Online pseudo Marginal Sequential Monte Carlo smoother for general state spaces. Application to recursive maximum likelihood estimation of stochastic differential equations

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Abstract

This paper focuses on the estimation of smoothing distributions in general state space models where the transition density of the hidden Markov chain or the conditional likelihood of the observations given the latent state cannot be evaluated pointwise. The consistency and asymptotic normality of a pseudo marginal online algorithm to estimate smoothed expectations of additive functionals when these quantities are replaced by unbiased estimators are established. A recursive maximum likelihood estimation procedure is also introduced by combining this online algorithm with an estimation of the gradient of the filtering distributions, also known as the tangent filters, when the model is driven by unknown parameters. The performance of this estimator is assessed in the case of a partially observed stochastic differential equation.

1 Introduction

The data considered in this paper originate from general state space models, usually defined as bivariate stochastic processes $\{(X_k, Y_k)\}_{1\leqslant k\leqslant n}$ where $\{Y_k\}_{1\leqslant k\leqslant n}$ are the observations and $\{X_k\}_{1\leqslant k\leqslant n}$ are the latent states comonly assumed to be a Markov chain. When both processes take values in general spaces, the estimation of the conditional distribution of a sequence of hidden states given a fixed observation record is a challenging task required for instance to perform maximum likelihood inference. Markov chain Monte Carlo (MCMC) and sequential Monte Carlo (SMC) methods (also known as particle filters or smoothers) are widespread solutions to propose consistent estimators of such distributions. This paper sets the focus on the special case where the conditional likelihood of an observation given the corresponding latent state (also known as the emission distribution) or the transition density of the hidden Markov chain cannot be evaluated pointwise, while they are pivotal tools of both MCMC and SMC approaches. The first objective of this paper is to prove that conditional expectations of additive functionals of the hidden states may still be estimated online with a consistent and asymptotically normal SMC algorithm. A recursive maximum likelihood estimation procedure based on this algorithm and using an approximation of the gradient of the filtering distributions, referred to as the tangent filters, is then introduced.

The use of latent data models is ubiquitous in time series analysis across a wide range of applied science and engineering domains such as signal processing [6], genomics [36, 35], target tracking [33], enhancement and segmentation of speech and audio signals [31], see also [32, 14, 37] and the numerous references therein. Statistical inference for such models is likely to require the computation of conditional expectations of sequences of hidden states given observations. In this Bayesian setting, one of the most challenging problems is the approximation of expectations under the joint smoothing distribution, i.e. the posterior distribution of the sequence of states $(X_1, \ldots, X_n)$ given the observations $(Y_1, \ldots, Y_n)$ for some $n \geqslant 1$. This computation is not tractable in the framework of this paper where it is assumed that the transtion density of the hidden process or the conditional likelihood of observations given states cannot be computed. This circumstance is somehow common for instance in the case of partially observed stochastic diferential equations (SDE), or in models where the emission distributions relies on a computationally prohibitive black-box routine.

Following [18, 21], this paper concentrates on SMC methods to approximate smoothing distributions with a random set of states, the particles, associated with importance weights by combining importance sampling and resampling steps. This allows to solve the filtering problem by combining an auxiliary particle filter with an unbiased estimate of the unknown densities. Then, the online smoother of [21] extends the particle-based rapid incremental smoother (PaRIS) of [28], to approximate, processing the data stream online, smoothed expectations of additive functionals when the unknown densities are replaced by unbiased estimates. This approach is an online version of the Forward Filtering Backward Simulation algorithm algorithm [11] specifically designed to approximate smoothed additive functionals. The crucial feature which makes the PaRIS algorithm appealing is the acceptance-rejection step which benefits from the unbiased estimation. The extension of the usual alternative, named the Forward Filtering Backward Smoothing algorithm [15], is more sensitive as it involves ratios of these unknown quantities. Other smoothing algorithms such as two-filter based approaches [2, 19, 25] could be extended similarly but they are intrisically not online procedures as they require the time horizon and all observations to be available to initialize a backward information filter.

In [21], the only theoretical guarantee is that the accept reject mechanism of the PaRIS algorithm is still correct when the transition densities are replaced by unbiased estimates. In this paper, the consistency of the algorithm as long as a central limit theorem (CLT) are established (see Proposition 4.2 and Proposition 4.3 in Section 4.2). This makes this pseudo marginal smoother the first algorithm to approximate such expectations in the general setting of this paper with theoretical guarantees and an explicit expression of the asymptotic variance. As a byproduct, the proofs of these results require to establish exponential deviation inequalities and a CLT for the PaRIS algorithm based on the auxiliary particle filter, see Section 4.1. This extends the result of [28], written only in the case of the bootstrap filter of [22]. This also extends the theoretical guarantees obtained for online sequential Monte Carlo smoothers given in [11, 9, 17, 20].

The second part of the paper is devoted to recursive maximum likelihood estimation when the emission distributions or the transition densities depend on an unknown parameter, see Section 5. Following the filter sensitivity approach of [5, Section 10.2.4], the pseudo marginal smoother is used to estimate online the gradient of the one-step predictive likelihood of an observation given past observations. This procedure allows to perform online estimation in complex frameworks and is applied in Section 6 to partially observed SDE.

2 Online Sequential Monte Carlo smoother

Let n be a positive integer and X and Y two general state spaces. Consider a distribution χ on $\mathcal{B}(\mathsf{X})$ and the Markov transition kernels $(\mathbf{Q}_k)_{0\leqslant k\leqslant n-1}$ on $\mathsf{X}\times\mathcal{B}(\mathsf{X})$ and $(\mathbf{G}_k)_{0\leqslant k\leqslant n-1}$ on $\mathsf{X}\times\mathsf{X}\times\mathcal{B}(\mathsf{Y})$. Throughout this paper, for all $0\leqslant k\leqslant n-1$, $\mathbf{G}_k$ has a density g_k with respect to a reference measure ν on $\mathcal{B}(\mathsf{Y})$. In the following, $\mathsf{F}(\mathsf{Z})$ denotes the set of real valued measurable functions defined on the set Z. Let $(Y_k)_{1\leqslant k\leqslant n}$ be a sequence of observations in Y and define the joint smoothing distributions, for any $0\leqslant k_1\leqslant k_2\leqslant n$ and any function $h\in\mathsf{F}(\mathsf{X}^{k_2-k_1+1})$, by:

$$\phi_{k_1:k_2|n}[h] := \mathcal{L}_n^{-1}(Y_{1:n})\int \chi(\mathrm{d}x_0)\prod_{k=0}^{n-1}\mathbf{Q}_k(x_k,\mathrm{d}x_{k+1})g_k(x_k,x_{k+1},Y_{k+1})h(x_{k_1:k_2})\;, \tag{1}$$

where $a_{u:v}$ is a short-hand notation for $(a_u,\ldots,a_v)$ and

$$\mathcal{L}_n(Y_{1:n}) = \int \chi(\mathrm{d}x_0)\prod_{k=0}^{n-1}\mathbf{Q}_k(x_k,\mathrm{d}x_{k+1})g_k(x_k,x_{k+1},Y_{k+1}) \tag{2}$$

is the observed data likelihood. For all $0\leqslant k\leqslant n-1$, $\mathbf{Q}_k$ has a density q_k with respect to a reference measure μ on $\mathcal{B}(\mathsf{X})$. The initial measure χ is also assumed to have a density with respect to μ which is also referred to as χ. For all $0\leqslant k\leqslant n$, $\phi_k=\phi_{k:k|k}$ are the filtering distributions, $\pi_{k+1}=\phi_{k+1:k+1|k}$ are the one-step predictive distributions, while $\phi_{k|n}=\phi_{k:k|n}$ are the marginal smoothing distributions.

Consider a latent Markov chain $(X_k)_{0\leqslant k\leqslant n}$ with initial distribution χ and Markov transition kernels $(\mathbf{Q}_k)_{0\leqslant k\leqslant n-1}$. The states $(X_k)_{0\leqslant k\leqslant n}$ are not available so that any statistical inference procedure is performed using the sequence of observations $(Y_k)_{1\leqslant k\leqslant n}$ only. The observations are assumed to be independent conditional on $(X_k)_{0\leqslant k\leqslant n}$ and such that for all $1\leqslant\ell\leqslant n$ the distribution of Y_ℓ given $(X_k)_{0\leqslant k\leqslant n}$ has distribution $\mathbf{G}_k(X_k,\cdot)$. In this case, (1) may be interpreted as:

$$\phi_{k_1:k_2|n}[h] = \mathbb{E}\left[h(X_{k_1:k_2})|Y_{1:n}\right]\;.$$

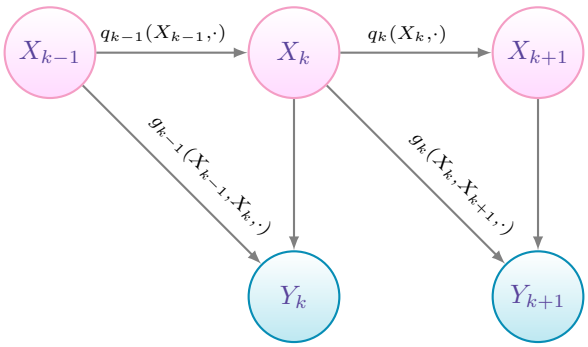

Figure 1: Graphical model of the general state space hidden Markov model

Figure 1 displays the graphical model associated with (2). Note that, when for all $0\leqslant k\leqslant n-1$ g_k only depends on its last two arguments, (2) is the likelihood of a standard hidden Markov model. In such models, computing (1) allows to solve classical problems such as:

i) path reconstruction, i.e. the reconstruction of the hidden states given the observations;

ii) parameter inference, i.e., when q_k and g_k depend on some unknown parameter θ, the design of a consistent estimator of θ from the observations.

As (1) is, in general, not available explicitly, this paper focuses on a sequential Monte Carlo based approximation specifically designed for cases where q_k and/or g_k cannot be evaluated pointwise. Partially observed diffusion processes (POD) [27], where the latent process is the solution to a stochastic differential equation are widespread examples where q_k is not tractable.

Recursive formulation of (1) **for additive functionals.** For all $0 \leqslant k \leqslant n-1$, define

$$\mathsf{r}_k(x_k, x_{k+1}) = q_k(x_k, x_{k+1}) g_k(x_k, x_{k+1}, Y_{k+1}) \; . \tag{3}$$

For all $0 \leqslant k \leqslant n-1$, define also the kernel $\mathbf{L}_k$ on $\mathsf{X} \times \mathcal{B}(\mathsf{X})$, for all $x \in \mathsf{X}$ and all $f \in \mathsf{F}(\mathsf{X})$ by

$$\mathbf{L}_k f(x) = \int \mathsf{r}_k(x, y) f(y) \mathrm{d}y \; .$$

In the following, $\mathbb{1}$ denotes the constant function which equals 1 for all $x \in \mathsf{X}$ so that

$$\mathbf{L}_k \mathbb{1}(x) = \int \mathsf{r}_k(x, y) \mathrm{d}y \; .$$

Following for instance [4], the joint smoothing distributions $\phi_{0:n|n}$ may be decomposed using the backward Markov kernels defined, for all $0 \leqslant k \leqslant n-1$, all $x_{k+1} \in \mathsf{X}$ and all $f \in \mathsf{F}(\mathsf{X})$, by:

$$\overleftarrow{\mathbf{Q}}_{\phi_k} f(x_{k+1}) := \frac{\int f(x_k) \mathsf{r}_k(x_k, x_{k+1}) \, \phi_k(\mathrm{d}x_k)}{\int \mathsf{r}_k(x'_k, x_{k+1}) \, \phi_k(\mathrm{d}x'_k)} \; . \tag{4}$$

Consequently, the joint-smoothing distribution $\phi_{0:n|n}$ may be expressed, for all $h \in \mathsf{F}(\mathsf{X}^{n+1})$, as

$$\phi_{0:n|n}[h] = \phi_n[\mathbf{T}_n h] \; , \tag{5}$$

where

$$\mathbf{T}_n := \begin{cases} \overleftarrow{\mathbf{Q}}_{\phi_{n-1}} \otimes \overleftarrow{\mathbf{Q}}_{\phi_{n-2}} \otimes \cdots \otimes \overleftarrow{\mathbf{Q}}_{\phi_0} & \text{for } n > 0 \; , \\ \mathrm{id} & \text{for } n = 0 \; , \end{cases} \tag{6}$$

where, for all Markov kernels $\mathbf{K_1}, \mathbf{K_2}$ on $\mathsf{X} \times \mathcal{B}(\mathsf{X})$, all $f \in \mathsf{F}(\mathsf{X}^2)$ and all $x \in \mathsf{X}$,

$$(\mathbf{K_1} \otimes \mathbf{K_2}) f(x) = \int f(y, z) \mathbf{K_1}(x, \mathrm{d}y) \mathbf{K_2}(y, \mathrm{d}z) \; .$$

In this paper, the focus is set on additive functionals of the form

$$h_{0:n}(x_{0:n}) = \sum_{k=0}^{n-1} \tilde{h}_k(x_k, x_{k+1}) \; , \tag{7}$$

with, for all $0 \leqslant k \leqslant n-1$, $\tilde{h}_k : \mathsf{X} \times \mathsf{X} \to \mathbb{R}^p$ for some $p \geqslant 1$. The additive form of the function h_n defined in (7) allows to update the backward statistics $(\mathbf{T}_k h_k)_{k \geqslant 0}$ recursively, see [3, 9]. For all $k \geqslant 0$,

$$\mathbf{T}_{k+1} h_{k+1}(x_{k+1}) = \int \{\mathbf{T}_k h_k(x_k) + \tilde{h}_k(x_{k:k+1})\} \overleftarrow{\mathbf{Q}}_{\phi_k}(x_{k+1}, \mathrm{d}x_k) \; . \tag{8}$$

By (5) and (8), the smoothed additive functional (5) can be updated recursively each time a new observation is available. However, its exact computation is not possible in general state spaces. In this paper, we propose to approximate $\phi_{0:n|n}[h_n]$ using SMC methods: ϕ_n in (5) and $\overleftarrow{\mathbf{Q}}_{\phi_k}$ in (8) are replaced by a set of random samples associated with nonnegative importance weights. These particle filters and smoothers approximations combine sequential importance sampling steps to update recursively ϕ_n and importance resampling steps to duplicate or discard particles according to their importance weights.

Sequential Monte Carlo for additive functionals. Let $(\xi_0^\ell)_{\ell=1}^N$ be independent and identically distributed according to the instrumental proposal density ρ_0 on X and define the importance weights:

$$\omega_0^\ell := \frac{\chi(\xi_0^\ell)}{\rho_0(\xi_0^\ell)} \; .$$

For any $f \in \mathsf{F}(\mathsf{X})$,

$$\phi_0^N[f] := \Omega_0^{-1} \sum_{\ell=1}^N \omega_0^\ell f(\xi_0^\ell) \; , \quad \text{where} \quad \Omega_0 := \sum_{\ell=1}^N \omega_0^\ell$$

is a consistent estimator of $\phi_0[f]$, see for instance [8]. Then, for all $k \geqslant 1$, once the observation Y_k is available, the weighted particle sample $\{(\omega_{k-1}^\ell, \xi_{k-1}^\ell)\}_{\ell=1}^N$ is transformed into a new weighted particle sample approximating ϕ_k. This update step is carried through in two steps, *selection* and *mutation*, using the auxiliary sampler introduced in [29]. New indices and particles $\{(I_k^\ell, \xi_k^\ell)\}_{\ell=1}^N$ are simulated independently from the instrumental distribution with density on $\{1, \ldots, N\} \times \mathsf{X}$:

$$\upsilon_k(\ell, x) \propto \omega_{k-1}^\ell \vartheta_{k-1}(\xi_{k-1}^\ell) p_{k-1}(\xi_{k-1}^\ell, x) \; , \tag{9}$$

where ϑ_{k-1} is an adjustment multiplier weight function and p_{k-1} a Markovian transition density. For any $\ell \in \{1, \ldots, N\}$, ξ_k^ℓ is associated with the importance weight defined by:

$$\omega_k^\ell := \frac{\mathsf{r}_{k-1}(\xi_{k-1}^{I_k^\ell}, \xi_k^\ell)}{\vartheta_{k-1}(\xi_{k-1}^{I_k^\ell}) p_{k-1}(\xi_{k-1}^{I_k^\ell}, \xi_k^\ell)} \tag{10}$$

to produce the following approximation of $\phi_k[f]$:

$$\phi_k^N[f] := \Omega_k^{-1} \sum_{\ell=1}^N \omega_k^\ell f(\xi_k^\ell) \; , \quad \text{where} \quad \Omega_k := \sum_{\ell=1}^N \omega_k^\ell \; .$$

For all $k \geqslant 0$ and all $(x, f) \in \mathsf{X} \times \mathsf{F}(\mathsf{X})$, replacing ϕ_k by ϕ_k^N in (4), $\overleftarrow{\mathbf{Q}}_{\phi_k} f(x)$ is approximated by:

$$\overleftarrow{\mathbf{Q}}_{\phi_k^N} f(x) = \sum_{i=1}^N \frac{\omega_k^i \mathsf{r}_k(\xi_k^i, x)}{\sum_{\ell=1}^N \omega_k^\ell \mathsf{r}_k(\xi_k^\ell, x)} f(\xi_k^i) \; . \tag{11}$$

The forward-filtering backward-smoothing (FFBS) algorithm proposed in [9] consists in replacing, in (8), $\overleftarrow{\mathbf{Q}}_{\phi_k}$ by the approximation $\overleftarrow{\mathbf{Q}}_{\phi_k^N}$. Proceeding recursively, this produces a sequence of estimates $(\tilde{\tau}_k^i)_{i=1}^N$ of $(\mathbf{T}_k h_k(\xi_k^i))_{i=1}^N$ for $0 \leqslant k \leqslant n$. Starting with $\tilde{\tau}_0^i = 0$ for all $1 \leqslant i \leqslant N$, this yields for all $0 \leqslant k \leqslant n-1$:

$$\tilde{\tau}_{k+1}^i = \sum_{j=1}^N \frac{\omega_k^j \mathsf{r}_k(\xi_k^j, \xi_{k+1}^i)}{\sum_{\ell=1}^N \omega_k^\ell \mathsf{r}_k(\xi_k^\ell, \xi_{k+1}^i)} \left(\tilde{\tau}_k^j + \tilde{h}_k(\xi_k^j, \xi_{k+1}^i) \right) . \tag{12}$$

Then, at each iteration $0 \leqslant k \leqslant n-1$, $\phi_{0:k+1|k}[h_{k+1}]$ and $\phi_{0:k+1|k+1}[h_{k+1}]$ are approximated by

$$\phi_{0:k+1|k}^{N,\mathrm{FFBS}}[h_{k+1}] := \frac{1}{N} \sum_{i=1}^N \tilde{\tau}_{k+1}^i \quad \text{and} \quad \phi_{0:k+1|k+1}^{N,\mathrm{FFBS}}[h_{k+1}] := \sum_{i=1}^N \frac{\omega_{k+1}^i}{\Omega_{k+1}} \tilde{\tau}_{k+1}^i .$$

The computational complexity of the update (12) grows *quadratically* with the number of particles N. This computational cost can be reduced following [28] by first replacing (12) by the Monte Carlo estimate

$$\tau_{k+1}^i = \frac{1}{\widetilde{N}} \sum_{j=1}^{\widetilde{N}} \left(\tau_k^{J_{k+1}^{(i,j)}} + \tilde{h}_k(\xi_k^{J_{k+1}^{(i,j)}}, \xi_{k+1}^i) \right) , \tag{13}$$

where the sample size $\widetilde{N} \geqslant 1$ is typically small compared to N and $(J_{k+1}^{(i,j)})_{j=1}^{\widetilde{N}}$ are i.i.d. samples in $\{1, \ldots, N\}$ with probabilities proportional to $(\omega_k^\ell \mathsf{r}_k(\xi_k^\ell, \xi_{k+1}^i))_{\ell=1}^N$. In the resulting Particle Rapid Incremental smoother (PaRIS) algorithm, the updated $(\tau_{k+1}^i)_{i=1}^N$, estimates of $\phi_{0:k+1|k}[h_{k+1}] = \pi_{k+1}[\mathbf{T}_{k+1} h_{k+1}]$ and $\phi_{0:k+1|k+1}[h_{k+1}] = \pi_{k+1}[\mathbf{T}_{k+1} h_{k+1}]$ are obtained as:

$$\phi_{0:k+1|k}^{N,\mathrm{PaRIS}}[h_k] := \frac{1}{N} \sum_{i=1}^N \tau_{k+1}^i \quad \text{and} \quad \phi_{0:k+1|k}^{N,\mathrm{PaRIS}}[h_{k+1}] := \sum_{i=1}^N \frac{\omega_{k+1}^i}{\Omega_{k+1}} \tau_{k+1}^i .$$

Acceptance-rejection procedure. The computational complexity of the described approach is still of order N^2 since it requires the normalising constant $\sum_{\ell=1}^N \omega_k^\ell \mathsf{r}_k(\xi_k^\ell, \xi_{k+1}^i)$ to sample $(J_{k+1}^{(i,j)})_{j=1}^{\widetilde{N}}$ for all particle ξ_{k+1}^i, $1 \leqslant i \leqslant N$. A faster algorithm is obtained by applying the accept-reject sampling approach proposed in [11] and illustrated in [16] which presupposes that there exists a constant $\overline{M} > 0$ such that $\mathsf{r}_k(x, x') \leqslant \overline{M}$ for all $(x, x') \in \mathsf{X} \times \mathsf{X}$. Then, in order to sample from $(\omega_k^\ell \mathsf{r}_k(\xi_k^\ell, \xi_{k+1}^i))_{\ell=1}^N$ a candidate $J^* \sim (\omega_k^i)_{i=1}^N$ is accepted with probability:

$$\Upsilon_k^{\overline{M}}(J^*, i) := \mathsf{r}_k(\xi_k^{J^*}, \xi_{k+1}^i)/\overline{M} . \tag{14}$$

This procedure is repeated until acceptance. Under strong mixing assumptions it can be shown, see for instance [11, Proposition 2] and [28, Theorem 10], that the expected number of trials needed for this approach to update $(\tau_k^i)_{i=1}^N$ to $(\tau_{k+1}^i)_{i=1}^N$ is $O(\widetilde{N}N)$.

3 Pseudo marginal Sequential Monte Carlo smoother

In many applications, Sequential Monte Carlo methods cannot be used as the transition densities q_k or g_k, $0 \leqslant k \leqslant n-1$, are unknown. The following crucial steps which rely on r_k are not tractable:

(a) computation of the importance weights ω_k^ℓ in (10) ;

(b) computation of the acceptance ratio (14).

To overcome these issues, following [21], consider the following algorithm.

Initialization. At time $k = 0$, set for all $1 \leqslant \ell \leqslant N$,

$$\widehat{\omega}_0^\ell = \omega_0^\ell \ , \quad \widehat{I}_0^\ell = 0 \quad \text{and} \quad \widehat{\tau}_\ell^0 = \tau_\ell^0 = 0 \ .$$

Propagation. Starting with weighted samples $\{(\xi_k^\ell, \widehat{\omega}_k^\ell)\}_{\ell=1}^N$, define

$$\widetilde{\mathcal{F}}_k^N = \sigma\left\{(\xi_u^\ell, \widehat{\omega}_u^\ell, \widehat{\tau}_u^\ell) \ ; \ 1 \leqslant \ell \leqslant N \ , \ 0 \leqslant u \leqslant k\right\} \quad \text{and} \quad \widetilde{\mathcal{G}}_k^N = \sigma\left\{(\widehat{I}_k^\ell, \xi_k^\ell) \ ; \ 1 \leqslant \ell \leqslant N\right\} \ .$$

New indices and particles $\{(\widehat{I}_{k+1}^\ell, \xi_{k+1}^\ell)\}_{\ell=1}^N$ are simulated independently from the instrumental distribution with density on $\{1, \ldots, N\} \times \mathsf{X}$:

$$\upsilon_{k+1}(\ell, x) \propto \widehat{\omega}_k^\ell \vartheta_k(\xi_k^\ell) p_k(\xi_k^\ell, x) \ , \tag{15}$$

Following [18, 27], weights update can be approximated by replacing $\mathsf{r}_k(\xi_k^\ell, \xi_{k+1}^i)$ by an unbiased estimator.

H1 There exist a Markov kernel R_k on $(\mathsf{X} \times \mathsf{X}, \mathcal{B}(\mathsf{Z}))$ where $(\mathsf{Z}, \mathcal{B}(\mathsf{Z}))$ is a general state space and a positive mapping $\widehat{\mathsf{r}}_k$ on $\mathsf{X} \times \mathsf{X} \times \mathsf{Z}$ such that, for all $(x, x') \in \mathsf{X}^2$,

$$\int \mathsf{R}_k(x, x'; \mathrm{d}z) \widehat{\mathsf{r}}_k(x, x'; z) = \mathsf{r}_k(x, x') \ .$$

Then, under H1, if conditionally on $\widetilde{\mathcal{F}}_k^N \vee \widetilde{\mathcal{G}}_{k+1}^N$, ζ_k^ℓ has distribution $\mathsf{R}_k(\xi_k^{\widehat{I}_{k+1}^\ell}, \xi_{k+1}^\ell; \cdot)$, then

$$\mathbb{E}\left[\widehat{\mathsf{r}}_k(\xi_k^{\widehat{I}_{k+1}^\ell}, \xi_{k+1}^\ell; \zeta_k^\ell) \middle| \widetilde{\mathcal{F}}_k^N \vee \widetilde{\mathcal{G}}_{k+1}^N\right] = \mathsf{r}_k(\xi_k^{\widehat{I}_{k+1}^\ell}, \xi_{k+1}^\ell) \ .$$

The filtering weights then become:

$$\widehat{\omega}_{k+1}^\ell := \frac{\widehat{\mathsf{r}}_k(\xi_k^{\widehat{I}_{k+1}^\ell}, \xi_{k+1}^\ell; \zeta_k^\ell)}{\vartheta_k(\xi_k^{\widehat{I}_{k+1}^\ell}) p_k(\xi_k^{\widehat{I}_{k+1}^\ell}, \xi_{k+1}^\ell)} \ . \tag{16}$$

For all $f \in \mathsf{F}(\mathsf{X})$ and all $0 \leqslant k \leqslant n$, $\phi_k[f]$ is approximated by

$$\widehat{\phi}_k^N[f] := \sum_{i=1}^N \frac{\widehat{\omega}_k^i}{\widehat{\Omega}_k} f(\xi_k^i) \ , \quad \widehat{\Omega}_k = \sum_{i=1}^N \widehat{\omega}_k^i \ .$$

To solve issue (b), [21] ensured that, under several assumptions, the acceptance-rejection mechanism introduced to implement PaRIS algorithm is still valid for stochastic differential equations. Consider the following assumption,

H2 For all $0 \leqslant k \leqslant n$, there exists a random variable $\overline{M}_k$ measurable with respect to $\widetilde{\mathcal{G}}_{k+1}^N$ such that

$$\sup_{x,y,\zeta} \widehat{\mathsf{r}}_k(x,y;\zeta) \leq \overline{M}_k \ .$$

If this assumption holds, the accept-reject mechanism of PaRIS algorithm is replaced by the following steps. For all $1 \leqslant i \leqslant N$ and all $1 \leqslant j \leqslant \widetilde{N}$, a candidate J^* is sampled in $\{1, \ldots, N\}$ with probabilities proportional to $(\widehat{\omega}_k^i)_{i=1}^N$ and is accepted with probability $\widehat{\mathsf{r}}_k(\xi_k^{J^*}, \xi_{k+1}^i; \zeta)/\overline{M}_k$, where ζ has distribution $\mathsf{R}_k(\xi_k^{J^*}, \xi_{k+1}^i; \cdot)$. Then, set

$$\widehat{J}_{k+1}^{(i,j)} = J^*$$

and

$$\widehat{\tau}_{k+1}^i = \frac{1}{\widetilde{N}} \sum_{j=1}^{\widetilde{N}} \left(\widehat{\tau}_k^{\widehat{J}_{k+1}^{(i,j)}} + \tilde{h}_k \left(\xi_k^{\widehat{J}_{k+1}^{(i,j)}}, \xi_{k+1}^i \right) \right) \ . \tag{17}$$

Lemma 3.1. *Assume that H1 and H2 hold. Then, for all $0 \leqslant k \leqslant n-1$ and all $1 \leqslant i \leqslant N$, $(\widehat{J}_{k+1}^{(i,j)})_{1 \leqslant j \leqslant \widetilde{N}}$ are i.i.d. and independent of $\widehat{\omega}_{k+1}^i$ given $\widetilde{\mathcal{F}}_k^N \vee \widetilde{\mathcal{G}}_{k+1}^N$ and such that for all $1 \leqslant \ell \leqslant N$,*

$$\mathbb{P}\left(\widehat{J}_{k+1}^{(i,j)} = \ell \middle| \widetilde{\mathcal{F}}_k^N \vee \widetilde{\mathcal{G}}_{k+1}^N \right) = \frac{\widehat{\omega}_k^\ell \mathsf{r}_k(\xi_k^\ell, \xi_{k+1}^i)}{\sum_{m=1}^N \widehat{\omega}_k^m \mathsf{r}_k(\xi_k^m, \xi_{k+1}^i)} \ ,$$

where $\widehat{\omega}_k^\ell$ is defined by (16)*.*

Proof. The proof follows the same lines as [21, Lemma 1]. □

The proposed algorithm therefore leads to an estimator of the expectation (1) in the general setting of this paper. The following section provides constistency and asymptotic normality results for this estimator.

4 Asymptotic results

4.1 Auxiliary Particle filter based PaRIS algorithms

In [26], the authors established the consistency and asymmptotic normality of PaRIS algorithm for the bootstrap filter, i.e. in the simple case where for all $0 \leqslant k \leqslant n-1$, ϑ_k is the constant function which equals 1 and $p_k = q_k$. This section extends these convergence results to the general auxiliary particle filter based PaRIS algorithm as such filters are required for the pseudo marginal smoother. Consider the following assumptions.

H3 For all $0 \leqslant k \leqslant n-1$, g_k is a positive function such that $\|g_k\|_\infty < \infty$. For all $0 \leqslant k \leqslant n-1$, $\|q_k\|_\infty < \infty$, $\|\vartheta_k\|_\infty < \infty$ and $\|\bar{\omega}_{k+1}\|_\infty < \infty$ where for all $(x,y) \in \mathsf{X} \times \mathsf{X}$,

$$\bar{\omega}_{k+1}(x,y) := \frac{\mathsf{r}_k(x,y)}{\vartheta_k(x) p_k(x,y)} \ . \tag{18}$$

Lemma 4.1. *Assume that H3 holds. Then, for all* $0 \leqslant k \leqslant n$, $(f_k, \widetilde{f}_k) \in \mathsf{F}(\mathsf{X})^2$ *and* $\widetilde{N} \geqslant 0$, *there exist* $(c_k, \widetilde{c}_k) \in (\mathbb{R}_+^\star)^2$ *such that for all* $N \in \mathbb{R}_+^\star$ *and all* $\varepsilon \in \mathbb{R}_+^\star$,

$$\mathbb{P}\left(\left|\sum_{i=1}^N \frac{\omega_k^i}{\Omega_k}\{\tau_k^i f_k(\xi_k^i) + \widetilde{f}_k(\xi_k^i)\} - \phi_k[\mathbf{T}_k h_k f_k + \widetilde{f}_k]\right| > \varepsilon\right) \leqslant c_k \mathrm{e}^{-\widetilde{c}_k N \varepsilon^2} .$$

Proof. The proof follows the same lines as the proof of [26, Theorem 1]. □

Lemma 4.2. *Assume that H3 holds. Then, for all* $0 \leqslant k \leqslant n$, $f_k \in \mathsf{F}(\mathsf{X})$ *and* $\widetilde{N} \geqslant 0$,

$$\sum_{i=1}^N \frac{\omega_k^i}{\Omega_k}(\tau_k^i)^2 f_k(\xi_k^i) \xrightarrow[N\to\infty]{\mathbb{P}} \eta_k[f_k] + \phi_k[\mathbf{T}_k^2 h_k f_k] ,$$

where $\eta_0[f_0] = 0$ *and for all* $0 \leqslant k \leqslant n-1$,

$$\eta_{k+1}[f_{k+1}] = \frac{\eta_k[\mathbf{L}_k f_{k+1}] + \phi_k[\mathbf{L}_k\{\overleftarrow{\mathbf{Q}}_{\phi_k}(\mathbf{T}_k h_k + \widetilde{h}_k - \mathbf{T}_{k+1}h_{k+1})^2 f_{k+1}\}]}{\widetilde{N}\phi_k[\mathbf{L}_k \mathbb{1}]} .$$

Proof. The proof is postponed to Section B.1. □

Following [26, Lemma 13], for all $0 \leqslant k \leqslant n$ and $f_k \in \mathsf{F}(\mathsf{X})$, the recursion given in Lemma 4.2 may also be expressed as

$$\eta_k[f_k] = \sum_{\ell=0}^{k-1} \frac{\phi_\ell[\mathbf{L}_\ell\{\overleftarrow{\mathbf{Q}}_{\phi_\ell}(\mathbf{T}_\ell h_\ell + \widetilde{h}_\ell - \mathbf{T}_{\ell+1}h_{\ell+1})^2 \mathbf{L}_{\ell+1}\ldots\mathbf{L}_{k-1} f_k\}]}{\widetilde{N}^{k-\ell}\phi_\ell[\mathbf{L}_\ell \ldots \mathbf{L}_{k-1}\mathbb{1}]} . \tag{19}$$

Establishing a central limit theorem for PaRIS algorithms requires to introduce the *retro-prospective* kernels, defined, for all $0 \leqslant k \leqslant m \leqslant n$, $x_k \in \mathsf{X}$ and $h \in \mathsf{F}(\mathsf{X}^{m+1})$, by

$$\begin{aligned}
\mathbf{D}_{k,m}h(x_k) &:= \int h(x_{0:m})\mathbf{T}_k(x_k, \mathrm{d}x_{0:k-1})\mathbf{L}_k \ldots \mathbf{L}_{m-1}(x_k, \mathrm{d}x_{k+1:m}) , \\
\widetilde{\mathbf{D}}_{k,m}h(x_k) &:= \mathbf{D}_{k,m}\{h - \phi_{0:m|m}[h]\}(x_k) .
\end{aligned}$$

Proposition 4.1. *Assume that H3 holds. Then, for all* $0 \leqslant k \leqslant n$, $(f_k, \widetilde{f}_k) \in \mathsf{F}(\mathsf{X})^2$,

$$\sqrt{N}\left(\sum_{i=1}^N \frac{\omega_k^i}{\Omega_k}\{\tau_k^i f_k(\xi_k^i) + \widetilde{f}_k(\xi_k^i)\} - \phi_k[\mathbf{T}_k h_k f_k + \widetilde{f}_k)]\right) \xrightarrow[N\to\infty]{\mathcal{D}} \sigma_k\langle f_k; \widetilde{f}_k\rangle Z ,$$

where Z *is a standard Gaussian random variable and for all* $0 \leqslant k \leqslant n-1$,

$$\begin{aligned}
\sigma_k^2\langle f_k; \widetilde{f}_k\rangle = &\sum_{s=0}^{k-1} \frac{\phi_s[\vartheta_s]\phi_s[\mathbf{L}_s\{\bar{\omega}_s \widetilde{\mathbf{D}}_{s+1,k}^2(h_k f_k + \widetilde{f}_k)\}]}{\phi_s[\mathbf{L}_s \ldots \mathbf{L}_{k-1}\mathbb{1}]^2} \\
&+ \sum_{s=0}^{k-1}\sum_{\ell=0}^{k} \frac{\phi_s[\vartheta_s]\phi_\ell[\mathbf{L}_\ell\{\overleftarrow{\mathbf{Q}}_{\phi_\ell}(\mathbf{T}_\ell h_\ell + \widetilde{h}_\ell - \mathbf{T}_{\ell+1}h_{\ell+1})^2 \mathbf{L}_{\ell+1}\ldots\mathbf{L}_s(\overleftarrow{\mathbf{Q}}_{\phi_s}\bar{\omega}_s\{\mathbf{L}_{s+1}\ldots\mathbf{L}_{k-1}f_k\}^2)]}{\widetilde{N}^{s+1-\ell}\phi_\ell[\mathbf{L}_\ell \ldots \mathbf{L}_{s-1}\mathbb{1}]\phi_s[\mathbf{L}_s \ldots \mathbf{L}_{k-1}\mathbb{1}]^2} .
\end{aligned}$$

Proof. The proof is postponed to Section B.2. □

Corollary 4.1. *Assume that H3 holds. Then, for all* $0 \leqslant k \leqslant n$, $(f_k, \widetilde{f}_k) \in \mathsf{F}(\mathsf{X})^2$,

$$\sqrt{N}\left(\sum_{i=1}^{N} \frac{\omega_k^i}{\Omega_k}\tau_k^i - \phi_k[\mathbf{T}_k h_k]\right) \xrightarrow[N\to\infty]{\mathcal{D}} \sigma_k(h_k) Z \ ,$$

where Z *is a standard Gaussian random variable and*

$$\sigma_k^2(h_k) = \sum_{s=0}^{k-1} \frac{\phi_s[\vartheta_s]\phi_s[\mathbf{L}_s\{\bar{\omega}_s \widetilde{\mathbf{D}}_{s+1,k}^2 h_k\}]}{\phi_s[\mathbf{L}_s \dots \mathbf{L}_{k-1}\mathbb{1}]^2}$$
$$+ \sum_{s=0}^{k-1}\sum_{\ell=0}^{k} \frac{\phi_s[\vartheta_s]\phi_\ell[\mathbf{L}_\ell\{\overleftarrow{\mathbf{Q}}_{\phi_\ell}(\mathbf{T}_\ell h_\ell + \widetilde{h}_\ell - \mathbf{T}_{\ell+1}h_{\ell+1})^2 \mathbf{L}_{\ell+1}\dots\mathbf{L}_s(\overleftarrow{\mathbf{Q}}_{\phi_s}\bar{\omega}_s\{\mathbf{L}_{s+1}\dots\mathbf{L}_{k-1}\mathbb{1}\}^2)]}{\widetilde{N}^{s+1-\ell}\phi_\ell[\mathbf{L}_\ell\dots\mathbf{L}_{s-1}\mathbb{1}]\phi_s[\mathbf{L}_s\dots\mathbf{L}_{k-1}\mathbb{1}]^2} \ .$$

4.2 Pseudo marginal PaRIS algorithms

Consider the following assumption.

H4 For all $0 \leqslant k \leqslant n-1$, $\|\widehat{\omega}_{k+1}\|_\infty < \infty$ where for all $(x, y, z) \in \mathsf{X}\times\mathsf{X}\times\mathsf{Z}$,

$$\widehat{\omega}_{k+1}(x, y; z) := \frac{\widehat{\mathsf{r}}_k(x, y; z)}{\vartheta_k(x) p_k(x, y)} \ . \tag{20}$$

Proposition 4.2. *Assume that H1, H2 and H4 hold. Then, for all* $0 \leqslant k \leqslant n$, $(f_k, \widetilde{f}_k) \in \mathsf{F}(\mathsf{X})^2$ *and* $\widetilde{N} \geqslant 0$, *there exist* $(c_k, \widetilde{c}_k) \in (\mathbb{R}_+^\star)^2$ *such that for all* $N \in \mathbb{R}_+^\star$ *and all* $\varepsilon \in \mathbb{R}_+^\star$,

$$\mathbb{P}\left(\left|\sum_{i=1}^{N} \frac{\widehat{\omega}_k^i}{\widehat{\Omega}_k}\{\widehat{\tau}_k^i f_k(\xi_k^i) + \widetilde{f}_k(\xi_k^i)\} - \phi_k[\mathbf{T}_k h_k f_k + \widetilde{f}_k]\right| > \varepsilon\right) \leqslant c_k \mathrm{e}^{-\widetilde{c}_k N\varepsilon^2} \ .$$

Proof. The proof follows the same lines as the proof of [26, Theorem 1]. □

Lemma 4.3. *Assume that H1, H2 and H4 hold. Then, for all* $0 \leqslant k \leqslant n$, $f_k \in \mathsf{F}(\mathsf{X})$ *and* $\widetilde{N} \geqslant 0$,

$$\sum_{i=1}^{N} \frac{\widehat{\omega}_k^i}{\widehat{\Omega}_k}(\widehat{\tau}_k^i)^2 f_k(\xi_k^i) \xrightarrow[N\to\infty]{\mathbb{P}} \eta_k[f_k] + \phi_k[\mathbf{T}_k^2 h_k f_k] \ ,$$

where for all $0 \leqslant k \leqslant n$, $\eta_k[f_k]$ *is defined in* (19).

Proof. The proof is postponed to Section C.1. □

Proposition 4.3. *Assume that H1, H2 and H4 hold. Then, for all* $0 \leqslant k \leqslant n$, $(f_k, \widetilde{f}_k) \in \mathsf{F}(\mathsf{X})^2$,

$$\sqrt{N}\left(\sum_{i=1}^{N} \frac{\widehat{\omega}_k^i}{\widehat{\Omega}_k}\{\widehat{\tau}_k^i f_k(\xi_k^i) + \widetilde{f}_k(\xi_k^i)\} - \phi_k[\mathbf{T}_k h_k f_k + \widetilde{f}_k)]\right) \xrightarrow[N\to\infty]{\mathcal{D}} \bar{\sigma}_k\langle f_k; \widetilde{f}_k\rangle Z \ ,$$

where Z *is a standard Gaussian random variable and for all* $0 \leqslant k \leqslant n-1$, $\bar{\sigma}_{k+1}^2\langle f_{k+1}; \widetilde{f}_{k+1}\rangle$ *can be computed using an explicit recursive formula given in Appendix C.2.*

Proof. The proof is potsponed to Section C.2. □

Corollary 4.2. *Assume that H1, H2 and H4 hold. Then, for all* $0 \leqslant k \leqslant n$,

$$\sqrt{N}\left(\sum_{i=1}^{N} \frac{\widehat{\omega}_k^i}{\widehat{\Omega}_k}\widehat{\tau}_k^i - \phi_k[\mathbf{T}_k h_k]\right) \xrightarrow[N\to\infty]{\mathcal{D}} \bar{\sigma}_k(h_k) Z \ ,$$

where Z *is a standard Gaussian random variable and* $\bar{\sigma}_k^2(h_k)$ *can be computed using an explicit recursive formula given in Appendix C.2.*

5 Tangent filters and online recursive maximum likelihood

Let Θ be a parameter space. This section considers a family of transition kernels $(\mathbf{Q}_{k;\theta})_{\theta\in\Theta;0\leqslant k\leqslant n-1}$ on $\mathsf{X}\times\mathcal{B}(\mathsf{X})$ and $(\mathbf{G}_{k;\theta})_{\theta\in\Theta;1\leqslant k\leqslant n}$ on $\mathsf{X}\times\mathcal{B}(\mathsf{Y})$ associated with densities $q_{k;\theta}$ and $g_{k;\theta}$ with respect to μ and ν. The joint smoothing distributions are then defined, for any $\theta\in\Theta$, $0\leqslant k_1\leqslant k_2\leqslant n$ and any function $h\in\mathsf{F}(\mathsf{X}^{k_2-k_1+1})$, by:

$$\phi_{k_1:k_2;\theta|n}[h] := \mathcal{L}_{n;\theta}^{-1}(Y_{1:n}) \int \chi(\mathrm{d}x_0) \prod_{k=0}^{n-1} \mathbf{Q}_{k;\theta}(x_k, \mathrm{d}x_{k+1}) g_{k+1;\theta}(x_{k+1}, Y_{k+1}) h(x_{k_1:k_2}) \ ,$$

where

$$\mathcal{L}_{n;\theta}(Y_{1:n}) = \int \chi(\mathrm{d}x_0) \prod_{k=0}^{n-1} \mathbf{Q}_{k;\theta}(x_k, \mathrm{d}x_{k+1}) g_{k+1;\theta}(x_{k+1}, Y_{k+1})$$

As noted for instance in [10, Section 2] and [26], for all $\theta\in\Theta$ and all $f_{0:n}\in\mathsf{F}(\mathsf{X}^{n+1})$,

$$\nabla_\theta \phi_{0:n;\theta|n-1}[f_{0:n}] = \phi_{0:n;\theta|n-1}[h_n f_{0:n}] - \phi_{0:n;\theta|n-1}[f_{0:n}] \times \phi_{0:n;\theta|n-1}[h_n] \ ,$$

where

$$h_n(x_{0:n}) = \sum_{k=0}^{n-1} \tilde{h}_{k;\theta}(x_k, x_{k+1}) \ ,$$

with, for all $0\leqslant k\leqslant n-1$,

$$\tilde{h}_{k;\theta}(x_k, x_{k+1}) = \nabla_\theta \log g_{k+1;\theta}(x_{k+1}, Y_{k+1}) + \nabla_\theta \log q_{k;\theta}(x_k, x_{k+1}) \ .$$

Considering an objective function $f_n\in\mathsf{F}(\mathsf{X})$ which depends on the last state x_n only, the tangent filter η_n is defined as the following signed measure:

$$\eta_{n;\theta}[f_n] := \nabla_\theta \pi_{n;\theta}[f_n] = \phi_{0:n;\theta|n-1}[h_{n;\theta} f_n] - \pi_{n;\theta}[f_n] \times \phi_{0:n;\theta|n-1}[h_n] \ ,$$

where $\pi_n = \phi_{n:n|n-1}$ is the predictive measure. The particle based estimator of $\pi_n[f]$ is given by:

$$\pi_n^N[f] = \frac{1}{N}\sum_{\ell=1}^{N} f(\xi_n^\ell) \ .$$

Using the tower property, (4) and the backward decomposition (6):

$$\eta_{n;\theta}[f_n] = \pi_{n;\theta}[(\mathbf{T}_n h_n - \pi_{n;\theta}[\mathbf{T}_n h_n]) f_n] \ . \tag{21}$$

Therefore, the tangent filter (21) can be approximated on-the-fly using the statistics $(\tilde{\tau}_n^i)_{i=1}^N$ and the weighted particles $\{(\xi_n^i, \omega_n^i)\}_{i=1}^N$:

$$\eta_{n;\theta}^{N,\mathrm{FFBS}}[f_n] = \frac{1}{N}\sum_{i=1}^N \tilde{\tau}_n^i f_n(\xi_i^n) - \left(\frac{1}{N}\sum_{i=1}^N \tilde{\tau}_n^i\right)\left(\frac{1}{N}\sum_{i=1}^N f_n(\xi_i^n)\right) . \tag{22}$$

In cases where r_k, $0 \leqslant k \leqslant n-1$, is unknown and replaced by an unbiased estimate, the associated pseudo marginal particle-based approximation of the tangent filter is given by:

$$\widehat{\eta}_{n;\theta}^N[f_n] = \frac{1}{N}\sum_{i=1}^N \widehat{\tau}_n^i f_n(\xi_i^n) - \left(\frac{1}{N}\sum_{i=1}^N \widehat{\tau}_n^i\right)\left(\frac{1}{N}\sum_{i=1}^N f_n(\xi_i^n)\right) . \tag{23}$$

Given a set of observations $Y_{1:n}$, maximum likelihood estimation amounts at obtaining a parameter $\hat{\theta}_n \in \Theta$ such that $\hat{\theta}_n = \arg\max_{\theta\in\Theta} \ell_{\theta;n}(Y_{1:n})$, where $\ell_{\theta;n}(Y_{1:n}) = \log \mathcal{L}_{\theta;n}(Y_{1:n})$ is the logarithm of the likelihood given in (2). There are many different approaches to compute an estimator of $\hat{\theta}_n$, see for instance [4, Chapter 10]. Following [12], under strong mixing assumptions, for all $\theta \in \theta$, the extended process $\{(X_n, Y_n, \pi_n, \eta_n)\}_{n\geqslant 0}$ is an ergodic Markov chain and for all $\theta \in \theta$, the normalized score $\nabla_\theta \ell_\theta(Y_{1:n})/n$ of the observations may be shown to converge where:

$$\frac{1}{n}\nabla_\theta \ell_\theta(Y_{1:n}) = \frac{1}{n}\sum_{k=1}^n \nabla_\theta \ell_\theta(Y_k \mid Y_{1:k-1}) = \frac{1}{n}\sum_{k=0}^n \frac{\pi_{k;\theta}[\nabla_\theta g_{k;\theta}] + \eta_{k;\theta}[g_{k;\theta}]}{\pi_{k;\theta}[g_{k;\theta}]} .$$

Assuming that the observations $Y_{1:n}$ are generated by a model driven by a true parameter $\theta_\star$ for all $\theta \in \theta$ this normalized score converges almost surely to a limiting quantity $\lambda(\theta, \theta_\star)$ such that, under identifiability constraints, $\lambda(\theta_\star, \theta_\star) = 0$. A gradient ascent algorithm cannot be designed as the limiting function $\theta \mapsto \lambda(\theta, \theta_\star)$ is not available explicitly. Solving the equation $\lambda(\theta_\star, \theta_\star) = 0$ may be cast into the framework of *stochastic approximation* to produce parameter estimates using the *Robbins-Monro algorithm*

$$\theta_{n+1} = \theta_n + \gamma_{n+1}\zeta_{n+1} , \quad n \geqslant 0 , \tag{24}$$

where ζ_{n+1} is a noisy observation of $\lambda(\theta_n, \theta_\star)$. Obtaining such an observation is not possible in practice and following [26] this noisy observation is approximated by

$$\zeta_{n+1} := \frac{\zeta_{n+1}^1 + \zeta_{n+1}^2}{\zeta_{n+1}^3} , \tag{25}$$

where

$$\zeta_{n+1}^1 := \pi_{n+1;\theta_n}\left[(\nabla_\theta g_{n+1;\theta})_{|\theta=\theta_n}\right] , \quad \zeta_{n+1}^2 := \eta_{n+1;\theta_n}[g_{n+1;\theta_n}] \quad \text{and} \quad \zeta_{n+1}^3 := \pi_{n+1;\theta_n}[g_{n+1;\theta_n}] . \tag{26}$$

In (26), the measures $\pi_{n+1;\theta_n}$ and $\eta_{n+1;\theta_n}$ depend on *all* the past parameter values. In the case of a finite state space X the algorithm was studied in [24], which also provides assumptions under which the sequence $\{\theta_n\}_{n\geqslant 0}$ converges towards the parameter $\theta_\star$ (see also [34] for refinements). In more general cases, these measures may be estimated online using the pseudo marginal smoother presented in this paper.

6 Application to partially observed SDE

Let $(X_t)_{t\geq 0}$ be defined as a weak solution to the following Stochastic Differential Equation (SDE) in $\mathbb{R}^d$:

$$X_0 = x_0 \quad \text{and} \quad \mathrm{d}X_t = \alpha_\theta(X_t)\mathrm{d}t + \mathrm{d}W_t \ , \tag{27}$$

where $(W_t)_{t\geq 0}$ is a standard Brownian motion, $\alpha_\theta : \mathsf{X} \to \mathsf{X}$ is the drift function . The inference procedure presented in this paper is applied in the case where the solution to (27) is supposed to be partially observed at times $t_0 = 0, \ldots, t_n$, for a given $n \geqslant 1$, through an observation process $(Y_k)_{0\leqslant k\leqslant n}$ taking values in $\mathbb{R}^m$. For all $0 \leqslant k \leqslant n$, the distribution of Y_k given $(X_t)_{t\geqslant 0}$ depends on $X_k = X_{t_k}$ only and has density $g_{k;\theta}$ with respect to ν. The distribution of X_0 has density χ with respect to μ and for all $0 \leqslant k \leqslant n-1$, the conditional distribution of X_{k+1} given $(X_t)_{0\leqslant t\leqslant k}$ has density $q_{k+1;\theta}(X_k, \cdot)$ with respect to μ. This unknown density can be expressed as an expectation of a Brownian Bridge functional [7].

Let $\omega = (\omega_s)_{0\leq s\leq t}$ be the realization of a Brownian Bridge starting at x at time 0 and ending in y at time Δ. The distribution of ω is denoted by $\mathbb{W}_x^{\Delta,y}$. Moreover, suppose that for all $\theta \in \Theta$, α_θ is of a gradient form $\alpha_\theta = \nabla_x A_\theta$ where $A_\theta : \mathsf{X} \to \mathbb{R}$ is a twice continuously differentiable function. Denoting, $\psi_\theta : \ x \mapsto \psi_\theta(x) = (\|\alpha_\theta(x)\|^2 + \Delta A_\theta(x))/2$, by Girsanov theorem, for all $x, y \in \mathbb{R}^d \times \mathbb{R}^d$

$$q_{k+1;\theta}(x,y) = \phi_{\Delta_k}(x-y)\exp\left(A_\theta(y) - A_\theta(x)\right)\mathbb{E}_{\mathbb{W}_x^{\Delta_k,y}}\left[\exp\left(-\int_0^{\Delta_k}\psi_\theta(\omega_s)\mathrm{d}s\right)\right] \ , \tag{28}$$

where $\Delta_k = t_{k+1} - t_k$, for all $a > 0$, ϕ_a is the probability density function of a centered Gaussian random variable with variance a.

The transition density then cannot be computed as it involves an integration over the whole path between x and y. To perform the algorithm proposed in this paper, we therefore have to design a positive an unbiased estimator of $q_{k+1;\theta}(x,y)$. Moreover, maximum likelihood estimation of θ requires an unbiased estimator of $\nabla_\theta \log q_{k+1;\theta}(x,y)$. Such two estimators can be obtained using the General Poisson Estimator (GPE, [18]).

Unbiased GPE estimator for $q_{k+1;\theta}(x,y;\zeta)$. Assume that there exist random variables $\underline{m}_\theta$ and $\overline{m}_\theta$ such that for all $0 \leqslant s \leqslant \Delta_k$, $\underline{m}_\theta \leqslant \psi_\theta(\omega_s) \leqslant \overline{m}_\theta$. Let κ be a random variable taking values in $\mathbb{N}$ with distribution μ, $\omega = (\omega_s)_{0\leq s\leq\Delta_k}$ be the realization of a Brownian Bridge, and $(U_j)_{1\leqslant j\leqslant\kappa}$ be independent uniform random variables on $(0,\Delta_k)$ and $\zeta = (\kappa, \omega, U_1, \ldots, U_\kappa)$. As shown in [18], equation (28) leads to a positive unbiased estimator given by

$$\widehat{q}_{k+1;\theta}(x,y;\zeta) = \phi_{\Delta_k}(x-y)\exp\left(A_\theta(y) - A_\theta(x) - \underline{m}_\theta\Delta_k\right)\prod_{j=1}^{\kappa}\frac{\overline{m}_\theta - \psi_\theta(\omega_{U_j})}{\overline{m}_\theta - \underline{m}_\theta} \ .$$

Unbiased GPE estimator of $\nabla_\theta \log q_{k+1;\theta}(x,y)$. Let's denote $\varphi_\theta : \ x \mapsto \psi_\theta(x) - \underline{m}_\theta$. By (28),

$$\nabla_\theta \log q_{k+1;\theta}(x,y) = \nabla_\theta A_\theta(y) - \nabla_\theta A_\theta(x) - \nabla_\theta\underline{m}_\theta\Delta_k - \frac{\mathbb{E}_{\mathbb{W}_x^{\Delta_k,y}}\left[\left(\int_0^{\Delta_k}\nabla_\theta\varphi_\theta(\omega_s)\mathrm{d}s\right)\exp\left(-\int_0^{\Delta_k}\varphi_\theta(\omega_s)\mathrm{d}s\right)\right]}{\mathbb{E}_{\mathbb{W}_x^{\Delta_k,y}}\left[\exp\left(-\int_0^{\Delta_k}\varphi_\theta(\omega_s)\mathrm{d}s\right)\right]} \ .$$

On the other hand, the diffusion bridge $\mathbb{S}_{\theta,x}^{\Delta_k,y}$ associated with the SDE (27) is absolutely continuous with respect to $\mathbb{W}_x^{\Delta_k,y}$ with Radon-Nikodym derivative given by

$$\frac{\mathrm{d}\mathbb{S}_{\theta,x}^{\Delta_k,y}}{\mathrm{d}\mathbb{W}_x^{\Delta_k,y}}(\omega) = [q_{k+1;\theta}(x,y)]^{-1}\,\phi_{\Delta_k}(x-y)\exp\left(A_\theta(y) - A_\theta(x) - \underline{m}_\theta\Delta_k - \int_0^{\Delta_k}\varphi_\theta(\omega_s)\mathrm{d}s\right) ,$$
$$= \mathbb{E}_{\mathbb{W}_x^{\Delta_k,y}}\left[\exp\left(-\int_0^{\Delta_k}\varphi_\theta(\omega_s)\mathrm{d}s\right)\right]^{-1}\exp\left(-\int_0^{\Delta_k}\varphi_\theta(\omega_s)\mathrm{d}s\right) .$$

This yields

$$\nabla_\theta \log q_{k+1;\theta}(x,y) = (\nabla_\theta A_\theta(y) - \nabla_\theta A_\theta(x) - \nabla_\theta\underline{m}_\theta\Delta_k) - \mathbb{E}_{\mathbb{S}_{\theta,x}^{\Delta_k,y}}\left[\int_0^{\Delta_k}\nabla_\theta\varphi_\theta(\omega_s)\mathrm{d}s\right]$$

and an unbiased estimator of $\nabla_\theta \log q_{k+1;\theta}(x,y)$ is given by

$$\mathsf{l}_{k+1;\theta}(x,y,\mathsf{s}_U^{\theta,x,y,\Delta_k}) = (\nabla_\theta A_\theta(y) - \nabla_\theta A_\theta(x) - \nabla_\theta\underline{m}_\theta\Delta_k) - \Delta_k\nabla_\theta\varphi_\theta(\mathsf{s}_U^{\theta,x,y,\Delta_k}) ,$$

where U is uniform on $(0,1)$ and independent of $\mathsf{s}^{\theta,x,y,\Delta_k} \sim \mathbb{S}_{\theta,x}^{\Delta_k,y}$. In the context of GPE, $\mathsf{s}^{\theta,x,y,\Delta_k}$ can be simulated exactly using exact algorithms for diffusion processes proposed in [1].

Experiments. Online recursive maximum likelihood using pseudo marginal SMC is illustrated when (27) has the specific form:

$$X_0 = x_0 \quad \text{and} \quad \mathrm{d}X_t = \sin(X_t - \theta)\mathrm{d}t + \mathrm{d}W_t , \tag{29}$$

where θ is an unknown parameter ranging between 0 and 2π. For this numerical experiments, we suppose that a realization of (29) is only observed at times $t_k = k$ for $0 \leqslant k \leqslant n$ with $n = 5000$ through a noisy observation process $(Y_k)_{0\leqslant k\leqslant n}$ such for all $0 \leqslant k \leqslant n$,

$$Y_k = X_{t_k} + \varepsilon_k ,$$

where $(\varepsilon_k)_{0\leqslant k\leqslant n}$ are i.i.d. standard Gaussian random variables, independent of $(W_t)_{t\geqslant 0}$. In this case $\alpha_\theta : x \mapsto \sin(x-\theta)$ and

$$\inf_{x\in\mathbb{R}}\,(\alpha_\theta^2(x) + \Delta A_\theta(x))/2 \geqslant -1/2$$

and for all $x \in \mathbb{R}$,

$$0 \leqslant \varphi_\theta(x) = (\alpha_\theta^2(x) + \Delta A_\theta(x))/2 + 1/2 \leqslant 9/8$$

and a GPE estimator of both the transition density and the gradient of its logarithm associated with the SINE model is straightforward to compute.

A simulated data set is displayed in Figure 2, where $\theta_* = \pi/4$. The solution to (29) is sampled at times $(t_k)_{0\leqslant k\leqslant n}$ using the Exact algorithm of [1]. For all $0 \leqslant k \leqslant n-1$, $\widehat{q}_{k,\theta}$ and the GPE unbiased estimator of $\nabla_\theta q_{k,\theta}(x,y)$ are estimated using $M = 30$ independent Monte Carlo replications of the general Poisson estimator. The estimations of θ_* are given for 50 independent runs started at random locations θ_0 with $N = 100$ particles and $\widetilde{N} = 2$ backward samples. Following [21], the

proposal distribution of the particle filter is obtained using an approximation of the fully adapted particle filter where $q_{k,\theta}$ is replaced by the its Euler scheme approximation.

Sensitivity to the starting point $\hat{\theta}_0$. The inference procedure was performed on the same data set from 50 different starting points uniformly chosen in $(0, 2\pi)$. The gradient step size γ_k of equation (24) was chosen constant (and equal to 0.5) for the first 300 time steps, and then decreasing with a rate proportional to $k^{-0.6}$. Results are given Figure 3. There is no sensitivity to the starting point of the algorithm, and after a couple of hundred observations, the estimates all concentrate around the true value. As the gradient step size decreases, the estimates stay around the true value following autocorrelated patterns that are common to all trajectories.

Asymptotic normality. The inference procedure was performed on 50 different data sets simulated with the same θ_*. The 50 estimates were obtained starting from the same starting point (fixed to θ_*, as Figure 3 shows no sensitivity to the starting point). Figure 4 shows the results for the raw and the averaged estimates. The averaged estimates $(\widetilde{\theta}_k)_{k\geqslant 0}$ consist in averaging the values produced by the estimation procedure after a burning phase of n_0 time steps (here $n_0 = 300$ time steps). This procedure allows to obtain an estimator whose convergence rate does not depend on the step sizes chosen by the user, see [30, 23]. For all $0 \leqslant k \leqslant n_0$, $\widetilde{\theta}_k = \widehat{\theta}_k$ and for all $k > n_0$,

$$\widetilde{\theta}_k = \frac{1}{k - n_0} \sum_{j=n_0+1}^{k} \widehat{\theta}_j \ .$$

As expected, the estimated distribution of the final estimates tends to be Gaussian, centered around the true value.

Step size influence. To illustrate the influence of the gradient step sizes, different settings are considered. In each scenario, the sequence $(\gamma_k)_{k\geqslant 0}$ is given by

$$\gamma_k = \gamma_0 \mathbb{1}_{\{k\leq n_0\}} + \frac{\gamma_0}{(k - n_0)^\kappa} \mathbb{1}_{\{k>n_0\}} \ ,$$

where $\gamma_0 = 0.5$. In this experiment $\kappa \in \{0.5, 0.6, 0.7, 0.8, 0.9, 1\}$. The results are shown in Figure 5. As expected, the raw estimator shows different rates of convergence depending on κ, whereas the averaged estimator has the same behavior in all cases.

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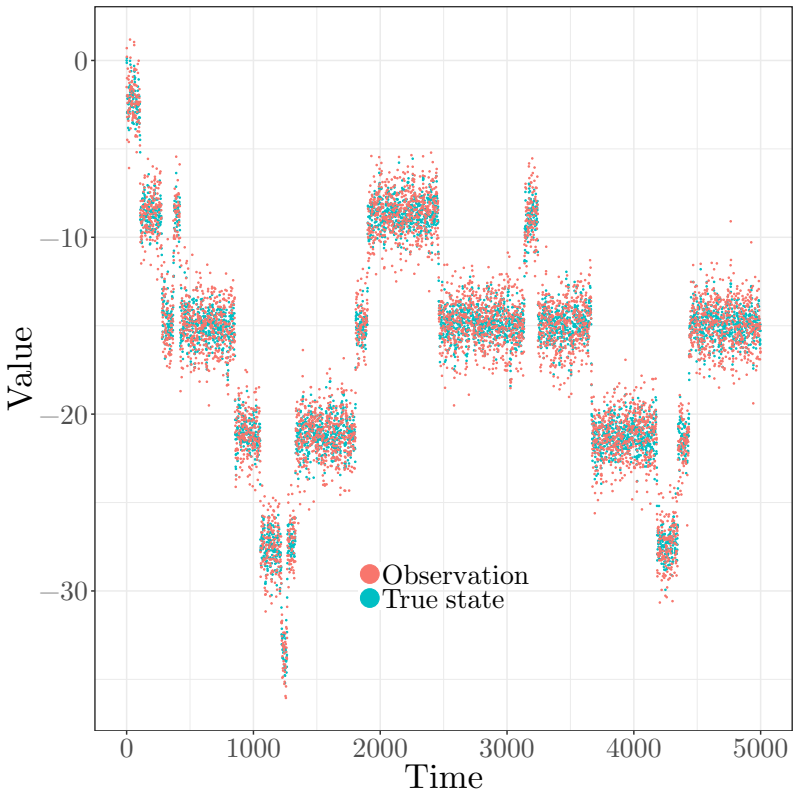

Figure 2: Data set simulated according to the SINE process, observed with noise at discrete time steps.

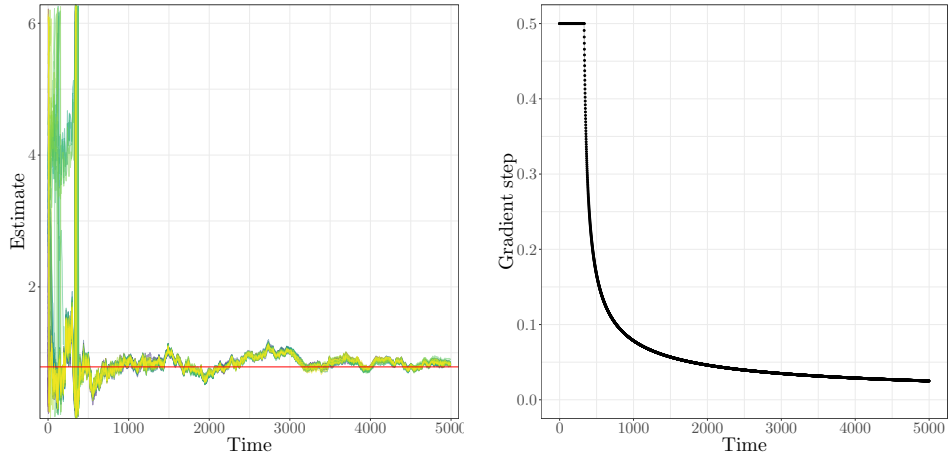

Figure 3: (*Left*) online estimation of θ for the data set presented in Figure 2. The algorithm is performed from 50 starting points. (*Right*) The gradient step sizes (defined in equation (24)).

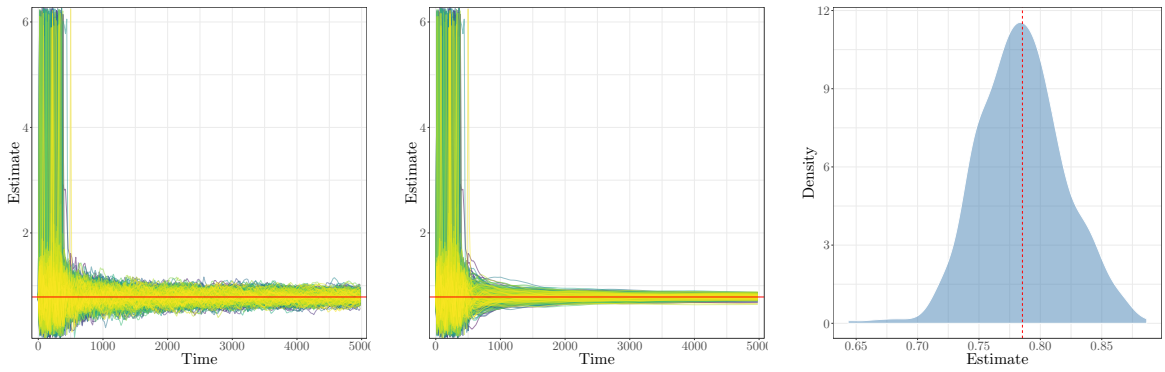

Figure 4: (*Left*) online estimation of θ for 50 different simulated data sets as presented in Figure 2. The algorithm is performed from 1 starting point with the gradient step size shown in Figure 3. (*Center*) Averaged estimator, where $\hat{\theta}$ is averaged after a burning phase of 300 time steps. (*Right*) Empirical distribution of $\hat{\theta}$. The red line is the value of θ^*.

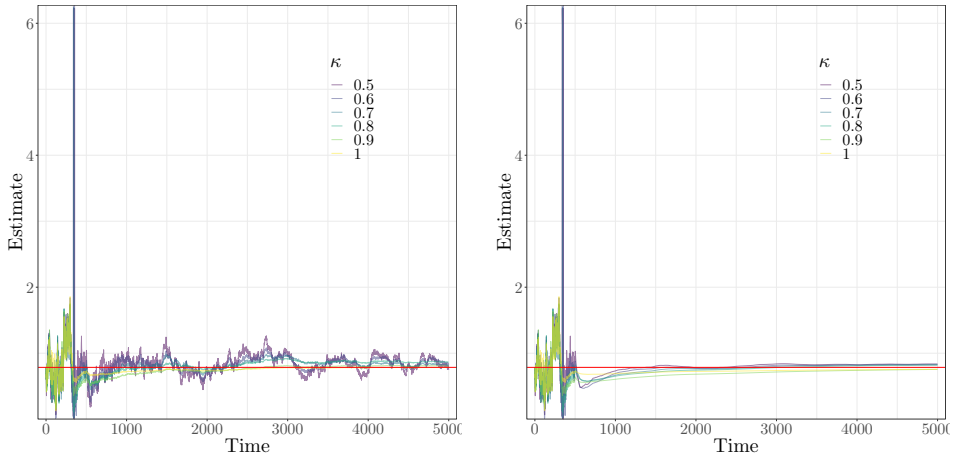

Figure 5: (*Left*) online estimation of θ for the data set presented in Figure 2, with different decreasing rates values κ. (*Right*) Averaged estimator, where $\hat{\theta}$ is averaged after a burning phase of 300 time steps.

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A Additional technical results

The proof of Lemma A.1 is given in [11].

Lemma A.1. *Assume that a_N, b_N, and b are random variables defined on the same probability space such that there exist positive constants β, B, C, and M satisfying*

(i) $|a_N/b_N| \leqslant M$, $\mathbb{P}$*-a.s. and* $b \geqslant \beta$, $\mathbb{P}$*-a.s.,*

(ii) For all $\varepsilon > 0$ *and all* $N \geqslant 1$, $\mathbb{P}\left(|b_N - b| > \varepsilon\right) \leqslant B\mathrm{e}^{-CN\varepsilon^2}$,

(iii) For all $\varepsilon > 0$ *and all* $N \geqslant 1$, $\mathbb{P}\left(|a_N| > \varepsilon\right) \leqslant B\mathrm{e}^{-CN(\varepsilon/M)^2}$.

Then,

$$\mathbb{P}\left(\left|\frac{a_N}{b_N}\right| > \varepsilon\right) \leqslant B\exp\left(-CN\left(\frac{\varepsilon\beta}{2M}\right)^2\right).$$

The proof of Theorem A.1 is given in [13, Theorem A.3].

Theorem A.1. *Let N be a positive integer, $(U_{N,i})_{1\leqslant i\leqslant N}$ be random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\mathcal{F}_{N,i})_{0\leqslant i\leqslant N}$ be a filtration on $(\Omega, \mathcal{F}, \mathbb{P})$. Assume that for all $1 \leqslant i \leqslant N$ the random variable $U_{N,i}$ is such that $\mathbb{E}[U_{N,i}^2|\mathcal{F}_{N,i-1}] < \infty$. Assume also that the two following conditions hold.*

(i) There exists $\sigma^2 > 0$ such that

$$\sum_{i=1}^{N}\left(\mathbb{E}[U_{N,i}^2|\mathcal{F}_{N,i-1}] - \mathbb{E}[U_{N,i}|\mathcal{F}_{N,i-1}]^2\right) \xrightarrow[N\to\infty]{\mathbb{P}} \sigma^2 .$$

(ii) For all $\varepsilon > 0$,

$$\sum_{i=1}^{N}\mathbb{E}[U_{N,i}^2\mathbb{1}_{|U_{N,i}|\geqslant\varepsilon}|\mathcal{F}_{N,i-1}] \xrightarrow[N\to\infty]{\mathbb{P}} 0 .$$

Then, for all $u > 0$,

$$\mathbb{E}\left[\exp\left(iu\sum_{i=1}^{N}\left\{U_{N,i} - \mathbb{E}[U_{N,i}|\mathcal{F}_{N,i-1}]\right\}\right)\middle|\mathcal{F}_{N,0}\right] \xrightarrow[N\to\infty]{\mathbb{P}} \mathrm{e}^{-u^2\sigma^2/2} .$$

The proof of Lemma A.2 follows the same lines as [26, Lemma 14].

Lemma A.2. *Assume that H3 holds. Let $\mathbf{K}$ be a transition kernel on $(\mathsf{X}, \mathcal{B}(\mathsf{X}))$ with transition density $\mathbf{k} \in \mathsf{F}(\mathsf{X}\times\mathsf{X})$ with respect to the reference measure μ. Assume that $(\varphi_N)_{N\geqslant 1}$ is a sequence of functions in $\mathsf{F}(\mathsf{X})$ such that*

i) there exists $\varphi \in \mathsf{F}(\mathsf{X})$ such that for all $x \in \mathsf{X}$, $\varphi_N(x) \xrightarrow[N\to\infty]{\mathbb{P}-\text{a.s.}} \varphi(x)$;

ii) there exists $0 < c_\infty < \infty$ such that for all $N \geqslant 1$, $\|\varphi_N\|_\infty \leqslant c_\infty$.

Then, for all $0 \leqslant k \leqslant n$,

$$\phi_k^N[\mathbf{K}\varphi_N] \xrightarrow[N\to\infty]{\mathbb{P}} \phi_k[\mathbf{K}\varphi] \quad \text{and} \quad \widehat{\phi}_k^N[\mathbf{K}\varphi_N] \xrightarrow[N\to\infty]{\mathbb{P}} \phi_k[\mathbf{K}\varphi] .$$

B Convergence results for PaRIS algorithms

For all $0 \leqslant k \leqslant n$, define the following σ-fields:

$$\mathcal{F}_k^N := \sigma\left\{(\xi_u^\ell, \omega_u^\ell, \tau_u^\ell)\ ;\ 1 \leqslant \ell \leqslant N\ ,\ 0 \leqslant u \leqslant k\right\} \quad \text{and} \quad \mathcal{G}_k^N := \sigma\left\{(\xi_k^\ell, \omega_k^\ell)\ ;\ 1 \leqslant \ell \leqslant N\right\}\ .$$

Lemma B.1. *For all $0 \leqslant k \leqslant n-1$, $(f_{k+1}, \widetilde{f}_{k+1}) \in \mathsf{F}(\mathsf{X})^2$ and $N, \widetilde{N} \geqslant 0$, the random variables $\{\omega_{k+1}^i(\tau_{k+1}^i f_{k+1}(\xi_{k+1}^i) + \widetilde{f}_{k+1}(\xi_{k+1}^i))\}_{i=1}^N$ are i.i.d. conditionally on $\mathcal{F}_k^N$ with*

$$\mathbb{E}\left[\omega_{k+1}^i(\tau_{k+1}^i f_{k+1}(\xi_{k+1}^i) + \widetilde{f}_{k+1}(\xi_{k+1}^i))\middle|\mathcal{F}_k^N\right] = \left(\phi_k^N[\vartheta_k]\right)^{-1} \sum_{\ell=1}^N \frac{\omega_k^\ell}{\Omega_k}\left\{\tau_k^\ell \mathbf{L}_k f_{k+1}(\xi_k^\ell) + \mathbf{L}_k(\widetilde{h}_k f_{k+1} + \widetilde{f}_{k+1})(\xi_k^\ell)\right\}\ .$$

Proof. The proof follows the same lines as [26, Lemma 12]. □

B.1 Proof of Lemma 4.2

Proof. The proof proceeds by induction. The case $k = 0$ is a direct consequence of the fact that $\mathbf{T}_0 h_0 = 0$ and $\tau_0^i = 0$ for all $1 \leqslant i \leqslant N$. Assume that the result holds for some $0 \leqslant k \leqslant n-1$ and write

$$\sum_{i=1}^N \frac{\omega_{k+1}^i}{\Omega_{k+1}}(\tau_{k+1}^i)^2 f_{k+1}(\xi_{k+1}^i) = \frac{a_N}{b_N}\ ,$$

where

$$a_N = \frac{1}{N}\sum_{i=1}^N \omega_{k+1}^i(\tau_{k+1}^i)^2 f_{k+1}(\xi_{k+1}^i) \quad \text{and} \quad b_N = \frac{1}{N}\sum_{i=1}^N \omega_{k+1}^i\ .$$

Then, using that $(\omega_{k+1}^i)_{1\leqslant i\leqslant N}$ are i.i.d. conditionally on $\mathcal{F}_k^N$ and

$$\mathbb{E}\left[\omega_{k+1}^1\middle|\mathcal{F}_k^N\right] = \frac{\phi_k^N[\mathbf{L}_k\mathbb{1}]}{\phi_k^N[\vartheta_k]}\ ,$$

by Hoeffding inequality, since for all $1 \leqslant i \leqslant N$, $0 \leqslant \omega_{k+1}^i \leqslant \|\bar{\omega}_{k+1}\|_\infty$,

$$\mathbb{P}\left(\left|b_N - \frac{\phi_k^N[\mathbf{L}_k\mathbb{1}]}{\phi_k^N[\vartheta_k]}\right| > \varepsilon\right) = \mathbb{E}\left[\mathbb{P}\left(\left|b_N - \frac{\phi_k^N[\mathbf{L}_k\mathbb{1}]}{\phi_k^N[\vartheta_k]}\right| > \varepsilon\middle|\mathcal{F}_k^N\right)\right] \leqslant 2\mathrm{e}^{-2N\varepsilon^2/\|\bar{\omega}_{k+1}\|_\infty^2}\ .$$

Therefore, by Lemma 4.1,

$$b_N \xrightarrow[N\to\infty]{\mathbb{P}-\text{a.s.}} \frac{\phi_k[\mathbf{L}_k\mathbb{1}]}{\phi_k[\vartheta_k]}\ .$$

Since $\phi_k[\mathbf{L}_k\mathbb{1}] > 0$ it remains to establish the convergence in probability of $(a_N)_{N\geqslant 1}$. On the other hand, by Hoeffding inequality, using that for all $1 \leqslant i \leqslant N$, $|\omega_{k+1}^i(\tau_{k+1}^i)^2 f_{k+1}(\xi_{k+1}^i)| \leqslant \|\bar{\omega}_{k+1}\|_\infty \|h_{k+1}\|_\infty^2 \|f_{k+1}\|_\infty$,

$$\mathbb{P}\left(\left|a_N - \mathbb{E}[a_N|\mathcal{F}_k^N]\right| > \varepsilon\right) \leqslant 2\mathrm{e}^{-N\varepsilon^2/(2\|\bar{\omega}_{k+1}\|_\infty \|h_{k+1}\|_\infty^2 \|f_{k+1}\|_\infty)}\ ,$$

and it is enough to obtain the limit of $\mathbb{E}[a_N|\mathcal{F}_k^N]$ as N grows to infinity. Then, write

$$\mathbb{E}[a_N|\mathcal{F}_k^N] = \mathbb{E}\left[\omega_{k+1}^1(\tau_{k+1}^1)^2 f_{k+1}(\xi_{k+1}^1)\middle|\mathcal{F}_k^N\right] = \widetilde{a}_N^1 + \widetilde{a}_N^2 \ ,$$

where

$$\widetilde{a}_N^1 = \widetilde{N}^{-1}\mathbb{E}\left[\omega_{k+1}^1 f_{k+1}(\xi_{k+1}^1)\mathbb{E}\left[\left(\tau_k^{J_{k+1}^{(1,1)}} + \widetilde{h}_k(\xi_k^{J_{k+1}^{(1,1)}}, \xi_{k+1}^1)\right)^2 \middle| \mathcal{F}_k^N \vee \mathcal{G}_{k+1}^N\right]\middle|\mathcal{F}_k^N\right] \ ,$$

$$\widetilde{a}_N^2 = (\widetilde{N}-1)\widetilde{N}^{-1}\mathbb{E}\left[\omega_{k+1}^1 f_{k+1}(\xi_{k+1}^1)\mathbb{E}\left[\tau_k^{J_{k+1}^{(1,1)}} + \widetilde{h}_k(\xi_k^{J_{k+1}^{(1,1)}}, \xi_{k+1}^1)\middle| \mathcal{F}_k^N \vee \mathcal{G}_{k+1}^N\right]^2\middle|\mathcal{F}_k^N\right] \ .$$

The first term is given by

$$\begin{aligned}\widetilde{a}_N^1 = \widetilde{N}^{-1}\sum_{j=1}^N \int \frac{\omega_k^j \vartheta_k(\xi_k^j)p_k(\xi_k^j, x)}{\sum_{m=1}^N \omega_k^m \vartheta_k(\xi_k^m)} \frac{\mathsf{r}_k(\xi_k^j, x)}{\vartheta_k(\xi_k^j)p_k(\xi_k^j, x)} f_{k+1}(x) \\ \times \sum_{\ell=1}^N \frac{\omega_k^\ell \mathsf{r}_k(\xi_k^\ell, x)}{\sum_{m=1}^N \omega_k^m \mathsf{r}_k(\xi_k^m, x)} \left(\tau_k^\ell + \widetilde{h}_k(\xi_k^\ell, x)\right)^2 \mu(\mathrm{d}x) \ , \\ = \widetilde{N}^{-1}(\phi_k^N[\vartheta_k])^{-1}\int f_{k+1}(x)\sum_{\ell=1}^N \frac{\omega_k^\ell}{\Omega_k}\mathsf{r}_k(\xi_k^\ell, x)\left(\tau_k^\ell + \widetilde{h}_k(\xi_k^\ell, x)\right)^2 \mu(\mathrm{d}x) \ .\end{aligned}$$

By the induction hypothesis and Lemma 4.1,

$$\begin{aligned}\widetilde{a}_N^1 \xrightarrow[N\to\infty]{\mathbb{P}} (\widetilde{N}\phi_k[\vartheta_k])^{-1} \\ \times \left\{\eta_k[\mathbf{L}_k f_{k+1}] + \phi_k[\mathbf{T}_k^2 h_k \mathbf{L}_k f_{k+1}] + \phi_k[\mathbf{L}_k(f_{k+1}\widetilde{h}_k^2)] + 2\phi_k[\mathbf{T}_k h_k \mathbf{L}_k(f_{k+1}\widetilde{h}_k)]\right\}\end{aligned}$$

which yields

$$\widetilde{a}_N^1 \xrightarrow[N\to\infty]{\mathbb{P}} (\widetilde{N}\phi_k[\vartheta_k])^{-1}\left\{\eta_k[\mathbf{L}_k f_{k+1}] + \phi_k[\mathbf{L}_k\{(\mathbf{T}_k h_k + \widetilde{h}_k)^2 f_{k+1}\}]\right\} \ .$$

The second term is given by

$$\begin{aligned}\widetilde{a}_N^2 = (\widetilde{N}-1)\widetilde{N}^{-1}\sum_{j=1}^N \int \frac{\omega_k^j \vartheta_k(\xi_k^j)p_k(\xi_k^j, x)}{\sum_{m=1}^N \omega_k^m \vartheta_k(\xi_k^m)} \frac{\mathsf{r}_k(\xi_k^j, x)}{\vartheta_k(\xi_k^j)p_k(\xi_k^j, x)} f_{k+1}(x) \\ \times \left(\sum_{\ell=1}^N \frac{\omega_k^\ell \mathsf{r}_k(\xi_k^\ell, x)}{\sum_{m=1}^N \omega_k^m \mathsf{r}_k(\xi_k^m, x)}\left\{\tau_k^\ell + \widetilde{h}_k(\xi_k^\ell, x)\right\}\right)^2 \mu(\mathrm{d}x) \ , \\ = (\widetilde{N}-1)\widetilde{N}^{-1}(\phi_k^N[\vartheta_k])^{-1}\phi_k^N[\mathbf{L}_k\varphi_N] \ ,\end{aligned}$$

with, for all $x \in \mathsf{X}$,

$$\varphi_N(x) = f_{k+1}(x)\left(\sum_{\ell=1}^N \frac{\omega_k^\ell \mathsf{r}_k(\xi_k^\ell, x)}{\sum_{m=1}^N \omega_k^m \mathsf{r}_k(\xi_k^m, x)}\left\{\tau_k^\ell + \widetilde{h}_k(\xi_k^\ell, x)\right\}\right)^2 \ .$$

For all $x \in \mathsf{X}$, by Lemma 4.1,

$$\varphi_N(x) \overset{\mathbb{P}-\text{a.s.}}{\underset{N\to\infty}{\longrightarrow}} f_{k+1}(x) \left(\frac{\phi_k[\mathbf{T}_k h_k \mathsf{r}_k(\cdot, x) + \mathsf{r}_k(\cdot, x)\widetilde{h}_k(\cdot, x)]}{\phi_k[\mathsf{r}_k(\cdot, x)]} \right)^2 .$$

In addition, for all $x \in \mathsf{X}$, by (8),

$$\frac{\phi_k[\mathbf{T}_k h_k \mathsf{r}_k(\cdot, x) + \mathsf{r}_k(\cdot, x)\widetilde{h}_k(\cdot, x)]}{\phi_k[\mathsf{r}_k(\cdot, x)]} = \overleftarrow{\mathbf{Q}}_{\phi_k} \left(\mathbf{T}_k h_k + \widetilde{h}_k \right)(x) = \mathbf{T}_{k+1} h_{k+1}(x)$$

so that

$$\varphi_N(x) \overset{\mathbb{P}-\text{a.s.}}{\underset{N\to\infty}{\longrightarrow}} f_{k+1}(x) \mathbf{T}_{k+1}^2 h_{k+1}(x) .$$

Therefore, as $\|\varphi_N\|_\infty \leqslant \|f_{k+1}\|_\infty \|h_{k+1}\|_\infty^2$, by the generalized Lebesgue dominated convergence theorem, see Lemma A.2,

$$\widetilde{a}_N^2 \overset{\mathbb{P}}{\underset{N\to\infty}{\longrightarrow}} (\widetilde{N} - 1)\widetilde{N}^{-1} (\phi_k[\vartheta_k])^{-1} \phi_k[\mathbf{L}_k\{f_{k+1}\mathbf{T}_{k+1}^2 h_{k+1}\}] .$$

Using that

$$\frac{\phi_k[\mathbf{L}_k f_{k+1}\mathbf{T}_{k+1}^2 h_{k+1}]}{\phi_k[\mathbf{L}_k \mathbb{1}]} = \phi_{k+1}[f_{k+1}\mathbf{T}_{k+1}^2 h_{k+1}] ,$$

yields

$$\frac{a_N}{b_N} \overset{\mathbb{P}}{\underset{N\to\infty}{\longrightarrow}} \phi_{k+1}[f_{k+1}\mathbf{T}_{k+1}^2 h_{k+1}] + \frac{\eta_k[\mathbf{L}_k f_{k+1}]}{\widetilde{N}\phi_k[\mathbf{L}_k \mathbb{1}]} + \frac{\phi_k[\mathbf{L}_k\{(\mathbf{T}_k h_k + \widetilde{h}_k)^2 f_{k+1}\}] - \phi_k[\mathbf{L}_k f_{k+1}\mathbf{T}_{k+1}^2 h_{k+1}]}{\widetilde{N}\phi_k[\mathbf{L}_k \mathbb{1}]} .$$

The proof is concluded upon noting that

$$\begin{aligned}\phi_k[\mathbf{L}_k\{(\mathbf{T}_k h_k + \widetilde{h}_k)^2 f_{k+1}\}] - \phi_k[\mathbf{L}_k f_{k+1}\mathbf{T}_{k+1}^2 h_{k+1}] \\ = \phi_k[\mathbf{L}_k\{\overleftarrow{\mathbf{Q}}_{\phi_k}(\mathbf{T}_k h_k + \widetilde{h}_k - \mathbf{T}_{k+1} h_{k+1})^2 f_{k+1}\}] .\end{aligned}$$

□

B.2 Proof of Proposition 4.1

Proof. The result is proved by induction on k. It holds for $k = 0$ as for all $1 \leqslant i \leqslant N$, $\tau_0^i = 0$. Assume now that the result holds for some $0 \leqslant k \leqslant n-1$ and that $\phi_{k+1}[\mathbf{T}_{k+1} h_{k+1} f_{k+1} + \widetilde{f}_{k+1}] = 0$. Write

$$\sqrt{N} \sum_{i=1}^N \frac{\omega_{k+1}^i}{\Omega_{k+1}} \{\tau_{k+1}^i f_{k+1}(\xi_{k+1}^i) + \widetilde{f}_{k+1}(\xi_{k+1}^i)\} = \Omega_{k+1}^{-1} \Delta_{k+1}^N ,$$

where $\Delta_{k+1}^N = \sqrt{N} \sum_{i=1}^N \omega_{k+1}^i \{\tau_{k+1}^i f_{k+1}(\xi_{k+1}^i) + \widetilde{f}_{k+1}(\xi_{k+1}^i)\}$ is decomposed as follows

$$\Delta_{k+1}^N = \Delta_{k+1,1}^N + \Delta_{k+1,2}^N ,$$

where

$$\Delta^N_{k+1,1} = \sqrt{N}\sum_{i=1}^N \mathbb{E}\left[\omega^i_{k+1}(\tau^i_{k+1}f_{k+1}(\xi^i_{k+1}) + \widetilde{f}_{k+1}(\xi^i_{k+1}))\middle|\mathcal{F}^N_k\right] ,$$

$$\Delta^N_{k+1,2} = \sqrt{N}\sum_{i=1}^N \left\{\omega^i_{k+1}\left(\tau^i_{k+1}f_{k+1}(\xi^i_{k+1}) + \widetilde{f}_{k+1}(\xi^i_{k+1})\right)\right.$$
$$\left. -\mathbb{E}\left[\omega^i_{k+1}(\tau^i_{k+1}f_{k+1}(\xi^i_{k+1}) + \widetilde{f}_{k+1}(\xi^i_{k+1}))\middle|\mathcal{F}^N_k\right]\right\} .$$

By Lemma B.1,

$$\Omega^{-1}_{k+1}\Delta^N_{k+1,1} = \frac{N}{\Omega_{k+1}}\left(\phi^N_k[\vartheta_k]\right)^{-1}\sqrt{N}\sum_{\ell=1}^N \frac{\omega^\ell_k}{\Omega_k}\left\{\tau^\ell_k \mathbf{L}_k f_{k+1}(\xi^\ell_k) + \mathbf{L}_k(\widetilde{h}_k f_{k+1} + \widetilde{f}_{k+1})(\xi^\ell_k)\right\}$$

As $\phi_{k+1}[\mathbf{T}_{k+1}h_{k+1}f_{k+1} + \widetilde{f}_{k+1}] = 0$,

$$\phi_k[\mathbf{T}_k h_k \mathbf{L}_k f_{k+1} + \mathbf{L}_k(\widetilde{h}_k f_{k+1} + \widetilde{f}_{k+1})] = 0 .$$

Therefore, using the induction hypothesis, Slutsky's lemma and

$$\frac{N}{\Omega_{k+1}}\left(\phi^N_k[\vartheta_k]\right)^{-1} \xrightarrow[N\to\infty]{\mathbb{P}} \left(\phi_k[\mathbf{L}_k\mathbb{1}]\right)^{-1}$$

yields

$$\Omega^{-1}_{k+1}\Delta^N_{k+1,1} \xrightarrow[N\to\infty]{\mathcal{D}} \frac{\sigma_k\langle \mathbf{L}_k f_{k+1}; \mathbf{L}_k(\widetilde{h}_k f_{k+1} + \widetilde{f}_{k+1})\rangle}{\phi_k[\mathbf{L}_k\mathbb{1}]} Z ,$$

where Z is a standard Gaussian random variable. By Lemma B.1,

$$\Omega^{-1}_{k+1}\Delta^N_{k+1,2} = \frac{N}{\Omega_{k+1}}\sum_{i=1}^N \upsilon^i_N ,$$

where for all $1 \leqslant i, j \leqslant N$ and all $x \in \mathsf{X}$,

$$\upsilon^i_N = \frac{1}{\sqrt{N}\widetilde{N}}\sum_{j=1}^{\widetilde{N}} \widetilde{\upsilon}_N(I^i_{k+1}, J^{(i,j)}_{k+1}, \xi^i_{k+1}) ,$$

$$\widetilde{\upsilon}_N(i,j,x) = \frac{\mathsf{r}_k(\xi^i_k, x)}{\vartheta_k(\xi^i_k)p_k(\xi^i_k, x)}\left\{\left(\tau^j_k + \widetilde{h}_k(\xi^j_k, x)\right)f_{k+1}(x) + \widetilde{f}_{k+1}(x)\right\}$$
$$- \left(\phi^N_k[\vartheta_k]\right)^{-1}\sum_{\ell=1}^N \frac{\omega^\ell_k}{\Omega_k}\left\{\tau^\ell_k \mathbf{L}_k f_{k+1}(\xi^\ell_k) + \mathbf{L}_k(\widetilde{h}_k f_{k+1} + \widetilde{f}_{k+1})(\xi^\ell_k)\right\} .$$

First, by Lemma 4.1,

$$\frac{N}{\Omega_{k+1}} \xrightarrow[N\to\infty]{\mathbb{P}} \frac{\phi_k[\vartheta_k]}{\phi_k[\mathbf{L}_k\mathbb{1}]} .$$

The proof is then concluded by applying Slutsky's Lemma and Theorem A.1 to the sequence $(\upsilon_N^i)_{1\leqslant i\leqslant N}$. By construction $\mathbb{E}[\upsilon_N^i|\mathcal{F}_k^N]=0$ so that the proof of (i) is based on

$$\sum_{i=1}^N \mathbb{E}[(\upsilon_N^i)^2|\mathcal{F}_k^N] = \widetilde{N}^{-1}\mathbb{E}[\mathbb{E}[(\widetilde{\upsilon}_N(I_{k+1}^1, J_{k+1}^{(1,1)}, \xi_{k+1}^1))^2|\mathcal{F}_k^N \vee \mathcal{G}_{k+1}^N|\mathcal{F}_k^N] + (\widetilde{N}-1)\widetilde{N}^{-1}\mathbb{E}[\mathbb{E}[\widetilde{\upsilon}_N(I_{k+1}^1, J_{k+1}^{(1,1)}, \xi_{k+1}^1)|\mathcal{F}_k^N \vee \mathcal{G}_{k+1}^N]^2|\mathcal{F}_k^N] \,. \quad (30)$$

The first term of (30) is given by

$$\begin{aligned}
&\mathbb{E}\left[\mathbb{E}\left[\widetilde{\upsilon}_N^2(I_{k+1}^1, J_{k+1}^{(1,1)}, \xi_{k+1}^1)\middle|\mathcal{F}_k^N \vee \mathcal{G}_{k+1}^N\right]\middle|\mathcal{F}_k^N\right] \\
&\quad = \mathbb{E}\left[\sum_{\ell=1}^N \frac{\omega_k^\ell \mathsf{r}_k(\xi_k^\ell, \xi_{k+1}^1)}{\sum_{m=1}^N \omega_k^m \mathsf{r}_k(\xi_k^m, \xi_{k+1}^1)} \widetilde{\upsilon}_N^2(I_{k+1}^1, \ell, \xi_{k+1}^1)\middle|\mathcal{F}_k^N\right] \,, \\
&\quad = \sum_{j=1}^N \int \frac{\omega_k^j \vartheta_k(\xi_k^j) p_k(\xi_k^j, x)}{\sum_{m=1}^N \omega_k^m \vartheta_k(\xi_k^m)} \sum_{\ell=1}^N \frac{\omega_k^\ell \mathsf{r}_k(\xi_k^\ell, x)}{\sum_{m=1}^N \omega_k^m \mathsf{r}_k(\xi_k^m, x)} \widetilde{\upsilon}_N(j,\ell,x)^2 \mu(\mathrm{d}x) \,, \\
&\quad = \left(\phi_k^N[\vartheta_k]\right)^{-1} \sum_{j=1}^N \int \frac{\omega_k^j}{\Omega_k} \vartheta_k(\xi_k^j) p_k(\xi_k^j, x) \sum_{\ell=1}^N \frac{\omega_k^\ell \mathsf{r}_k(\xi_k^\ell, x)}{\sum_{m=1}^N \omega_k^m \mathsf{r}_k(\xi_k^m, x)} \widetilde{\upsilon}_N(j,\ell,x)^2 \mu(\mathrm{d}x) \,, \\
&\quad = \left(\phi_k^N[\vartheta_k]\right)^{-1} \int A_N(x) B_N(x) \mu(\mathrm{d}x) \\
&\qquad - \left(\phi_k^N[\vartheta_k]\right)^{-2} \left(\sum_{\ell=1}^N \frac{\omega_k^\ell}{\Omega_k} \left\{\tau_k^\ell \mathbf{L}_k f_{k+1}(\xi_k^\ell) + \mathbf{L}_k(\widetilde{h}_k f_{k+1} + \widetilde{f}_{k+1})(\xi_k^\ell)\right\}\right)^2 \,,
\end{aligned}$$

where

$$\begin{aligned}
A_N(x) &= \sum_{j=1}^N \frac{\omega_k^j}{\Omega_k} \vartheta_k(\xi_k^j) p_k(\xi_k^j, x) \left(\frac{\mathsf{r}_k(\xi_k^j, x)}{\vartheta_k(\xi_k^j) p_k(\xi_k^j, x)}\right)^2 = \sum_{j=1}^N \frac{\omega_k^j}{\Omega_k} \mathsf{r}_k(\xi_k^j, x) \bar{\omega}(\xi_k^j, x) \,, \\
B_N(x) &= \sum_{\ell=1}^N \frac{\omega_k^\ell \mathsf{r}_k(\xi_k^\ell, x)}{\sum_{m=1}^N \omega_k^m \mathsf{r}_k(\xi_k^m, x)} \left\{\left(\tau_k^\ell + \widetilde{h}_k(\xi_k^\ell, x)\right) f_{k+1}(x) + \widetilde{f}_{k+1}(x)\right\}^2 \,.
\end{aligned}$$

By Lemma 4.1,

$$\begin{aligned}
&\left(\phi_k^N[\vartheta_k]\right)^{-2} \left(\sum_{\ell=1}^N \frac{\omega_k^\ell}{\Omega_k} \left\{\tau_k^\ell \mathbf{L}_k f_{k+1}(\xi_k^\ell) + \mathbf{L}_k(\widetilde{h}_k f_{k+1} + \widetilde{f}_{k+1})(\xi_k^\ell)\right\}\right)^2 \\
&\qquad \xrightarrow[N\to\infty]{\mathbb{P}} \left(\phi_k[\vartheta_k]\right)^{-2} \left(\phi_k[\mathbf{T}_k h_k \mathbf{L}_k f_{k+1} + \mathbf{L}_k(\widetilde{h}_k f_{k+1} + \widetilde{f}_{k+1})]\right)^2 = 0 \,,
\end{aligned}$$

since by assumption and [26, Lemma 11], $\phi_k[\mathbf{T}_k h_k \mathbf{L}_k f_{k+1} + \mathbf{L}_k(\widetilde{h}_k f_{k+1} + \widetilde{f}_{k+1})] = 0$. Then, write

$$\int A_N(x) B_N(x) \mu(\mathrm{d}x) = \widetilde{a}_N^1 + \widetilde{a}_N^2 + \widetilde{a}_N^3 \,,$$

where

$$\varphi_N : x \mapsto \left\{ \sum_{j=1}^N \frac{\omega_k^j}{\Omega_k} \mathsf{r}_k(\xi_k^j, x) \bar{\omega}_k(\xi_k^j, x) \right\} \left(\sum_{m=1}^N \frac{\omega_k^m}{\Omega_k} \mathsf{r}_k(\xi_k^m, x) \right)^{-1} = \overleftarrow{\mathbf{Q}}_{\phi_k^N} \bar{\omega}_k(x) \; ,$$

$$\widetilde{a}_N^1 = \sum_{\ell=1}^N \frac{\omega_k^\ell}{\Omega_k} (\tau_k^\ell)^2 \int \mathsf{r}_k(\xi_k^\ell, x) f_{k+1}^2(x) \varphi_N(x) \mu(\mathrm{d}x) \; ,$$

$$\widetilde{a}_N^2 = \sum_{j=1}^N \frac{\omega_k^j}{\Omega_k} \int \mathsf{r}_k(\xi_k^j, x) \bar{\omega}_k(\xi_k^j, x) \sum_{\ell=1}^N \frac{\omega_k^\ell \mathsf{r}_k(\xi_k^\ell, x)}{\sum_{m=1}^N \omega_k^m \mathsf{r}_k(\xi_k^m, x)} \left(\widetilde{h}_k(\xi_k^\ell, x) f_{k+1}(x) + \widetilde{f}_{k+1}(x) \right)^2 \mu(\mathrm{d}x) \; ,$$

$$\widetilde{a}_N^3 = 2 \sum_{\ell=1}^N \frac{\omega_k^\ell}{\Omega_k} \tau_k^\ell \int \mathsf{r}_k(\xi_k^\ell, x) \varphi_N(x) f_{k+1}(x) \left(\widetilde{h}_k(\xi_k^\ell, x) f_{k+1}(x) + \widetilde{f}_{k+1}(x) \right) \mu(\mathrm{d}x) \; .$$

By Lemma 4.1, for all $x \in \mathsf{X}$, $|\varphi_N(x) - \overleftarrow{\mathbf{Q}}_{\phi_k} \bar{\omega}_k(x)| \overset{\mathbb{P}-\text{a.s.}}{\underset{N\to\infty}{\longrightarrow}} 0$, and note that

$$\left| \widetilde{a}_N^1 - \sum_{\ell=1}^N \frac{\omega_k^\ell}{\Omega_k} (\tau_k^\ell)^2 \int \mathsf{r}_k(\xi_k^\ell, x) f_{k+1}^2(x) \overleftarrow{\mathbf{Q}}_{\phi_k} \bar{\omega}_k(x) \mu(\mathrm{d}x) \right|$$
$$\leqslant \|h_k\|_\infty^2 \sum_{\ell=1}^N \frac{\omega_k^\ell}{\Omega_k} \int \mathsf{r}_k(\xi_k^\ell, x) f_{k+1}^2(x) \left| \varphi_N(x) - \overleftarrow{\mathbf{Q}}_{\phi_k} \bar{\omega}_k(x) \right| \mu(\mathrm{d}x) \; .$$

Since $\|f_{k+1}^2 |\varphi_N - \overleftarrow{\mathbf{Q}}_{\phi_k} \bar{\omega}_k|\|_\infty \leqslant 2\|\bar{\omega}_k\|_\infty \|f_{k+1}^2\|_\infty < \infty$, by the generalized Lebesgue dominated convergence theorem, see also Lemma A.2,

$$\left| \widetilde{a}_N^1 - \sum_{\ell=1}^N \frac{\omega_k^\ell}{\Omega_k} (\tau_k^\ell)^2 \int \mathsf{r}_k(\xi_k^\ell, x) f_{k+1}^2(x) \overleftarrow{\mathbf{Q}}_{\phi_k} \bar{\omega}_k(x) \mu(\mathrm{d}x) \right| \overset{\mathbb{P}}{\underset{N\to\infty}{\longrightarrow}} 0$$

and by Lemma 4.2,

$$\widetilde{a}_N^1 \overset{\mathbb{P}}{\underset{N\to\infty}{\longrightarrow}} \eta_k[\mathbf{L}_k\{f_{k+1}^2 \overleftarrow{\mathbf{Q}}_{\phi_k} \bar{\omega}_k\}] + \phi_k[\mathbf{T}_k^2 h_k \mathbf{L}_k\{f_{k+1}^2 \overleftarrow{\mathbf{Q}}_{\phi_k} \bar{\omega}_k\}] \; .$$

On the other hand, by Lemma A.2 applied to

$$\psi_N : x \mapsto \sum_{\ell=1}^N \frac{\omega_k^\ell \mathsf{r}_k(\xi_k^\ell, x)}{\sum_{m=1}^N \omega_k^m \mathsf{r}_k(\xi_k^m, x)} \left(\widetilde{h}_k(\xi_k^\ell, x) f_{k+1}(x) + \widetilde{f}_{k+1}(x) \right)^2$$

which is such that $\|\psi_N\|_\infty \leqslant 2(\|\widetilde{h}_k f_{k+1}\|_\infty^2 + \|\widetilde{f}_{k+1}\|_\infty)$,

$$\widetilde{a}_N^2 \overset{\mathbb{P}}{\underset{N\to\infty}{\longrightarrow}} \int \phi_k \left[\mathsf{r}_k(\cdot, x) \bar{\omega}(\cdot, x) \right] \phi_k[\mathsf{r}_k(\cdot, x)(\widetilde{h}_k(\cdot, x) f_{k+1}(x) + \widetilde{f}_{k+1}(x))^2] (\phi_k[\mathsf{r}_k(\cdot, x)])^{-1} \mu(\mathrm{d}x) \; ,$$

which yields

$$\widetilde{a}_N^2 \overset{\mathbb{P}}{\underset{N\to\infty}{\longrightarrow}} \phi_k[\mathbf{L}_k\{(\overleftarrow{\mathbf{Q}}_{\phi_k} \bar{\omega}_k)(\widetilde{h}_k f_{k+1} + \widetilde{f}_{k+1})^2\}] \; .$$

Finally,

$$\left|\widetilde{a}_N^3 - 2\sum_{\ell=1}^N \frac{\omega_k^\ell}{\Omega_k}\tau_k^\ell \int \mathsf{r}_k(\xi_k^\ell, x) f_{k+1}(x) \overleftarrow{\mathbf{Q}}_{\phi_k}\bar{\omega}_k(x) \left(\widetilde{h}_k(\xi_k^\ell, x) f_{k+1}(x) + \widetilde{f}_{k+1}(x)\right) \mu(\mathrm{d}x)\right|$$

$$\leqslant \|h_k\|_\infty (\|f_{k+1}\|_\infty \|\widetilde{h}_k\|_\infty + \|\widetilde{f}_{k+1}\|_\infty) \sum_{\ell=1}^N \frac{\omega_k^\ell}{\Omega_k} \int \mathsf{r}_k(\xi_k^\ell, x) f_{k+1}(x) \left|\varphi_N(x) - \overleftarrow{\mathbf{Q}}_{\phi_k}\bar{\omega}_k(x)\right| \mu(\mathrm{d}x) \,,$$

so that using again Lemma A.2 and Lemma 4.1,

$$\widetilde{a}_N^3 \xrightarrow[N\to\infty]{\mathbb{P}} 2\phi_k \left[\mathbf{T}_k h_k \mathbf{L}_k \left\{(\overleftarrow{\mathbf{Q}}_{\phi_k}\bar{\omega}_k) f_{k+1} \left(\widetilde{h}_k f_{k+1} + \widetilde{f}_{k+1}\right)\right\}\right] \,.$$

Therefore, the first term of (30) satisfies

$$\widetilde{N}^{-1}\mathbb{E}[\mathbb{E}[(\widetilde{\upsilon}_N(I_{k+1}^1, J_{k+1}^{(1,1)}, \xi_{k+1}^1))^2 | \mathcal{F}_k^N \vee \mathcal{G}_{k+1}^N | \mathcal{F}_k^N]$$

$$\xrightarrow[N\to\infty]{\mathbb{P}} \frac{\eta_k[\mathbf{L}_k\{f_{k+1}^2 \overleftarrow{\mathbf{Q}}_{\phi_k}\bar{\omega}_k\}]}{\widetilde{N}\phi_k[\vartheta_k]} + \frac{\phi_k[\mathbf{L}_k\{(\overleftarrow{\mathbf{Q}}_{\phi_k}\bar{\omega}_k)\left[(\mathbf{T}_k h_k + \widetilde{h}_k) f_{k+1} + \widetilde{f}_{k+1}\right]^2\}]}{\widetilde{N}\phi_k[\vartheta_k]} \,,$$

which concludes the proof for the first term of (30). The second term of (30) is given by

$$\begin{aligned}
&\mathbb{E}[\mathbb{E}[\widetilde{\upsilon}_N(I_{k+1}^1, J_{k+1}^{(1,1)}, \xi_{k+1}^1) | \mathcal{F}_k^N \vee \mathcal{G}_{k+1}^N]^2 | \mathcal{F}_k^N] \\
&\quad = \mathbb{E}\left[\left(\sum_{\ell=1}^N \frac{\omega_k^\ell \mathsf{r}_k(\xi_k^\ell, \xi_{k+1}^1)}{\sum_{m=1}^N \omega_k^m \mathsf{r}_k(\xi_k^m, \xi_{k+1}^1)} \widetilde{\upsilon}_N(I_{k+1}^1, \ell, \xi_{k+1}^1)\right)^2 \middle| \mathcal{F}_k^N\right] \,, \\
&\quad = \sum_{j=1}^N \int \frac{\omega_k^j \vartheta_k(\xi_k^j) p_k(\xi_k^j, x)}{\sum_{m=1}^N \omega_k^m \vartheta_k(\xi_k^m)} \left(\sum_{\ell=1}^N \frac{\omega_k^\ell \mathsf{r}_k(\xi_k^\ell, x)}{\sum_{m=1}^N \omega_k^m \mathsf{r}_k(\xi_k^m, x)} \widetilde{\upsilon}_N(j, \ell, x)\right)^2 \mu(\mathrm{d}x) \,, \\
&\quad = \left(\phi_k^N[\vartheta_k]\right)^{-1} \phi_k^N[\vartheta_k \varphi_k^N] \,,
\end{aligned}$$

where, for all $x \in \mathsf{X}$,

$$\varphi_k^N(x) = \int p_k(x,z) \left(\frac{\mathsf{r}_k(x,z)}{\vartheta_k(x) p_k(x,z)} f_{k+1}(z) \sum_{\ell=1}^N \frac{\omega_k^\ell \mathsf{r}_k(\xi_k^\ell, z)}{\sum_{m=1}^N \omega_k^m \mathsf{r}_k(\xi_k^m, z)} \left(\tau_k^\ell + \widetilde{h}_k(\xi_k^\ell, z)\right)\right.$$

$$\left. + \frac{\mathsf{r}_k(x,z)}{\vartheta_k(x) p_k(x,z)} \widetilde{f}_{k+1}(z) - \left(\phi_k^N[\vartheta_k]\right)^{-1} \sum_{\ell=1}^N \frac{\omega_k^\ell}{\Omega_k} \left\{\tau_k^\ell \mathbf{L}_k f_{k+1}(\xi_k^\ell) + \mathbf{L}_k(\widetilde{h}_k f_{k+1} + \widetilde{f}_{k+1})(\xi_k^\ell)\right\}\right)^2 \mu(\mathrm{d}z) \,.$$

By assumption and [26, Lemma 11], $\phi_k[\mathbf{T}_k h_k \mathbf{L}_k f_{k+1} + \mathbf{L}_k(\widetilde{h}_k f_{k+1} + \widetilde{f}_{k+1})] = 0$ so that by Lemma A.2,

$$\phi_k^N[\vartheta_k \varphi_k^N] \xrightarrow[N\to\infty]{\mathbb{P}} \phi_k[\vartheta_k \varphi_k] \,,$$

where

$$\begin{aligned}
\varphi_k(x) &= \int p_k(x,z) \left(\frac{\mathsf{r}_k(x,z)}{\vartheta_k(x) p_k(x,z)}\right)^2 \left(f_{k+1}(z) \overleftarrow{\mathbf{Q}}_{\phi_k}(\mathbf{T}_k h_k + \widetilde{h}_k)(z) + \widetilde{f}_{k+1}(z)\right)^2 \mu(\mathrm{d}z) \,, \\
&= \int p_k(x,z) \left(\frac{\mathsf{r}_k(x,z)}{\vartheta_k(x) p_k(x,z)}\right)^2 \left(f_{k+1}(z) \mathbf{T}_{k+1} h_{k+1}(z) + \widetilde{f}_{k+1}(z)\right)^2 \mu(\mathrm{d}z) \,.
\end{aligned}$$

Therefore,

$$\mathbb{E}[\mathbb{E}[\widetilde{v}_N(I_{k+1}^1, J_{k+1}^{(1,1)}, \xi_{k+1}^1)|\mathcal{F}_k^N \vee \mathcal{G}_{k+1}^N]^2|\mathcal{F}_k^N] \\ \xrightarrow[N\to\infty]{\mathbb{P}} \left(1 - \frac{1}{\widetilde{N}}\right)(\phi_k[\vartheta_k])^{-1}\phi_k\left[\mathbf{L}_k\left\{\bar{\omega}_k\left(f_{k+1}\mathbf{T}_{k+1}h_{k+1} + \bar{f}_{k+1}\right)^2\right\}\right] .$$

The proof of (ii) is an immediate consequence of H3 since for all $1 \leqslant i \leqslant N$,

$$v_N^i \leqslant 2\|\bar{\omega}_{k+1}\|_\infty \left(\|h_{k+1}\|_\infty \|\widetilde{f}_{k+1}\|_\infty + \|\widetilde{f}_{k+1}\|_\infty\right) N^{-1/2} .$$

Then, defining $c_k = 2\|\bar{\omega}_{k+1}\|_\infty(\|h_{k+1}\|_\infty\|\widetilde{f}_{k+1}\|_\infty + \|\widetilde{f}_{k+1}\|_\infty)$, for all $\varepsilon > 0$,

$$\sum_{i=1}^N \mathbb{E}[(v_N^i)^2 \mathbb{1}_{|v_N^i|\geqslant\varepsilon}|\mathcal{F}_k^N] \leqslant c_k^2 \mathbb{1}_{c_k \geqslant \varepsilon\sqrt{N}} \xrightarrow[N\to\infty]{\mathbb{P}} 0 .$$

Writing

$$\bar{f}_{k+1} = \widetilde{f}_{k+1} - \phi_{k+1}[\mathbf{T}_{k+1}h_{k+1}f_{k+1} + \widetilde{f}_{k+1})],$$

yields

$$\begin{aligned}\sigma_{k+1}^2\langle f_{k+1}; \widetilde{f}_{k+1}\rangle = {} & \frac{\sigma_k^2\langle \mathbf{L}_k f_{k+1}; \mathbf{L}_k\{\widetilde{h}_k f_{k+1} + \bar{f}_{k+1}\}\rangle}{\phi_k[\mathbf{L}_k\mathbb{1}]^2} + \frac{\phi_k[\vartheta_k]\eta_k[\mathbf{L}_k\{f_{k+1}^2 \overleftarrow{\mathbf{Q}}_{\phi_k}\bar{\omega}_k\}]}{\widetilde{N}\phi_k[\mathbf{L}_k\mathbb{1}]^2} \\ & + \frac{\phi_k[\vartheta_k]\phi_k\left[\mathbf{L}_k\left\{\bar{\omega}_k\left(f_{k+1}\mathbf{T}_{k+1}h_{k+1} + \bar{f}_{k+1}\right)^2\right\}\right]}{\phi_k[\mathbf{L}_k\mathbb{1}]^2} \\ & + \frac{\phi_k[\vartheta_k]\phi_k\left[\mathbf{L}_k\{\bar{\omega}_k f_{k+1}^2 \overleftarrow{\mathbf{Q}}_{\phi_k}\left(\mathbf{T}_k h_k + \widetilde{h}_k - \mathbf{T}_{k+1}h_{k+1}\right)^2\}\right]}{\widetilde{N}\phi_k[\mathbf{L}_k\mathbb{1}]^2}\end{aligned}$$

and then, by (19),

$$\begin{aligned}\sigma_{k+1}^2\langle f_{k+1}; \widetilde{f}_{k+1}\rangle = {} & \frac{\sigma_k^2\langle \mathbf{L}_k f_{k+1}; \mathbf{L}_k\{\widetilde{h}_k f_{k+1} + \bar{f}_{k+1}\}\rangle}{\phi_k[\mathbf{L}_k\mathbb{1}]^2} \\ & + \frac{\phi_k[\vartheta_k]\phi_k\left[\mathbf{L}_k\left\{\bar{\omega}_k\left(f_{k+1}\mathbf{T}_{k+1}h_{k+1} + \bar{f}_{k+1}\right)^2\right\}\right]}{\phi_k[\mathbf{L}_k\mathbb{1}]^2} \\ & + \frac{\phi_k[\vartheta_k]}{\phi_k[\mathbf{L}_k\mathbb{1}]^2}\sum_{\ell=0}^k \frac{\phi_\ell[\mathbf{L}_\ell\{\overleftarrow{\mathbf{Q}}_{\phi_\ell}(\mathbf{T}_\ell h_\ell + \widetilde{h}_\ell - \mathbf{T}_{\ell+1}h_{\ell+1})^2\mathbf{L}_{\ell+1}\ldots\mathbf{L}_k\{f_{k+1}^2\overleftarrow{\mathbf{Q}}_{\phi_k}\bar{\omega}_k\}\}]}{\widetilde{N}^{k+1-\ell}\phi_\ell[\mathbf{L}_\ell\ldots\mathbf{L}_{k-1}\mathbb{1}]} .\end{aligned}$$

By definition of the kernel $\widetilde{\mathbf{D}}_{k+1,k+1}$,

$$\phi_k\left[\mathbf{L}_k\bar{\omega}_k\left(f_{k+1}\mathbf{T}_{k+1}h_{k+1} + \bar{f}_{k+1}\right)^2\right] = \phi_k\left[\mathbf{L}_k\bar{\omega}_k\widetilde{\mathbf{D}}_{k+1,k+1}^2\left\{h_{k+1}f_{k+1} + \widetilde{f}_{k+1}\right\}\right] .$$

It remains to prove the explicit expression of $\sigma_{k+1}^2\langle f_{k+1}; \widetilde{f}_{k+1}\rangle$ from this recursion formula. First, following the proof of [26, Theorem 3], for all $0 \leqslant s < k$,

$$\widetilde{\mathbf{D}}_{s+1,k}\left(\mathbf{L}_k f_{k+1} + \mathbf{L}_k\{\widetilde{h}_k f_{k+1} + \bar{f}_{k+1}\right) = \widetilde{\mathbf{D}}_{s+1,k+1}\left(h_{k+1}f_{k+1} + \widetilde{f}_{k+1}\right) .$$

In addition, $0 \leqslant s < k$,

$$\phi_k[\mathbf{L}_k \mathbb{1}] = \frac{\phi_s[\mathbf{L}_s \dots \mathbf{L}_k \mathbb{1}]}{\phi_s[\mathbf{L}_s \dots \mathbf{L}_{k-1} \mathbb{1}]} ,$$

which concludes the proof. □

C Convergence results for Pseudo marginal PaRIS algorithms

Lemma C.1. *Assume that H1 and H2 hold. The, for all* $0 \leqslant k \leqslant n-1$, $(f_{k+1}, \widetilde{f}_{k+1}) \in \mathsf{F}(\mathsf{X})^2$ *and* $N, \widetilde{N} \geqslant 0$, *the random variables* $\{\widehat{\omega}^i_{k+1}(\widehat{\tau}^i_{k+1} f_{k+1}(\xi^i_{k+1}) + \widetilde{f}_{k+1}(\xi^i_{k+1}))\}_{i=1}^N$ *are i.i.d. conditionally on* $\widetilde{\mathcal{F}}^N_k$ *with*

$$\mathbb{E}\left[\widehat{\omega}^i_{k+1}(\widehat{\tau}^i_{k+1} f_{k+1}(\xi^i_{k+1}) + \widetilde{f}_{k+1}(\xi^i_{k+1}))\middle|\widetilde{\mathcal{F}}^N_k\right] = \left(\widehat{\phi}^N_k[\vartheta_k]\right)^{-1} \sum_{\ell=1}^N \frac{\widehat{\omega}^\ell_k}{\widehat{\Omega}_k} \left\{\widehat{\tau}^\ell_k \mathbf{L}_k f_{k+1}(\xi^\ell_k) + \mathbf{L}_k(\widetilde{h}_k f_{k+1} + \widetilde{f}_{k+1})(\xi^\ell_k)\right\} .$$

Proof. The proof follows the same lines as [21, Lemma 2]. Note first that

$$\mathbb{E}\left[\widehat{\omega}^1_{k+1}\middle|\widetilde{\mathcal{F}}^N_k \vee \widetilde{\mathcal{G}}^N_{k+1}\right] = \bar{\omega}_{k+1}(\xi_k^{\widehat{I}^1_{k+1}}, \xi^1_{k+1}) ,$$
$$\mathbb{E}\left[\widehat{\tau}^1_{k+1}\middle|\widetilde{\mathcal{F}}^N_k \vee \widetilde{\mathcal{G}}^N_{k+1}\right] = \sum_{\ell=1}^N \frac{\widehat{\omega}^\ell_k \mathsf{r}_k(\xi^\ell_k, \xi^1_{k+1})}{\sum_{m=1}^N \widehat{\omega}^m_k \mathsf{r}_k(\xi^m_k, \xi^1_{k+1})} \left(\widehat{\tau}^\ell_k + \widetilde{h}_k(\xi^\ell_k, \xi^1_{k+1})\right) ,$$

where $\bar{\omega}_k$ is defined by (18) in H3. Then, since conditionally on $\widetilde{\mathcal{F}}^N_k \vee \widetilde{\mathcal{G}}^N_{k+1}$, $\widehat{\tau}^1_{k+1}$ is independent of $\widehat{\omega}^1_{k+1}$,

$$\begin{aligned}
&\mathbb{E}\left[\widehat{\omega}^1_{k+1}(\widehat{\tau}^i_{k+1} f_{k+1}(\xi^1_{k+1}) + \widetilde{f}_{k+1}(\xi^1_{k+1}))\middle|\widetilde{\mathcal{F}}^N_k\right] \\
&\quad = \mathbb{E}\left[\mathbb{E}\left[\widehat{\omega}^1_{k+1}\middle|\widetilde{\mathcal{F}}^N_k \vee \widetilde{\mathcal{G}}^N_{k+1}\right] \mathbb{E}\left[\widehat{\tau}^1_{k+1}\middle|\widetilde{\mathcal{F}}^N_k \vee \widetilde{\mathcal{G}}^N_{k+1}\right] f_{k+1}(\xi^1_{k+1})\middle|\widetilde{\mathcal{F}}^N_k\right] \\
&\qquad\qquad + \mathbb{E}\left[\mathbb{E}\left[\widehat{\omega}^1_{k+1}\middle|\widetilde{\mathcal{F}}^N_k \vee \widetilde{\mathcal{G}}^N_{k+1}\right] \widetilde{f}_{k+1}(\xi^1_{k+1})\middle|\widetilde{\mathcal{F}}^N_k\right] , \\
&\quad = \mathbb{E}\left[\bar{\omega}_{k+1}(\xi_k^{\widehat{I}^1_{k+1}}, \xi^1_{k+1}) \sum_{\ell=1}^N \frac{\widehat{\omega}^\ell_k \mathsf{r}_k(\xi^\ell_k, \xi^i_{k+1})}{\sum_{m=1}^N \widehat{\omega}^m_k \mathsf{r}_k(\xi^m_k, \xi^i_{k+1})} \left(\widehat{\tau}^\ell_k + \widetilde{h}_k(\xi^\ell_k, \xi^i_{k+1})\right) f_{k+1}(\xi^1_{k+1})\middle|\widetilde{\mathcal{F}}^N_k\right] \\
&\qquad\qquad + \mathbb{E}\left[\bar{\omega}_{k+1}(\xi_k^{\widehat{I}^1_{k+1}}, \xi^1_{k+1}) \widetilde{f}_{k+1}(\xi^1_{k+1})\middle|\widetilde{\mathcal{F}}^N_k\right] , \\
&\quad = \left(\widehat{\phi}^N_k[\vartheta_k]\right)^{-1} \sum_{\ell=1}^N \frac{\widehat{\omega}^\ell_k}{\widehat{\Omega}_k} \left\{\widehat{\tau}^\ell_k \mathbf{L}_k f_{k+1}(\xi^\ell_k) + \mathbf{L}_k(\widetilde{h}_k f_{k+1} + \widetilde{f}_{k+1})(\xi^\ell_k)\right\} ,
\end{aligned}$$

which concludes the proof. □

C.1 Proof of Lemma 4.3

Proof. The proof proceeds by induction and follows the same lines as [26, Lemma 13]. The case $k = 0$ is a direct consequence of the fact that $\mathbf{T}_0 h_0 = 0$ and $\widehat{\tau}_0^i = 0$ for all $1 \leqslant i \leqslant N$. Assume that the result holds for some $0 \leqslant k \leqslant n-1$ and write

$$\sum_{i=1}^{N} \frac{\widehat{\omega}_{k+1}^i}{\widehat{\Omega}_{k+1}} (\widehat{\tau}_{k+1}^i)^2 f_{k+1}(\xi_{k+1}^i) = \frac{a_N}{b_N} ,$$

where

$$a_N = \frac{1}{N} \sum_{i=1}^{N} \widehat{\omega}_{k+1}^i (\widehat{\tau}_{k+1}^i)^2 f_{k+1}(\xi_{k+1}^i) \quad \text{and} \quad b_N = \frac{1}{N} \sum_{i=1}^{N} \widehat{\omega}_{k+1}^i .$$

The random variables $(\widehat{\omega}_{k+1}^i)_{1 \leqslant i \leqslant N}$ are i.i.d. conditionally on $\widetilde{\mathcal{F}}_k^N$ with

$$\mathbb{E}\left[\widehat{\omega}_{k+1}^1 \middle| \widetilde{\mathcal{F}}_k^N\right] = \mathbb{E}\left[\mathbb{E}\left[\widehat{\omega}_{k+1}^1 \middle| \widetilde{\mathcal{F}}_k^N \vee \widetilde{\mathcal{G}}_{k+1}^N\right] \middle| \widetilde{\mathcal{F}}_k^N\right] = \mathbb{E}\left[\bar{\omega}_{k+1}(\xi_k^{\widehat{I}_{k+1}^1}, \xi_{k+1}^1) \middle| \widetilde{\mathcal{F}}_k^N\right] ,$$

where $\bar{\omega}_k$ is defined by (18) in H3. Noting that by H4 for all $1 \leqslant i \leqslant N$ $|\widehat{\omega}_{k+1}^i| \leqslant \|\widehat{\omega}_k\|_\infty$ and

$$\mathbb{E}\left[\bar{\omega}_{k+1}(\xi_k^{\widehat{I}_{k+1}^1}, \xi_{k+1}^1) \middle| \widetilde{\mathcal{F}}_k^N\right] = \frac{\sum_{i=1}^N \widehat{\omega}_k^i \mathbf{L}_k \mathbb{1}(\xi_{k+1}^i)}{\sum_{i=1}^N \widehat{\omega}_k^i \vartheta_k(\xi_{k+1}^i)} = \frac{\widehat{\phi}_k^N[\mathbf{L}_k \mathbb{1}]}{\widehat{\phi}_k^N[\vartheta_k]} ,$$

by Hoeffding's inequality, there exist positive constants c_k and $\widetilde{c}_k$ such that

$$\mathbb{P}\left(\left| b_N - \frac{\widehat{\phi}_k^N[\mathbf{L}_k \mathbb{1}]}{\widehat{\phi}_k^N[\vartheta_k]} \right| > \varepsilon\right) \leqslant c_k \mathrm{e}^{-\widetilde{c}_k N \varepsilon^2} .$$

Therefore, by Proposition 4.2 and Lemma A.1,

$$\mathbb{P}\left(\left| b_N - \frac{\phi_k[\mathbf{L}_k \mathbb{1}]}{\phi_k[\vartheta_k]} \right| > \varepsilon\right) \leqslant c_k \mathrm{e}^{-\widetilde{c}_k N \varepsilon^2}$$

so that,

$$b_N \xrightarrow[N \to \infty]{\mathbb{P}-\text{a.s.}} \frac{\phi_k[\mathbf{L}_k \mathbb{1}]}{\phi_k[\vartheta_k]} .$$

Since $\phi_k[\mathbf{L}_k \mathbb{1}] > 0$ it remains to establish the convergence in probability of $(a_N)_{N \geqslant 1}$. On the other hand, by Hoeffding inequality, using that for all $1 \leqslant i \leqslant N$, $|\widehat{\omega}_{k+1}^i (\widehat{\tau}_{k+1}^i)^2 f_{k+1}(\xi_{k+1}^i)| \leqslant \|\widehat{\omega}_{k+1}\|_\infty \|h_{k+1}\|_\infty^2 \|f_{k+1}\|_\infty$,

$$\mathbb{P}\left(\left| a_N - \mathbb{E}[a_N | \widetilde{\mathcal{F}}_k^N] \right| > \varepsilon\right) \leqslant c_k \mathrm{e}^{-\widetilde{c}_k N \varepsilon^2} ,$$

Then, write

$$\begin{aligned}
\mathbb{E}[a_N | \widetilde{\mathcal{F}}_k^N] &= \mathbb{E}\left[\widehat{\omega}_{k+1}^1 (\widehat{\tau}_{k+1}^1)^2 f_{k+1}(\xi_{k+1}^1) \middle| \widetilde{\mathcal{F}}_k^N\right] , \\
&= \mathbb{E}\left[\mathbb{E}\left[\widehat{\omega}_{k+1}^1 \middle| \widetilde{\mathcal{F}}_k^N \vee \widetilde{\mathcal{G}}_{k+1}^N\right] \mathbb{E}\left[(\widehat{\tau}_{k+1}^1)^2 \middle| \widetilde{\mathcal{F}}_k^N \vee \widetilde{\mathcal{G}}_{k+1}^N\right] f_{k+1}(\xi_{k+1}^1) \middle| \widetilde{\mathcal{F}}_k^N\right] , \\
&= \mathbb{E}\left[\bar{\omega}_{k+1}(\xi_k^{\widehat{I}_{k+1}^1}, \xi_{k+1}^1) \mathbb{E}\left[(\widehat{\tau}_{k+1}^1)^2 \middle| \widetilde{\mathcal{F}}_k^N \vee \widetilde{\mathcal{G}}_{k+1}^N\right] f_{k+1}(\xi_{k+1}^1) \middle| \widetilde{\mathcal{F}}_k^N\right] , \\
&= \widetilde{a}_N^1 + \widetilde{a}_N^2 ,
\end{aligned}$$

where

$$\widetilde{a}_N^1 = \widetilde{N}^{-1}\mathbb{E}\left[\bar{\omega}_{k+1}(\xi_k^{\widehat{I}_{k+1}^1}, \xi_{k+1}^1) f_{k+1}(\xi_{k+1}^1)\mathbb{E}\left[\left(\widehat{\tau}_k^{\widehat{J}_{k+1}^{(1,1)}} + \widetilde{h}_k(\xi_k^{\widehat{J}_{k+1}^{(1,1)}}, \xi_{k+1}^1)\right)^2 \middle| \widetilde{\mathcal{F}}_k^N \vee \widetilde{\mathcal{G}}_{k+1}^N\right] \middle| \widetilde{\mathcal{F}}_k^N\right] ,$$

$$\widetilde{a}_N^2 = (\widetilde{N}-1)\widetilde{N}^{-1}\mathbb{E}\left[\bar{\omega}_{k+1}(\xi_k^{\widehat{I}_{k+1}^1}, \xi_{k+1}^1) f_{k+1}(\xi_{k+1}^1)\mathbb{E}\left[\tau_k^{\widehat{J}_{k+1}^{(1,1)}} + \widetilde{h}_k(\xi_k^{\widehat{J}_{k+1}^{(1,1)}}, \xi_{k+1}^1) \middle| \widetilde{\mathcal{F}}_k^N \vee \widetilde{\mathcal{G}}_{k+1}^N\right]^2 \middle| \widetilde{\mathcal{F}}_k^N\right] .$$

By Lemma 3.1, the first term is given by

$$\begin{aligned}\widetilde{a}_N^1 = \widetilde{N}^{-1}\sum_{j=1}^N \int \frac{\widehat{\omega}_k^j \vartheta_k(\xi_k^j) p_k(\xi_k^j, x)}{\sum_{m=1}^N \widehat{\omega}_k^m \vartheta_k(\xi_k^m)} \frac{\mathsf{r}_k(\xi_k^j, x)}{\vartheta_k(\xi_k^j) p_k(\xi_k^j, x)} f_{k+1}(x) \\ \times \sum_{\ell=1}^N \frac{\widehat{\omega}_k^\ell \mathsf{r}_k(\xi_k^\ell, x)}{\sum_{m=1}^N \widehat{\omega}_k^m \mathsf{r}_k(\xi_k^m, x)} \left(\widehat{\tau}_k^\ell + \widetilde{h}_k(\xi_k^\ell, x)\right)^2 \mu(\mathrm{d}x) , \\ = \widetilde{N}^{-1}(\widehat{\phi}_k^N[\vartheta_k])^{-1} \int f_{k+1}(x) \sum_{\ell=1}^N \frac{\widehat{\omega}_k^\ell}{\widehat{\Omega}_k} \mathsf{r}_k(\xi_k^\ell, x) \left(\widehat{\tau}_k^\ell + \widetilde{h}_k(\xi_k^\ell, x)\right)^2 \mu(\mathrm{d}x) ,\end{aligned}$$

By the induction hypothesis and Proposition 4.2,

$$a_N^1 \xrightarrow[N\to\infty]{\mathbb{P}} (\widetilde{N}\phi_k[\vartheta_k])^{-1}\left\{\eta_k[\mathbf{L}_k f_{k+1}] + \phi_k[\mathbf{T}_k^2 h_k \mathbf{L}_k f_{k+1}] + \phi_k[\mathbf{L}_k(f_{k+1}\widetilde{h}_k^2)] + 2\phi_k[\mathbf{T}_k h_k \mathbf{L}_k(f_{k+1}\widetilde{h}_k)]\right\} ,$$

which yields

$$a_N^1 \xrightarrow[N\to\infty]{\mathbb{P}} (\widetilde{N}\phi_k[\vartheta_k])^{-1}\left\{\eta_k[\mathbf{L}_k f_{k+1}] + \phi_k[\mathbf{L}_k\{(\mathbf{T}_k h_k + \widetilde{h}_k)^2 f_{k+1}\}]\right\} .$$

The second term is given by

$$\begin{aligned}\widetilde{a}_N^2 = (\widetilde{N}-1)\widetilde{N}^{-1}\sum_{j=1}^N \int \frac{\widehat{\omega}_k^j \vartheta_k(\xi_k^j) p_k(\xi_k^j, x)}{\sum_{m=1}^N \widehat{\omega}_k^m \vartheta_k(\xi_k^m)} \frac{\mathsf{r}_k(\xi_k^j, x)}{\vartheta_k(\xi_k^j) p_k(\xi_k^j, x)} f_{k+1}(x) \\ \times \left(\sum_{\ell=1}^N \frac{\widehat{\omega}_k^\ell \mathsf{r}_k(\xi_k^\ell, x)}{\sum_{m=1}^N \widehat{\omega}_k^m \mathsf{r}_k(\xi_k^m, x)} \left\{\widehat{\tau}_k^\ell + \widetilde{h}_k(\xi_k^\ell, x)\right\}\right)^2 \mu(\mathrm{d}x) , \\ = (\widetilde{N}-1)\widetilde{N}^{-1}(\widehat{\phi}_k^N[\vartheta_k])^{-1}\widehat{\phi}_k^N[\mathbf{L}_k\varphi_N] ,\end{aligned}$$

with, for all $x \in \mathsf{X}$,

$$\varphi_N(x) = f_{k+1}(x)\left(\sum_{\ell=1}^N \frac{\widehat{\omega}_k^\ell \mathsf{r}_k(\xi_k^\ell, x)}{\sum_{m=1}^N \widehat{\omega}_k^m \mathsf{r}_k(\xi_k^m, x)} \left\{\widehat{\tau}_k^\ell + \widetilde{h}_k(\xi_k^\ell, x)\right\}\right)^2 .$$

For all $x \in \mathsf{X}$, by Proposition 4.2,

$$\varphi_N(x) \xrightarrow[N\to\infty]{\mathbb{P}-\text{a.s.}} f_{k+1}(x)\left(\frac{\phi_k[\mathbf{T}_k h_k \mathsf{r}_k(\cdot, x) + \mathsf{r}_k(\cdot, x)\widetilde{h}_k(\cdot, x)]}{\phi_k[\mathsf{r}_k(\cdot, x)]}\right)^2 .$$

Therefore, as $\|\varphi_N\|_\infty \leqslant \|f_{k+1}\|_\infty \|h_{k+1}\|_\infty^2$, by the generalized Lebesgue dominated convergence theorem, see Lemma A.2,

$$\widetilde{a}_N^2 \xrightarrow[N\to\infty]{\mathbb{P}} (\widetilde{N}-1)\widetilde{N}^{-1}(\phi_k[\vartheta_k])^{-1}\phi_k[\mathbf{L}_k\{f_{k+1}\mathbf{T}_{k+1}^2 h_{k+1}\}] \, .$$

This concludes the proof following the same steps as in the proof of Lemma 4.2. □

C.2 Proof of Proposition 4.3

Proof. The result is proved by induction on k. It holds for $k=0$ as for all $1 \leqslant i \leqslant N$, $\widehat{\tau}_0^i = 0$. Assume now that the result holds for some $0 \leqslant k \leqslant n-1$ and that $\phi_{k+1}[\mathbf{T}_{k+1}h_{k+1}f_{k+1} + \widetilde{f}_{k+1}] = 0$. Write

$$\sqrt{N}\sum_{i=1}^N \frac{\widehat{\omega}_{k+1}^i}{\widehat{\Omega}_{k+1}}\{\widehat{\tau}_{k+1}^i f_{k+1}(\xi_{k+1}^i) + \widetilde{f}_{k+1}(\xi_{k+1}^i)\} = \widehat{\Omega}_{k+1}^{-1}\Delta_{k+1}^N \, ,$$

where $\Delta_{k+1}^N = \sqrt{N}\sum_{i=1}^N \widehat{\omega}_{k+1}^i\{\widehat{\tau}_{k+1}^i f_{k+1}(\xi_{k+1}^i) + \widetilde{f}_{k+1}(\xi_{k+1}^i)\}$ is decomposed as follows

$$\Delta_{k+1}^N = \Delta_{k+1,1}^N + \Delta_{k+1,2}^N \, ,$$

where

$$\Delta_{k+1,1}^N = \sqrt{N}\sum_{i=1}^N \mathbb{E}\left[\widehat{\omega}_{k+1}^i(\widehat{\tau}_{k+1}^i f_{k+1}(\xi_{k+1}^i) + \widetilde{f}_{k+1}(\xi_{k+1}^i))\middle|\widetilde{\mathcal{F}}_k^N\right] \, ,$$

$$\Delta_{k+1,2}^N = \sqrt{N}\sum_{i=1}^N \left\{\widehat{\omega}_{k+1}^i\left(\widehat{\tau}_{k+1}^i f_{k+1}(\xi_{k+1}^i) + \widetilde{f}_{k+1}(\xi_{k+1}^i)\right)\right.$$
$$\left. - \mathbb{E}\left[\widehat{\omega}_{k+1}^i(\widehat{\tau}_{k+1}^i f_{k+1}(\xi_{k+1}^i) + \widetilde{f}_{k+1}(\xi_{k+1}^i))\middle|\widetilde{\mathcal{F}}_k^N\right]\right\} \, .$$

By Lemma B.1,

$$\widehat{\Omega}_{k+1}^{-1}\Delta_{k+1,1}^N = \frac{N}{\widehat{\Omega}_{k+1}}\left(\widehat{\phi}_k^N[\vartheta_k]\right)^{-1}\sqrt{N}\sum_{\ell=1}^N \frac{\widehat{\omega}_k^\ell}{\widehat{\Omega}_k}\left\{\widehat{\tau}_k^\ell \mathbf{L}_k f_{k+1}(\xi_k^\ell) + \mathbf{L}_k(\widetilde{h}_k f_{k+1} + \widetilde{f}_{k+1})(\xi_k^\ell)\right\}$$

As $\phi_{k+1}[\mathbf{T}_{k+1}h_{k+1}f_{k+1} + \widetilde{f}_{k+1}] = 0$,

$$\phi_k[\mathbf{T}_k h_k \mathbf{L}_k f_{k+1} + \mathbf{L}_k(\widetilde{h}_k f_{k+1} + \widetilde{f}_{k+1})] = 0 \, .$$

Therefore, using the induction hypothesis, Slutsky's lemma and

$$\frac{N}{\widehat{\Omega}_{k+1}}\left(\widehat{\phi}_k^N[\vartheta_k]\right)^{-1} \xrightarrow[N\to\infty]{\mathbb{P}} (\phi_k[\mathbf{L}_k\mathbb{1}])^{-1}$$

yields

$$\widehat{\Omega}_{k+1}^{-1}\Delta_{k+1,1}^N \xrightarrow[N\to\infty]{\mathcal{D}} \frac{\bar{\sigma}_k\langle \mathbf{L}_k f_{k+1}; \mathbf{L}_k(\widetilde{h}_k f_{k+1} + \widetilde{f}_{k+1})\rangle}{\phi_k[\mathbf{L}_k\mathbb{1}]} Z \, ,$$

where Z is a standard Gaussian random variable. By Lemma B.1,

$$\widehat{\Omega}_{k+1}^{-1}\Delta_{k+1,2}^N = \frac{N}{\widehat{\Omega}_{k+1}}\sum_{i=1}^N \upsilon_N^i \ ,$$

where for all $1 \leqslant i,j \leqslant N$ and all $x \in \mathsf{X}$,

$$\upsilon_N^i = \frac{1}{\sqrt{N}\widetilde{N}}\sum_{j=1}^{\widetilde{N}} \widetilde{\upsilon}_N(\widehat{I}_{k+1}^i, \widehat{J}_{k+1}^{(i,j)}, \xi_{k+1}^i) \ ,$$

$$\widetilde{\upsilon}_N(i,j,x) = \frac{\widehat{\mathsf{r}}_k(\xi_k^i, x; \zeta_k^i)}{\vartheta_k(\xi_k^i)p_k(\xi_k^i, x)}\left\{\left(\widehat{\tau}_k^j + \widetilde{h}_k(\xi_k^j, x)\right) f_{k+1}(x) + \widetilde{f}_{k+1}(x)\right\}$$
$$- \left(\widehat{\phi}_k^N[\vartheta_k]\right)^{-1}\sum_{\ell=1}^N \frac{\widehat{\omega}_k^\ell}{\widehat{\Omega}_k}\left\{\widehat{\tau}_k^\ell \mathbf{L}_k f_{k+1}(\xi_k^\ell) + \mathbf{L}_k(\widetilde{h}_k f_{k+1} + \widetilde{f}_{k+1})(\xi_k^\ell)\right\} \ .$$

First,

$$\frac{N}{\widehat{\Omega}_{k+1}} \xrightarrow[N\to\infty]{\mathbb{P}} \frac{\phi_k[\vartheta_k]}{\phi_k[\mathbf{L}_k\mathbb{1}]} \ .$$

Then, by construction, $\mathbb{E}[\upsilon_N^i|\widetilde{\mathcal{F}}_k^N] = 0$ and

$$\sum_{i=1}^N \mathbb{E}[(\upsilon_N^i)^2|\widetilde{\mathcal{F}}_k^N] = \widetilde{N}^{-1}\mathbb{E}[\mathbb{E}[(\widetilde{\upsilon}_N(\widehat{I}_{k+1}^1, \widehat{J}_{k+1}^{(1,1)}, \xi_{k+1}^1))^2|\widetilde{\mathcal{F}}_k^N \vee \widetilde{\mathcal{G}}_{k+1}^N|\widetilde{\mathcal{F}}_k^N]$$
$$+ (\widetilde{N}-1)\widetilde{N}^{-1}\mathbb{E}[\mathbb{E}[\widetilde{\upsilon}_N(\widehat{I}_{k+1}^1, \widehat{J}_{k+1}^{(1,1)}, \xi_{k+1}^1)|\widetilde{\mathcal{F}}_k^N \vee \widetilde{\mathcal{G}}_{k+1}^N]^2|\widetilde{\mathcal{F}}_k^N] \ . \quad (31)$$

The first term of (31) is given by

$$\mathbb{E}\left[\mathbb{E}\left[\widetilde{\upsilon}_N^2(\widehat{I}_{k+1}^1, \widehat{J}_{k+1}^{(1,1)}, \xi_{k+1}^1)\middle|\widetilde{\mathcal{F}}_k^N \vee \widetilde{\mathcal{G}}_{k+1}^N\right]\middle|\widetilde{\mathcal{F}}_k^N\right]$$
$$= \left(\widehat{\phi}_k^N[\vartheta_k]\right)^{-1}\int A_N(x)B_N(x)\mu(\mathrm{d}x)$$
$$- \left(\widehat{\phi}_k^N[\vartheta_k]\right)^{-2}\left(\sum_{\ell=1}^N \frac{\widehat{\omega}_k^\ell}{\widehat{\Omega}_k}\left\{\widehat{\tau}_k^\ell \mathbf{L}_k f_{k+1}(\xi_k^\ell) + \mathbf{L}_k(\widetilde{h}_k f_{k+1} + \widetilde{f}_{k+1})(\xi_k^\ell)\right\}\right)^2 \ ,$$

where, for all $(x,y) \in \mathsf{X}\times\mathsf{X}$,

$$\varpi_k(x,y) = \int \widehat{\mathsf{r}}_k(x,y;z)\widehat{\omega}_{k+1}(x,y;z)\mathsf{R}_k(x,y,\mathrm{d}z) \ ,$$
$$A_N(x) = \sum_{j=1}^N \frac{\widehat{\omega}_k^j}{\widehat{\Omega}_k}\frac{\int \widehat{\mathsf{r}}_k(\xi_k^j, x; u)^2\mathsf{R}_k(\xi_k^j, x; \mathrm{d}u)}{\vartheta_k(\xi_k^j)p_k(\xi_k^j, x)} = \sum_{j=1}^N \frac{\widehat{\omega}_k^j}{\widehat{\Omega}_k}\varpi_k(\xi_k^j, x) \ ,$$
$$B_N(x) = \sum_{\ell=1}^N \frac{\widehat{\omega}_k^\ell \mathsf{r}_k(\xi_k^\ell, x)}{\sum_{m=1}^N \widehat{\omega}_k^m \mathsf{r}_k(\xi_k^m, x)}\left\{\left(\widehat{\tau}_k^\ell + \widetilde{h}_k(\xi_k^\ell, x)\right) f_{k+1}(x) + \widetilde{f}_{k+1}(x)\right\}^2 \ .$$

By Proposition 4.2,

$$\left(\widehat{\phi}_k^N[\vartheta_k]\right)^{-2}\left(\sum_{\ell=1}^N \frac{\widehat{\omega}_k^\ell}{\widehat{\Omega}_k}\left\{\widehat{\tau}_k^\ell \mathbf{L}_k f_{k+1}(\xi_k^\ell) + \mathbf{L}_k(\widetilde{h}_k f_{k+1} + \widetilde{f}_{k+1})(\xi_k^\ell)\right\}\right)^2 \xrightarrow[N\to\infty]{\mathbb{P}} (\phi_k[\vartheta_k])^{-2}\left(\phi_k[\mathbf{T}_k h_k \mathbf{L}_k f_{k+1} + \mathbf{L}_k(\widetilde{h}_k f_{k+1} + \widetilde{f}_{k+1})]\right)^2 = 0\,,$$

where by assumption $\phi_{k+1}[\mathbf{T}_{k+1}h_{k+1}f_{k+1} + \widetilde{f}_{k+1}] = 0$, so that $\phi_k[\mathbf{T}_k h_k \mathbf{L}_k f_{k+1} + \mathbf{L}_k(\widetilde{h}_k f_{k+1} + \widetilde{f}_{k+1})] = 0$. Then, write

$$\int A_N(x)B_N(x)\mu(\mathrm{d}x) = \widetilde{a}_N^1 + \widetilde{a}_N^2 + \widetilde{a}_N^3\,,$$

where

$$\begin{aligned}
\varphi_N &: x \mapsto \left\{\sum_{j=1}^N \frac{\widehat{\omega}_k^j}{\widehat{\Omega}_k}\varpi_k(\xi_k^j, x)\right\}\left(\sum_{m=1}^N \frac{\widehat{\omega}_k^m}{\widehat{\Omega}_k}\mathsf{r}_k(\xi_k^m, x)\right)^{-1}\,,\\
\widetilde{a}_N^1 &= \sum_{\ell=1}^N \frac{\widehat{\omega}_k^\ell}{\widehat{\Omega}_k}(\widehat{\tau}_k^\ell)^2 \int \mathsf{r}_k(\xi_k^\ell, x) f_{k+1}^2(x)\varphi_N(x)\mu(\mathrm{d}x)\,,\\
\widetilde{a}_N^2 &= \sum_{j=1}^N \frac{\widehat{\omega}_k^j}{\widehat{\Omega}_k}\int \varpi_k(\xi_k^j, x)\sum_{\ell=1}^N \frac{\widehat{\omega}_k^\ell \mathsf{r}_k(\xi_k^\ell, x)}{\sum_{m=1}^N \widehat{\omega}_k^m \mathsf{r}_k(\xi_k^m, x)}\left(\widetilde{h}_k(\xi_k^\ell, x)f_{k+1}(x) + \widetilde{f}_{k+1}(x)\right)^2\mu(\mathrm{d}x)\,,\\
\widetilde{a}_N^3 &= 2\sum_{\ell=1}^N \frac{\widehat{\omega}_k^\ell}{\widehat{\Omega}_k}\widehat{\tau}_k^\ell \int \mathsf{r}_k(\xi_k^\ell, x)\varphi_N(x) f_{k+1}(x)\left(\widetilde{h}_k(\xi_k^\ell, x)f_{k+1}(x) + \widetilde{f}_{k+1}(x)\right)\mu(\mathrm{d}x)\,.
\end{aligned}$$

Following the same steps as in the proof of Proposition 4.1,

$$\begin{aligned}
\widetilde{a}_N^1 &\xrightarrow[N\to\infty]{\mathbb{P}} \eta_k[\mathbf{L}_k\{f_{k+1}^2\widehat{\mathbf{Q}}_{\phi_k}\varpi_k\}] + \phi_k[\mathbf{T}_k^2 h_k \mathbf{L}_k\{f_{k+1}^2\widehat{\mathbf{Q}}_{\phi_k}\varpi_k\}]\,,\\
\widetilde{a}_N^2 &\xrightarrow[N\to\infty]{\mathbb{P}} \phi_k\left[\int \varpi_k(\cdot, x)\phi_k[\mathsf{r}_k(\cdot, x)(\widetilde{h}_k(\cdot, x)f_{k+1}(x) + \widetilde{f}_{k+1}(x))^2](\phi_k[\mathsf{r}_k(\cdot, x)])^{-1}\mu(\mathrm{d}x)\right]\,,\\
\widetilde{a}_N^3 &\xrightarrow[N\to\infty]{\mathbb{P}} 2\phi_k\left[\mathbf{T}_k h_k \mathbf{L}_k\left\{(\widehat{\mathbf{Q}}_{\phi_k}\varpi_k) f_{k+1}\left(\widetilde{h}_k f_{k+1} + \widetilde{f}_{k+1}\right)\right\}\right]\,,
\end{aligned}$$

where $\widehat{\mathbf{Q}}_{\phi_k}\varpi_k : x \mapsto \phi_k[\varpi_k(., x)]/\phi_k[\mathsf{r}_k(., x)]$. Therefore, the first term of (31) satisfies

$$\begin{aligned}
\widetilde{N}^{-1}\mathbb{E}[\mathbb{E}[(\widetilde{\upsilon}_N(I_{k+1}^1, J_{k+1}^{(1,1)}, \xi_{k+1}^1))^2|\mathcal{F}_k^N \vee \mathcal{G}_{k+1}^N|\mathcal{F}_k^N]&\\
\xrightarrow[N\to\infty]{\mathbb{P}} \int \frac{\eta_k[f_{k+1}^2(x)\mathsf{r}_k(\cdot, x)]\phi_k\left[\varpi_k(\cdot, x)\right]}{\widetilde{N}\phi_k[\vartheta_k]\phi_k[\mathsf{r}_k(\cdot, x)]}\mu(\mathrm{d}x)&\\
+ \int \frac{\phi_k[\mathsf{r}_k(\cdot, x)\{(\mathbf{T}_k h_k + \widetilde{h}_k(\cdot, x))f_{k+1}(x) + \widetilde{f}_{k+1}(x)\}^2]\phi_k\left[\varpi_k(\cdot, x)\right]}{\widetilde{N}\phi_k[\vartheta_k]\phi_k[\mathsf{r}_k(\cdot, x)]}\mu(\mathrm{d}x)\,,&
\end{aligned}$$

which concludes the proof for the first term of (31). The second term of (31) is given by

$$\mathbb{E}[\mathbb{E}[\widetilde{\upsilon}_N(I_{k+1}^1, J_{k+1}^{(1,1)}, \xi_{k+1}^1)|\mathcal{F}_k^N \vee \mathcal{G}_{k+1}^N]^2|\mathcal{F}_k^N] = \left(\widehat{\phi}_k^N[\vartheta_k]\right)^{-1}\widehat{\phi}_k^N[\vartheta_k\varphi_k^N]\,,$$

where, for all $x \in \mathsf{X}$,

$$\varphi_k^N(x) = \int p_k(x,z) \left(\frac{\mathsf{r}_k(x,z)}{\vartheta_k(x) p_k(x,z)} f_{k+1}(z) \sum_{\ell=1}^N \frac{\widehat{\omega}_k^\ell \mathsf{r}_k(\xi_k^\ell, z)}{\sum_{m=1}^N \widehat{\omega}_k^m \mathsf{r}_k(\xi_k^m, z)} \left(\widehat{\tau}_k^\ell + \widetilde{h}_k(\xi_k^\ell, z) \right) \right.$$
$$\left. + \frac{\mathsf{r}_k(x,z)}{\vartheta_k(x) p_k(x,z)} \widetilde{f}_{k+1}(z) - \left(\phi_k^N[\vartheta_k]\right)^{-1} \sum_{\ell=1}^N \frac{\widehat{\omega}_k^\ell}{\widehat{\Omega}_k} \left\{ \widehat{\tau}_k^\ell \mathbf{L}_k f_{k+1}(\xi_k^\ell) + \mathbf{L}_k(\widetilde{h}_k f_{k+1} + \widetilde{f}_{k+1})(\xi_k^\ell) \right\} \right)^2 \mu(\mathrm{d}z) \; .$$

By assumption, $\phi_k[\mathbf{T}_k h_k \mathbf{L}_k f_{k+1} + \mathbf{L}_k(\widetilde{h}_k f_{k+1} + \widetilde{f}_{k+1})] = 0$ so that by Lemma A.2,

$$\widehat{\phi}_k^N[\vartheta_k \varphi_k^N] \xrightarrow[N\to\infty]{\mathbb{P}} \phi_k[\vartheta_k \varphi_k] \; ,$$

where

$$\begin{aligned} \varphi_k(x) &= \int p_k(x,z) \left(\frac{\mathsf{r}_k(x,z)}{\vartheta_k(x) p_k(x,z)} \right)^2 \left(f_{k+1}(z) \overleftarrow{\mathbf{Q}}_{\phi_k}(\mathbf{T}_k h_k + \widetilde{h}_k)(z) + \widetilde{f}_{k+1}(z) \right)^2 \mu(\mathrm{d}z) \; , \\ &= \int p_k(x,z) \left(\frac{\mathsf{r}_k(x,z)}{\vartheta_k(x) p_k(x,z)} \right)^2 \left(f_{k+1}(z) \mathbf{T}_{k+1} h_{k+1}(z) + \widetilde{f}_{k+1}(z) \right)^2 \mu(\mathrm{d}z) \; . \end{aligned}$$

Therefore,

$$\begin{aligned} &\mathbb{E}[\mathbb{E}[\widetilde{v}_N(I_{k+1}^1, J_{k+1}^{(1,1)}, \xi_{k+1}^1) | \mathcal{F}_k^N \vee \mathcal{G}_{k+1}^N]^2 | \mathcal{F}_k^N] \\ &\quad \xrightarrow[N\to\infty]{\mathbb{P}} \left(1 - \frac{1}{\widetilde{N}}\right) (\phi_k[\vartheta_k])^{-1} \phi_k \left[\int \mathsf{r}_k(\cdot, z) \bar{\omega}_k(\cdot, z) \left(f_{k+1}(z) \mathbf{T}_{k+1} h_{k+1}(z) + \widetilde{f}_{k+1}(z) \right)^2 \mu(\mathrm{d}z) \right] \; . \end{aligned}$$

The proof of (ii) is an immediate consequence of H4 since for all $1 \leqslant i \leqslant N$,

$$v_N^i \leqslant 2 \|\widehat{\omega}_{k+1}\|_\infty \left(\|h_{k+1}\|_\infty \|\widetilde{f}_{k+1}\|_\infty + \|\widetilde{f}_{k+1}\|_\infty \right) N^{-1/2} \; .$$

Then, defining $c_k = 2\|\widehat{\omega}_{k+1}\|_\infty (\|h_{k+1}\|_\infty \|\widetilde{f}_{k+1}\|_\infty + \|\widetilde{f}_{k+1}\|_\infty)$, for all $\varepsilon > 0$,

$$\sum_{i=1}^N \mathbb{E}[(v_N^i)^2 \mathbb{1}_{|v_N^i| \geqslant \varepsilon} | \mathcal{F}_k^N] \leqslant c_k^2 \mathbb{1}_{c_k \geqslant \varepsilon \sqrt{N}} \xrightarrow[N\to\infty]{\mathbb{P}} 0 \; .$$

Writing

$$\bar{f}_{k+1} = \widetilde{f}_{k+1} - \phi_{k+1}[\mathbf{T}_{k+1} h_{k+1} f_{k+1} + \widetilde{f}_{k+1})],$$

yields

$$\begin{aligned}\bar{\sigma}^2_{k+1}\langle f_{k+1};\widetilde{f}_{k+1}\rangle = &\frac{\bar{\sigma}^2_k\langle \mathbf{L}_k f_{k+1};\mathbf{L}_k\{\widetilde{h}_k f_{k+1}+\bar{f}_{k+1}\}\rangle}{\phi_k[\mathbf{L}_k\mathbb{1}]^2}+\frac{\phi_k[\vartheta_k]\int \eta_k[\mathsf{r}_k(\cdot,x)]f^2_{k+1}(x)\widehat{\mathbf{Q}}_{\phi_k}\varpi_k(x)\mu(\mathrm{d}x)}{\widetilde{N}\phi_k[\mathbf{L}_k\mathbb{1}]^2}\\
&+\frac{\phi_k[\vartheta_k]\phi_k\left[\int \mathsf{r}_k(\cdot,z)\bar{\omega}_k(\cdot,z)\left(f_{k+1}(z)\mathbf{T}_{k+1}h_{k+1}(z)+\bar{f}_{k+1}(z)\right)^2\mu(\mathrm{d}z)\right]}{\phi_k[\mathbf{L}_k\mathbb{1}]^2}\\
&+\frac{\phi_k[\vartheta_k]\phi_k\left[\int \varpi_k(\cdot,z)f^2_{k+1}(z)\overleftarrow{\mathbf{Q}}_{\phi_k}\left(\mathbf{T}_k h_k+\widetilde{h}_k-\mathbf{T}_{k+1}h_{k+1}\right)^2(z)\mu(\mathrm{d}z)\right]}{\widetilde{N}\phi_k[\mathbf{L}_k\mathbb{1}]^2},\\
&+\frac{\phi_k[\vartheta_k]\phi_k\left[\int \mathrm{Cov}\{\widehat{\mathsf{r}}_k(\cdot,z;\zeta_k)\widehat{\omega}_k(\cdot,z;\zeta_k)\}\left(f_{k+1}(z)\mathbf{T}_{k+1}h_{k+1}(z)+\bar{f}_{k+1}(z)\right)^2\mu(\mathrm{d}z)\right]}{\widetilde{N}\phi_k[\mathbf{L}_k\mathbb{1}]^2},\end{aligned}$$

which concludes the proof.

□