On the robust stability of some parameter-dependent linear systems Solutions via matrix pencil techniques

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Abstract— This note focuses on deriving stability conditions for a class of linear parameter-dependent systems in a state-space representation. More precisely, we will compute the set of parameters for which the characteristic roots are located on the imaginary axis, and next we will give the characterization of the way such critical roots are crossing the imaginary axis. The methodology considered makes use of the computation of the generalized eigenvalues of an appropriate matrix pencil combined with an operator perturbation approach for deriving the crossing direction. Finally, the particular case of parameter-dependent polynomials will be also considered, and the stability analysis of time-delay systems is also revisited in this perspective.

I. INTRODUCTION

The aim of the paper is to discuss in extent the asymptotic behavior of imaginary eigenvalues of some matrix polynomials. The main interest of this discussion lies in deriving less conservative methods for dynamical systems which are *marginally stable*. We adopt an *operator perturbation approach* (see, e.g., [9], [8]) to the corresponding switch problem. Indeed, the approach seeks to recast the zero asymptotic analysis problem as one of eigenvalue perturbation. This enables us to obtain readily computable results that fully characterize the variation properties of the critical zeros at the critical roots of the corresponding system. A special attention will be paid to the cases when the critical characteristic root on the imaginary axis is *not simple*. Thus, both semi-simple and multiple, but not semi-simple eigenvalue cases will be largely discussed, and explicit formula of the corresponding crossing directions evaluation will be derived.

As a byproduct of the analysis proposed in the note several classical problems are revised and discussed in this new perspective. In particular, we will emphasize the connections with the Routh-Hurwitz criterion and root locus method. Finally, connections to delay systems, and in particular the so-called *pseudo-delay* technique are also considered.

The paper is organized as follows: Section II includes the Problem formulation. Some preliminary results are briefly outlined in Section III. The main results are presented in both state-space representation and parameter-dependent polynomial cases in Section IV, and V respectively. Various

comments and useful observations in the context of time-delay systems are given in Section VI. Some comments and remarks end the paper.

The following notations will be used throughout the paper. Let $\mathbb{R}$ be the set of real numbers, $\mathbb{C}$ the set of complex numbers, and $\mathbb{R}_+$ the set of nonnegative real numbers. Denote the open right half plane by $\mathbb{C}_+ := \{s : \Re(s) > 0\}$, the closed right half plane by $\overline{\mathbb{C}}_+$, and the imaginary axis by $\partial\mathbb{C}_+$. Similarly, denote the open unit disc by $\mathbb{D}$, the unit circle by $\partial\mathbb{D}$, and the closed exterior of the unit disc by $\mathbb{D}^c$. For a matrix A, denote its spectrum by $\sigma(A)$, and the ith eigenvalue by $\lambda_i(A)$. For a matrix pair (A, B), denote the set of all its generalized eigenvalues by $\sigma(A, B)$, i.e.,

$$\sigma(A, B) := \{\lambda \in \mathbb{C} : \det(A - \lambda B) = 0\}.$$

II. PROBLEM FORMULATION

Consider the following class of dynamical systems:

$$\dot{x}(t) = A(p)x(t), \tag{1}$$

where $x \in \mathbb{R}^n$, and the mapping $A : [p_{min}, p_{max}] \mapsto \mathbb{R}^{n \times n}$ is given by the matrix polynomial:

$$A(p) := \sum_{k=0}^{q} A_k p^k. \tag{2}$$

In the sequel, a parameter-dependent system defined by (1)-(2) is called *marginally stable* for some parameter $p = p_c$ if all the eigenvalues of the corresponding characteristic equation are located in $\mathbb{C}_-$ or on the imaginary axis $\partial\mathbb{C}_+$ (not any strictly unstable roots).

The problem considered in this paper is to give a complete characterization of the *stability* of (1)-(2) with respect to the parameter p. We shall exploit the particular structure of the parameter-dependence in order to compute *all* the *parameter-crossing values* for which a change in the stability may take place. Furthermore, the way the eigenvalues are crossing the imaginary axis will be explicitly derived. Define now the set $\mathcal{P}$ as the set of all parameters for which the original system is marginally stable. Such a set will be called the *parameter-crossing set*. As discussed in the sequel, the computation of $\mathcal{P}$ is reduced to the computation of the generalized eigenvalue of some appropriate matrix pencil of finite dimension, and a simple matrix eigenvalue test. All the cases (simple, multiple semi-simple, and not semi-simple eigenvalues) will be discussed in some extent. Various results from robust stability and robust control will be revisited in this formalism.

III. PRELIMINARY RESULTS

In this section, we introduce the operator perturbation theory for matrix eigenvalue problems, which concerns how the eigenvalues of a matrix function may vary with respect to a small perturbation. The development is based on the classical treatise of Kato [9], but goes further beyond.

A. First-order analysis

Consider a matrix operator $T(x)$ of a real variable x. Suppose that in the neighborhood of $x = 0$, the perturbed operator $T(x)$ is holomorphic, or equivalently, can be expanded into the power series,

$$T(x) = T(0) + xT'(0) + x^2T''(0) + \cdots. \quad (3)$$

Let D_0 be a small disk near $x = 0$ but excluding $x = 0$. It is known that for $x \in D_0$, any semi-simple eigenvalue of $T(x)$, namely the semi-simple root of the characteristic equation

$$\det(T(x) - \xi I) = 0, \quad (4)$$

is an analytic functions of x and can also be expressed as a power series in x; here by a semi-simple eigenvalue, we mean a repeated but diagonalizable eigenvalue. This fact is summarized in Lemma 1.

Lemma 1: [9] Let $\lambda^{(0)}$ be a semi-simple eigenvalue of $T(0)$ with multiplicity m, and P be the eigenprojection for $\lambda^{(0)}$, that is,

$$P = \frac{1}{2\pi j}\oint_\Gamma (\xi I - T(0))^{-1} d\xi, \quad (5)$$

where Γ is a positively-oriented closed contour enclosing $\lambda^{(0)}$ but no other eigenvalues of $T(0)$. Then the corresponding eigenvalues of $T(x)$ are analytic in x and have the form

$$\mu_i(x) = \lambda^{(0)} + \lambda_i^{(1)}x + o(x^2), \quad i = 1, \ldots, m, \quad (6)$$

where $\lambda_i^{(1)}$ are the eigenvalues of $PT'(0)P$.

With no loss of generality, let $\lambda^{(0)}$ be ordered as the first eigenvalue of $T(0)$ with multiplicity m. Then $T(0)$ can be decomposed as

$$T(0) = Q\Sigma R = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}, \quad (7)$$

where Σ_1 is diagonal with diagonal entries as $\lambda^{(0)}$, $R = Q^{-1} = \begin{bmatrix} r_1^T & \cdots & r_n^T \end{bmatrix}^T$, and $Q = \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix}$ consist of the eigenvectors and generalized eigenvectors of of $T(0)$. The following lemma shows how $\lambda_i^{(1)}$ may be computed.

Lemma 2: [4] Let $T(0)$ be partitioned as in (7). Then $\lambda_i^{(1)}$, $i = 1, \ldots, m$, in (6) are the eigenvalues of $R_1T'(0)Q_1$.

Consider next the case that $\lambda^{(0)}$ is not a semi-simple but repeated eigenvalue of $T(0)$ with multiplicity m. In this case, $T(0)$ admits a Jordan decomposition in which Σ is block diagonal with diagonal Jordan blocks, and Q_1 consists of the generalized eigenvectors associated with $\lambda^{(0)}$. In particular,

$$\Sigma_1 = \begin{bmatrix} \lambda^{(0)} & 1 & \cdots & 0 \\ 0 & \ddots & \ddots & 0 \\ \vdots & & \ddots & 1 \\ 0 & \cdots & \cdots & \lambda^{(0)} \end{bmatrix}.$$

The eigenvalue of $T(x)$ can no longer be expanded in the form of (6), but instead as a Puiseux series:

Lemma 3: [4] Let $\lambda^{(0)}$ be a non-semi-simple eigenvalue of $T(0)$ with multiplicity m. Then the corresponding eigenvalues of $T(x)$ have the form

$$\mu_i(x) = \lambda^{(0)} + \left(\gamma_i^{(1)}\right)^{\frac{1}{m}} x^{\frac{1}{m}} + \cdots, \quad i = 1, \ldots, m, \quad (8)$$

where $\gamma_i^{(1)} = r_m T'(0) q_1$.

B. Second-Order Asymptotic Expansion

The first-order asymptotic series introduced in the preceding subsection can be further developed to include terms of higher orders. In this section we present formulas for computing the coefficients of the asymptotic series up to the second-order as presented in [6] by using results by Kato [9].

Define the operator-valued function

$$\Upsilon(\xi) = (T(0) - \xi I)^{-1}, \quad (9)$$

which is known as the resolvent of $T(0)$. It is obvious that the singularities of $\Upsilon(\xi)$ are the eigenvalues of $T(0)$. Let $\lambda^{(0)}$ be a semi-simple eigenvalue of $T(0)$. Then $\Upsilon(\xi)$ can be expanded as a Laurent series at $\xi = \lambda^{(0)}$, that is,

$$\begin{aligned} \Upsilon(\xi) &= -(\xi - \lambda^{(0)})^{-1}P - \sum_{n=1}^\infty (\xi - \lambda^{(0)})^{-n-1}D_n \\ &+ \sum_{n=0}^\infty (\xi - \lambda^{(0)})^n S_{n+1}, \end{aligned} \quad (10)$$

where P, D_n and S_{n+1} are corresponding coefficient matrices. Evidently, the matrix P, known as the eigenprojection for $\lambda^{(0)}$, can be found as

$$P = -\frac{1}{2\pi j}\oint_\Gamma \Upsilon(\xi)d\xi = \frac{1}{2\pi j}\oint_\Gamma (\xi I - T(0))^{-1} d\xi,$$

where Γ is a positively-oriented closed contour enclosing $\lambda^{(0)}$ but no other eigenvalues of $T(0)$. The holomorphic part in the Laurent expansion is called the reduced resolvent of $T(0)$ with respect to the eigenvalue $\lambda^{(0)}$, denoted as

$$S(\xi) = \sum_{n=0}^\infty \left(\xi - \lambda^{(0)}\right)^n S_{n+1}.$$

Let $S = S(\lambda^{(0)})$, namely the value of the reduced resolvent of $T(0)$ at $\xi = \lambda^{(0)}$. Then it is obvious that $S = S_1$.

Lemma 4: [6] For any matrix $T(0)$ decomposed in the form of (7), where Σ_1 is in Jordan form with diagonal entries as $\lambda^{(0)}$, the reduced resolvent at $\xi = \lambda^{(0)}$ is equal to

$$S = Q\begin{bmatrix} 0 & 0 \\ 0 & (\Sigma_2 - \lambda^{(0)}I)^{-1} \end{bmatrix} R = Q_2(\Sigma_2 - \lambda^{(0)}I)^{-1}R_2.$$

The following Lemma given in [9] provides the result on the second order perturbation of $T(x)$ when all the eigenvalues of $T(0)$ are semi-simple.

Lemma 5: [9] Let $\lambda^{(0)}$ be a semi-simple eigenvalue of $T(0)$, $\lambda_i^{(1)}$ be a semi-simple eigenvalue of $PT'(0)P$ with the eigen-projection $P_i^{(1)}$, that is

$$P_i^{(1)} = \oint_{\Gamma_i} (\xi I - PT'(0)P)^{-1} d\xi, \quad (11)$$

where Γ_i is a positively-oriented closed contour enclosing $\lambda_i^{(1)}$ but no other eigenvalues of $PT'(0)P$. Then $T(x)$ has $d = \dim P_i^{(1)}$ repeated eigenvalues of the form

$$\mu_{ip}(x) = \lambda^{(0)} + x\lambda_i^{(1)} + x^2\mu_{ip}^{(2)} + o(x^2), \quad p = 1, \cdots, d, \quad (12)$$

where $\mu_{ip}^{(2)}$ are the repeated eigenvalues of $P_i^{(1)}T^{(2)}P_i^{(1)}$ with $T^{(2)} = T''(0) - T'(0)ST'(0)$, and

$$P_i^{(1)}T^{(2)}P_i^{(1)} = P_i^{(1)}T''(0)P_i^{(1)} - P_i^{(1)}T'(0)ST'(0)P_i^{(1)}.$$

The eigenvalues of $P_i^{(1)}T^{(2)}P_i^{(1)}$ can be computed in a manner similar to that in the first-order analysis.

Lemma 6: [6] Let $\lambda^{(0)}$ be a semi-simple eigenvalue of $T(0)$, $\lambda_i^{(1)}$ be a semi-simple eigenvalue of $PT'(0)P$. Let also $T(0)$ be decomposed as in (7), and $R_1T'(0)Q_1$ be decomposed as

$$
\begin{aligned}
R_1T'(0)Q_1 \\
&= Q^{(2)}\Sigma^{(2)}R^{(2)} \\
&= \begin{bmatrix} Q_1^{(2)} & Q_2^{(2)} \end{bmatrix} \begin{bmatrix} \Sigma_1^{(2)} & 0 \\ 0 & \Sigma_2^{(2)} \end{bmatrix} \begin{bmatrix} R_1^{(2)} \\ R_2^{(2)} \end{bmatrix}, (13)
\end{aligned}
$$

where $\Sigma_1^{(2)}$ is the Jordan block corresponding to the eigenvalue $\lambda_i^{(1)}$. Then the eigenvalues of $P_i^{(1)}T^{(2)}P_i^{(1)}$ are those of the matrix $R^{(2)}R_1T^{(2)}Q_1Q_1^{(2)}$.

IV. MAIN RESULTS

Introduce now the following matrix pencil $\Lambda : \mathbb{C} \mapsto \mathbb{C}^{qn^2 \times qn^2}$, $\Lambda(\lambda) := \lambda U + V$, where the matrices U, and V are given by:

$$
U = \begin{bmatrix} I_{n^2} & 0 & 0 & \dots & 0 \\ 0 & I_{n^2} & 0 & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & B_q \end{bmatrix}, \quad (14)
$$

$$
V = \begin{bmatrix} 0 & -I_{n^2} & 0 & \dots & 0 \\ 0 & 0 & -I_{n^2} & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ B_0 & B_1 & B_2 & \dots & B_{q-1} \end{bmatrix}, \quad (15)
$$

where $B_k := A_k \oplus A_k^T$, for all $k = \overline{0, q}$.

A. Stability crossing set

With the definitions, and the notations above, we have:

Lemma 7: Consider the system (1)-(2) for some $p_0 \in [p_{min}, p_{max}]$. Then p_0 is a parameter-crossing value if and only if the following conditions are satisfied simultaneously:

(i) $p_0 \in \sigma(U, V)$, that is p_0 is a generalized eigenvalue of the matrix pencil Λ, and

(ii) $\sigma(A(p_0)) \cap j\mathbb{R} \neq \varnothing$, that is the real matrix $A(p_0)$ has at least one eigenvalue on the imaginary axis.

Remark 1 (Symmetric eigenvalues): The condition (i) in the Lemma 7 is *not sufficient* for crossing existence. Indeed, by construction, the spectrum of Λ includes all the eigenvalues of the matrix $A(p)$ which are *symmetric* with respect to the origin, and thus we need to exclude the symmetric roots which are not located on $\partial\mathbb{C}_+$.

Since, excepting $\mathcal{P}$, there are no other crossings with respect to $\partial\mathbb{C}_+$, let $\text{card}(\mathcal{P}) = \ell$, and let $p_1, p_2, \ldots, p_\ell$ be corresponding *ordered* crossing parameters. The continuity properties of the characteristic roots with respect to the parameters [8] allows concluding that for all $p \in (p_k, p_{k+1})$, $k = 1, \ldots \ell - 1$, the *number of strictly unstable characteristic roots (i.e. the instability degree) of (1)-(2) is constant.* In conclusion, the characterization of the *parameter-crossing set* $\mathcal{P}$ is complete.

B. Crossing direction characterization

1) Simple crossing characteristic roots: Consider now that the root $s = j\omega_0$ corresponding to some parameter $p_0 \in \mathcal{P}$ is *simple*. Let u_0 (v_0^*) be the corresponding right- (left-) eigenvectors, that is:

$$
\begin{cases} (j\omega_0 I_n - A(p_0))u_0 = 0, \\ v_0^*(j\omega_0 I_n - A(p_0)) = 0. \end{cases}
$$

Without any loss of generality, we can assume that the right- and left-eigenvectors u_0, v_0^* satisfy the *regularity* condition $v_0^*u_0 = 1$. Then:

Proposition 1: Let $j\omega_0$ be a simple eigenvalue of $A(p_0)$. Thus, for p sufficiently close to p_0 but $p > p_0$, there exist a characteristic zero entering the right-half plane (or vice versa) if the following condition is satisfied:

$$\text{Re}\left\{ v_0^*\left(\sum_{k=1}^q kA_kp_0^{k-1}\right)u_0 \right\} < 0 \ (> 0). \quad (16)$$

Additionally, if

$$\text{Re}\left\{ v_0^*\left(\sum_{k=1}^q kA_kp_0^{k-1}\right)u_0 \right\} = 0,$$

then for any p sufficiently close to p_0 but $p > p_0$, $j\omega_0$ enters the right-half plane (or vice versa) if

$$\text{Re}\left\{ v_0^*\left(\sum_{k=2}^q k(k-1)A_kp_0^{k-2}\right)u_0 \right\} < 0 \ (> 0), \quad (17)$$

In other words, the crossing direction for simple roots is given by the sign of (16) easy to evaluate since u_0, v_0^*, A_k, $k = 1, \ldots, m$ and $p_0 = j\omega_0$ are known. It is important to point out that such a quantity is *non-zero* since the corresponding characteristic root is simple, but, as mentioned in Proposition 1, it can be on the imaginary axis.

2) Semi-simple crossing characteristic roots: Consider now the case of a *multiple*, but *semi-simple* eigenvalue $j\omega_0$ on the imaginary axis. Using the arguments proposed in the previous section, simple computations lead to the following:

Proposition 2: Let $j\omega_0$ be a semi-simple eigenvalue of $A(p_0)$ with multiplicity m and $\lambda_i^{(1)}$ be a semi-simple eigenvalue of

$$-R_1\left(\sum_{k=1}^q kA_kp_0^{k-1}\right)Q_1$$

with multiplicity d. Then for any p sufficiently close to p_0, the characteristic zeros corresponding to $j\omega_0$ can be expanded by the power series

$$j\omega_0 + \lambda_i^{(1)}(p - p_0) + \mu_{il}^{(2)}(p - p_0)^2 + o\left((p - p_0)^3\right), \quad (18)$$

with

$$\lambda_i^{(1)} = -\lambda_i \left[R_1 \left(\sum_{k=1}^{q} k A_k p_0^{k-1} \right) Q_1 \right],$$
$$i = 1, 2, \cdots, m,$$

$$\mu_{il}^{(2)} = -\lambda_l \left[R_1^{(2)} R_1 \left(\sum_{k=2}^{q} k(k-1) A_k p_0^{k-2} \right) Q_1 Q_1^{(2)} \right],$$
$$l = 1, 2, \cdots, d,$$

where $Q_1^{(2)} = [\tilde{q}_1 \ \tilde{q}_2 \ \cdots \ \tilde{q}_d]$ and $R_1^{(2)} = [\tilde{r}_1^T \ \tilde{r}_2^T \ \cdots \ \tilde{r}_d^T]^T$, with $\tilde{q}_p$ and $\tilde{r}_p$ the right and left eigenvectors of

$$-R_1 \left(\sum_{k=1}^{q} k A_k p_0^{k-1} \right) Q_1$$

associated with $\lambda_i^{(1)}$.

(i) For p sufficiently close to p_0 but $p > p_0$, there are at least h ($h \leq m$) of characteristic zeros entering the right-half plane (or vice versa) if h of the eigenvalues satisfy the condition

$$\text{Re}\left\{ \lambda_i \left[R_1 \left(\sum_{k=1}^{q} k A_k p_0^{k-1} \right) Q_1 \right] \right\} < 0 \ (> 0),$$
$$i = 1, 2, \cdots, m.$$

(ii) If

$$\text{Re}\left\{ \lambda_i \left[R_1 \left(\sum_{k=1}^{q} k A_k p_0^{k-1} \right) Q_1 \right] \right\} = 0, \quad (19)$$

then for p sufficiently close to p_0 but $p > p_0$, there are at least h' ($h' \leq d$) of characteristic zeros entering the right-half plane (or vice versa) if h' of the eigenvalues $\mu_{il}^{(2)}$ satisfy the condition

$$\text{Re}\left\{ \lambda_l \left[R_1^{(2)} R_1 \left(\sum_{k=2}^{q} k(k-1) A_k p_0^{k-2} \right) Q_1 Q_1^{(2)} \right] \right\}$$
$$< 0 \ (> 0), \ l = 1, 2, \cdots, d. \quad (20)$$

Finally, the last case of interest in this situation is given by the case when the eigenvalues $\lambda_i^{(1)}$ in Proposition 2 are not semi-simple:

Proposition 3: Let $j\omega_0$ be a semi-simple eigenvalue of $A(p_0)$ with multiplicity m and $\lambda_i^{(1)}$ be a repeated eigenvalue of

$$-R_1 \left(\sum_{k=1}^{q} k A_k p_0^{k-1} \right) Q_1$$

with multiplicity d. Suppose that $\lambda_i^{(1)}$ is not semi-simple.

Then for any p sufficiently close to p_0 but $p > p_0$, the characteristic zeros corresponding to $j\omega_0$ can be expanded by the Puiseux series

$$j\omega_0 + \lambda_i^{(1)}(p - p_0) + \left| \tilde{r}_d \tilde{T} \tilde{q}_1 \right|^{\frac{1}{d}} e^{j\frac{2h\pi + \pi + \theta}{d}} (p - p_0)^{1 + \frac{1}{d}}$$
$$+ \cdots, \quad h = 0, 1, \cdots, d - 1, \quad (21)$$

where

$$\tilde{T} = R_1^{(2)} R_1 \left(\sum_{k=2}^{q} k(k-1) A_k p_0^{k-2} \right) Q_1 Q_1^{(2)},$$

where $\theta \in [0, 2\pi]$ is the phase angle of $\tilde{r}_d \tilde{T} \tilde{q}_1$, and $\tilde{q}_1$ and $\tilde{r}_d$ are obtained from the Jordan decomposition

$$Q_1 R_1 \left(-\sum_{k=1}^{q} k A_k p_0^{k-1} \right) Q_1 R_1 = \tilde{Q} \tilde{\Sigma} \tilde{R},$$

with $\tilde{Q} = [\tilde{q}_1 \ \cdots \tilde{q}_d \ \cdots \ \tilde{q}_m]$ and $\tilde{R} = \tilde{Q}^{-1} = [\tilde{r}_1^T \ \cdots \ \tilde{r}_d^T \ \cdots \ \tilde{r}_m^T]^T$.

Hence, for p sufficiently close to p_0 but $p > p_0$, the number of critical zeros entering the right-half plane (or vice versa) can be determined by the condition

$$\cos\left(\frac{2h\pi + \pi + \theta}{d} \right) > 0 \ (< 0), \quad h = 0, \cdots, d-1. \quad (22)$$

Propositions 2 and 3 reveal a fundamental difference between the asymptotic behaviors of a semi-simple eigenvalue and one that is not. Take for instance Proposition 3, for a repeated eigenvalue that is not semi-simple, the result shows that, precluding the case that

$$r_m \left(\sum_{k=1}^{q} k A_k p_0^{k-1} \right) q_1 = 0,$$

whether the zero will enter the right-half plane is solely determined by the multiplicity of the zero and the phase angle θ. In this case, the branches of the zero will almost generically enter the right-half plane. Note that in the degenerate case $m = 1$, Proposition 3 also reduces to Proposition 2.

3) Multiple non semi-simple characteristic roots: Finally, consider the case of a *multiple*, but *not semi-simple* eigenvalue $j\omega_0$ on the imaginary axis. Then:

Proposition 4: Let $j\omega^*$ be a repeated eigenvalue of $\sum_{k=0}^{q} A_k e^{-j\omega^* k \tau^*}$ with multiplicity m. Suppose that $j\omega_0$ is not semi-simple.

Then for any p sufficiently close to p_0 but $p > p_0$, the characteristic zeros corresponding to p_0 can be expanded by the Puiseux series

$$j\omega_0 + \left| r_m \left(\sum_{k=1}^{q} k A_k p_0^{k-1} \right) q_1 \right|^{\frac{1}{m}} e^{j\frac{2h\pi + \pi + \theta}{m}}$$
$$\times (p - p_0)^{\frac{1}{m}} + \cdots, \quad h = 0, 1, \cdots, m - 1,$$

where $\theta \in [0, 2\pi]$ is the phase angle of

$$r_m \left(\sum_{k=1}^{q} k A_k p_0^{k-1} \right) q_1,$$

and q_1 and r_m are obtained from the Jordan decomposition

$$\sum_{k=0}^{q} A_k p_0^k = Q \Sigma R,$$

with $Q = [q_1 \ q_2 \ \cdots \ q_n]$ and $R = Q^{-1} = [r_1^T \ r_2^T \ \cdots \ r_n^T]^T$.

Hence, for p sufficiently close to p_0 but $p > p_0$, the number of critical zeros entering the right-half plane (or vice versa) can be determined by the condition

$$\cos\left(\frac{2h\pi + \pi + \theta}{m} \right) > 0 \ (< 0), \quad h = 0, \cdots, m - 1. \quad (23)$$

V. PARAMETER-DEPENDENT POLYNOMIALS

In the sequel, we will consider the following class of parameter-dependent polynomials:

$$f(s,\ p)\ :=\ P_0(s) + \sum_{k=1}^{n_q} P_k(s)p^k, \qquad (24)$$

with the parameter $p \in \mathbb{R}$. In principle, there are several ways to compute the stability crossing set $\mathcal{P}$ by using the methodology presented in the previous sections. The standard method consists in finding a state-space representation that has the characteristic function given by (24)[1]. Such a method was largely used in the context of robust analysis (see, e.g., [8], [1], [2] and the references therein). For the sake of brevity, such an approach will be omitted here.

In the sequel, we will propose however a different method which is inspired by the static output feedback control problem for SISO systems. Introduce now the following Hurwitz matrix associated to some polynomial

$$A(s) = \sum_{i=0}^{n_a} a_i s^{n_a - i}$$

and denoted by $H(A)$:

$$H(A) := \begin{bmatrix} a_1 & a_3 & a_5 & \dots & a_{2n_a-1} \\ a_0 & a_2 & a_4 & \dots & a_{2n_a-2} \\ 0 & a_1 & a_3 & \dots & a_{2n_a-3} \\ 0 & a_0 & a_2 & \dots & a_{2n_a-4} \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \dots & a_{n_a} \end{bmatrix} \in \mathbb{R}^{n_a \times n_a}, \tag{25}$$

where the coefficients $a_l = 0$, for all $l > n_a$. Next, define the matrix pencil:

$$\Lambda(\lambda) := \det(\lambda U + V),$$

with U, V given by:

$$U\ =\ \begin{bmatrix} I & & & \\ & \ddots & & \\ & & I & \\ & & & H(P_{n_q}) \end{bmatrix} \tag{26}$$

$$V\ =\ \begin{bmatrix} 0 & -I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -I \\ H(P_0) & H(P_1) & \cdots & H(P_{n_q-1}) \end{bmatrix}, \tag{27}$$

where the identity, and the zero-blocks matrices have appropriate dimension, and $H(P_i) \in \mathbb{R}^{(n+n_q)\times(n+n_q)}$, for all $i = 1, 2, \dots, n_q$ represents the corresponding Hurwitz matrix[2] associated to the polynomial $P_i(s)$ defined above.

The following result gives the characterization of the characteristic roots of f given by (24) on the imaginary

axis as a function of T, and represents a generalization of the method proposed by [3] in the context of static output feedback for SISO systems:

Proposition 5: Let $\lambda_1 < \lambda_2 < \dots \lambda_h$, with $h \le n + n_q$ be the real eigenvalues of the matrix pencil

$$\Lambda(\lambda) = \det(\lambda U + V).$$

Then the parameter-dependent polynomial f has some characteristic roots on the imaginary axis if and only if $T = \lambda_i$, $i = 1, 2, \dots h$. Furthermore, if there are r unstable roots $(0 \le r \le n + n_d)$ for $p = p^*$, $p^* \in (\lambda_i, \lambda_{i+1})$, then, there are r unstable roots for any parameter $p \in (\lambda_i, \lambda_{i+1})$. The same property holds for the intervals $(0, \lambda_1)$ and (λ_h, ∞).

In conclusion, as derived in the Proposition 5 above, the set of crossing parameters $\mathcal{P}$ is given by the set of generalized eigenvalue of the corresponding matrix pencil Λ. For each $p_0 \in \mathcal{P}$ a simple test will give the set of corresponding characteristic roots on the imaginary axis. Let $j\omega_0$ one such a critical root. The remaining point to be clarified is the characterization of the crossing direction in some neighborhood of p_0. The crossing characterization derived in the previous section rewrites as follows in the simple and multiple root case respectively:

Proposition 6: Let $j\omega_0$ be a simple zero of $f(s,p)$ corresponding to the parameter $p = p_0$. If

$$\text{Re}\left\{ \frac{\frac{d\ f(j\omega_0,\ p)}{dp}\big|_{p=p_0}}{\frac{d\ f(s,p_0)}{ds}\big|_{s=j\omega_0}} \right\} < 0,\ \ (>0)$$

then for any p sufficiently close to p_0 but $p > p_0$, $j\omega_0$ enters the right-half plane (or vice versa).

Finally, if

$$\text{Re}\left\{ \frac{\frac{d\ f(j\omega_0,\ p)}{dp}\big|_{p=p_0}}{\frac{d\ f(s,p_0)}{ds}\big|_{s=j\omega_0}} \right\} = 0,$$

then for any p sufficiently close to p_0 but $p > p_0$, $j\omega_0$ enters the right-half plane (or vice versa) if

$$\text{Re}\left\{ \frac{\frac{d^2\ f(j\omega_0,\ p)}{dp^2}\big|_{p=p_0}}{\frac{d\ f(s,\ p_0)}{ds}\big|_{s=j\omega_0}} \right\} < 0\ \ (>0). \tag{28}$$

Proposition 7: Let $j\omega_0$ be a repeated zero of $f(s,\ p))$ with multiplicity m, corresponding to the parameter $p = p_0$. If

$$m!\frac{\frac{d\ f(j\omega_0,\ p)}{dp}\big|_{p=p_0}}{\frac{d^m\ f(s,\ p_0)}{ds^m}\big|_{s=j\omega_0}} = 0,$$

then for any p sufficiently close to p_0 but $p > p_0$, the zeros corresponding to $j\omega_0$ can be expanded by the Puiseux series

$$j\omega_0 + \left| m!\frac{\frac{d^2\ f(j\omega_0,\ p)}{d\tau^2}\big|_{p=p_0}}{\frac{d^m\ f(s,\ p_0)}{ds^m}\big|_{s=j\omega_0}} \right|^{\frac{1}{m}} e^{j\frac{2h\pi+\pi+\theta}{m}}(p-p_0)^{\frac{1}{m}}$$

$$+\cdots,\ \ h = 0,\ 1,\cdots,\ m-1,$$

where $\theta \in [0,\ 2\pi]$ is the phase angle of

$$\frac{\frac{d^2\ f(j\omega_0,p)}{dp^2}\big|_{p=p_0}}{\frac{d^m\ f(s,\ p_0)}{ds^m}\big|_{s=j\omega_0}}.$$

---

[1]by using, for example, appropriate companion forms and defining all the coefficients of s^k as appropriate polynomials in the variable p, etc.

[2]Independently of the order of the polynomial P_i ($i = 1,\dots,n_q$), $H(P_i)$ will be constructed as a $(n+n_q)\times(n+n_q)$ matrix by setting the coefficients of high-order terms as zeroes, that is $q_h = 0$, for all $h > \deg(P_i)$.

Hence, for p sufficiently close to p_0 but $p > p_0$, the number of critical zeros entering the right-half plane (or vice versa) can be determined by the condition

$$\cos\left(\frac{2h\pi + \pi + \theta}{m}\right) > 0 \quad (< 0), \quad h = 0, 1, \cdots, m - 1.$$

VI. Pseudo-delay and parameter-dependent polynomials

There exists several frequency-domain methods for handling the stability analysis of linear systems with commensurate delays (see, for instance, [7], [10] and the references therein). In the sequel, we shall focus on the *pseudo-delay technique* (see, e.g., [11], [12] for further details).

The characteristic function of a linear delay system:

$$\dot{x}(t) = \sum_{i=0}^{n_c} x(t - i\tau),$$

with $x \in \mathbb{R}^n$ is given by the quasipolynomial:

$$f(s; e^{-s\tau}) := P_0(s) + \sum_{i=1}^{n_d} P_i(s) e^{-is\tau}. \tag{29}$$

Roughly speaking, the idea of the approach is to use the bilinear transformation associating to $z = e^{-s\tau}$ the "quantity":

$$z := \frac{1 - \lambda T}{1 + \lambda T}, \quad T > 0.$$

and the parameter-dependent polynomial:

$$\begin{aligned}
\tilde{f}(s, T) &:= (1 + sT)^n f\left(s, \frac{1 - sT}{1 + sT}\right) \\
&= \sum_{i=0}^{n_d} P_i(s)(1 - Ts)^i (1 + Ts)^{n_d - i}. \tag{30}
\end{aligned}$$

It is easy to see that the imaginary value $s = j\omega_0$ with $\omega_0 > 0$ is a characteristic root of $f(s, e^{-s\tau})$ for some $\tau_0 > 0$ *iff* $s = j\omega_0$ is also a root of $\tilde{f}(s, T_0)$ for some $T_0 > 0$, and the relation between τ_0 and T_0 can be explicitly derived (see, e.g. [13], [12]).

Based on the theory presented in the paper, it is easy to see that the corresponding set of T for which, at least, one characteristic root of $\tilde{f}$ lies on $\partial\mathbb{C}_+$ will be given by computing the generalized eigenvalues of some appropriate matrix pencil associated to $\tilde{f}$. In other words, crossing detection for commensurate delays systems is reduced to finding parameters T such that $\tilde{f}$ has characteristic roots on the imaginary axis. In conclusion, the generalized eigenvalues of Λ constructed for $\tilde{f}$ will define the values of the parameter T for which crossing exists. Next, for each $T_0 \in \{\lambda_1, \ldots, \lambda_h\}$, we will compute the characteristic roots of the polynomial $\tilde{f}(s, T_0)$ on $\partial\mathbb{C}_+$. The set of frequencies corresponding to all these roots on the imaginary axis will define the *frequency crossing set* Ω. Next, consider some frequency $\omega_0 \in \Omega$, which corresponds to T_0. Then, the corresponding delay crossing set $\mathcal{T}_{\omega_0}$ will be derived by solving the equation:

$$e^{-j\omega_0\tau} = \frac{1 - j\omega_0 T_0}{1 + j\omega_0 T_0}.$$

Finally, the crossing direction characterization follows the same procedures as in [4], [6], and it is omitted since it is out of the scope of the paper.

Remark 2: The detection of the crossing frequency set proposed here is quite distinct from the procedures proposed by [11], [12].

VII. Concluding remarks

This paper addressed a robust stability analysis of a class of linear systems depending on some real parameter (polynomial dependence). The approach considered here makes use of two ingredients: the computation of the generalized eigenvalues of an appropriate matrix pencil, combined with an operator perturbation methodology. A complete crossing characterization is proposed in the simple, semi-simple, and multiple, but not semi-simple characteristic root cases. Both state-representation and parameter-dependent polynomial cases have been presented. Some connections with some method encountered in the analysis of time-delay systems was also proposed.

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