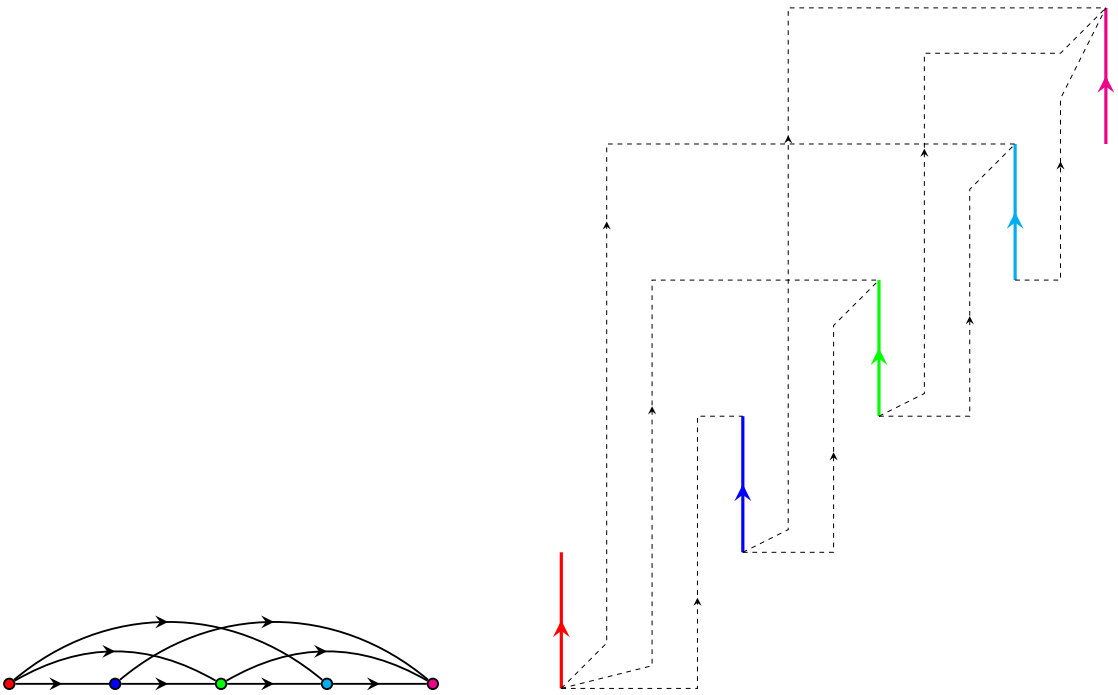
Observation 3.1. *For every oriented graph $\vec{G}$, we have $\chi(\text{und}(\vec{G})) = BMRN^*(D_{\vec{G}}, M_{\vec{G}})$.*

Note that $(D_{\vec{G}}, B_{\vec{G}})$ can be far from planar, in particular if $\vec{G}$ itself is far from planar. The other way around, the authors of [3] proved that, given that $\vec{G}$ is planar, also $(D_{\vec{G}}, B_{\vec{G}})$ is planar. Since the Planar 3-Colouring problem is NP-hard [5], this implies that Planar 3-BMRN*-Colouring is NP-hard. A simple connecting operation, to be described later, implies that this problem remains NP-hard for planar spanned digraphs as well.

Since all planar graphs are 4-colourable [1, 2], the same arguments cannot be used to prove that Planar k-BMRN*-Colouring is NP-hard for any $k > 3$, as it would require the corresponding Planar k-Colouring problem to be NP-hard as well. To overcome this point, we come up with an improved reduction scheme, which allows us to establish the NP-hardness of Planar k-BMRN*-Colouring for every $k \in \{3, 4, 5, 6\}$, even when restricted to planar spanned digraphs. This is by means of an adaptation of so-called *crossover gadgets*, which were, to the best of our knowledge, first used by Garey, Johnson and Stockmeyer to establish the NP-hardness of the Planar 3-Colouring problem [5]. Crossover gadgets are graphs with certain colouring properties that can be used to "replace" edge crossings in a non-planar embedding of a graph, with preserving the k-colourability of the whole graph.

Before defining what is a crossover gadget in our context, we first need to introduce a particular way to draw a matched digraph defined over an acyclic oriented graph. Let $\vec{G}$ be an acyclic orientation of an undirected graph G. This orientation defines an ordering $w_1, ..., w_n$ of the vertices of $\vec{G}$, such that all arcs are directed "to the right" (i.e., if (w_i, w_j) is an arc, then $i < j$). We now consider the matched digraph $(D_{\vec{G}}, M_{\vec{G}})$ associated to $\vec{G}$, which we here draw in the plane in a specific way (see Figure 5), which we call a *good drawing*. Let us assume (u_i, v_i) denotes the backbone arc of $(D_{\vec{G}}, M_{\vec{G}})$ associated to vertex w_i of $\vec{G}$. We first draw (u_1, v_1) vertically, having some length ℓ. We then draw (u_2, v_2) vertically too, of length ℓ, but positioned at some horizontal distance x to the right of (u_1, v_1) and at some vertical distance x above (u_1, v_1). We draw all (u_i, v_i)'s this way, i.e., each (u_i, v_i) is drawn vertically, of length ℓ, positioned at horizontal distance x to the right of (u_{i-1}, v_{i-1}) and at vertical distance x above (u_{i-1}, v_{i-1}).

We now draw the interference arcs. By the orientation $\vec{G}$, for every arc (w_i, w_j) of $\vec{G}$, we want to add to $(D_{\vec{G}}, M_{\vec{G}})$ an interference arc (u_i, v_j), where v_j is located somewhere



(a) An acyclic oriented graph $\vec{G}$.

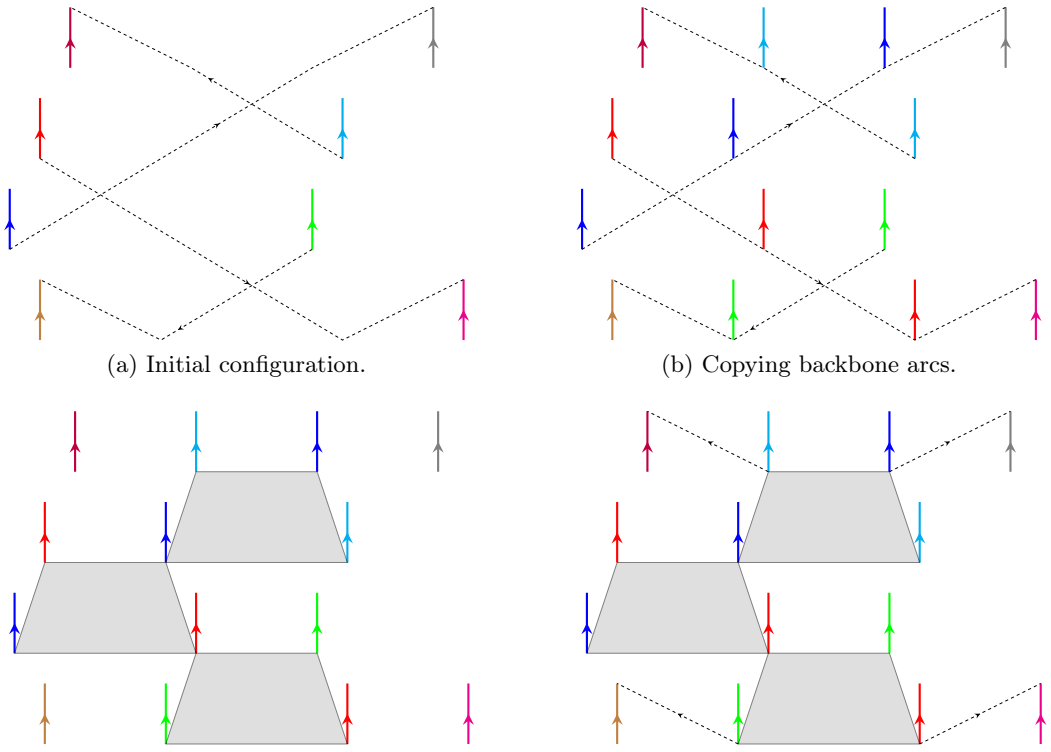
(b) A good drawing of $(D_{\vec{G}}, M_{\vec{G}})$.

Figure 5: An acyclic oriented graph $\vec{G}$, and a good drawing of the matched digraph $(D_{\vec{G}}, M_{\vec{G}})$. The colours show, for every vertex of $\vec{G}$, the corresponding backbone arc of $(D_{\vec{G}}, M_{\vec{G}})$. In the picture (b), thick solid arcs are backbone arcs. Thin dashed arcs are interference arcs. Note that the tails of the backbone arcs of $(D_{\vec{G}}, M_{\vec{G}})$ lie on the outer face, and that the arc crossings involve interference arcs originating from different backbone arcs.

above and at the right of u_i. We draw this interference arc (u_i, v_j) in the following way: We make the arc leave u_i from the right, then immediately go vertically until the altitude of v_j is attained, and then going straight horizontally to v_j. Note that this way we may have lots of arc crossings involving at least three arcs, and many pairs of interference arcs intersecting on more than just one point. However, we can make the interference arcs go to their destination in a clean way, as follows (see Figure 5). For every two interference arcs (u_i, v_j) and (u_i, v_k) out-going from u_i, we make them leave u_i with different angles so that they do not intersect and there is some horizontal "delay" before they start going vertically, in a non-intersecting parallel way. This delay, to avoid any further crossing involving these two arcs, is as follows: if $j < k$, i.e., the destination of (u_i, v_k) is farther to the right than that of (u_i, v_j), then we grant more horizontal delay to (u_i, v_j), as this arc will stop its ascension and "turn" right first. Reversely, we do not make every two interference arcs $(u_i, v_k), (u_j, v_k)$, both supposed to reach v_k, attain v_k following a same horizontal line. Instead, assuming $i < j$, i.e., u_i is somewhere on the left of u_j, we add some vertical delay to (u_i, v_k) to the moment it stops its ascension and turns right.

By this good drawing of $(D_{\vec{G}}, M_{\vec{G}})$, all interference arcs are drawn following an "S" shape, and it is thus easy to see that all u_i's lie on the "outer face" (to be more formal, in the digraph obtained by replacing all arc crossings by new vertices). Also, all arc crossings are perpendicular, involve exactly two interference arcs originating from different vertices, and the intersection between any two interference arcs is a set of points.

Let us now define what a crossover gadget is. In our context, a *k-crossover gadget* (for some $k \geq 3$) will be a backboned digraph (D, B) with the following properties:



(a) Initial configuration.

(b) Copying backbone arcs.

(c) Adding crossover gadgets.

(d) Replacing interference arcs.

Figure 6: Getting rid of arc crossings via crossover gadgets. Thick solid arcs are backbone arcs. Thin dashed arcs are interference arcs. Backbone arcs with the same colour are copies of a same initial backbone arc. Gray boxes are crossover gadgets, where the four incident backbone arcs are their corner arcs.

1. B has four particular pending *corner arcs* e, e', f, f' (i.e., they are each incident to a degree-1 vertex), were e, e' (and similarly f, f') are said to be *opposite*.

2. D has planar embeddings such that all of e, e', f, f' have their both sides being incident to the outer face, and, as going along the outer face, no two opposite corner arcs appear consecutively (i.e., the sequence of appearance must be e, f, e', f', or reversely).

3. In every k-BMRN*-colouring ϕ of (D, B), we have $\phi(e) = \phi(e')$ and $\phi(f) = \phi(f')$.

4. There exist k-BMRN*-colourings ϕ of (D, B) where $\phi(e) = \phi(e') = \phi(f) = \phi(f')$. Also, there exist k-BMRN*-colourings ϕ of (D, B) where $\phi(e) = \phi(e') \neq \phi(f) = \phi(f')$.

Such gadgets will be used in the following way (see Figure 6). Assume we have a non-planar matched digraph (D, M) that we want to k-BMRN*-colour. To ease the following explanations, let us consider a good drawing of (D, M) on the plane. As pointed out earlier, all arcs crossing involve two interference arcs crossing perpendicularly, and originating from different vertices. Furthermore, no three arcs cross on a same point, and the intersection (if any) between any two arcs is a point. Let us now consider every interference arc (u_a, v_b) in turn (as in the definition of matched digraphs, for convenience we here write all backbone arcs of (D, M) under the form (u_a, v_a)). Since (u_a, v_b) is an interference arc, it means that (u_a, v_a) and (u_b, v_b) are backbone arcs, drawn vertically by assumption. Let us now go along (u_a, v_b), from u_a towards v_b. Each time we encounter an arc crossing (involving

(u_a, v_b)), let us add to the backboned digraph a *copy* of (u_a, v_a), being a new backbone arc (u'_a, v'_a) drawn vertically in such a way that u'_a is located on (u_a, v_b) right after the crossing, before the next crossing involving (u_a, v_b) (if any), and before v_b. Free to make (u'_a, v'_a) as small as desired, we might assume that this new backbone arc does not intersect anything (with the exception of its tail lying on (u_a, v_b) at the moment).

We perform this transformation for all interference arcs of (D, M), resulting in an auxiliary backboned digraph (D', M') (Figure 6 (b)). For every original backbone arc (u_a, v_a) of (D, M) and each interference arc (u_a, v_b) out-going from u_a, we have thus added x copies of (u_a, v_a) to the digraph, where x denotes the number of arc crossings in which (u_a, v_b) was involved. There are thus a certain number of copies of (u_a, v_a) in (D', M'), including the original copy of (u_a, v_a), that are "surrounding" the crossings (meaning that, for every interference arc (u_a, v_b) that is crossed, there is a copy of (u_a, v_a) lying on (u_a, v_b) before and after every arc crossing involving (u_a, v_b)).

Let us now modify (D', M') further. To each arc crossing of (D, M) involving two interference arcs (u_a, v_b) and (u_c, v_d), we associate, in (D', M'), four "surrounding" backbone arcs, being, as going from u_a to v_b, the copy of (u_a, v_a) located before the crossing and the copy of (u_a, v_a) located after the crossing, and, as going from u_c to v_d, the copy of (u_c, v_c) located before the crossing and the copy of (u_c, v_c) located after the crossing. We are now ready to get rid of the conflicts of (D, M). In (D', M'), remove all interference arcs that are, in (D, M), involved in arc conflicts. Then, for every arc crossing of (D, M) that involves two interference arcs (u_a, v_b) and (u_c, v_d), add, in (D', M'), a new k-crossover gadget where the crossing was occurring, and embed that gadget in such a way that its four corner arcs fully lie in the face surrounding the gadget, and so that no two opposite corner arcs appear consecutively in that face. Next, identify two opposite corner arcs of the gadget with the two copies of (u_a, v_a) associated to the crossing, and identify the two other opposite corner arcs with the two copies of (u_c, v_c) associated to the crossing (Figure 6 (c)). Once this has been done for all arc crossings, finally consider every interference arc (u_a, v_b) that is involved in some crossings in (D, M), and, denoting (u'_a, v'_a) the last copy of (u_a, v_a) (as going from u_a to v_b) in (D', M'), add the interference arc (u'_a, v_b) to (D', M') (Figure 6 (d)). Assuming we do have a k-crossover in hand, we denote by $\mathrm{UC}(D, M)$ the backboned digraph obtained from (D, M) in this way, where that k-crossover gadget is implicitly used to replace the arc crossings.

We now prove that, assuming we do have a k-crossover gadget, using it to construct $\mathrm{UC}(D, M)$ from (D, M) results in $\mathrm{UC}(D, M)$ having the desired properties.

Theorem 3.2. *Let (D, M) be a matched digraph drawn in a good way, and assume we have a k-crossover gadget for some $k \geq 3$. Then, the backboned digraph $(D', M') = \mathrm{UC}(D, M)$, constructed as described above using copies of that gadget, admits a planar embedding. Furthermore, $BMRN^*(D, M) \leq k$ if and only if $BMRN^*(D', M') \leq k$.*

Proof. The first part of the claim follows from arguments used to describe the construction of $\mathrm{UC}(D, M)$. When adding to (D', M') the backbone arcs that will later become the corner arcs of the crossover gadgets, we note that, by how these arcs are positioned, when removing all interference arcs involved into arc crossings we get a plane digraph. Adding the gadgets does not create new arc crossings, as these gadgets admit planar embeddings with their four corner arcs being fully in the outer face (by definition), and, when identifying their corner arcs to four existing arcs, we can "shape" the gadget so that it is roughly drawn like the arc crossing it is locally replacing. This way, we make sure that no arc of the gadget is involved in a new arc crossing.

We now focus on proving the last part of the statement.

- First assume that we have a k-BMRN*-colouring ϕ of (D, M). We derive one ϕ' of (D', M'). For every backbone arc (u_a, v_a) of (D, M), we set $\phi'((u_a, v_a)) = \phi((u_a, v_a))$. Now, for every other copy (u'_a, v'_a) of (u_a, v_a) in (D', M'), we set $\phi'((u'_a, v'_a)) = \phi((u_a, v_a))$. Note that this does not create any indirect conflict involving two backbone arcs (a, b) and (c, d), via, say, the interference arc (a, d). Indeed, either (a, b) and (c, d) both belong to (D, M), in which case (a, d) also does and we have $\phi((a, b)) = \phi((c, d))$, an indirect conflict that is a contradiction to the definition of ϕ. Otherwise, we have, say, that (a, b) is actually a copy of an original backbone arc (a', b'). By the existence of (a, d), we deduce that, in (D, M), there is an interference arc (a', d), which is involved in arc crossings that resulted in the addition of (a, b) to (D', M'). By construction of ϕ', we have $\phi'((a', b')) = \phi'((a, b))$, while, if (a, b) and (c, d) are in conflict, $\phi'((a, b)) = \phi'((c, d))$. Thus, we have $\phi((a', b')) = \phi((c, d))$, while (a', d) exists in (D, M), which is an indirect conflict by ϕ. This is a contradiction.

  The only backbone arcs of (D', M') that remain to be coloured are those of the k-crossover gadgets. Consider such a gadget in (D', M'), where $(u'_a, v'_a), (u''_a, v''_a)$ are its first two opposite corner arcs and $(u'_b, v'_b), (u''_b, v''_b)$ are its two last opposite corner arcs, being copies of some original backbone arcs (u_a, v_a) and (u_b, v_b), respectively, of (D, M). By construction of ϕ' so far, we have $\phi'((u'_a, v'_a)) = \phi'((u''_a, v''_a))$ and $\phi'((u'_b, v'_b)) = \phi'((u''_b, v''_b))$. By the definition of a k-crossover gadget, ϕ' can be extended to the gadget provided each pair of its opposite corner arcs are assigned the same colour, which is the case here. Thus, ϕ' can be extended to all gadgets used to construct (D', M'), which thus admits ϕ' as a k-BMRN*-colouring.

- Now assume that we have a k-BMRN*-colouring ϕ' of (D', M'), which we wish to extend to one ϕ of (D, M). Every backbone arc (u_a, v_a) of (D, M) also exists in (D', M'); then we simply set $\phi((u_a, v_a)) = \phi'((u_a, v_a))$. Now consider, in (D, M), two backbone arcs (u_a, v_a) and (u_b, v_b) such that (u_a, v_b) is an interference arc. If that interference arc is also present in (D', M'), then we have $\phi'((u_a, v_a)) \neq \phi'((u_b, v_b))$ and so, by ϕ, the backbone arcs (u_a, v_a) and (u_b, v_b) are not involved in an indirect conflict. Now, if (u_a, v_b) is not an interference arc of (D', M'), then it must be that this arc is involved in arc crossings in (D, M). By construction, there is a copy (u'_a, v'_a) of (u_a, v_a) in (D', M') (located right after the last arc crossing involving (u_a, v_b) in (D, M)) such that (u'_a, v_b) is an interference arc (replacing (u_a, v_b)). Then $\phi'(((u'_a, v'_a))) \neq \phi'((u_b, v_b))$. By the definition of a k-crossover gadget, we have $\phi'((u_a, v_a)) = \phi'((u'_a, v'_a))$. From this, we deduce that $\phi((u_a, v_a)) \neq \phi((u_b, v_b))$, and that there cannot be indirect conflicts by ϕ. □

We now show how to deduce the NP-hardness of Planar k-BMRN*-Colouring from the previous results and observations (assuming a k-crossover gadget exists).

Theorem 3.3. *If k-crossover gadgets exist, then Planar k-BMRN*-Colouring is NP-hard.*

Proof. The proof is by reduction from the k-Colouring problem, which is NP-hard. Let G be an undirected graph; we build a planar backboned digraph (D, B) such that G admits a proper k-vertex-colouring if and only if (D, B) admits a k-BMRN*-colouring.

Let $\vec{G}$ be an acyclic orientation of G, and let $(D_{\vec{G}}, M_{\vec{G}})$ be the matched digraph of $\vec{G}$. Consider a good drawing of $(D_{\vec{G}}, M_{\vec{G}})$, and, from it, build (D, B) the backboned digraph $\mathrm{UC}(D_{\vec{G}}, M_{\vec{G}})$ obtained from $(D_{\vec{G}}, M_{\vec{G}})$ by removing arc crossings using copies of a k-crossover gadget. Note that the whole construction is achieved in polynomial time; in particular, by the properties of a good drawing of a matched digraph, at most a quadratic number of k-crossover gadgets must be used to get rid of all conflicts.

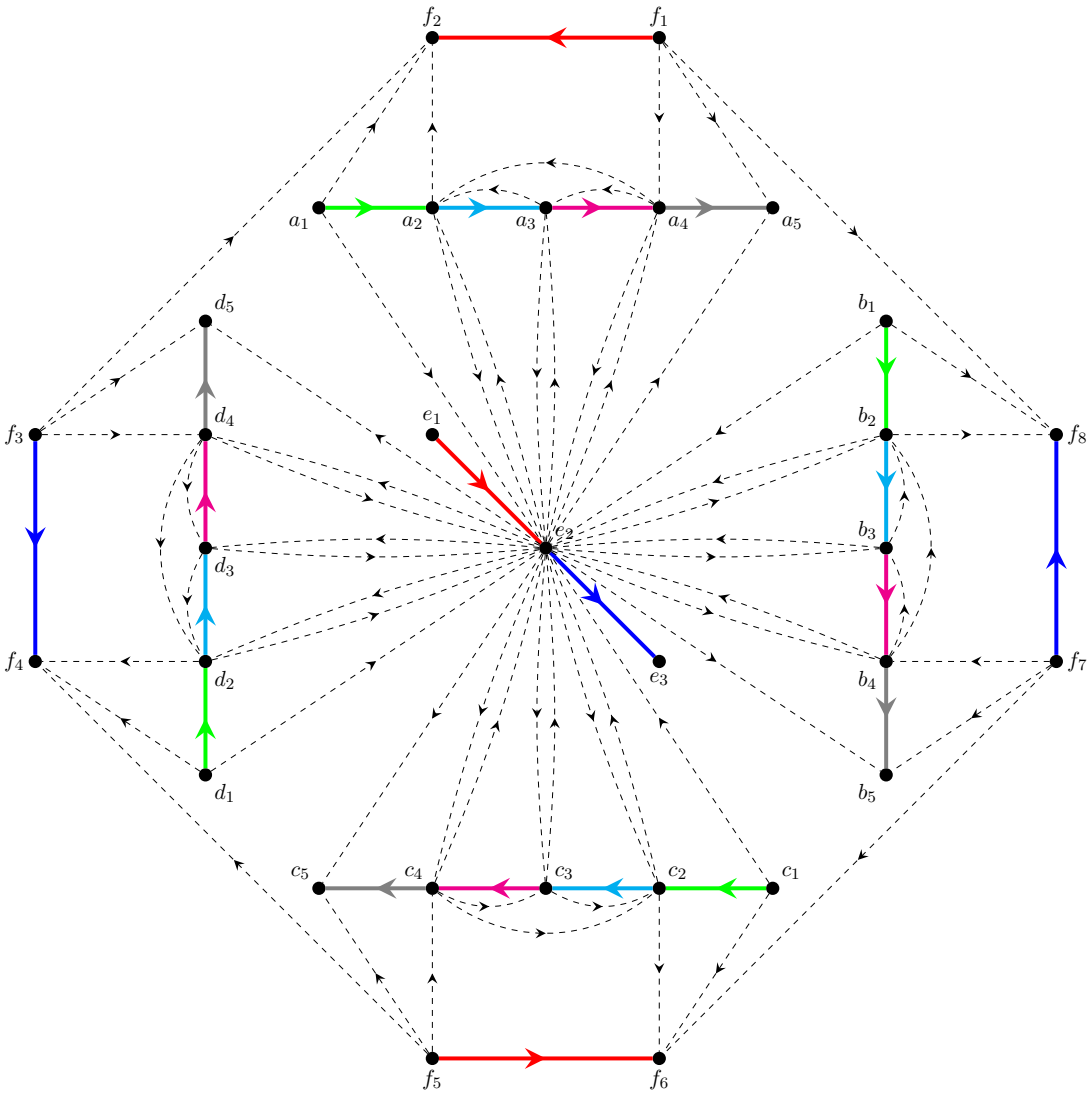


Figure 7: The core of the 6-crossover gadget, and one of its 6-BMRN*-colourings. Thick solid arcs are backbone arcs. Thin dashed arcs are interference arcs.

By Theorem 3.2, (D, B) is planar. Furthermore, it preserves the BMRN*-colourability of $(D_{\vec{G}}, M_{\vec{G}})$, while, in $(D_{\vec{G}}, M_{\vec{G}})$, finding a k-BMRN*-colouring is equivalent to finding a proper k-vertex-colouring of G (Observation 3.1). Thus, G admits a proper k-vertex-colouring if and only if (D, B) admits a k-BMRN*-colouring. □

In what follows, we prove that 6-crossover gadgets exist. At the end of this section, we will show how to derive, from that gadget, k-crossover gadgets for any $k \in \{3, 4, 5\}$.

3.2. A 6-crossover gadget

The 6-crossover gadget we exhibit is made of several pieces with particular colouring properties, which we introduce little by little to ease the understanding.

The *core* of our 6-crossover gadget is the backboned digraph depicted in Figure 7. Its main property of interest is the following one, where, in the statement and further, we deal with its vertices and arcs using the terminology from the figure:

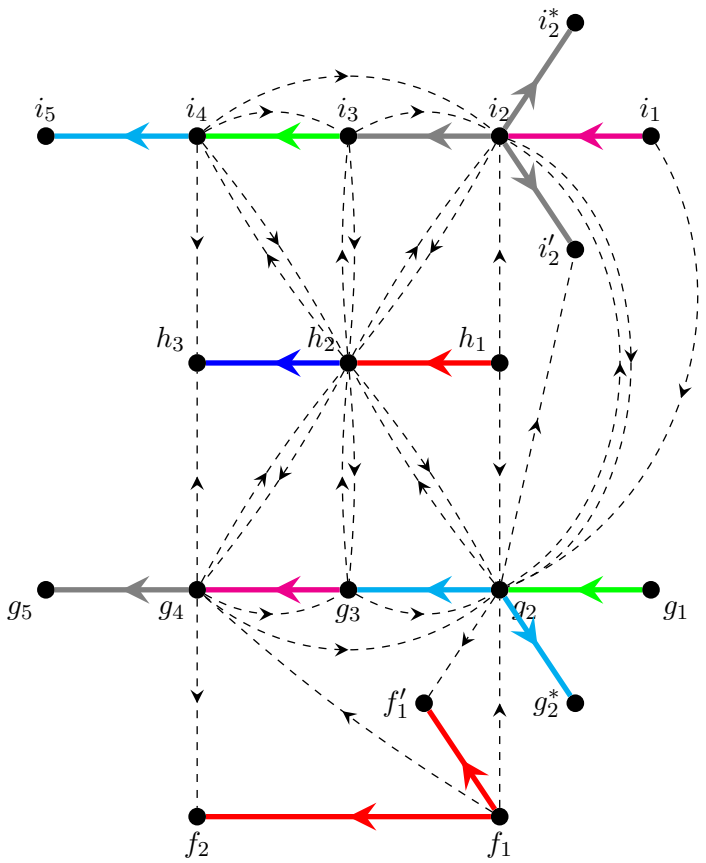


Figure 8: A tower of the 6-crossover gadget, and one of its 6-BMRN*-colourings. Thick solid arcs are backbone arcs. Thin dashed arcs are interference arcs.

Theorem 3.4. *Let (D, B) be the backboned digraph depicted in Figure 7. In every 6-BMRN*-colouring ϕ of (D, B), we have $\phi((f_1, f_2)) = \phi((f_5, f_6)) \neq \phi((f_3, f_4)) = \phi((f_7, f_8))$.*

Proof. Let ϕ be a 6-BMRN*-colouring of (D, B). To avoid a direct conflict, we must have $\phi((e_1, e_2)) \neq \phi((e_2, e_3))$. Similarly, no two of $(a_1, a_2), (a_2, a_3), (a_3, a_4), (a_4, a_5)$ can be assigned the same colour: either to avoid a direct conflict (case of two consecutive arcs), or an indirect conflict (otherwise). Furthermore, the colour of (e_1, e_2) cannot be assigned to any (a_i, a_{i+1}) of these four arcs because of the interference arc (a_i, e_2), while the colour of (e_2, e_3) cannot be assigned to (a_i, a_{i+1}) because of the interference arc (e_2, a_{i+1}). Thus, all six arcs $(e_1, e_2), (e_2, e_3), (a_1, a_2), (a_2, a_3), (a_3, a_4), (a_4, a_5)$ are assigned different colours by ϕ. Now, we note that none of the colours of $(a_1, a_2), (a_2, a_3), (a_3, a_4), (a_4, a_5)$ can be assigned to (f_1, f_2), because of the four interference arcs joining these arcs. A consequence is that $\phi((f_1, f_2)) \in \{\phi((e_1, e_2)), \phi((e_2, e_3))\}$.

Repeating these arguments towards $(f_3, f_4), (f_5, f_6), (f_7, f_8)$, we get that each of the arcs $(f_1, f_2), (f_3, f_4), (f_5, f_6), (f_5, f_6)$ must be assigned a colour from $\{\phi((e_1, e_2)), \phi((e_2, e_3))\}$. Because of the interference arcs $(f_1, f_8), (f_3, f_2), (f_5, f_4), (f_7, f_6)$, we must have $\phi((f_1, f_2)) \neq \phi((f_3, f_4)) \neq \phi((f_5, f_6)) \neq \phi((f_7, f_8))$. Thus, we must have $\phi((f_1, f_2)) = \phi((f_5, f_6)) \neq \phi((f_3, f_4)) = \phi((f_7, f_8))$. □

Our 6-crossover gadget will also be made of *towers*, being copies of the backboned digraph depicted in Figure 8. The backbone arcs $(i_3, i_4), (i_4, i_5), (g_3, g_4), (g_4, g_5)$ are its four *left-side arcs*, while the backbone arcs $(i_1, i_2), (i_2, i_2^*), (g_1, g_2), (g_2, g_2^*)$ are its four *right-side*

arcs. Its backbone arc (f_1, f_2) is the *base arc.* It has the following properties (where, again, we deal with its vertices and arcs following the terminology used in the figure):

Theorem 3.5. *Let (D, B) be the backboned digraph depicted in Figure 8. In every 6-BMRN*-colouring ϕ of (D, B), we have:*

1. *all of $(g_1, g_2), (g_2, g_3), (g_3, g_4), (g_4, g_5)$ have different colours by ϕ, and similarly for all of $(i_1, i_2), (i_2, i_3), (i_3, i_4), (i_4, i_5)$;*
2. $\{\phi((g_1, g_2)), \phi((g_2, g_3)), \phi((g_3, g_4)), \phi((g_4, g_5))\} = \{\phi((i_1, i_2)), \phi((i_2, i_3)), \phi((i_3, i_4)), \phi((i_4, i_5))\}$;
3. *all of $(g_1, g_2), (g_2, g_3), (i_1, i_2), (i_2, i_3)$ have different colours by ϕ, and similarly for all of $(g_3, g_4), (g_4, g_5), (i_3, i_4), (i_4, i_5)$;*
4. $\phi((f_1, f_2)) \notin \{\phi((g_1, g_2)), \phi((g_2, g_3)), \phi((g_3, g_4)), \phi((g_4, g_5))\}$.

Proof. The first item is because no two arcs from these sets of four arcs can have the same colour by ϕ, either because they are consecutive (direct conflict), or because they are joined by an interference arc (indirect conflict).

The second item is because $\phi((h_1, h_2))$ and $\phi((h_2, h_3))$ must be different from all of $\phi((g_1, g_2)), \phi((g_2, g_3)), \phi((g_3, g_4)), \phi((g_4, g_5))$, and similarly $\phi((h_1, h_2))$ and $\phi((h_2, h_3))$ must be different from all of $\phi((i_1, i_2)), \phi((i_2, i_3)), \phi((i_3, i_4)), \phi((i_4, i_5))$. This is because of all interference arcs between h_1, h_2, h_3 and these eight arcs. Thus, by the first item, $\phi((g_1, g_2)), \phi((g_2, g_3)), \phi((g_3, g_4)), \phi((g_4, g_5))$ are four different colours, and $\phi((h_1, h_2)), \phi((h_2, h_3))$ are the last two colours. All of $\phi((i_1, i_2)), \phi((i_2, i_3)), \phi((i_3, i_4)), \phi((i_4, i_5))$ must be distinct colours (still by the first item), and must be different from $\phi((h_1, h_2)), \phi((h_2, h_3))$.

The third item is because of the four interference arcs $(g_2, i'_2), (g_2, i_2), (i_2, g_2), (i_1, g_2)$. These interference arcs force the four colours $\phi((g_2, i'_2)), \phi((g_2, i_2)), \phi((i_2, g_2)), \phi((i_1, g_2))$ to be different. By the first and second items, we then deduce that $\{\phi((i_3, i_4)), \phi((i_4, i_5))\} = \{\phi((g_1, g_2)), \phi((g_2, g_3))\}$, and similarly that $\{\phi((i_1, i_2)), \phi((i_2, i_3))\} = \{\phi((g_3, g_4)), \phi((g_4, g_5))\}$. Thus, the four left-side arcs receive different colours, that are exactly the colours received by the four right-side arcs.

The fourth item is because of previous arguments, and because of the presence of the interference arcs $(f_1, g_2), (g_2, f'_1), (f_1, g_4), (g_4, f_2)$. That is, we have $\phi((g_1, g_2)) \neq \phi((f_1, f_2))$ because of (f_1, g_2), we have $\phi((g_2, g_3)) \neq \phi((f_1, f_2))$ because of (g_2, f'_1), we have $\phi((g_3, g_4)) \neq \phi((f_1, f_2))$ because of (f_1, g_4), and we have $\phi((g_4, g_5)) \neq \phi((f_1, f_2))$ because of (g_4, f_2). □

Our 6-crossover gadget is depicted in Figure 9. The central octagon C is the core from Figure 7. Each of the four peripheral backbone arcs of C serves as the base arc of a tower from Figure 8. Note that, going anticlockwise, all four towers $T_0, ..., T_3$ are "oriented" the same way (with respect to the four base arcs). The four backbone arcs $(y_0, z_0), ..., (y_3, z_3)$ with both sides on the outer face are the corner arcs. The tail y_i of each such arc (y_i, z_i) is joined, via out-going interference arcs only, to the base arc of a tower T_i, to the four left-side arcs from that tower T_i, and to the four right-side arcs from the next tower T_{i+1} (modulo 4).

Theorem 3.6. *The backboned digraph (D, B) depicted in Figure 9 is a 6-crossover gadget.*

Proof. Figure 9 shows that (D, B) indeed admits planar embeddings where its four corner arcs have both sides on the outer face. It remains to show that (D, B) has the desired colouring properties, i.e.,:

- (D, B) has 6-BMRN*-colourings;

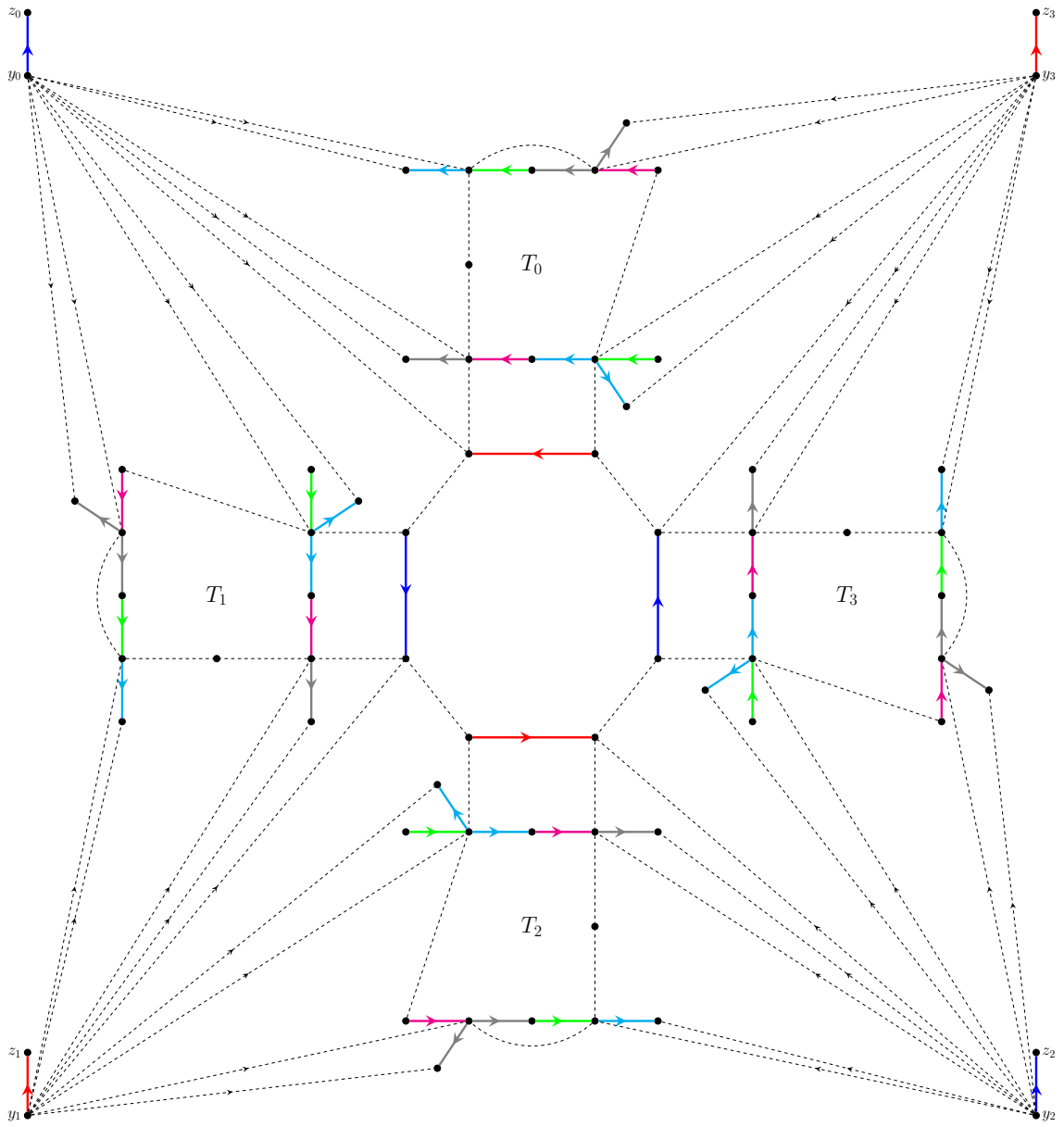


Figure 9: The 6-crossover gadget, and one of its 6-BMRN*-colourings. Thick solid arcs are backbone arcs. Thin dashed arcs are interference arcs. The central octagon is the core of the gadget, to which four towers are attached. Dashed edges show the outer face in the planar embedding of the core and towers we are considering.

- in every 6-BMRN*-colouring of (D, B), every two opposite corner arcs are assigned the same colour;
- there exist 6-BMRN*-colourings of (D, B) where all four corner arcs are assigned the same colour;
- there exist 6-BMRN*-colourings of (D, B) where two opposite corner arcs are assigned some colour, that is different from the colour assigned to the remaining two opposite corner arcs.

Figure 9 shows an example of an arc-colouring which is a 6-BMRN*-colouring of (D, B). Let us now prove thoroughly that (D, B) has the desired properties, by describing how

a 6-BMRN*-colouring ϕ of (D, B) behaves, starting from the core C, continuing with its attached towers $T_0, ..., T_3$, and propagating to the corner arcs $(y_0, z_0), ..., (y_3, z_3)$. For simplicity, we assume below that ϕ assigns colours in $\{1, ..., 6\}$.

By Theorem 3.4, the four backbone arcs of the core are assigned two different colours only, in such a way that no two consecutive of them are assigned the same colour. Let us thus assume that the backbone arcs of the core that are the base arcs of T_0 and T_2 are assigned colour 1, while the backbone arcs of the core that are the base arcs of T_1 and T_3 are assigned colour 2. Let us now focus on T_0. By Theorem 3.5, the four left-side arcs of T_0 must be assigned four different colours different from 1. We distinguish two cases, depending on whether 2 is one of these colours or not.

- Assume 2 is not one of these colours. Then the four colours assigned to the left-side arcs of T_0 are $3, 4, 5, 6$. Still by Theorem 3.5, we deduce that the four right-side arcs of T_0 are also assigned colours $3, 4, 5, 6$. By construction, we note that (y_0, z_0) cannot be assigned a colour assigned to the four left-side arcs of T_0 and to its base arc. Thus, we must have $\phi((y_0, z_0)) = 2$. Similarly, (y_3, z_3) cannot be assigned a colour assigned to the four right-side arcs of T_0 and to the base arc of T_3; the only available colour for (y_3, z_3) is thus 1. In the tower T_3, the four left-side arcs must be assigned different colours, that must be different from that of (y_3, z_3) (because of interference arcs) and that of the base of T_3. Thus, the four left-side arcs of T_3 are assigned colours $3, 4, 5, 6$, which are also the colours of the four right-side arcs of T_3 by Theorem 3.5.

  Repeating these last arguments to T_2 and then T_1, we successively deduce that (y_2, z_2) must be assigned colour 2, the left-side arcs (and right-side arcs) of T_2 must be assigned colours $3, 4, 5, 6$, the arc (y_1, z_1) must be assigned colour 1, and the left-side arcs (and right-side arcs) of T_1 must be assigned colours $3, 4, 5, 6$. Then ϕ is a 6-BMRN*-colouring where $1 = \phi((y_1, z_1)) = \phi((y_3, z_3)) \neq \phi((y_0, z_0)) = \phi((y_2, z_2)) = 2$.

- Assume 2 is one of the four colours assigned to the four left-side arcs of T_0. Without loss of generality, we may assume that these colours are $2, 3, 4, 5$. By Theorem 3.5, these colours are also those of the right-side arcs of T_0. By the interference arcs leaving from y_0 to T_0, the colour assigned to (y_0, z_0) must be different from $1, 2, 3, 4, 5$, and it must thus be 6. We now consider T_1: its four right-side arcs must be assigned distinct colours different from that of (y_0, z_0) and that of the base arc of T_1. Then the four right-side arcs of T_1 are assigned colours $1, 3, 4, 5$, and these are also the colours of the four left-side arcs of T_1. Then (y_1, z_1) must be assigned a colour different from that of the four left-side arcs of T_1 and that of the base arc of T_1. Then $\phi((y_1, z_1)) = 6$. That colour cannot be assigned to the right-side arcs of T_2; since these colours must be different from that of the base arc of T_2, they are $2, 3, 4, 5$. These colours are also the colours of the left-side arcs of T_2 by Theorem 3.5.

  Going on that way, we deduce that (y_2, z_2) must be assigned colour 6, the left-side arcs (and right-side arcs) of T_3 must be assigned colours $1, 3, 4, 5$, and the arc (y_3, z_3) must be assigned colour 6. This results in ϕ being a 6-BMRN*-colouring where $\phi((y_0, z_0)) = \phi((y_1, z_1)) = \phi((y_2, z_2)) = \phi((y_3, z_3)) = 6$. □

3.3. Summarizing and going farther

From our 6-crossover gadget we can easily deduce, for any $k \in \{3, 4, 5\}$, a k-crossover gadget in which the same colouring mechanisms apply. For instance:

- For $k = 5$, remove the arcs $(a_4, a_5), (b_4, b_5), (c_4, c_5), (d_4, d_5)$ from the core, and the arcs $(i_4, i_5), (g_1, g_2)$ from the tower.

- For $k = 4$, additionally remove the arcs $(a_3, a_4), (b_3, b_4), (c_3, c_4), (d_3, d_4)$ from the core, and the arcs $(i_1, i_2), (g_4, g_5)$ from the tower.
- For $k = 3$, additionally remove the arcs $(a_2, a_3), (b_2, b_3), (c_2, c_3), (d_2, d_3)$ from the core, and the arc (h_1, h_2) from the tower.

It can easily be seen that, under those modifications, we do end up with a k-crossover gadget in each case, essentially because some particular sets of arcs (with size $k - 2$ decreasing as k gets smaller) must be assigned different colours. From this and Theorem 3.3, we immediately get that Planar k-BMRN*-Colouring is NP-hard for every $k \in \{3, 4, 5\}$.

Theorem 3.7. *Planar k-BMRN*-Colouring is NP-hard for every $k \in \{3, 4, 5, 6\}$.*

With some extra effort, we can also prove that, for $k \in \{3, 4, 5, 6\}$, Planar k-BMRN*-Colouring remains NP-hard when restricted to planar spanned digraphs. Note that the reduced backboned digraphs we construct in the proof of Theorem 3.7 are far from being spanned, as the backbones we get have many connected components (already note that the core of each copy of the 6-crossover gadget generates nine such connected components). In the next result, we explain how to make these backbones connected without altering the general colouring properties. We show this for $k = 6$ below, which is the most intricate case, but the arguments also apply for the modified k-crossover gadgets with $k \in \{3, 4, 5\}$ mentioned earlier.

Theorem 3.8. *Planar 6-BMRN*-Colouring is NP-hard when restricted to planar spanned digraphs.*

Proof. Consider the reduction from the proof of Theorem 3.3, performed using copies of the 6-crossover gadget exhibited in Theorem 3.6. Then (D, B) is a planar backboned digraph (obtained from G in polynomial time), and G has a proper 6-vertex-colouring if and only if (D, B) has a 6-BMRN*-colouring. Furthermore, in the good drawing of (D, B), by shaping all crossover gadgets similarly to the crossings they replace, we still retain the property that the u_i's lie on the outer face.

We now explain how to turn (D, B) into a planar spanned digraph (D', T), in such a way that the colouring equivalence with G is preserved. We first add a vertex r at the very bottom of the drawing, under all u_i's. This r will be the root of our eventual out-tree T. Our goal now, is to repeatedly add, starting from r, directed paths from a connected component of B to another one, so that a bigger connected component (actually an out-tree) results, until all connected components are absorbed to a unique out-tree T. The crucial point is the following. Assume $(a, b), (c, d)$ are backbone arcs; if we add a long directed path (made up of backbone arcs only) from, say, b to c, then we note that these added arcs do not interfere with the colouring of (a, b) and (c, d) in a 6-BMRN*-colouring. This is because the inner arcs of such a directed path, if long enough, are subject to only two colour constraints. In other words, assuming (a, b) and (c, d) are coloured, we can easily extend the colouring to the joining directed path (assuming again it is long enough).

So the question now is whether, starting from r, we can add long directed paths (made up of new backbone arcs) going to all connected components of B (more precisely to their unique vertex with no in-coming backbone arc), thus making only one out-tree, without breaking planarity. Since all u_i's belong to the outer face, we can freely add, for every u_i, a long directed path from r to u_i, so that all (u_i, v_i)'s now belong to a single connected component (being an out-tree with root r) of T. It now remains to reach the connected components of the crossover gadgets. As can be seen in the drawing of Figure 9, assuming

the corner arc (y_0, z_0) belongs to T, from y_0 we can easily add three long directed paths navigating in faces and going to the vertices i_1, g_1, f_1 of the next tower T_1, thus adding three connected components of that tower to T. Once the vertex i_2 of T_1 is part of T, we can then easily add a long directed path to its vertex h_1 of that tower to add the last connected component of T_1 to T. After that, assuming (y_1, z_1) (the other corner arc adjacent to T_1) is not already part of T, we can freely add a long directed path from, say, the vertex i_5 of T_1 (navigating in a common face) to y_1, so that (y_1, z_1) is added as well. From (y_1, z_1), we can easily reach the next tower T_2, and so on. From this, we deduce that all copies of the original backbone arcs of $(D_{\vec{G}}, M_{\vec{G}})$ can be added to T, and similarly the connected components of B belonging to the towers of the crossover gadgets. For every crossover gadget, it just remains to connect to T the five inner connected components of its core, which is easy to do assuming f_1, f_3, f_5, f_7 (which are incident to the base arcs of the towers) are already part of T. Namely, navigating inside faces to preserve planarity, we can add a long directed path from f_2 to a_1, from f_4 to d_1, from f_6 to c_1, from f_8 to b_1, and finally from a_1 to e_1.

Repeating this procedure to all crossover gadgets, we can make sure that T eventually is an out-tree with root r, and we end up with a planar spanned digraph (D', T) such that G has a proper 6-vertex-colouring if and only if (D', T) has a 6-BMRN*-colouring. In particular, note that the number of connecting directed paths we must add is polynomial. Thus, the whole construction is achieved in polynomial time. □

4. Connection between BMRN*-index and girth

The planar backboned digraph depicted in Figure 3 has "large" BMRN*-index (for a planar one), mainly due to its several short cycles (with length 2 or 3). In this section, we investigate the effects on the BMRN*-index of forbidding small cycles in a planar backboned digraph.

We prove that, as one could expect, the BMRN*-index of planar spanned digraphs decreases as the girth grows. We prove this in the general case, i.e., when no further backbone restrictions are imposed. Under an additional structural condition (backbone with bounded maximum degree), we give a result involving a stronger girth assumption. To summarize, our results are as follows:

Theorem 4.1. *Let (D, B) be a planar backboned digraph. Then:*

- *if $g(D) \geq 5$ and $\Delta^+(B) \leq 1$, then BMRN*$(D, B) \leq 7$;*
- *if $g(D) \geq 7$, then BMRN*$(D, B) \leq 6$;*
- *if $g(D) \geq 16$, then BMRN*$(D, B) \leq 4$;*
- *if $g(D) \geq 21$, then BMRN*$(D, B) \leq 3$;*
- *there is no k such that if $g(D) \geq k$, then BMRN*$(D, B) < 3$.*

4.1. General case

Throughout this section, we deal with vertices having certain degrees. A *k-vertex* is a vertex having degree precisely k. A *k^--vertex* (resp. *k^+-vertex*) is a vertex having degree at most (resp. at least) k. For some $\ell \geq 1$, an *ℓ-thread* refers to a path $(v_1, ..., v_{\ell+2})$ where the ℓ inner vertices $v_2, ..., v_{\ell+1}$ are 2-vertices. Under a mild minimum degree assumption, threads are well-known to exist in planar graphs with large enough girth:

Theorem 4.2 (e.g. [4]). *Every planar graph with with minimum degree at least* 2 *and girth at least* $5\ell + 1$ *contains an* ℓ*-thread.*

In some of the upcoming proofs, we will need the fact that trees have low BMRN*-index:

Theorem 4.3. *Let* (D, B) *be a backboned digraph. If* D *is a tree, then* $BMRN^*(D, B) \leq 2$.

Proof. The proof is by induction on $|V(D)| + |A(D)|$. As the claim can easily be verified for trees with small order, we focus on the general case. In particular, we may assume that all vertices of D are part of the backbone, as otherwise we could remove "useless" vertices, and a 2-BMRN*-colouring of the remaining backboned digraph would also be one of (D, B). Also, we may assume that no interference arc (u, v) is "useless", i.e., we have $d_B^+(u) \geq 1$ and $d_B^-(v) = 1$.

If B has only one connected component, then (D, B) has no interference arcs (as otherwise D would have cycles). In this case, we only have to deal with direct conflicts, which can be done by simply considering the bipartition (V_1, V_2) of D, and, for $i = 1, 2$, assigning colour i to all backbone arcs originating from a vertex in V_i. Thus, we may assume that B has several connected components, connected via interference arcs. If we contract, in D, all connected components of B to vertices, resulting in an oriented multigraph $\vec{G}$, then $\text{und}(\vec{G})$ cannot have cycles as otherwise D would as well. This implies that $\vec{G}$ is actually an oriented graph, and that two connected components of B are joined by at most one interference arc.

Since (D, B) is assumed to have no useless interference arcs, $\text{und}(\vec{G})$ is actually connected, and is thus an oriented tree. For every vertex v of $\vec{G}$, let $C(v)$ denote the connected component of B corresponding to v. We deduce a 2-BMRN*-colouring of each connected component of B, one after another, such that all colourings comply with each other and give a 2-BMRN*-colouring of the whole (D, B). This is done by considering the connected components of B as the corresponding vertices of $\text{und}(\vec{G})$ are encountered during a BFS performed from an arbitrary root r of $\vec{G}$.

We thus start with r. Since nothing is coloured yet, we can freely choose, as a 2-BMRN*-colouring of $C(r)$, any 2-BMRN*-colouring, which exists by the induction hypothesis. Let us now consider the general case, i.e., that during the BFS of $\text{und}(\vec{G})$, we are now dealing with a vertex v, whose parent in $\text{und}(\vec{G})$ (by the BFS ordering) is u. Recall that either (u, v) or (v, u) can be the corresponding arc in $\vec{G}$. We assume in what follows that (u, v) is the arc, but the arguments can be symmetrized in case the arc is (v, u). Let x_u denote the vertex of $C(u)$ from which the corresponding interference arc originates in (D, B), and x_v denote the vertex of $C(v)$ at which the interference arc terminates in (D, B). By the induction hypothesis, $C(v)$ admits a 2-BMRN*-colouring. Free to permute the colours, we may assume that the colour assigned to the unique backbone arc in-coming to x_v is different from the unique colour assigned to the backbone arcs out-going from x_u. Then, there is no indirect conflict raised, and the whole partial colouring is a partial 2-BMRN*-colouring of (D, B).

Going on like this until all vertices of $\vec{G}$ have been treated by the BFS, we end up with a 2-BMRN*-colouring of (D, B). □

We now prove all results (but the first one, postponed to the next subsection) in Theorem 4.1, by dedicating a theorem or observation to each item. We voluntarily modify the order in which these results are delivered, as some of the proofs depend on other ones.

We start off by observing that, in general, planar spanned digraphs with arbitrarily large girth might require at least three colours in a BMRN*-colouring:

Observation 4.4. *There is no k such that spanned digraphs (D, T) with girth at least k have BMRN$^*(D, T) < 3$.*

Proof. This is because large girth does not prevent a spanned digraph to have an odd number of backbone arcs being constraining each other in a "cyclic" way (i.e., which would correspond to an odd-length cycle for proper vertex-colouring of undirected graphs). Such backbone arcs force the use, in a BMRN*-colouring, of at least three colours. This occurs, for instance, when a planar spanned digraph (D, T) contains the following configuration. Let $k \geq 3$ be any odd integer. For every $i \in \{0, ..., k-1\}$, assume (D, T) has a backbone arc (v_i, v'_i), and, modulo k, the interference arc (v_i, v'_{i+1}). Then it is easy to see that two "consecutive" backbone arcs (v_i, v'_i) and (v_{i+1}, v'_{i+1}) must receive different colours by a BMRN*-colouring. Since there is an odd number of arcs in that "cycle", at least three colours are needed. □

We now prove that, in general, if the girth of a planar backboned digraph is larger than some threshold, then its BMRN*-index becomes less than some value.

Theorem 4.5. *Let (D, B) be a planar backboned digraph. If D has girth at least 21, then BMRN$^*(D, B) \leq 3$.*

Proof. Assume the statement is wrong, and let (D, B) be a planar backboned digraph with girth at least 21 verifying BMRN$^*(D, B) > 3$. We consider a such (D, B) that is minimum in terms of $|V(D)|+|A(D)|$. This property implies that D is connected. It also implies that, for every interference arc (u, v), we must have $d^+_B(u) \geq 1$ and $d^-_B(v) = 1$, as otherwise this arc could not be involved in any indirect conflict, and we could just remove it from (D, B), and deduce a 3-BMRN*-colouring of the remaining backboned digraph (in which every connected component is either a backboned oriented tree, which has 3-BMRN*-colourings by Theorem 4.3, or a smaller planar backboned digraph with girth at least 21) that is also one of (D, B), a contradiction. This in turn implies that every vertex v of D must be part of B, as otherwise we could find, in (D, B), interference arcs that are useless.

Our aim is to find a 4-thread in D by means of Theorem 4.2. That theorem tells us that the existence of such threads is, despite the girth of D, not guaranteed if $\delta(D) = 1$. We thus first need to investigate how 1-vertices behave in D.

First off, we note that if v is a vertex of D adjacent to 1-vertices, then v is adjacent to at most two 1-vertices. Indeed, if v is adjacent to three 1-vertices u_1, u_2, u_3, then we must have two arcs, say $(v, u_1), (v, u_2)$, being backbone arcs. This is because every vertex must be incident to a backbone arc (by minimality of (D, B)), and every vertex has in-degree at most 1 in the backbone. In that case, we consider (D', B') the backboned digraph obtained when removing u_1 from D. Since (D', B') is a planar backboned digraph with girth at least 21 that is smaller than (D, B), it has a 3-BMRN*-colouring which we can extend to (v, u_1) by simply assigning the colour of (v, u_2) to (v, u_1). This is correct since these two backbone arcs are subject to the same colour constraints. Thus, we get a 3-BMRN*-colouring of (D, B), a contradiction. Consequently, if a vertex v of D is adjacent to 1-vertices, then it is adjacent to at most two 1-vertices. Furthermore, if u_1, u_2 are two 1-vertices adjacent to v, then, without loss of generality, (u_1, v) and (v, u_2) are backbone arcs.

Assume now that D has a 1-vertex u, and let v be its unique neighbour. By minimality of (D, B), vertices v and u must be joined by a backbone arc. Assume (v, u) is that backbone arc. We claim that v must be a 4^+-vertex. Indeed, consider the backboned digraph (D', B') obtained from (D, B) by removing u; again, (D', B') admits a 3-BMRN*-colouring. If we cannot extend it to (v, u), thus to (D, B), then this means that the three

colours appear "around" v. The colours that cannot be assigned to (v, u) are the following: that assigned to the unique backbone arc (w, v) (if any), and, for every interference arc (v, w) incident to v, that assigned to the unique backbone arc in-coming to w (if any). In other words, every arc incident to v in (D', B') prevents us from assigning at most one colour to (v, u). Thus, for the three colours to be not assignable to (v, u), it must be that v has degree at least 3 in (D', B'), and thus at least 4 in (D, B). These arguments also hold in the case where (u, v) is a backbone arc ((u, v) cannot be assigned the unique colour assigned to the backbone arcs out-going from v (if any), and, for every interference arc (w, v), the unique colour assigned to the backbone arcs out-going from w).

Now, assume that D has a vertex v adjacent to two 1-vertices u_1, u_2. By arguments above, we may assume that (u_1, v) and (v, u_2) are backbone arcs of (D, B). We claim that v is a 6^+-vertex. Indeed, first consider the backboned digraph (D', B') obtained when removing u_1 from (D, B). Again, (D', B') has a 3-BMRN*-colouring. We try to extend it to (u_1, v). The colour assigned to (u_1, v) must be different from that of (v, u_2). It must also be different, for every interference arc (w, v), from the unique colour assigned to the backbone arcs out-going from w, if any. Thus, so that the colouring cannot be extended to (u_1, v), in which case we would get a contradiction, there must be at least two interference arcs in-coming to v. Now consider (D', B') the backboned digraph obtained from (D, B) by removing u_2. By symmetric arguments, for a 3-BMRN*-colouring of (D', B') to be not extendable to (v, u_2), there must be at least two interference arcs out-going from v. We thus deduce that v has degree at least 6 in D.

We are now ready to combine all these arguments for deducing the existence of a 4-thread in D. We apply the following simple iterative procedure: As long as D has a 1-vertex, we just remove it. Let D' denote the resulting digraph once all 1-vertices have been peeled off (and thus no 1-vertex remains). It is easy to see that D' cannot be a tree, unless D was one (as the presence of any cycle in D makes it impossible for D' to be a tree, and removing 1-vertices from a graph cannot disconnect it). Also, D' cannot be empty. Then D' is a (connected) planar digraph with girth at least 21 and $\delta(D') \geq 2$. Then, by Theorem 4.2, D' has a 4-thread $(v_1, ..., v_6)$, where $v_2, ..., v_5$ are its 2-vertices. The crucial point is that this 4-thread is also a 4-thread in D. This is because, although seemingly iterative, the process of removing 1-vertices from D while possible actually only removes vertices that were already of degree 1 in D. This is because vertices of D are adjacent to at most two 1-vertices, and vertices v of D adjacent to 1-vertices are of large degree. Indeed, either v is adjacent to only one 1-vertex in D, in which case v is a 4^+-vertex in D and it becomes a 3^+-vertex once its adjacent 1-vertex has been removed, or v is adjacent to two 1-vertices in D, in which case v is a 6^+-vertex in D and thus it becomes a 4^+-vertex once its two adjacent 1-vertices have been removed. Hence, all 2-vertices $v_2, ..., v_5$ of the 4-thread in D' are also 2-vertices in D, and this thread is thus also a 4-thread in D.

We now deal with this 4-thread $(v_1, ..., v_6)$ in (D, B). Assume first that (v_3, v_4) is a backbone arc. If (v_3, v_2) is not a backbone arc, then we consider the backboned digraph (D', B') obtained by removing (v_3, v_4) from (D, B). By arguments similar to that above, (D', B') admits a 3-BMRN*-colouring, which we wish to extend to (v_3, v_4). This is possible because (v_3, v_4) is subject to at most two colour constraints. Indeed, since (v_3, v_2) is not a backbone arc, by minimality of (D, B) this arc (v_3, v_2) must be an interference arc, and (v_1, v_2) must be a backbone arc. Then (v_3, v_4) must be assigned a colour different from that unique one assigned to the backbone arcs out-going from v_1 (that of (v_1, v_2)), and that of (v_4, v_5) (case where this is a backbone arc), or that of (v_5, v_6) (otherwise: if (v_4, v_5) is not a backbone arc, then (v_5, v_4) must be an interference arc, and thus (v_5, v_6) must be a backbone arc). This is a contradiction.

Now assume that both (v_3, v_2) and (v_3, v_4) are backbone arcs. We here consider (D', B') to be the backboned digraph obtained after removing $(v_3, v_2), (v_3, v_4)$. Again, it admits a 3-BMRN*-colouring which we wish to extend to the two removed arcs, assigning them a same colour. We claim that, here as well, at most two colours only must be avoided. Indeed, we cannot assign to $(v_3, v_2), (v_3, v_4)$ the colour of (v_2, v_1) (case where that arc is a backbone arc), or the unique colour assigned to the backbone arcs out-going from v_1 (otherwise: if (v_2, v_1) is not a backbone arc, then (v_1, v_2) must be an interference arc by minimality of (D, B)). Similarly, we cannot assign to $(v_3, v_2), (v_3, v_4)$ the colour of (v_4, v_5) (case where it is a backbone arc), or the colour of (v_5, v_6) (otherwise: if (v_4, v_5) is not a backbone arc, then (v_5, v_4) must be an interference arc, and (v_5, v_6) must be a backbone arc). Thus one of the three colours is not used around, and we can freely assign it to $(v_3, v_2), (v_3, v_4)$, resulting in a 3-BMRN*-colouring of (D, B), a contradiction.

The last case to consider is when the arc joining v_3 and v_4 is an interference arc. By symmetry, we may assume that (v_3, v_4) is an interference arc. Then, by minimality of (D, B), both (v_3, v_2) and (v_5, v_4) are backbone arcs. In that case, we consider (D', B') the backboned digraph obtained when removing (v_3, v_2) from (D, B). A 3-BMRN*-colouring of (D', B') can be extended to (v_3, v_2), thus to (D, B), because this arc is subject to at most two colour constraints only: either the colour of (v_2, v_1) (if it is a backbone arc) or the unique colour assigned to the backbone arcs out-going from v_1 (otherwise), and the colour of (v_5, v_4). This is yet another contradiction, which concludes the proof. □

Theorem 4.6. *Let (D, B) be a planar backboned digraph. If D has girth at least 16, then BMRN*$(D, B) \leq 4$.*

Proof. The proof starts similarly as that of Theorem 4.5. Let (D, B) be a minimum counterexample to the claim. Since we are now working with four colours, from arguments we have used earlier the following properties of D and D' (the digraph obtained when removing all 1-vertices from D) can be deduced:

- every vertex of D is adjacent to at most two 1-vertices;
- every vertex of D adjacent to a 1-vertex is a 5^+-vertex;
- every vertex of D adjacent to two 1-vertices is an 8^+-vertex;
- D' has 3-threads, each of which is a 3-thread of D.

Let $(v_1, ..., v_5)$ be a 3-thread of D, where $v_2, ..., v_4$ are its 2-vertices. Note that we may assume that v_1, v_5 are 3^+-vertices, as otherwise D would have a 4-thread, from which we could deduce a 4-BMRN*-colouring of (D, B) just as in the proof of Theorem 4.5. In most cases, a 4-BMRN*-colouring of (D, B) can be deduced from this 3-thread $(v_1, ..., v_5)$. Actually, only one case is not reducible. Let us first prove that all other cases are indeed reducible.

Assume first that v_3 is incident to out-going backbone arcs. There are two cases to consider. First, assume that both (v_3, v_2) and (v_3, v_4) are backbone arcs. We here remove v_3 from (D, B); the remaining backboned digraph admits a 4-BMRN*-colouring that can be extended to (v_3, v_2) and (v_3, v_4), a contradiction. Indeed, we must avoid either the colour of (v_2, v_1) (if it is a backbone arc) or the unique colour assigned to the backbone arcs out-going from v_1 (otherwise), and the colour of (v_4, v_5) (if it is a backbone arc) or the unique colour assigned to the backbone arcs out-going from v_5 (otherwise). Second, assume that only (v_3, v_2) is a backbone arc. By minimality of (D, B), (v_3, v_4) is an interference

arc, and (v_5, v_4) is a backbone arc. We here remove only (v_3, v_2) from (D, B). Here, for extending a 4-BMRN*-colouring of the remaining backboned digraph to (v_3, v_2), we must avoid either the colour of (v_2, v_1) (if it is a backbone arc) or the colour assigned to the backbone arcs out-going from v_1 (otherwise), and the unique colour assigned to the backbone arcs out-going from v_5 (i.e., the colour of (v_5, v_4)). Since we have four colours in hand, we can find an open colour for (v_3, v_2), a contradiction.

Thus, we now assume that v_3 is incident to only one in-coming backbone arc. Assume (v_2, v_3) is that backbone arc. Then (v_4, v_3) must be an interference arc, and, in turn, (v_4, v_5) is a backbone arc. Again, if (v_2, v_1) is an interference arc, then we are done, because we can just remove (v_2, v_3) from (D, B), deduce a 4-BMRN*-colouring, an extend it to (v_2, v_3) as this arc is subject to at most two colour constraints (the colour of the unique backbone arc in-coming to v_1 (if any), and the colour of (v_4, v_5).

It can be checked that the remaining type of 3-thread $(v_1, ..., v_5)$, which is when $(v_2, v_1), (v_2, v_3), (v_4, v_5)$ are backbone arcs while (v_4, v_3) is an interference arc, cannot be reduced that simply, via counting arguments only. Such 3-threads, we call *bad threads*. Before going on, we need to exhibit a few properties of these bad threads in (D, B).

A first important property is deduced from the fact that, for a bad 3-thread $(v_1, ..., v_5)$, we have the backbone arcs (v_2, v_1) and (v_4, v_5). This means that, in (D, B), a vertex v can be incident to at most one bad thread, as otherwise v would have two backbone arcs coming in. Another property is that v_1 and v_5 must be 4^+-vertices; this is again deduced by removing either (v_2, v_1) and (v_2, v_3) (for establishing the bound on the degree v_1), or (v_4, v_5) (for establishing the bound on the degree of v_5), and counting the number of colour constraints around. Lastly, for a 3-thread $(v_1, ..., v_5)$ of (D, B), the vertex v_1 (and similarly v_5) can be adjacent to at most one 1-vertex (via an out-going backbone arc), and, in that case, we recall that v_1 (resp. v_5) is a 5^+-vertex.

We are now ready to conclude. Start from (D, B), and, as in the proof of Theorem 4.5, remove 1-vertices as long as possible. The remaining digraph (D', B') cannot be a tree, remains connected, it is planar with girth at least 16, and, as mentioned earlier, it has minimum degree 2. (D', B') thus has a 3-thread $T_1 = (v_1, ..., v_5)$, which is also a 3-thread in (D, B). If T_1 is not bad, then we are done. So assume that T_1 is a bad 3-thread. Back in (D', B'), we remove the vertices v_2, v_3, v_4, resulting in a backboned digraph (D'', B''). Recall that, in (D, B), both v_1 and v_5 are 4^+-vertices, and, if any of them is also adjacent to a unique 1-vertex, then it is even a 5^+-vertex. This implies that, in (D'', B''), both v_1 and v_5 are 3^+-vertices, and they do not have any backbone arc coming in. Also, D'' remains of minimum degree at least 2.

Although D'' might have several connected components, they all have girth at least 16. Thus, in D'', by Theorem 4.2 we can again find a 3-thread T_2, which is also a 3-thread in D. Again, if T_2 is not a bad 3-thread in (D, B), then we are done. Thus we may assume that T_2 is bad, and, by the properties we have on the first and last vertices of bad threads, we know that T_1 and T_2 do not intersect. Next, we again remove the inner degree-2 vertices of T_2 from the current digraph; it remains of minimum degree at least 2, and it remains a planar digraph whose all connected components have girth at least 16.

We continue this process as long as possible. That is, for a backboned digraph obtained from D after removing the 1-vertices and the inner vertices from some bad 3-threads $T_1, ..., T_k$ of (D, B), we deduce another 3-thread T_{k+1} by Theorem 4.2, which is also present in (D, B). If T_{k+1} is not a bad thread, then we are done. Otherwise, we remove its three inner vertices from the current digraph, and we go on. We know that the two ends of T_{k+1} remain of degree strictly more than 2, and thus so do all vertices. Furthermore, all

resulting connected components remain of girth at least 16. Also, recall that no two threads T_i, T_j can share vertices (because any vertex of (D, B) can be incident to at most one bad thread, and after removing a bad thread no new 2-vertex can be created). The process will thus end up with finding a 3-thread that is no bad. Back in (D, B), this is a 3-thread that can be reduced, and from its existence we get that (D, B) has 4-BMRN*-colourings; a contradiction. □

Theorem 4.7. *Let (D, B) be a planar backboned digraph. If D has girth at least 7, then $BMRN^*(D, B) \leq 6$.*

Proof. This is proved by designing a particular vertex-colouring ϕ of D. To every vertex v of D with $d_B^+(v) \geq 1$, we assign a 2-element colour $\phi(v) = \{c_1(v), c_2(v)\}$ so that, when assigning colour $\phi(v)$ to all backbone arcs out-going from v, the resulting "derived" arc-colouring forms a BMRN*-colouring of (D, B). The function c_1 will take value in $\{1, 2\}$, while c_2 will take value in $\{1, 2, 3\}$, so that ϕ is a 6-colouring.

We construct ϕ in the following way. Let (V_1, V_2) denote the bipartition of B. For every vertex $v \in V_1$, we set $c_1(v) = 1$, while we set $c_1(v) = 2$ for every $v \in V_2$. This way, note that, in (D, B), already we cannot have direct conflicts by the arc-colouring derived from ϕ (because no two consecutive backbone $(u, v), (v, w)$ arcs originate from vertices in the same partite set). It remains to deal with indirect conflicts, which is done by defining c_2 appropriately.

For $i = 1, 2$, we denote by D_i the digraph obtained from (D, B) and c_1 as follows. The vertices of D_i are the vertices v of D verifying $c_1(v) = i$. The arcs of D_i model the potential indirect conflicts in (D, B) between vertices v with $c_1(v) = i$. That is, for every two vertices u, v with $c_1(u) = c_1(v) = i$, we add the arc (u, v) to D_i if (u, v') is an interference arc of (D, B), where (v, v') is a backbone arc. That is, the presence of the arc (u, v) in D_i indicates that u and v should receive distinct colours by a BMRN*-colouring of (D, B) (to avoid an indirect conflict), while its direction indicates the direction of the corresponding interference arc (it goes from u to an out-neighbour of v). Note that D_i, in general, is a multidigraph.

It can be noted that each D_i is a planar digraph. Indeed, seen differently, D_i was obtained from D by deleting all arcs out-going from vertices in V_{3-i}, and contracting all backbone arcs out-going from vertices in V_i. This means that D_i is a minor of D; since D was assumed planar, so is D_i, as planar graphs form a minor-closed family of graphs.

Let us now focus on D_1. Recall that its arcs join vertices (with the same colour by c_1) whose out-going backbone arcs must not be assigned the same colour in a BMRN*-colouring of (D, B) (to avoid some indirect conflict). Thus, by the arc-colouring derived from ϕ, no colour conflict involving two vertices of D_1 will arise as soon as c_2, when restricted to D_1, is a proper vertex-colouring. We claim that, because D has girth at least 7, D_1 itself has girth at least 4, and thus admits a proper 3-vertex-colouring by Grötzsch Theorem [6].

Assume the contrary, i.e., that D_1 has a 2-cycle or 3-cycle C.

- If $C = (u, v, u)$ has length 2, then there are two possible orientations for C in D_1.
  - On the one hand, assume there are two arcs from u to v in D_1. Since D is simple, this means that there are two interference arcs $(u, v_1), (u, v_2)$ in (D, B), where $(v, v_1), (v, v_2)$ are two backbone arcs. Then (u, v_1, v, v_2, u) is a 4-cycle of D, a contradiction to it having girth at least 7.
  - On the second hand, assume D_1 has two arcs (u, v) and (v, u). This means that (D, B) has two interference arcs $(u, v'), (v, u')$, where $(u, u'), (v, v')$ are backbone arcs. Then (u, u', v, v', u) is a 4-cycle, a contradiction.

- If $C = (u, v, w, u)$ has length 3, then there are two possible (non-isomorphic) orientations in D_1:
  - On the one hand, assume the three arcs are $(u, v), (v, w), (w, u)$. This means that (D, B) has three interference arcs $(u, v'), (v, w'), (w, u')$, where $(u, u'), (v, v'), (w, w')$ are three backbone arcs. Then (u, v', v, w', w, u', u) is a 6-cycle in D, a contradiction.
  - On the other hand, assume that the three arcs are $(u, v), (v, w), (u, w)$. This means that, in (D, B), one of the following two situations occurs:
    * (D, B) has three interference arcs $(u, v'), (u, w'), (v, w')$, where $(v, v'), (w, w')$ are two backbone arcs. Then (u, v', v, w', u) is a 4-cycle, a contradiction.
    * (D, B) has three interference arcs $(u, v'), (u, w'), (v, w'')$, where $(v, v'), (w, w'), (w, w'')$ are two backbone arcs. Then $(u, v', v, w'', w, w', u)$ is a 6-cycle, a contradiction.

Thus, D_1 has girth at least 4, meaning it admits a proper 3-vertex-colouring. Similarly, D_2 admits a proper 3-vertex-colouring as well. These two colourings yield our c_2. □

4.2. Bounded-degree backbone

One notable property of the spanned digraph (D, T) in Figure 2 is that T is a directed path, i.e., it has maximum outdegree 1. This indicates that having, in a planar backboned digraph, the backbone arcs inducing a very simple topology is sometimes sufficient to have "large" BMRN*-index. The other way around, in planar spanned digraphs the interference arcs are sufficient to make the number of colours large. Yet, in the following result we show how to take advantage of a simple backbone to prove that Question 1.1 is true for planar spanned digraphs with even smaller girth, and that eight colours are not necessary to colour them.

Theorem 4.8. *Let (D, B) be a planar backboned digraph. If D has girth at least 5 and $\Delta^+(B) \leq 1$, then BMRN*$(D, B) \leq 7$.*

Proof. Assume the claim is wrong, and consider (D, B) a smallest counterexample to the claim (in terms of $|V(D)| + |A(D)|$). By minimality, we may suppose that D is connected, that all vertices are part of the backbone, and that there are no useless interference arcs.

We might as well suppose that D has no bridge, say (u, v), such that none of u, v is a 1-vertex. Indeed, on the one hand, if (u, v) is a backbone arc, then we can remove (u, v) from (D, B), resulting in the disjoint union of two backboned digraphs $(D_1, B_1), (D_2, B_2)$, where B_1, B_2 are the restrictions of B to D_1, D_2. Because none of u, v is a 1-vertex, note that both $(D_1+(u, v), B_1+(u, v))$ and $(D_2+(u, v), B_2+(u, v))$ admit a 7-BMRN*-colouring (either by induction or Theorem 4.3). It is then easy to see that, when permutting the colours of these two colourings so that (u, v) gets the same colour in each of them, a 7-BMRN*-colouring of (D, B) is obtained. On the second hand, if (u, v) is an interference arc, then we can just, by freely permutting the colours, consider 7-BMRN*-colourings of (D_1, B_1) and (D_2, B_2) such that the unique colour assigned to the backbone arcs outgoing from u is different from the colour of the unique backbone arc in-coming to v. This forms a 7-BMRN*-colouring of (D, B), a contradiction. Thus, omitting its arcs incident to 1-vertices, we may assume that D has no bridge.

We get to a contradiction to the existence of (D, B) through the use of the so-called *discharging method*, which consists in the following steps. We first prove that some configurations (subdigraphs with certain properties) are *reducible* in (D, B), meaning that if (D, B) had one of these reducible configurations, then we would find a way to deduce a 7-BMRN*-colouring of (D, B) from it, a contradiction. Next, through the discharging phase itself, we will supply a precise amount of charge to elements (vertices and faces) of (D, B) and, because of the reducible configurations, show that, upon only moving charges between elements in a very specific way, a contradiction to the initial provided amount of charges is obtained.

1. First reducible configurations

As a general tool, we start off by proving that, in (D, B), two vertices with small degree cannot be joined by a backbone arc.

Claim 1. *Let (u, v) be a backbone arc of (D, B). Then*

$$d_B^-(u) + d_{D-B}^+(u) + d_B^+(v) + d_{D-B}^-(v) \geq 7.$$

Proof of the claim. Assume to the contrary that (D, B) has a backbone arc (u, v) where $d_B^-(u) + d_{D-B}^+(u) + d_B^+(v) + d_{D-B}^-(v) < 7$. Let (D', B') be the planar backboned digraph obtained by removing (u, v) from (D, B). By minimality of (D, B), this (connected, by previous arguments) digraph (D', B'), which is still of girth at least 5 with $\Delta^+(B') \leq 1$, admits a 7-BMRN*-colouring ϕ. We show below that ϕ can be extended to (u, v), thus to (D, B), a contradiction.

By the definition of BMRN*-colouring, those colours around (u, v) that cannot be assigned to (u, v) are the following (assuming they all exist):

- the colour of the backbone arc in-coming to u;
- the colour of the backbone arc out-going from v;
- for every interference arc (u, w) out-going from u, the colour of the backbone arc in-coming to w;
- for every interference arc (w, v) in-coming to v, the colour of the backbone arc out-going from w.

It can easily be seen that any other arc incident to u or v does not yield any colouring constraint for extending ϕ to (u, v). Also, any constraining arc incident to u or v constrains the assignation of at most one colour to (u, v). Since each of the seven colours we are playing with must be not assignable to (u, v), the result follows. $\Diamond$

In particular through Claim 1, we deduce properties of small-degree vertices in (D, B).

Claim 2. *Let v be a vertex of (D, B) being adjacent to 1-vertices; then:*

1. *v is adjacent to at most two 1-vertices;*
2. *every arc between v and a 1-vertex is a backbone arc;*
3. *if v is adjacent to a one 1-vertex, then $d(v) \geq 8$;*
4. *if v is adjacent to exactly two 1-vertices, then $d(v) \geq 14$; moreover, the two backbone arcs joining v and these two 1-vertices have opposite directions.*

Proof of the claim. The first and second items, as well as the last part of the fourth item, are because every vertex of (D, B) must be incident to a backbone arc, and we have $\Delta^+(B) \leq 1$. The third item and the first part of the fourth item follow from a direct application of Claim 1. Recall in particular that removing a backbone arc from (D, B) results in a connected digraph (since bridges are incident to 1-vertices, by previous arguments), which implies that the girth restriction is preserved upon removing single arcs. $\Diamond$

Claim 3. *D has no 3-thread.*

Proof of the claim. Assume D has a 2-vertex v adjacent to two 2-vertices u, w. By minimality of (D, B), v must be incident to a backbone arc; assume (u, v) is that backbone arc. Then we get a contradiction to Claim 1, since both u and v have degree 2. $\Diamond$

From Claim 3, we know that all ℓ-threads in (D, B), if there are any, verify $\ell \in \{1, 2\}$. Some of these threads are reducible because of Claim 1. The remaining *irreducible threads*, which we cannot reduce immediately via the degree condition in the claim, have the following properties:

Claim 4. *An irreducible thread of (D, B) is either:*

- *A 2-thread (v_1, v_2, v_3, v_4); in that case, (v_1, v_2) and (v_3, v_4) (resp. (v_4, v_3) and (v_2, v_1)) are backbone arcs, while (v_3, v_2) (resp. (v_2, v_3)) is an interference arc. Furthermore, both v_1 and v_4 are 7^+-vertices.*
- *A 1-thread (v_1, v_2, v_3); in that case, at least one of (v_1, v_2), (v_2, v_1), (v_3, v_2), and (v_2, v_3) is a backbone arc. Furthermore, each of v_1, v_3 incident to such a backbone arc is a 7^+-vertex.*

Proof of the claim. First assume (v_1, v_2, v_3, v_4) is a 2-thread of (D, B). Note that the arc joining v_2 and v_3 cannot be a backbone arc, as otherwise Claim 1 would yield a contradiction. Thus that arc must be an interference arc, say (v_2, v_3) without loss of generality. By minimality of (D, B), this arc cannot be removed, which means that (v_2, v_1) and (v_4, v_3) must be backbone arcs. Now, knowing that both (v_2, v_1) and (v_4, v_3) are backbone arcs, and v_2, v_3 are 2-vertices, the last part of the first item follows from Claim 1.

Now assume (v_1, v_2, v_3) is a 1-thread. If none of its two arcs is a backbone arc, then v_2 could be just removed from (D, B), contradicting its minimality. So one of its two arcs is a backbone arc, and its end different from v_2 must be a 7^+-vertex by Claim 1. $\Diamond$

For an irreducible 2-thread (v_1, v_2, v_3, v_4), we define v_1 (resp. v_4) as the *support vertex* of v_2 (resp. v_3). For an irreducible 1-thread (v_1, v_2, v_3), we know that at least one of the two arcs is a backbone arc; the end different from v_2 of that arc (if the two of v_1, v_3 are candidate, we choose any of them) we define as the *support vertex* of v_2. For every 2-vertex of D, we have thus defined a support vertex, which is a 7^+-vertex.

Similarly as for the previous claims, Claim 1 can be used to prove the following:

Claim 5. *Let v be a vertex of (D, B) supporting some 2-vertices; then:*

1. *v supports at most two 2-vertices;*
2. *the number of 1-vertices adjacent to v plus the number of 2-vertices supported by v is at most 2;*

3. *if v is adjacent to a 1-vertex and a 2-vertex it supports, then $d(v) \geq 13$;*

4. *if v is adjacent to two 2-vertices it supports, then $d(v) \geq 12$.*

Proof of the claim. The first item is because $\Delta^+(B) \leq 1$ and, by definition, vertices support adjacent 2-vertices via backbone arcs. The second item follows from these reasons, and the fact that the only arc incident to a 1-vertex must be a backbone arc, by the minimality of (D, B). The third and fourth items follow from Claim 1, by considering the two backbone arcs incident to v going to the two vertices with degree 1 or 2 (recall that these two backbone arcs have different directions with respect to v, due to the backbone restrictions). $\Diamond$

2. First discharging process

To every vertex v of D, we assign an initial charge $\omega(v) = 2d(v) - 6$. To every face f, we assign an initial charge $\omega(f) = d(f) - 6$. Playing with Euler's formula, we get

$$\sum_{v \in V(D)} \omega(v) + \sum_{f \in F(D)} \omega(f) = \sum_{v \in V(D)} (2d(v) - 6) + \sum_{f \in F(D)} (d(f) - 6) = -12$$

This means that the total amount of charge is strictly negative. Throughout this proof, the goal is, without creating any new charge, to move charge from elements to elements, before proving that the total amount of charge eventually gets negative, a contradiction.

The discharging rules of the first discharging phase are the following:

R1. Every face sends 2 to each of its incident 1-vertices.

R2. Every vertex sends 2 to each of its adjacent 1-vertices.

R3. Every vertex sends 2 to each of its adjacent 2-vertices it supports.

For every element (vertex or face) e of D, let us denote by $\omega'(e)$ the charge of e once rules R1 to R3 above have been performed. Let us now study the value of $\omega'(e)$ for each element e.

First assume that $e = v$ is some vertex. Note that, by rules R1 to R3, only 1-vertices and 2-vertices receive some charge. According to Claims 2 and 4, only 7^+-vertices can send charge, because only these types of vertices neighbour 2^--vertices. By Claim 5, vertices can send charge to at most two adjacent vertices, and, from Claims 2 and 5, only 12^+-vertices send charge to two adjacent vertices. Thus:

- If v is a 1-vertex, then v does not send any charge. It however receives 2 from its unique neighbour, via rule R2, and 2 from the unique face to which v is incident (rule R1). Thus $\omega'(v) = \omega(v) + 2 \times 2 = -4 + 4 = 0$.

- If v is a 2-vertex, then, through rule R3, it receives 2 from its supporting vertex, while it does not send any charge. Thus $\omega'(v) = \omega(v) + 2 = -2 + 2 = 0$.

- If v is a k-vertex for $k \in \{3, ..., 6\}$, then v does not send nor receive charge; thus $\omega'(v) = \omega(v) = 2k - 6 \geq 0$.

- If v is a k-vertex for $k \in \{7, ..., 11\}$, then v sends 2 to at most one adjacent 1-vertex or 2-vertex it supports (through rule R2 or R3), while v does not receive any charge. Thus $\omega'(v) = \omega(v) - 2 \geq (2k - 6) - 2 = 2k - 8 \geq 14 - 8 = 6$.

- If v is a k-vertex with $k \geq 12$, then v sends 2 to at most two adjacent 1-vertices or 2-vertices it supports (through rule R2 or R3). Also v does not receive any charge. Thus $\omega'(v) = \omega(v) - 2 \times 2 \geq (2k-6) - 4 = 2k - 10 \geq 24 - 10 = 14$.

Let us finally assume that $e = f$ is a face. Note that, through rules R1 to R3, f does not receive any charge; but f sends 2 to each 1-vertex incident to it. Let f' denote the face obtained from f by deleting all its incident 1-vertices, say there are x such. Since D has girth at least 5, we have $d(f') \geq 5$. Actually, we have $d(f) = d(f') + 2x$. Thus, $\omega'(f) = \omega(f) - 2x$, and f, after applying the discharging rules, eventually has the same charge $\omega'(f)$ that a face with length $d(f')$ would get by ω. Thus, $\omega'(f) < 0$ only when f' is a 5-face, in which case we actually have $\omega'(f) = -1$.

We note that, at this point, the only elements e of D that verify $\omega'(e) < 0$ are faces f whose *support*, i.e., the cycle obtained after removing the 1-vertices incident to f, is a 5-cycle. More precisely, $\omega'(f) = -1$ for such a face. In what follows, we apply additional discharging rules to make sure that also such faces have non-negative charge, to get our final contradiction. The main argument we will use is the fact that, for every vertex v with $d(v) > 3$, its remaining charge $\omega'(v)$ is rather large. That is, all vertices v with $d(v) \geq 4$ verify $\frac{\omega'(v)}{d(v)} > 0$. More specifically, we note that $\frac{\omega'(v)}{d(v)} \geq 1$ whenever $d(v) \in \{6, 8, 9, ...\}$, while, when $d(v) \in \{4, 5, 7\}$, we have $\frac{\omega'(v)}{d(v)} \geq \frac{1}{2}$. Even more specifically, we note that a 7-vertex v verifies $\frac{\omega'(v)}{d(v)} < 1$ only when v is the support of exactly one adjacent 2-vertex (recall Claim 4), in which case $\frac{\omega'(v)}{d(v)} = \frac{6}{7}$. Such a 7-vertex is said *weak* below.

3. More reducible configurations

We focus on the 5-faces of D (which cannot be incident to 1-vertices, since D has girth at least 5) having most of their vertices being of small degree. More precisely, we say that a 5-face is *bad* if it has at most one vertex that is not a 3^--vertex, and, if that vertex exists, it is a 4-vertex, 5-vertex or weak 7-vertex. In other words, a 5-face is not bad as soon as it has at least two 4^+-vertices, or whenever it has a 6-vertex, non-weak 7-vertex, or any k-vertex with $k \geq 8$. Conversely, a face (of any length) of D is said *heavy* if it contains at least three 4^+-vertices, including two 6^+-vertices. A face of D is said *almost heavy* if it contains at least three 4^+-vertices, including a 6^+-vertex and a 5-vertex that does not belong to a bad 5-face.

Claim 6. *(D, B) has no bad 5-face $f = (v_1, ..., v_5, v_1)$ where v_1 is a 2-vertex, v_2 is a weak 7-vertex, and v_3, v_4, v_5 are 3-vertices.*

Proof of the claim. Assume (D, B) has such a bad 5-face f. For each v_i of v_3, v_4, v_5, we denote by v_i' its unique neighbour not on f.

By minimality of (D, B) and by Claim 1, the arc joining v_1 and v_2 must be a backbone arc (as otherwise the one joining v_2 and v_3 would be one, and Claim 1 would bring a contradiction). Let us assume that (v_1, v_2) is a backbone arc. By minimality of (D, B), (v_1, v_5) is an interference arc. By the same arguments, we deduce sequentially that (v_5', v_5) and (v_4, v_4') are backbone arcs, while (v_4, v_5) and (v_4, v_3) are interference arcs (this is because two 3-vertices cannot be joined by a backbone arc, as otherwise Claim 1 would yield a contradiction). Again by minimality, we know that one of the two arcs incident to v_3 must be a backbone arc directed toward v_3; there are two possibilities:

- (v_3', v_3) is a backbone arc, in which case (v_2, v_3) must be an interference arc, by minimality of (D, B). Then we get a contradiction when applying Claim 1 onto (v_1, v_2), as this arc is subject to at most six colour constraints.

- (v_2, v_3) is a backbone arc, in which case, applying Claim 1 onto (v_1, v_2), we deduce that the remaining five arcs incident to v_2 must be interference arcs directed toward v_2. We then get a contradiction when applying Claim 1 onto (v_2, v_3).

The case where (v_2, v_1) is a backbone arc can dealt with in a very similar way, using symmetric arguments. $\diamondsuit$

Claim 7. *(D, B) has no bad 5-face $f = (v_1, ..., v_5, v_1)$ where all v_i's are 3-vertices.*

Proof of the claim. Assume (D, B) has such a bad 5-face f. According to Claim 1, none of the arcs of f can be a backbone arc. By minimality of (D, B), this means that, for each v_i, its unique incident arc not belonging to f must be a backbone arc. For each v_i, let us denote by v'_i its unique neighbour not in f. Assume without loss of generality that (v_1, v'_1) is a backbone arc. By minimality of (D, B), we deduce that the two interference arcs incident to v_1 (thus on f) are directed towards v_5 and v_2. Again by minimality, we deduce that both (v'_2, v_2) and (v'_5, v_5) are backbone arcs, and thus that (v_3, v_2) and (v_4, v_5) are interference arcs. Going on this way, we deduce that (v_3, v'_3) and (v_4, v'_4) are backbone arcs. We finally get to a contradiction, because the interference arc joining v_3 and v_4, whatever be its direction, cannot be involved in an indirect conflict. This is a contradiction to the minimality of (D, B). $\diamondsuit$

Claim 8. *(D, B) has no bad 5-face $f = (v_1, ..., v_5, v_1)$ where v_1 is a weak 7-vertex, and $v_2, ..., v_5$ are 3-vertices.*

Proof of the claim. For $i = 2, 3, 4, 5$, we denote by v'_i the neighbour of v_i not on f. Similarly as in the proof of Claim 7, without loss of generality we may assume that (v_3, v'_3) and (v'_4, v_4) are backbone arcs, while (v_3, v_2), (v_3, v_4) and (v_5, v_4) are interference arcs. Since v_1 is weak, it is adjacent to a 2-vertex v'_1, thus not on f. Also, the arc joining v_1 and v'_1 is a backbone arc, since v_1 supports v'_1.

Assume (v_1, v'_1) is a backbone arc. By Claim 1, at least five of the other six arcs incident to v_1 must be interference arcs out-going from v_1. If (v_5, v_1) or (v_2, v_1) is a backbone arc, then we get a contradiction by applying Claim 1 onto it. Thus, both (v_1, v_2) and (v_1, v_5) are interference arcs. In that situation, no matter what direction the backbone arc joining v_5 and v'_5 is, we get that one of (v_5, v_4) and (v_1, v_5) is useless, contradicting the minimality of (D, B).

Lastly, if (v'_1, v_1) is a backbone arc, then reversing the arguments used to deal with the previous case also gives a contradiction in all cases. $\diamondsuit$

Claim 9. *If (D, B) has a bad 5-face $f = (v_1, ..., v_5, v_1)$ where v_1 is a 4-vertex and $v_2, ..., v_5$ are 3-vertices, then v_1 is also incident to a heavy face.*

Proof of the claim. For $i = 2, 3, 4, 5$, we denote by v'_i the neighbour of v_i not on f. Similarly as in the proof of Claim 7, without loss of generality we may assume that (v_3, v'_3) and (v'_4, v_4) are backbone arcs, while (v_3, v_2), (v_3, v_4) and (v_5, v_4) are interference arcs. Also, since v_1 is is a 4-vertex, still by Claim 1 none of the arcs joining v_5 and v_1, and v_2 and v_1 can be a backbone arc. By minimality of (D, B), we deduce that (v_5, v'_5) and (v'_2, v_2) must be backbone arcs, while (v_5, v_1) and (v_1, v_2) are interference arcs. Still by minimality of (D, B), the presence of these two arcs imply that the two remaining arcs incident to v_1 must be backbone arcs, one (v_1, v'_1) being directed away from v_1, one (v''_1, v_1)

being directed toward v_1. Claim 1, applied to these arcs, now implies that v_1' and v_1'' are both 6^+-vertices. The face that contains v_1, v_1', v_1'' is then the desired heavy face. $\Diamond$

Claim 10. *If (D, B) has a bad 5-face $f = (v_1, ..., v_5, v_1)$ where v_1 is a 5-vertex and $v_2, ..., v_5$ are 3-vertices, then v_1 is also incident to a heavy face or to an almost-heavy face.*

Proof of the claim. As in the proof of Claim 9, we may assume that (v_2', v_2), (v_3, v_3'), (v_4', v_4) and (v_5, v_5') are backbone arcs, while (v_3, v_2), (v_3, v_4), (v_1, v_2), (v_5, v_4) and (v_5, v_1) are interference arcs. Again by minimality of (D, B), two of the three remaining arcs incident to v_1 must be backbone arcs, one being directed toward v_1, one being directed away from v_1. Let us thus denote (v_1, v_1') and (v_1'', v_1) the two backbone arcs incident to v_1. Without loss of generality, we may assume that (v_1, v_1''') is an interference arc, where $v_1''' \neq v_1', v_1''$ does not belong to f.

Note that, for any $i \in \{2, ..., 5\}$, applying Claim 1 on the arc joining v_i and v_i' yields that v_i' must be a 6^+-vertex. Applying Claim 1 on (v_1'', v_1), we deduce that v_1'' is a 6^+-vertex. Applying the claim on (v_1, v_1'), we deduce that v_1' is a 5^+-vertex. If v_1' is a 6^+ vertex, then we are done, because either v_1, v_5' and one of v_1', v_1'' all belong to a same heavy face, or v_1, v_2' and one of v_1', v_1'' all belong to a same heavy face. Thus, we may assume that v_1' is a 5-vertex. Again, we note that the two faces containing the arc (v_1'', v_1) cannot include one or v_2' or v_5', as otherwise we would get a heavy face including v_1, v_1'' and v_2' or v_5'. Thus, we may assume that one face incident to (v_1, v_1') contains v_5', while one face incident to (v_1, v_1''') contains v_2'. Now, after removing (v_1, v_1') from (D, B), if a 7-BMRN*-colouring of the remaining backboned digraph cannot be extended to (v_1, v_1'), then it means that each of the other four arcs incident to v_1' must be either a backbone arc out-going from v_1', or an interference arc in-coming to v_1'. This prevents v_1' from being part of a bad 5-face, as, as seen so far, a 5-vertex being the only 4^+-vertex of a bad 5-face must be incident to an out-going interference arc and an in-coming interference arc (just as v_1 in f). Since v_1' does not belong to any bad 5-face, we have that the face containing v_1, v_1', v_5' is an almost-heavy face containing v_1. $\Diamond$

The arguments used in the proofs of Claims 9 and 10 yield another property of bad 5-faces having a 4-vertex or a 5-vertex.

Claim 11. *If (D, B) has a bad 5-face $f = (v_1, ..., v_5, v_1)$ where v_1 is a 4-vertex or 5-vertex and $v_2, ..., v_5$ are 3-vertices, then v_1 is not incident to another bad 5-face.*

Proof. By the arguments used in the proof of Claim 10, if v_1 is a 4-vertex or 5-vertex being the only 4^+-vertex of a bad 5-face f, then the two arcs incident to v_1 on f must be one out-going interference arc and one in-coming interference arc. Furthermore, two of the other arcs incident to v_1 must be one out-going backbone arc and one in-coming backbone arc. Also, f must have its five arcs being interference arcs, and the four arcs incident to the four 3-vertices of f must be backbone arcs. From all these arguments, it is easy to check that the statement is true. In particular, one way to see this is that, under all these assumptions, all faces (different from f) containing v_1 include a backbone arc. $\square$

4. Second discharging process

For the second discharging phase, we apply, from the charge function ω', these rules:

R4. Every 4^+-vertex v that is not a 4-vertex or a 5-vertex incident to a bad 5-face sends $\frac{\omega'(v)}{d(v)}$ to each of the at most $d(v)$ faces it is incident to.

R5. Every 4-vertex or 5-vertex v incident to a bad 5-face sends $\frac{\omega'(v)}{d(v)}$ to each of the at most $d(v)$ non-heavy and non-almost-heavy faces it is incident to. Furthermore, v sends another $\frac{\omega'(v)}{d(v)}$ to each bad 5-face it is incident to.

For every vertex or face e of D, let us denote by $\omega''(e)$ the charge of e once rules R4 and R5 have been applied. We now analyze how $\omega'(e)$ was altered to $\omega''(e)$, for each element e.

First, assume $e = v$ is a vertex. Note that, by the rules, no vertex receives charge, and thus we necessarily have $\omega''(v) \leq \omega'(v)$. If $d(v) \leq 3$, then no charge is sent by v, which means that $\omega''(v) = \omega'(v) = 0$. If v is not a 4-vertex or 5-vertex incident to a bad 5-face, then, by rule R4, v splits its charge between the at most $d(v)$ faces it is incident to. Thus $\omega''(v) \geq \omega'(v) - d(v) \times \frac{\omega'(v)}{d(v)} = 0$. Now, if v is a 4-vertex or 5-vertex incident to a bad 5-face, then, by Claims 9, 10 and 11, v is incident to at most one bad 5-face f and at least one heavy or almost-heavy face f'. By rule R5, v sends $\frac{\omega'(v)}{d(v)}$ to each incident face (including f) but f', and instead sends another $\frac{\omega'(v)}{d(v)}$ to f. Thus v sends $\frac{\omega'(v)}{d(v)}$ at most $d(v)$ times, and $\omega''(v) \geq \omega'(v) - d(v) \times \frac{\omega'(v)}{d(v)} = 0$.

Second, assume $e = f$ is a face. By rules R4 and R5, no face sends charge. Thus $\omega''(f) \geq \omega'(f)$. Furthermore, as stated earlier we already have $\omega'(f) \geq 0$ whenever the support of f is not a 5-cycle. So, we may now focus on those cases where the support of f is a 5-cycle, in which case $\omega'(f) = -1$. If f is not a 5-face, then f is incident to a 1-vertex u. By Claim 2, the neighbour v of u is an 8^+-vertex, and v also is incident to f. By rule R4, vertex v sends $\frac{\omega'(v)}{d(v)} \geq 1$ to f, and thus $\omega''(f) \geq \omega'(f) + \frac{\omega'(v)}{d(v)} \geq -1 + 1 = 0$.

The last case to consider is when f is actually a 5-face. First assume that f is not bad. If f is not heavy and almost-heavy, then f contains at least two 4^+-vertices v, v' which, through rule R4, send $\frac{\omega'(v)}{d(v)}, \frac{\omega'(v')}{d(v')} \geq \frac{1}{2}$ to f. Thus $\omega''(f) \geq \omega'(f) + \frac{\omega'(v)}{d(v)} + \frac{\omega'(v')}{d(v')} \geq -1 + 2 \times \frac{1}{2} = 0$. If f is heavy or almost-heavy, then f contains either two 6^+-vertices v, v', or a 6^+-vertex v and a 5-vertex v' that does not belong to any bad 5-face. By rule R4, v and v' send $\frac{\omega'(v)}{d(v)}, \frac{\omega'(v')}{d(v')} \geq \frac{1}{2}$ to f. Thus $\omega''(f) \geq \omega'(f) + \frac{\omega'(v)}{d(v)} + \frac{\omega'(v')}{d(v')} \geq -1 + 2 \times \frac{1}{2} = 0$.

We may thus lastly assume that f is a bad 5-face. By definition, f thus has at most one 4^+-vertex. By Claim 7, f cannot have only 3-vertices. By Claim 4, f cannot have two consecutive 2-vertices (a 2-thread), as otherwise f would have at least two 7^+-vertices, and would thus not be bad. If f has a 2-vertex, then, by the same claim, f must have a 7^+-vertex v, which we may assume is a 7-vertex by definition of a bad 5-face. We may also assume that v is weak, as otherwise, by rule R4, v would send $\frac{\omega'(v)}{d(v)} \geq 1$ to f, and thus $\omega''(f) \geq \omega'(f) + \frac{\omega'(v)}{d(v)} \geq -1 + 1 = 0$. Now, because v is the only 4^+-vertex of f, a second 2-vertex of f must be adjacent to v (still by Claim 4). Furthermore, v must be the support of these two 2-vertices (by definition of a supporting vertex); this is a contradiction to Claim 5, which states that a vertex adjacent to two 2-vertices must be a 12^+-vertex. Consequently, if f is a bad 5-face containing a 2-vertex, then this 2-vertex is adjacent to a weak 7-vertex (that supports it) in f, and the remaining three vertices of f are 3-vertices; this is the configuration described in Claim 6, which is forbidden.

We may thus assume that f is a bad 5-face having exactly one 4-vertex, 5-vertex or weak 7-vertex v, while the other four vertices of f are 3-vertices. By Claim 8, actually v cannot be a weak 7-vertex. Thus v is a 4-vertex or 5-vertex, and, by Claims 9 and 10, it sends $2 \times \frac{\omega'(v)}{d(v)}$ to f, while $\frac{\omega'(v)}{d(v)} \geq \frac{1}{2}$. Then $\omega''(f) \geq \omega'(f) + 2 \times \frac{\omega'(v)}{d(v)} \geq -1 + 2 \times \frac{1}{2} = 0$.

Thus, we now have

$$-12 = \sum_{v \in V(D)} \omega(v) + \sum_{f \in F(D)} \omega(f) = \sum_{v \in V(D)} \omega''(v) + \sum_{f \in F(D)} \omega''(f) \geq 0,$$

the desired contradiction. Thus, (D, B) cannot exist and the claim is true. $\square$

5. Conclusion

Following [3], we have investigated, in this work, the behaviour of the BMRN*-index of planar backboned digraphs, answering some open questions from that work. We have exhibited planar spanned digraphs with BMRN*-index 8, which meets the upper bound for that class of digraphs. We have proved that the Planar k-BMRN*-Colouring is NP-hard for every $k \in \{3, ..., 6\}$, even when restricted to planar spanned digraphs. Finally, we have investigated how the BMRN*-index of a planar backboned digraph behaves in front of its girth.

We however leave a few aspects open, which we believe might be interesting to study in a later work. First, we have proved that the BMRN*-index of a planar backboned digraph can be as large as 8, and it thus makes sense wondering about the structure of planar backboned digraphs with large BMRN*-index. In particular:

Question 5.1. *What is the complexity of Planar 7-BMRN*-Colouring?*

Our approach of using crossover gadgets is of course still applicable here. However, we were not able to design 7-crossover gadgets. Designing such gadgets indeed requires lots of interference arcs, which hardly comply with the planarity requirement. Nevertheless, our bet is that Planar 7-BMRN*-Colouring should also be NP-hard.

Another remaining algorithmic question is about the complexity of Planar k-BMRN-Colouring for planar spanned digraphs, the variant of Planar k-BMRN*-Colouring for BMRN-colouring. Recall that the difference between BMRN-colouring and BMRN*-colouring is that, by the former, it is not mandatory, for every vertex, that all incident out-going backbone arcs are assigned the same colour. Thus, the only context where BMRN-colouring and BMRN*-colouring coincide is when considering backbones B with $\Delta^+(B) \leq 1$. As can be noted in Figure 8, the tower we have designed for our 6-crossover gadget has vertices incident to multiple out-going backbone arcs, which are crucial to ensure the planarity of the whole graph. We were unfortunately unsuccessful in designing a similar 6-crossover gadget (D, B) with $\Delta^+(B) \leq 1$; however, we feel that Planar k-BMRN-Colouring should be NP-hard when restricted to planar spanned digraphs.

Question 5.2. *For every $k \in \{4, 5, 6, 7\}$, what is the complexity of Planar k-BMRN-Colouring?*

Regarding planar backboned digraphs with large girth, we were not able to exhibit some better threshold above which 5-BMRN*-colourings always exist. For instance, we think the following question could be an interesting first step to consider:

Question 5.3. *Is it true that every planar backboned digraph (D, B) with girth at least 11 has BMRN*-index at most 5?*

We believe the value 11 in that question would be a nice value, as this is the girth threshold guaranteeing the existence of 2-threads in planar graphs with minimum degree 2 (recall Theorem 4.2). Using this fact would be a nice enhancement of our proof of Theorems 4.5 and 4.6. However, we were not successful with this approach, as several types of 2-threads cannot be reduced when five colours are allowed. It is likely that more sophisticated arguments are needed here, or even new approaches.

Another related aspect is how much can the girth conditions in Theorem 4.1 be lowered, namely for a given $k \in \{3, 4, 5, 6, 7\}$, what is the smallest $g(k)$ such that planar backboned

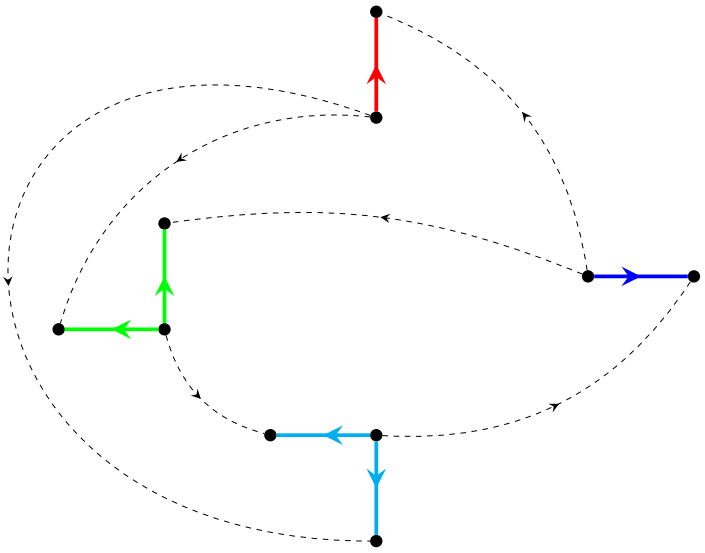


Figure 10: A planar backboned digraph (D, B) with girth 6 and BMRN*-index 4, and a 4-BMRN*-colouring of (D, B). Thick solid arcs are backbone arcs. Thin dashed arcs are interference arcs.

digraphs (D, B) with $g(D) \geq g(k)$ have BMRN*-index at most k. This is a legitimate question, as the bounds we have exhibited seem far from optimal in general.

Question 5.4. *For every $k \in \{3, ..., 7\}$, what is the smallest $g(k)$ such that planar backboned digraphs (D, B) with $g(D) \geq g(k)$ have BMRN*-index at most k?*

As a first step towards the first case, that of $k = 3$, let us mention that there exist planar backboned digraphs with girth 6 and BMRN*-index 4, such as the one depicted in Figure 10. Thus, $7 \leq g(3) \leq 21$.

Finally, it might be interesting investigating how all these concerns behave for particular classes of planar digraphs. For instance, in [3], the authors proved that the BMRN*-index of an outerplanar backboned digraph is at most 5, which is tight. Another interesting class of planar digraphs to consider could be that of grids (triangular, square, etc.). An intriguing case is that of square grids. It is fairly easy to see that the BMRN*-index of a backboned directed square grid (D, B) is at most 5, because grids admit distance-2 5-colourings (which can be derived to 5-BMRN*-colourings of (D, B), as mentioned in the introduction). Regarding lower bounds, one can easily come up with example of backboned directed square grids with BMRN*-index 4. However, even via lots of computer experimentations, we were not able to find backboned directed square grids with BMRN*-index 5. This makes us wonder about the following question, which would be interesting to answer towards understanding better the connection between BMRN*-colourings and distance-2 colourings.

Question 5.5. *Is it true that every backboned directed square grid has BMRN*-index at most 4?*

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