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# Guaranteed optimal reachability control of reaction-diffusion equations using one-sided Lipschitz constants and model reduction

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**Abstract.** We show that, for any spatially discretized system of reaction-diffusion, the approximate solution given by the explicit Euler time-discretization scheme converges to the exact time-continuous solution, provided that diffusion coefficient be sufficiently large. By “sufficiently large”, we mean that the diffusion coefficient value makes the *one-sided Lipschitz constant* of the reaction-diffusion system negative. We apply this result to solve a finite horizon control problem for a 1D reaction-diffusion example. We also explain how to perform *model reduction* in order to improve the efficiency of the method.

## 1 Introduction

### 1.1 Guaranteed reachability analysis

Given a system of Ordinary Differential equations (ODEs) of dimension  $n$  satisfying standard conditions of existence and uniqueness of the solution, the area of *Numerical Analysis* makes use of numerical tools in order to compute the approximate value of the solution, starting at an initial point of  $\mathbb{R}^n$ , with high accuracy: 1st order methods (explicit/implicit Euler method, trapezoid rule), higher-order Runge-Kutta methods, etc. In contrast, the area of *Guaranteed (or Symbolic) Analysis* is devoted to the construction of an overapproximation of the set of solutions that start, not at a single point of  $\mathbb{R}^n$ , but from a dense compact set of initial points. Guaranteed analysis, in its modern form, has been initiated in the 60’s by R.E. Moore and his creation of *Interval Arithmetic* [40]: the set of solutions (or trajectories) are overapproximated by a sequence of “rectangular sets”, i.e., cross-product of intervals of  $\mathbb{R}$ . A set of arithmetic and differential calculus has been created for manipulating such sets. An overapproximation of the set of trajectories is computed using a Taylor development up to some order and an overestimation of the “Lagrange remainder”. The method has been considerably refined in the 90’s [11, 12, 35, 43, 44]. These recent techniques make

use of different convex data structures such as parallelepipeds [35] or zonotopes [21, 29] instead of rectangular sets in order to enclose the flow of ODEs.

Such methods are typically applied to the formal *proof* of correctness of ODE integration, and more generally, to guarantee that the solutions of the ODEs satisfy some desired properties. Guaranteed reachability analysis generally treats *linear* systems. Extensions to nonlinear systems have been proposed, e.g., in [4], using local linearizations (see also [38, 39]).

## 1.2 Guaranteed optimal control

In presence of inputs, we can use guaranteed analysis to describe a law that allows the system to satisfy a desired property. This corresponds to the topic of *guaranteed (or correct-by-design) control synthesis*. Several works have recently applied guaranteed analysis to *optimal* control synthesis. Thus, in [49, 50], the authors focus on a (finite time-horizon) optimal control procedure with a formal guarantee of safety constraint satisfaction, using zonotopes as state set representations. In [16], the authors focus on (periodically) sampled systems, and perform reachability analysis using convex polytopes as state set representations. In [27, 37, 19, 46, 47], the authors construct an over-approximation of the set of trajectories using a growth bound (bounding the distance of neighboring trajectories) exploiting the notion of *one-sided Lipschitz constant* (also called “logarithmic norm” or “matrix norm”). The notion of “one-sided Lipschitz (OSL) constant” has been introduced independently by Dahlquist [17] and Lozinskii [36] in order to derive error bounds in initial value problems (see survey in [51]). We used ourselves OSL constants in the context of symbolic optimal control in [14]. The main difference with previous work [27, 37, 19, 46, 47] is that our method makes use of explicit Euler’s algorithm for ODE integration (cf. [32, 33]) instead of sophisticated algorithms such as Lohner’s algorithm [27] or interval Taylor series methods [44]. This leads us to a simple implementation of just a few hundred lines of Octave (see [31]).

As explained in [48], using the Dynamic Programming (DP) [10] one can approximate the “value” of the solution of Hamilton-Jacobi-Bellman (HJB) equations. In [18, 48], the authors thus show how to use finite difference schemes, Euler time integration and DP for solving finite horizon control problems. Furthermore, they give *a priori* errors estimates which are first-order in the size  $\Delta t$  of the time discretization step; however, the error involves a constant  $C(T)$  which depends *exponentially* on the length  $T$  of the finite horizon<sup>3</sup>. We solve here finite horizon control problems along the same lines (using finite difference, explicit Euler and DP) but, under the hypothesis of OSL *negativity* (see section 1.3), we obtain an error upper bound that is *linear* in  $T$  (see Section 2.4, Theorem 2).

## 1.3 Reaction-diffusion equations

It is natural to adapt the optimal control methods of ODEs to the control of Partial Differential Equations (PDEs). This can be done by transforming the PDE

<sup>3</sup>  $C(T) = O(e^{L_f T})$  where  $L_f$  is the Lipschitz constant associated with vector field  $f$ .

into (a vast system of) ODEs, using space discretization techniques such as finite difference or finite element methods. In the present work, we focus on a particular class of *non-linear* PDEs called “reaction-diffusion” equations. Reaction-diffusion equations cover a variety of particular cases with important applications in mathematical physics, and in biological models such as the Schlögl model or the FitzHugh-Nagumo system [13]. The problem of optimal control of reaction-diffusion equations has been recently the topic of many works of (classical) numerical analysis: see, e.g., [9, 15, 20, 22, 41, 42].

The notion OSL constant can be naturally extended to PDEs and reaction-diffusion equations in particular, as shown in [8, 6, 5, 7]. In these works, the authors focus on the case where the OSL constant associated with the reaction-diffusion equation is *negative*. In this case, the system has a *contractivity* (or “incremental stability”) property which expresses the fact that all solutions converge exponentially to each other (see [52]).

In this work, we also study reaction-diffusion equations with negative OSL constants, but the equations are equipped with *control inputs*, and the problem of controlling these inputs in an optimal way is here considered.

#### 1.4 Model reduction

In order to reduce the large dimension of ODE systems originating from the PDE space discretization, Model Order Reduction (MOR) techniques are often used in conjunction with the analysis of ODE systems. The idea is to first infer the optimal control at a *reduced level*, then apply it at the original level. In the field of guaranteed analysis, the MOR technique of “balanced truncation” was used to treat *linear* systems (e.g., [3, 23, 24, 34]). In [25], a MOR technique based on spectral element method was coupled to an HJB approach for application to advection-reaction-diffusion systems (cf. [26] for application to semilinear parabolic PDEs). The MOR technique of “Proper Orthogonal Decomposition (POD)” was coupled to an HJB approach in [1, 2, 30]. Here, we couple our HJB-based method to a simple *ad hoc* reduction method (see Section 2.5).

The plan of the paper is as follows: We explain how to convert the reaction-diffusion equation into a system of ODEs by domain discretization in Section 2.1, and how to approximate the solution of the latter system using the explicit Euler scheme of time integration in Section 2.2. Our procedure for solving finite horizon control problems is explained in Section 2.3. In Section 2.4, we give an upper bound to the error between the approximate value thus computed and the exact optimal value. In Section 2.5, we explain how to perform MOR in order to treat systems of larger dimension. We conclude in Section 3.

## 2 Optimal Reachability Control of Reaction-Diffusion Equations

Let us consider the special class of PDEs called “reaction-diffusion” equations. For the sake of notation simplicity, we focus on 1D reaction-diffusion equations

with Dirichlet boundary conditions (the domain  $\Omega$  is of the form  $[0, L] \subset \mathbb{R}$ ), but the method applies to 2D or 3D reaction-diffusion equations with other boundary conditions. A 1D *reaction-diffusion* system with Dirichlet boundary conditions is of the form:

$$\begin{aligned} \frac{\partial \mathbf{y}(t, x)}{\partial t} &= \sigma \frac{\partial^2 \mathbf{y}(t, x)}{\partial x^2} + f(\mathbf{y}(t, x)), \quad t \in [0, T], \quad x \in \Omega \equiv [0, L]. \\ \mathbf{y}(t, 0) &= u_0(t), \quad \mathbf{y}(t, L) = u_L(t), \quad t \in [0, T], \\ \mathbf{y}(0, x) &= \mathbf{y}_0(x), \quad x \in \Omega \equiv [0, L]. \end{aligned}$$

Here,  $\mathbf{y} = \mathbf{y}(t, x)$  is an  $\mathbb{R}$ -valued unknown function,  $\Omega$  is a bounded domain in  $\mathbb{R}$  with boundary  $\partial\Omega := \{0, L\}$ , and  $f$  is a function from  $[0, T] \times \Omega$  to  $[0, 1]$ . Also  $\mathbf{y}_0(x)$  is a given function called “initial condition”, and  $\sigma$  a positive constant, called “diffusion constant”.

The boundary control  $u(\cdot) := (u_0(\cdot), u_L(\cdot))$  that we consider here, is a *piece-wise constant* (or “staircase”) function from  $[0, T]$  to a *finite* set  $U \subset [0, 1] \times [0, 1]$ . The control  $u(t)$  changes its value *periodically* at  $t = \tau, 2\tau, \dots$ . We assume that  $T = k\tau$  for some positive integer  $k$ . The constant  $\tau$  is called the “switching (or sampling) period”.

Given an initial condition  $\mathbf{y}_0(\cdot)$  such that  $\mathbf{y}_0(x) \in [0, 1]$  for all  $x \in [0, L]$ , we assume that, for any boundary control  $u(\cdot)$ , the solution  $\mathbf{y}(\cdot, \cdot)$  of the system exists, is unique, and  $\mathbf{y}(t, x) \in [0, 1]$  for all  $(t, x) \in [0, T] \times [0, L]$ .

## 2.1 Domain discretization

A well-known approach in numerical analysis of PDEs (see, e.g., [28]) is to discretize in space by finite difference or finite element methods in order to transform the PDE into a system of ODEs.

Let  $M$  be a positive integer,  $h = L/(M+1)$ , and let  $\Omega_h$  be a uniform grid with nodes  $x_j = jh$ ,  $j = 1, \dots, M$ . By replacing the 2nd order spatial derivative with the second order centered difference, we obtain a space-discrete approximation:

$$\frac{dy}{dt} = \sigma \mathcal{L}_h y + \sigma \varphi_h(t, u) + f(t, y),$$

with  $y(t) = [y^1(t), \dots, y^M(t)]^T$ ,  $y^j(t) \approx \mathbf{y}(t, x_j)$ , and

$$\mathcal{L}_h = \frac{1}{h^2} \begin{bmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ 0 & 1 & -2 & \cdots & 0 \\ & & & \cdots & \\ 0 & 0 & \cdots & 1 & -2 \end{bmatrix}$$

$$\varphi_h(t, u) = \frac{1}{h^2} [u_0(t), 0, \dots, 0, u_L(t)]^\top.$$

The point  $y(t)$ , often abbreviated as  $y$ , is thus an element of  $S = [0, 1]^M$ .

## 2.2 Explicit Euler time integration

Let us abbreviate the equation

$$\frac{dy}{dt} = \sigma \mathcal{L}_h y + \sigma \varphi_h(t, u) + f(t, y)$$

by:

$$\frac{dy}{dt} = f_u(t, y).$$

We denote by  $Y_{t,y_0}^u$ , the solution  $y$  of the system at time  $t \in [0, \tau)$  controlled by mode  $u \in U$ , for initial condition  $y_0$ . Given a sequence of modes (or “pattern”)  $\pi := u_k \cdots u_1 \in U^k$ , we denote by  $Y_{t,y_0}^\pi$  the solution of the system for mode  $u_k$  on  $t \in [0, \tau)$  with initial condition  $y_0$ , extended continuously with the solution of the system for mode  $u_{k-1}$  on  $t \in [\tau, 2\tau)$ , and so on iteratively until mode  $u_1$  on  $t \in [(k-1)\tau, k\tau]$ .

Let us now approximate the solution of the system by performing time integration with the *explicit Euler* scheme. This yields:

$$y_{n+1} = y_n + \tau f_u(t_n, y_n),$$

Here  $y_n$  is an approximate value of  $y(t_n)$ . Given a starting point  $z \in \mathcal{X}$  and a mode  $u \in U$ , we denote by  $\tilde{Y}_{t,z}^u$  the Euler-based image of  $z$  at time  $t$  via  $u$  for  $t \in [0, \tau)$ . We have:  $\tilde{Y}_{t,z}^u := z + t f_u(z)$ . We denote similarly by  $\tilde{Y}_{t,z}^\pi$  the Euler-based image of  $z$  via pattern  $\pi \in U^k$  at time  $t \in [0, k\tau]$ .

## 2.3 Finite horizon control problems

Let us now explain the principle of the method of optimal control of ODEs used in [14], in the present context. We consider the *cost function*:  $J_k : [0, 1]^M \times U^k \rightarrow \mathbb{R}_{\geq 0}$  defined by:

$$J_k(y, \pi) = \|Y_{k\tau,y}^\pi - y_f\|,$$

where  $\|\cdot\|$  denotes the Euclidean norm in  $\mathbb{R}^M$ , and  $y_f \in [0, 1]^M$  is a given “target” state.

We consider the *value function*  $\mathbf{v}_k : [0, 1]^M \rightarrow \mathbb{R}_{\geq 0}$  defined by:

$$\mathbf{v}_k(y) := \min_{\pi \in U^k} \{J_k(y, \pi)\} \equiv \min_{\pi \in U^k} \{\|Y_{k\tau,y}^\pi - y_f\|\}.$$

Given  $k \in \mathbb{N}$  and  $\tau \in \mathbb{R}_{>0}$ , we consider the following *finite time horizon optimal control problem*: Find for each  $y \in [0, 1]^M$

– the *value*  $\mathbf{v}_k(y)$ , i.e.

$$\min_{\pi \in U^k} \{\|Y_{k\tau,y}^\pi - y_f\|\},$$

– and an *optimal pattern*:

$$\pi_k(y) := \arg \min_{\pi \in U^k} \{\|Y_{k\tau,y}^\pi - y_f\|\}.$$

In order to solve such optimal control problems, a classical “direct” method consists in *spatially discretizing* the state space  $S = [0, 1]^M$  (i.e., the space of values of  $y$ ). We consider here a uniform partition of  $S$  into a finite number  $N$  of cells of equal size: in our case, this means that interval  $[0, 1]$  is divided into  $K$  subintervals of equal size, and  $N = K^M$ . A cell thus corresponds to a  $M$ -tuple of subintervals. The center of a cell corresponds to the  $M$ -tuple of the subinterval midpoints. The associated grid  $\mathcal{X}$  is the set of centers of the cells of  $S$ . The center  $z \in \mathcal{X}$  of a cell  $C$  is considered as the  $\varepsilon$ -representative of all the points of  $C$ . We suppose that the cell size is such that  $\|y - z\| \leq \varepsilon$ , for all  $y \in C$  (i.e.  $K \geq \sqrt{M}/2\varepsilon$ ). In this context, the direct method proceeds as follows (cf. [14]): we consider the points of  $\mathcal{X}$  as the vertices of a finite oriented graph; there is a connection from  $z \in \mathcal{X}$  to  $z' \in \mathcal{X}$  if  $z'$  is the  $\varepsilon$ -representative of the Euler-based image ( $z + \tau f_u(z)$ ) of  $z$ , for some  $u \in U$ . We then compute using dynamic programming the “path of length  $k$  with minimal cost” starting at  $z$ : such a path is a sequence of  $k + 1$  connected points  $z, z_k, z_{k-1}, \dots, z_1$  of  $\mathcal{X}$  which minimizes the distance  $\|z_1 - y_f\|$ . This procedure allows us to compute a pattern  $\pi_k^\varepsilon(z)$  of length  $k$ , which approximates the optimal pattern  $\pi_k(y)$ .

**Definition 1.** *The function  $next^u : \mathcal{X} \rightarrow \mathcal{X}$  is defined by:*

$$- next^u(z) = z', \text{ where } z' \text{ is the } \varepsilon\text{-representative of } \tilde{Y}_{\tau, z}^u.$$

**Definition 2.** *For all point  $x \in \mathcal{X}$ , the spatially discrete value function  $\mathbf{v}_k^\varepsilon : \mathcal{X} \rightarrow \mathbb{R}_{\geq 0}$  is defined by:*

$$\begin{aligned} - \text{for } k = 0, \mathbf{v}_k^\varepsilon(z) &= \|z - y_f\|, \\ - \text{for } k \geq 1, \mathbf{v}_k^\varepsilon(z) &= \min_{u \in U} \{ \mathbf{v}_{k-1}^\varepsilon(next^u(z)) \}. \end{aligned}$$

**Definition 3.** *The approximate optimal pattern of length  $k$  associated to  $z \in \mathcal{X}$ , denoted by  $\pi_k^\varepsilon(z) \in U^k$ , is defined by:*

$$\begin{aligned} - \text{if } k = 0, \pi_k^\varepsilon(z) &= nil, \\ - \text{if } k \geq 1, \pi_k^\varepsilon(z) &= \mathbf{u}_k(z) \cdot \pi' \text{ where} \end{aligned}$$

$$\mathbf{u}_k(z) = \arg \min_{u \in U} \{ \mathbf{v}_{k-1}^\varepsilon(next^u(z)) \}$$

$$\text{and } \pi' = \pi_{k-1}^\varepsilon(z') \quad \text{with } z' = next^{\mathbf{u}_k(z)}(z).$$

It is easy to construct a procedure  $PROC_k^\varepsilon$  which takes a point  $z \in \mathcal{X}$  as input, and returns an approximate optimal pattern  $\pi_k^\varepsilon \in U^k$ .

*Remark 1.* The complexity of  $PROC_k^\varepsilon$  is  $O(m \times k \times N)$  where  $m$  is the number of modes ( $|U| = m$ ),  $k$  the time-horizon length ( $T = k\tau$ ) and  $N$  the number of cells of  $\mathcal{X}$  ( $N = K^M$  with  $K = \sqrt{M}/2\varepsilon$ ).

## 2.4 Error upper bound

Given a point  $y \in S$  of  $\varepsilon$ -representative  $z \in \mathcal{X}$ , and a pattern  $\pi_k^\varepsilon$  returned by  $PROC_k^\varepsilon(z)$ , we are now going to show that the distance  $\|\tilde{Y}_{k\tau,z}^{\pi_k^\varepsilon} - y_f\|$  converges to  $\mathbf{v}_k(y)$  as  $\varepsilon \rightarrow 0$ . We first consider the ODE:  $\frac{dy}{dt} = f_u(y)$ , and give an upper bound to the error between the exact solution of the ODE and its Euler approximation (see [33]).

**Definition 4.** Let  $\mu$  be a given positive constant. Let us define, for all  $u \in U$  and  $t \in [0, \tau]$ ,  $\delta_{t,\mu}^u$  as follows:

$$\text{if } \lambda_u < 0 : \quad \delta_{t,\mu}^u = \left( \mu^2 e^{\lambda_u t} + \frac{C_u^2}{\lambda_u^2} \left( t^2 + \frac{2t}{\lambda_u} + \frac{2}{\lambda_u^2} (1 - e^{\lambda_u t}) \right) \right)^{\frac{1}{2}}$$

$$\text{if } \lambda_u = 0 : \quad \delta_{t,\mu}^u = (\mu^2 e^t + C_u^2 (-t^2 - 2t + 2(e^t - 1)))^{\frac{1}{2}}$$

$$\text{if } \lambda_u > 0 : \quad \delta_{t,\mu}^u = \left( \mu^2 e^{3\lambda_u t} + \frac{C_u^2}{3\lambda_u^2} \left( -t^2 - \frac{2t}{3\lambda_u} + \frac{2}{9\lambda_u^2} (e^{3\lambda_u t} - 1) \right) \right)^{\frac{1}{2}}$$

where  $C_u$  and  $\lambda_u$  are real constants specific to function  $f_u$ , defined as follows:

$$C_u = \sup_{y \in S} L_u \|f_u(y)\|,$$

where  $L_u$  denotes the Lipschitz constant for  $f_u$ , and  $\lambda_u$  is the OSL constant associated to  $f_u$ , i.e., the minimal constant such that, for all  $y_1, y_2 \in S$ :

$$\langle f_u(y_1) - f_u(y_2), y_1 - y_2 \rangle \leq \lambda_u \|y_1 - y_2\|^2,$$

where  $\langle \cdot, \cdot \rangle$  denotes the scalar product of two vectors of  $S$ .

**Proposition 1.** [33] Consider the solution  $Y_{t,y_0}^u$  of  $\frac{dy}{dt} = f_u(y)$  with initial condition  $y_0$  of  $\varepsilon$ -representative  $z_0$  (hence such that  $\|y_0 - z_0\| \leq \varepsilon$ ), and the approximate solution  $\tilde{Y}_{t,z_0}^u$  given by the explicit Euler scheme. For all  $t \in [0, \tau]$ , we have:

$$\|Y_{t,y_0}^u - \tilde{Y}_{t,z_0}^u\| \leq \delta_{t,\varepsilon}^u.$$

**Proposition 2.** Consider the system  $\frac{dy}{dt} = f_u(y)$  with  $f_u(y) := \sigma \mathcal{L}_h y + \sigma \varphi_h(t, u) + f(y)$ . For a diffusion coefficient  $\sigma > 0$  sufficiently large, the OSL constant  $\lambda_u$  associated to  $f_u$  is such that:  $\lambda_u < 0$ .

*Proof.* Consider the ODE:  $\frac{dy}{dt} = f_u(y) = \sigma \mathcal{L}_h y + \sigma \varphi_h(t, u) + f(y)$ . For all  $y_1, y_2 \in S$ , we have:  $\langle f(y_2) - f(y_1), y_2 - y_1 \rangle \leq \lambda_f \|y_2 - y_1\|^2$ , where  $\lambda_f$  is the OSL constant of  $f$ . Hence:

$$\begin{aligned} \langle f_u(y_2) - f_u(y_1), y_2 - y_1 \rangle &= \langle \sigma \mathcal{L}_h (y_2 - y_1) + f(y_2) - f(y_1), y_2 - y_1 \rangle \\ &\leq (y_2 - y_1)^\top (\sigma \mathcal{L}_h + \lambda_f) (y_2 - y_1). \end{aligned}$$



Since  $y^\top \mathcal{L}_h y < 0$  for all  $y \in S$  (negativity of the quadratic form associated to  $\mathcal{L}_h$ ), we have:

$$\lambda_u \|y_1 - y_2\|^2 \leq (y_2 - y_1)^\top (\sigma \mathcal{L}_h + \lambda_f)(y_2 - y_1) < 0,$$

for  $\sigma > 0$  sufficiently large. Hence  $\lambda_u < 0$ .  $\square$

**Lemma 1.** Consider the system  $\frac{dy}{dt} = f_u(y)$  where the OSL constant  $\lambda_u$  associated to  $f_u$  is negative, and initial error  $e_0 := \|y_0 - z_0\| > 0$ . Let  $G_u := \frac{\sqrt{3}e_0|\lambda_u|}{C_u}$ . Consider the (smallest) positive root

$$\alpha_u := 1 + |\lambda_u|G_u/4 - \sqrt{1 + (\lambda_u G_u/4)^2}$$

of equation:  $-\frac{1}{2}|\lambda_u|G_u + (2 + \frac{1}{2}|\lambda_u|G_u)\alpha - \alpha^2 = 0$ .

Suppose:  $\frac{|\lambda_u|G_u}{4} < 1$ . Then we have  $0 < \alpha_u < 1$ , and, for all  $t \in [0, \tau]$  with  $\tau \leq G_u(1 - \alpha_u)$ :

$$\delta_{e_0}^u(t) \leq e_0.$$

*Proof.* See Appendix 1.

*Remark 2.* In practical case studies  $|\lambda_u|$  is often small, and the term  $(\lambda_u G_u/4)^2$  can be neglected, leading to  $\alpha_u \approx |\lambda_u|G_u/4$  and  $G_u(1 - \alpha_u) \approx G_u(1 - \frac{|\lambda_u|G_u}{4}) \approx G_u$ .

*Remark 3.* It follows that, for  $\tau \leq G_u(1 - \alpha_u)$ , the Euler explicit scheme is *stable*, in the sense that initial errors are damped out.

*Remark 4.* If  $\tau > G_u(1 - \alpha_u)$ , we can make use of *subsampling*, i.e., decompose  $\tau$  into a sequence of elementary time steps  $\Delta t$  with  $\Delta t \leq G_u(1 - \alpha_u)$  in order to be still able to apply Lemma 1 (see Example 1). Let us point out that Lemma 1 (and the use of subsampling) allows to ensure set-based reachability with the use of procedure  $PROC_k^\varepsilon$ . Indeed, in this setting, the explicit Euler scheme leads to decreasing errors, and thus, point based computations performed with the center of a cell can be applied to the entire cell.

We suppose henceforth that the system  $\frac{dy}{dt} = f_u(y)$  satisfies:

$$(H) : \quad \lambda_u < 0, \quad \frac{|\lambda_u|G_u}{4} < 1 \quad \text{and} \quad \tau \leq G_u(1 - \alpha_u), \quad \text{for all } u \in U.$$

From Proposition 1 and Lemma 1, it easily follows:

**Theorem 1.** Consider a system  $\frac{dy}{dt} = f_u(y)$  satisfying (H), and a point  $y \in S$  of  $\varepsilon$ -representative  $z \in \mathcal{X}$ . We have:

$$\|Y_{t,y}^\pi - \tilde{Y}_{t,z}^\pi\| \leq \varepsilon, \quad \text{for all } \pi \in U^k \text{ and } t \in [0, k\tau].$$

**Proposition 3.** Let  $z \in \mathcal{X}$  and  $\pi_k^\varepsilon$  be the pattern of  $U^k$  returned by  $PROC_k^\varepsilon(z)$ . For all  $\pi \in U^k$ , we have:

$$\|\tilde{Y}_{k\tau,z}^{\pi_k^\varepsilon} - y_f\| \leq \|\tilde{Y}_{k\tau,z}^\pi - y_f\| + 2k\varepsilon.$$

*Proof.* W.l.o.g., let us suppose that  $y_f$  is the origin  $O$ . Let us prove by induction on  $k$ :

$$\|\tilde{Y}_{k\tau,z}^{\pi_k^\varepsilon}\| \leq \|\tilde{Y}_{k\tau,z}^\pi\| + 2k\varepsilon.$$

Let  $\pi_k^\varepsilon := u_k \cdots u_1$ . The base case  $k = 1$  is easy. For  $k \geq 2$ , we have:

$$\begin{aligned} \|\tilde{Y}_{k\tau,z}^{\pi_k^\varepsilon}\| &= \|\tilde{Y}_{(k-1)\tau,z_k}^{u_{k-1}\cdots u_1}\| \text{ with } z_k = \tilde{Y}_{\tau,z}^{u_k} \text{ with } u_k = \operatorname{argmin}_{u \in U} \{\mathbf{v}_{k-1}^\varepsilon(\operatorname{next}^u(z))\} \\ &\leq \|\tilde{Y}_{(k-1)\tau,\operatorname{next}^{u_k}(z_k)}^{u_{k-1}\cdots u_1}\| + \varepsilon \\ &\leq \|\tilde{Y}_{(k-1)\tau,\operatorname{next}^{u_k}(z_k)}^{\pi'}\| + (2k-1)\varepsilon \text{ for all } \pi' \in U^{k-1} \text{ by induction hypothesis,} \\ &\leq \|\tilde{Y}_{(k-1)\tau,z'}^{\pi'}\| + 2k\varepsilon \text{ for all } \pi' \in U^{k-1} \text{ and all } z' \in \{\operatorname{next}^u(z) \mid u \in U\} \\ &\leq \|\tilde{Y}_{\tau,z}^\pi\| + 2k\varepsilon \text{ for all } \pi \in U^k. \end{aligned}$$

□

**Theorem 2.** Let  $y \in S$  be a point of  $\varepsilon$ -representative  $z \in \mathcal{X}$ . Let  $\pi_k^\varepsilon$  be the pattern returned by  $\operatorname{PROC}_k^\varepsilon(z)$ , and  $\pi^* := \operatorname{argmin}_{\pi \in U^k} \|Y_{k\tau,y}^\pi - y_f\|$ . The discretization error  $E_\varepsilon(T) := \|\tilde{Y}_{k\tau,z}^{\pi_k^\varepsilon} - y_f\| - \mathbf{v}_k(y)$ , with  $\mathbf{v}_k(y) := \|Y_{k\tau,y}^{\pi^*} - y_f\|$  and  $T = k\tau$ , satisfies:

$$E_\varepsilon(T) \leq (2k+1)\varepsilon.$$

It follows that  $\|\tilde{Y}_{k\tau,z}^{\pi_k^\varepsilon} - y_f\|$  converges to  $\mathbf{v}_k(y)$  as  $\varepsilon \rightarrow 0$ .

*Proof.* W.l.o.g., let us suppose that  $y_f$  is the origin  $O$ . For all  $\pi \in U^k$ , we have by Proposition 3 and Theorem 1:

$$\|\tilde{Y}_{k\tau,z}^{\pi_k^\varepsilon}\| \leq \|\tilde{Y}_{k\tau,z}^\pi\| + 2k\varepsilon \leq \|Y_{k\tau,y}^\pi\| + (2k+1)\varepsilon.$$

Hence

$$\|\tilde{Y}_{k\tau,z}^{\pi_k^\varepsilon}\| \leq \min_{\pi \in U^k} \|Y_{k\tau,y}^\pi\| + (2k+1)\varepsilon = \|Y_{k\tau,y}^{\pi^*}\| + (2k+1)\varepsilon.$$

On the other hand, for all  $\pi \in U^k$ , it follows from Theorem 1:

$$\|Y_{k\tau,y}^{\pi^*}\| \leq \|Y_{k\tau,y}^\pi\| \leq \|\tilde{Y}_{k\tau,z}^\pi\| + \varepsilon.$$

Hence:

$$\|Y_{k\tau,y}^{\pi^*}\| \leq \|\tilde{Y}_{k\tau,z}^{\pi_k^\varepsilon}\| + \varepsilon.$$

Therefore we have:  $\|\tilde{Y}_{k\tau,z}^{\pi_k^\varepsilon}\| - \|Y_{k\tau,y}^{\pi^*}\| \leq (2k+1)\varepsilon$ . □

*Remark 5.* The error bound  $E_\varepsilon(T)$  is thus *linear* in  $k = T/\tau$ . In order to decrease  $k$ , one can apply consecutively  $p \geq 2$  modes *in a row* (without intermediate  $\varepsilon$ -approximation); this is equivalent to divide  $k$  by  $p$ , at the price of considering  $m^p$  “extended” modes instead of just  $m$  modes. (see Example 1, Figure 2). An alternative for decreasing  $k$  is to increase  $\tau$  (which may require in turn to decrease  $\Delta t$  for preserving assumption  $\Delta t \leq G_u(1 - \alpha_u)$ , see Remark 4).

*Example 1.* Consider the 1D reaction-diffusion system with Dirichlet boundary condition (see [45], bistable case):

$$\begin{aligned}\frac{\partial y(t, x)}{\partial t} &= \sigma \frac{\partial^2 y(t, x)}{\partial x^2} + f(y(t, x)), \quad t \in [0, T], \quad x \in [0, L] \\ y(t, 0) &= u_0, \quad y(t, L) = u_L, \\ y(0, x) &= y_0(x), \quad x \in [0, L]\end{aligned}$$

with  $\sigma = 1$ ,  $L = 4$  and  $f(y) = y(1-y)(y-\theta)$  with  $\theta = 0.3$ . The control switching period is  $\tau = 0.1$ . The values of the boundary control  $u = (u_0, u_L)$  are in

$$U = \{(0, 0), (0.2, 0.2), (0.4, 0.4), (0.6, 0.6), (0.8, 0.8), (1, 1)\}^4$$

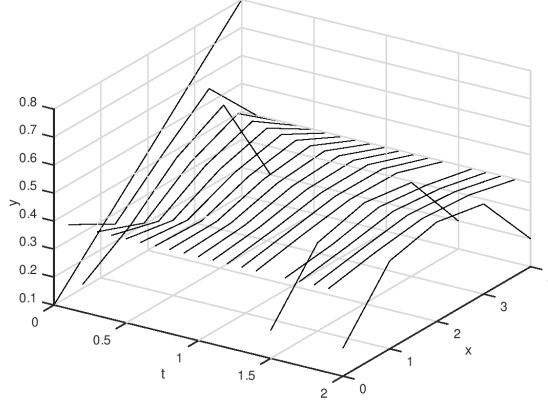
We discretize the domain  $\Omega = [0, L]$  of the system with  $M_1 = 5$  discrete points, using a finite difference scheme. Our program returns an OSL constant  $\lambda_u = -0.322$  for all  $u \in U$ . Constant  $C_u$  varies between 10.33 and 11.85 depending on the values of  $u$ .

We then discretize each interval component of the space  $S = [0, 1]^{M_1}$  of values of  $y$  into 15 points with spacing  $\eta = 1/15 \approx 0.066$ . The grid  $\mathcal{X}$  is of the form  $\{0, \eta, 2\eta, \dots, 15\eta\}^{M_1}$ , and the initial error  $e_0$  equal to  $\varepsilon = \sqrt{M_1}\eta/2$ . This leads to  $G_u$  varying between 0.00155 and 0.00178 depending on the value of  $u \in U$ . One checks:  $\frac{|\lambda_u|G_u}{4} < 1$  for all  $u \in U$ . The time step upper bound required by Theorem 1 for ensuring numeric stability is 0.00155. Since the switching period is  $\tau = 0.1$ , we perform *subsampling* (see, e.g., [33]) by decomposing every time step  $[i\tau, (i+1)\tau)$  ( $1 \leq i \leq k-1$ ) into a sequence of elementary Euler steps of length  $\Delta t = \tau/100 < 0.00155$ . This ensures that the system satisfies (H), hence, by Theorem 1, the explicit Euler scheme is stable and error  $\|Y_{t, y_0}^\pi - \tilde{Y}_{t, z_0}^\pi\|$  never exceeds  $\varepsilon$ .

For objective with  $y_f = (0.3, 0.3, 0.3, 0.3, 0.3)$  and horizon time  $T = k\tau = 2$  (i.e.,  $k = 20$ ), our program<sup>5</sup> returns an approximate optimal controller in 2 minutes. Let  $z_0$  be the  $\varepsilon$ -representative of  $y_0 = 0.8x/L + 0.1(1-x/L)$ . Let  $\pi_k^\varepsilon$  be the pattern output by  $PROC_k^\varepsilon(z_0)$ . A simulation of  $z(t) := \tilde{Y}_{t, z_0}^{\pi_k^\varepsilon}$  is given in Figure 1 with  $T = 2$ ,  $\tau = 0.1$  ( $k = 20$ ),  $\Delta t = \frac{\tau}{100}$ . We have  $\|z(T) - y_f\| \approx 0.276$ . The simulation presents some similarity with simulations displayed in [45] (see, e.g., lower part of Figure 6), with a phase control  $u_0 = u_L > \theta$  (here,  $u_0 = u_L = 0.4$ ) alternating with a phase control  $u_0 = u_L < \theta$  (here,  $u_0 = u_L = 0.2$ ). The discretization error  $E_\varepsilon(T)$  is smaller than  $(2k+1)\varepsilon = 41\sqrt{5}/30 < 3.1$ .

<sup>4</sup> Note that, in [45], the values of the boundary control are in the full interval  $[0, 1]$ , not in a finite set  $U$  as here. In [45], they focus, not on the bounding of computation errors during integration as here, but on a formal proof that the objective state  $y_f = \theta$  ( $0 < \theta < 1$ ) is reachable in finite time iff  $L < L^*$  for some threshold value  $L^*$ .

<sup>5</sup> The program, called ‘‘OSLator’’ [31], is implemented in Octave. It is composed of 10 functions and a main script totalling 600 lines of code. The computations are realised in a virtual machine running Ubuntu 18.06 LTS, having access to one core of a 2.3GHz Intel Core i5, associated to 3.5 GB of RAM memory.



**Fig. 1.** Simulation of the system of Example 1 discretized with  $M_1 = 5$  points, for initial condition  $y_0 = 0.8x/L + 0.1(1 - x/L)$ , objective  $y_f = 0.3$  and horizon time  $T = 2$  ( $\tau = 0.1$ ,  $\Delta t = \frac{\tau}{100}$ ).

Let us now proceed with extended modes of length  $p = 2$  and  $p = 4$ , as explained in Remark 5. For  $p = 2$  (i.e.,  $k = 10$ ), the control is synthesized in 7mn of CPU time. The controller simulation is given in the left part of Figure 2; we have:  $\|z(T) - y_f\| \approx 0.445$  with  $E_\varepsilon(T) < 1.57$ . For  $p = 4$  (i.e.,  $k = 5$ ), the computation of the control requires 8h of CPU time. The corresponding simulation is given in the right part of Figure 2; we now have:  $\|z(T) - y_f\| \approx 0.164$  with  $E_\varepsilon(T) < 0.82$ .

## 2.5 Model reduction

Let us consider the system  $\mathcal{S}_2$  on space  $S_{h_2} = [0, 1]^{M_2}$  (with  $M_2$  even). The differential equation can be written under the form:

$$\frac{dy_2}{dt} = \sigma \mathcal{L}_{h_2} y_2 + \varphi_{h_2}(u) + f(y_2).$$

where  $\mathcal{L}_{h_2}$  corresponds to the  $(M_2 \times M_2)$  Laplacian matrix, and  $h_2 = \frac{L}{M_2+1}$ .

Let us consider the “reduced” system  $\mathcal{S}_1$  defined on  $S_{h_1} = [0, 1]^{M_1}$  with  $M_1 = M_2/2$ , defined by:

$$\frac{dy_1}{dt} = \sigma \mathcal{L}_{h_1} y_1 + \varphi_{h_1}(u) + f(y_1),$$

where  $\mathcal{L}_{h_1}$  is the  $(M_1 \times M_1)$  Laplacian matrix and  $h_1 = \frac{L}{M_1+1}$ .

With  $M_1 = M_2/2$ , we have  $h_2 = \frac{L}{2M_1+1}$  ( $= \frac{h_1(M_1+1)}{2M_1+1}$ ). Let us consider the  $(M_1 \times M_2)$  reduction matrix:

$$\Pi := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 1 & \cdots & 0 \\ & & & \cdots & & \\ 0 & 0 & \cdots & 0 & 1 & 1 \end{bmatrix}$$

Note that  $\Pi\Pi^\top = \mathcal{I}_M$ . Let us consider a point  $w_0 \in S_{h_2}$ , and let  $z_0 = \Pi w_0 \in S_{h_1}$ .

**Theorem 3.** Consider the system  $\mathcal{S}_2$  and a point  $w_0 \in S_{h_2}$ , and let  $z_0 = \Pi w_0 \in S_{h_1}$ . Let  $Y_{w_0}^{h_2}$  and  $Y_{z_0}^{h_1}$  be the solutions of  $\mathcal{S}_2$  and  $\mathcal{S}_1$  with initial conditions  $w_0 \in S_{h_2}$  and  $z_0 \in S_{h_1}$  respectively. We have:

$$\forall t \geq 0 \quad \|\Pi Y_{w_0}^{h_2}(t) - Y_{z_0}^{h_1}(t)\| \leq \frac{K_2 \sigma}{|\lambda_{h_1}|},$$

where

$$K_2 := \sup_{w \in S_{h_2}} \|( \Pi \mathcal{L}_{h_2} - \mathcal{L}_{h_1} \Pi ) w \|,$$

and  $\mathcal{L}_{h_2}$  (resp.  $\mathcal{L}_{h_1}$ ) is the Laplacian matrix of size  $M_2 \times M_2$  (resp.  $M_1 \times M_1$ ).

*Proof.* Let us consider the system  $\mathcal{S}_2$ :

$$\frac{dy_2}{dt} = \sigma \mathcal{L}_{h_2} y_2 + \varphi_{h_2}(u) + f(y_2).$$

By application of the projection matrix  $\Pi$ , we get:

$$\frac{d\Pi y_2}{dt} = \sigma \Pi \mathcal{L}_{h_2} y_2 + \varphi_{h_1}(u) + f(\Pi y_2).$$

By subtracting pairwise with the sides of  $\mathcal{S}_1$ , we have:

$$\begin{aligned} \frac{d\Pi y_2}{dt} - \frac{dy_1}{dt} &= \sigma(\Pi \mathcal{L}_{h_2} y_2 - \mathcal{L}_{h_1} y_1) + f(\Pi y_2) - f(y_1) \\ &= F_{h_1}(\Pi y_2) - F_{h_1}(y_1) + \sigma(\Pi \mathcal{L}_{h_2} - \mathcal{L}_{h_1} \Pi) y_2, \end{aligned}$$

where  $F_{h_1}(y) = \sigma \mathcal{L}_{h_1}(y) + f(y)$  for  $y \in S_{h_1}$ . On the other hand, we have:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\|\Pi y_2 - y_1\|^2) &= \left\langle \frac{d}{dt} (\Pi y_2 - y_1), \Pi y_2 - y_1 \right\rangle \\ &= \left\langle F_{h_1}(\Pi y_2) - F_{h_1}(y_1) + \sigma(\Pi \mathcal{L}_{h_2} - \mathcal{L}_{h_1} \Pi) y_2, \Pi y_2 - y_1 \right\rangle \\ &= \left\langle F_{h_1}(\Pi y_2) - F_{h_1}(y_1), \Pi y_2 - y_1 \right\rangle \\ &\quad + \sigma \left\langle (\Pi \mathcal{L}_{h_2} - \mathcal{L}_{h_1} \Pi) y_2, \Pi y_2 - y_1 \right\rangle \\ &\leq \lambda_{h_1} \|\Pi y_2 - y_1\|^2 + \sigma \left\langle (\Pi \mathcal{L}_{h_2} - \mathcal{L}_{h_1} \Pi) y_2, \Pi y_2 - y_1 \right\rangle \\ &\leq \lambda_{h_1} \|\Pi y_2 - y_1\|^2 + K_2 \sigma \|\Pi y_2 - y_1\| \end{aligned}$$

$$\text{with } K_2 := \sup_{w \in S_{h_2}} \|( \Pi \mathcal{L}_{h_2} - \mathcal{L}_{h_1} \Pi ) w \||$$

$\leq \lambda_{h_1} \| \Pi y_2 - y_1 \|^2 + K_2 \sigma \frac{1}{2} (\alpha \| \Pi y_2 - y_1 \|^2 + \frac{1}{\alpha})$ ,  
for all  $\alpha > 0$ . Choosing  $\alpha > 0$  such that  $K_2 \sigma \alpha = -\lambda_{h_1}$ , i.e.:  $\alpha = -\frac{\lambda_{h_1}}{K_2 \sigma}$ , we have:

$$\frac{1}{2} \frac{d}{dt} (\| \Pi y_2 - y_1 \|^2) \leq \frac{\lambda_{h_1}}{2} \| \Pi y_2 - y_1 \|^2 - \frac{(K_2 \sigma)^2}{2 \lambda_{h_1}}.$$

Since  $y_2(0) = w_0$  and  $y_1(0) = z_0$ , we get by integration:

$$\| \Pi y_2(t) - y_1(t) \|^2 \leq \frac{(K_2 \sigma)^2}{\lambda_{h_1}^2} (1 - e^{\lambda_{h_1} t}) \leq \frac{(K_2 \sigma)^2}{\lambda_{h_1}^2}.$$

Hence:  $\| \Pi Y_{w_0}^{h_2}(t) - Y_{z_0}^{h_1}(t) \| \leq \frac{K_2 \sigma}{|\lambda_{h_1}|}$  for all  $t \geq 0$ .  $\square$

This proposition expresses that the reduction error is bounded by constant  $\frac{K_2 \sigma}{|\lambda_{h_1}|}$  when the same control modes are applied to both systems.<sup>6</sup>

Let  $y_2^0 \in \mathcal{S}_2$  and  $y_2^f \in \mathcal{S}_2$  be an initial and objective point respectively. Let  $y_1^0 := \Pi y_2^0 \in \mathcal{S}_1$  and  $y_1^f := \Pi y_2^f \in \mathcal{S}_1$  denote their projections. Suppose that  $\pi^\varepsilon$  is the pattern returned by  $PROC_k^\varepsilon(y_1^0)$  for the reduced system  $\mathcal{S}_1$ . Then, from Theorem 3, it follows that, when the *same* control  $\pi^\varepsilon$  is applied to the original system  $\mathcal{S}_2$  with  $y_2(0) = y_2^0 \in \mathcal{S}_2$ , it makes the projection  $\Pi y_2^{\pi^\varepsilon}(t) \in \mathcal{S}_1$  reach a *neighborhood* of  $y_1^f$  at time  $t = T$ . Formally, we have:

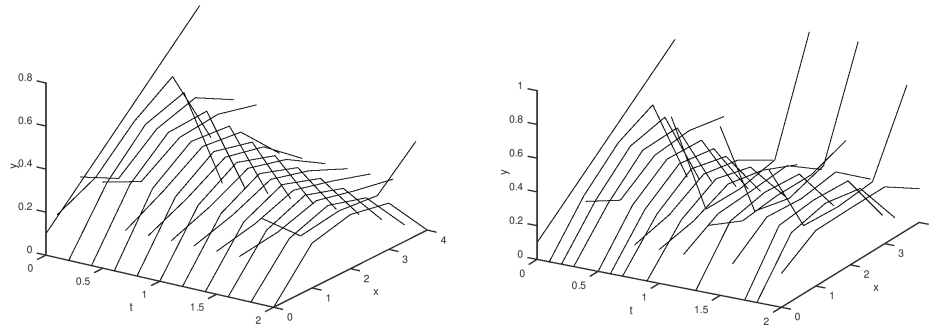
$$\| \Pi y_2^{\pi^\varepsilon}(T) - y_1^f \| \leq \| y_1^{\pi^\varepsilon}(T) - y_1^f \| + \frac{K_2 \sigma}{|\lambda_{h_1}|}.$$

*Example 2.* Let us take the system defined in Example 1 as reduced system  $\mathcal{S}_1$  ( $M_1 = 5$ ), and let us take as “full-size” system  $\mathcal{S}_2$  the system corresponding to  $M_2 = 10$ . Since the size of the grid  $\mathcal{X}_2$  associated to  $\mathcal{S}_2$  is exponential in  $M_2$ , the size  $\mathcal{X}_2$  is multiplied by  $(1/\eta)^{M_2 - M_1} = 15^5 \approx 7.6 \cdot 10^5$  w.r.t. the size of the grid  $\mathcal{X}_1$  associated to  $\mathcal{S}_1$ . The complexity for synthesizing directly the optimal control of  $\mathcal{S}_2$  thus becomes intractable. On the other hand, if we apply to  $\mathcal{S}_2$  the optimal strategy  $\pi^\varepsilon \in U^k$  found for  $\mathcal{S}_1$  in Example 1, we obtain a simulation depicted in Figure 3 for extended mode of length 1, which is the counterpart of Figure 1 with  $M_2 = 10$  (instead of  $M_1 = 5$ ), and has a very similar form. Likewise, if we apply to  $\mathcal{S}_2$  the optimal strategy  $\pi^\varepsilon \in U^k$  found for  $\mathcal{S}_1$  in Example 1, we obtain a simulation depicted in Figure 4 for extended modes of length 2 and 4, which is the counterpart of Figure 2, and very similar to it. As seen above, we have:

$$\| \Pi y_2^{\pi^\varepsilon}(T) - y_1^f \| \leq \| y_1^{\pi^\varepsilon}(T) - y_1^f \| + \frac{K_2 \sigma}{|\lambda_{h_1}|},$$

where  $y_1^f = (0.3, 0.3, 0.3, 0.3, 0.3)$ , and the reduction error is bounded by  $\frac{K_2 \sigma}{|\lambda_{h_1}|} = 17.9 \sigma$ .

<sup>6</sup> By comparison, in [2], the error term originating from the POD model reduction is *exponential* in  $T$  (see  $C_1(T, |x|)$  in the proof of Theorem 5.1).



**Fig. 2.** Simulation of the system of Example 1 discretized with  $M_1 = 5$  points, with extended modes of length 2 (left) and extended modes of length 4 (right).

The subexpression  $\|y_1^{\pi^\varepsilon}(T) - y_1^f\|$  can be computed *a posteriori* by simulation: see Table 1 of Appendix 2, with  $\sigma = 1$ ,  $\sigma = 0.5$ . The value of  $\|y_2^{\pi^\varepsilon}(T) - y_2^f\|$  for  $\mathcal{S}_2$  is also given in Table 1 for comparison.

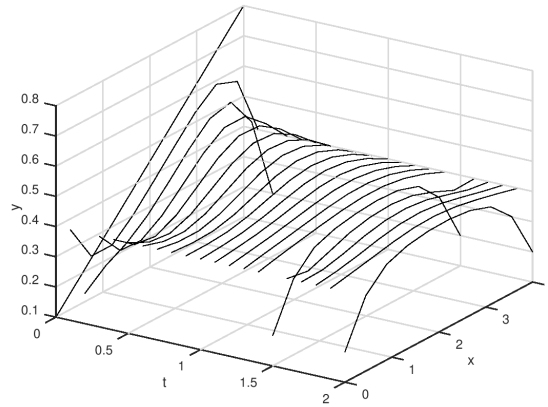
The upper bound  $\|y_1^{\pi^\varepsilon}(T) - y_1^f\| + \frac{K_2\sigma}{|\lambda_{h_1}|}$  of the distance  $\|Py_2^{\pi^\varepsilon}(T) - y_1^f\|$  is very conservative, due to *a priori* error bound  $\frac{K_2\sigma}{|\lambda_{h_1}|}$ . One can obtain *a posteriori* a much sharper estimate of  $\|Py_2^{\pi^\varepsilon}(T) - y_1^f\|$  by simulation: see Table 2, Appendix 2.

### 3 Final Remarks

Using the notion of OSL constant, we have shown how to use the finite difference and explicit Euler methods in order to solve finite horizon control problems for reaction-diffusion equations. Furthermore, we have quantified the deviation of this control with the optimal strategy, and proved that the error upper bound is *linear* in the horizon length. We have applied the method to a 1D bi-stable reaction-diffusion equation, and have found experimental results similar to those of [45]. We have also given a simple and specific model reduction method which allows to apply the method to equations of larger size. In future work, we plan to apply the method to 2D reaction-diffusion equations (e.g., Test 1 of [2]).

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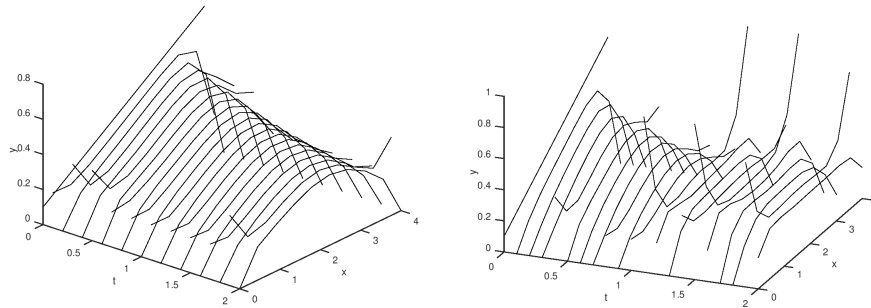
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**Fig. 3.** Simulation of the system of Ex. 1, discretized with  $M_2 = 10$  points, with extended mode of length 1.

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**Fig. 4.** Simulation of the system of Ex. 1, discretized with  $M_2 = 10$  points, with extended modes of length 2 (left) and extended modes of length 4 (right).

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## Appendix 1: Proof of Lemma 1

*Proof.* It is easy to check that  $0 < \alpha_u < 1$  when  $\frac{|\lambda_u|G_u}{4} < 1$ .

Let  $t^* := G_u(1 - \alpha_u)$ . Let us first prove  $\delta_{e_0}(t) \leq e_0$  for  $t = t^*$ . We have:

$$-\frac{1}{2}|\lambda_u|G_u + (2 + \frac{1}{2}|\lambda_u|G_u)\alpha_u - \alpha_u^2 = 0.$$

Hence:

$$\frac{1}{2G_u(1 - \alpha_u)}\lambda_u G_u^2(1 - \alpha_u)^2 + 2\alpha_u - \alpha_u^2 = 0,$$

i.e.

$$\frac{1}{2t^*}\lambda_u(t^*)^2 + 2\alpha_u - \alpha_u^2 = 0.$$

We have:  $-\frac{1}{4G_u^2 t^*}\lambda_u(t^*)^4 e^{\lambda_u t^*} \geq 0$ . It follows:

$$\frac{1}{2t^*}\lambda_u(t^*)^2 + 2\alpha_u - \alpha_u^2 - \frac{1}{4G_u^2 t^*}\lambda_u(t^*)^4 e^{\lambda_u t^*} \geq 0.$$

Hence:

$$1 + \frac{1}{2t^*}\lambda_u(t^*)^2 - \frac{1}{G_u^2}((t^*)^2 + \frac{1}{4t^*}\lambda_u(t^*)^4 e^{\lambda_u t^*}) \geq 0.$$

By multiplying by  $t^*$ :

$$(t^* + \frac{1}{2}\lambda_u(t^*)^2) - \frac{1}{G_u^2}((t^*)^3 + \frac{1}{4}\lambda_u(t^*)^4 e^{\lambda_u t^*}) \geq 0.$$

Since  $G = \sqrt{3}|\lambda_u|e_0/C_u$ :

$$e_0^2(t^* + \frac{1}{2}\lambda_u(t^*)^2) + \frac{C_u^2}{\lambda_u^2}(-\frac{1}{3}(t^*)^3 - \frac{1}{12}\lambda_u(t^*)^4 e^{\lambda_u t^*}) \geq 0.$$

By multiplying by  $\lambda_u$ :

$$e_0^2(\lambda_u t^* + \frac{1}{2}\lambda_u^2(t^*)^2) + \frac{C_u^2}{\lambda_u^2}(-\frac{1}{3}\lambda_u(t^*)^3 - \frac{1}{12}\lambda_u^2(t^*)^4 e^{\lambda_u t^*}) \leq 0.$$

Note that, in the above formula, the subexpression  $\lambda_u t^* + \frac{1}{2}\lambda_u^2(t^*)^2$  is such that:

$$\lambda_u t^* + \frac{1}{2}\lambda_u^2(t^*)^2 \geq e^{\lambda_u t^*} - 1$$

since  $e^{\lambda_u t^*} - 1 = \lambda_u t^* + \frac{1}{2}\lambda_u^2(t^*)^2 e^{\lambda\theta} \leq \lambda_u t^* + \frac{1}{2}\lambda_u^2(t^*)^2$ .

On the other hand, the subexpression  $-\frac{1}{3}\lambda_u(t^*)^3 - \frac{1}{12}\lambda_u^2(t^*)^4 e^{\lambda_u t^*}$  is such that:

$$-\frac{1}{3}\lambda_u(t^*)^3 - \frac{1}{12}\lambda_u^2(t^*)^4 e^{\lambda_u t^*} \geq \frac{2t^*}{\lambda_u} + (t^*)^2 + \frac{2}{\lambda_u^2}(1 - e^{\lambda_u t^*})$$

since

$$\begin{aligned} & \frac{2t^*}{\lambda_u} + (t^*)^2 + \frac{2}{\lambda_u^2}(1 - e^{\lambda_u t^*}) \\ &= \frac{2t^*}{\lambda_u} + (t^*)^2 + \frac{2}{\lambda_u^2}(-\lambda_u t^* - \frac{1}{2}\lambda_u^2(t^*)^2 - \frac{1}{6}\lambda_u^3(t^*)^3 - \frac{1}{24}\lambda_u^4(t^*)^4 e^{\lambda_u \theta}) \\ &= \frac{2}{\lambda_u^2}(-\frac{1}{6}\lambda_u^3(t^*)^3 - \frac{1}{24}\lambda_u^4(t^*)^4 e^{\lambda_u \theta}) \text{ for some } 0 \leq \theta \leq t^* \\ &= -\frac{1}{3}\lambda_u(t^*)^3 - \frac{1}{12}\lambda_u^2(t^*)^4 e^{\lambda_u \theta} \\ &\leq -\frac{1}{3}\lambda_u(t^*)^3 - \frac{1}{12}\lambda_u^2(t^*)^4 e^{\lambda_u t^*}. \end{aligned}$$

It follows:

$$e_0^2(e^{\lambda_u t^*} - 1) + \frac{C_u^2}{\lambda_u^2}(\frac{2t^*}{\lambda_u} + (t^*)^2 + \frac{2}{\lambda_u^2}(1 - e^{\lambda_u t^*})) \leq 0.$$

$$e_0^2 e^{\lambda_u t^*} + \frac{C_u^2}{\lambda_u^2}(\frac{2t^*}{\lambda_u} + (t^*)^2 + \frac{2}{\lambda_u^2}(1 - e^{\lambda_u t^*})) \leq e_0^2.$$

i.e.

$$(\delta_{e_0}^u(t^*))^2 \leq e_0^2.$$

Hence:  $\delta_{e_0}^u(t^*) \leq e_0$ . It remains to show:  $\delta_{e_0}^u(t) \leq e_0$  for  $t \in [0, t^*]$ .

Consider the 1st and 2nd derivative  $\delta'(\cdot)$  and  $\delta''(\cdot)$  of  $\delta(\cdot)$ . We have:

$$\delta'(t) = \lambda_u e_0^2 e^{\lambda_u t} + \frac{C_u^2}{\lambda_u^2}(2t + \frac{2}{\lambda_u} - \frac{2}{\lambda_u} e^{\lambda_u t})$$

$$\delta''(t) = \lambda_u^2 e_0^2 e^{\lambda_u t} + \frac{C_u^2}{\lambda_u^2}(2 - 2e^{\lambda_u t}).$$

Hence  $\delta''(t) > 0$  for all  $t \geq 0$ . On the other hand, for  $t = 0$ ,  $\delta'(t) = \lambda_u e_0^2 < 0$ , and for  $t$  sufficiently large,  $\delta'(t) > 0$ . Hence,  $\delta'(\cdot)$  is strictly increasing and has a unique root. It follows that the equation  $\delta(t) = e_0$  has a unique solution  $t^{**}$  for  $t > 0$ . Besides,  $\delta(t) \leq e_0$  for  $t \in [0, t^{**}]$ , and  $\delta(t) \geq e_0$  for  $t \in [t^{**}, +\infty)$ . Since we have shown:  $\delta(t^*) \leq e_0$ , it follows  $t^* \leq t^{**}$  and  $\delta(t) \leq e_0$  for  $t \in [0, t^*]$ .  $\square$

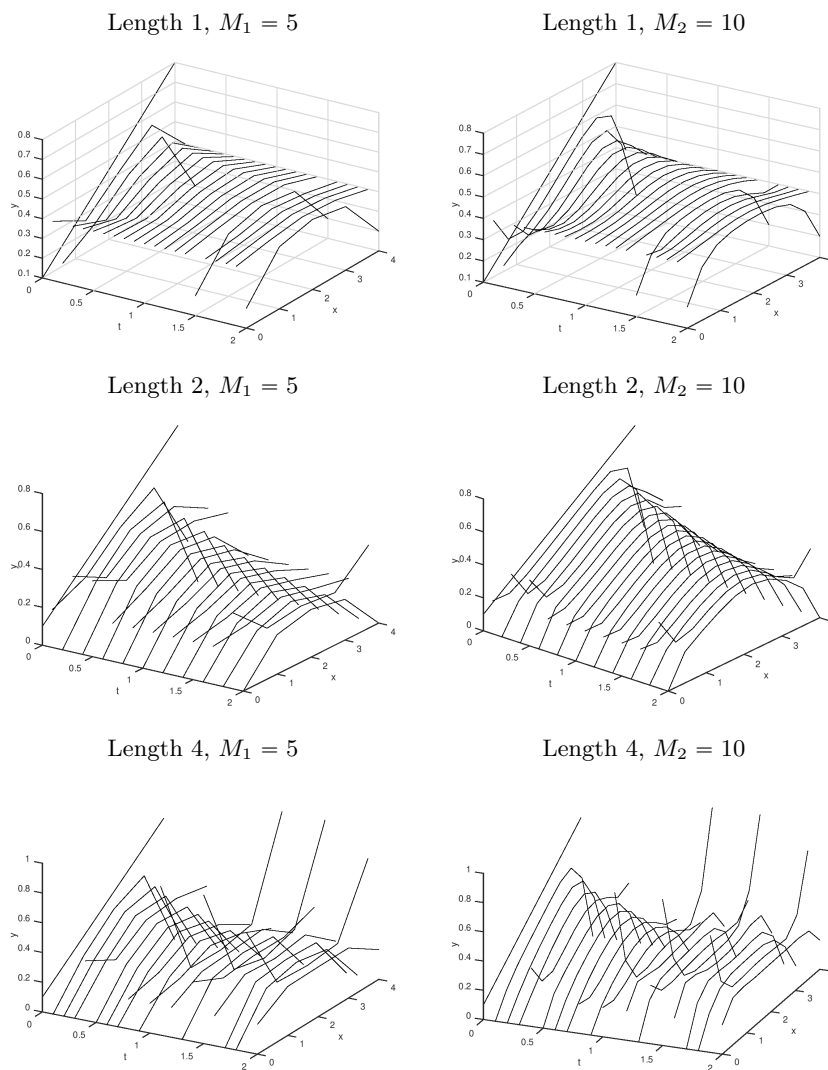
## Appendix 2: Numerical results

Dimension	Extended mode length	$\ y_i^{\pi^\varepsilon}(T) - y_i^f\ $ for $\sigma = 1$	$\ y_i^{\pi^\varepsilon}(T) - y_i^f\ $ for $\sigma = 0.5$
$i = 1$ ( $M_i = 5$ )	1	0.27642	0.33869
	2	0.44496	0.39068
	4	0.15294	0.22024
$i = 2$ ( $M_i = 10$ )	1	0.39904	0.50251
	2	0.50092	0.58500
	4	0.16738	0.31440

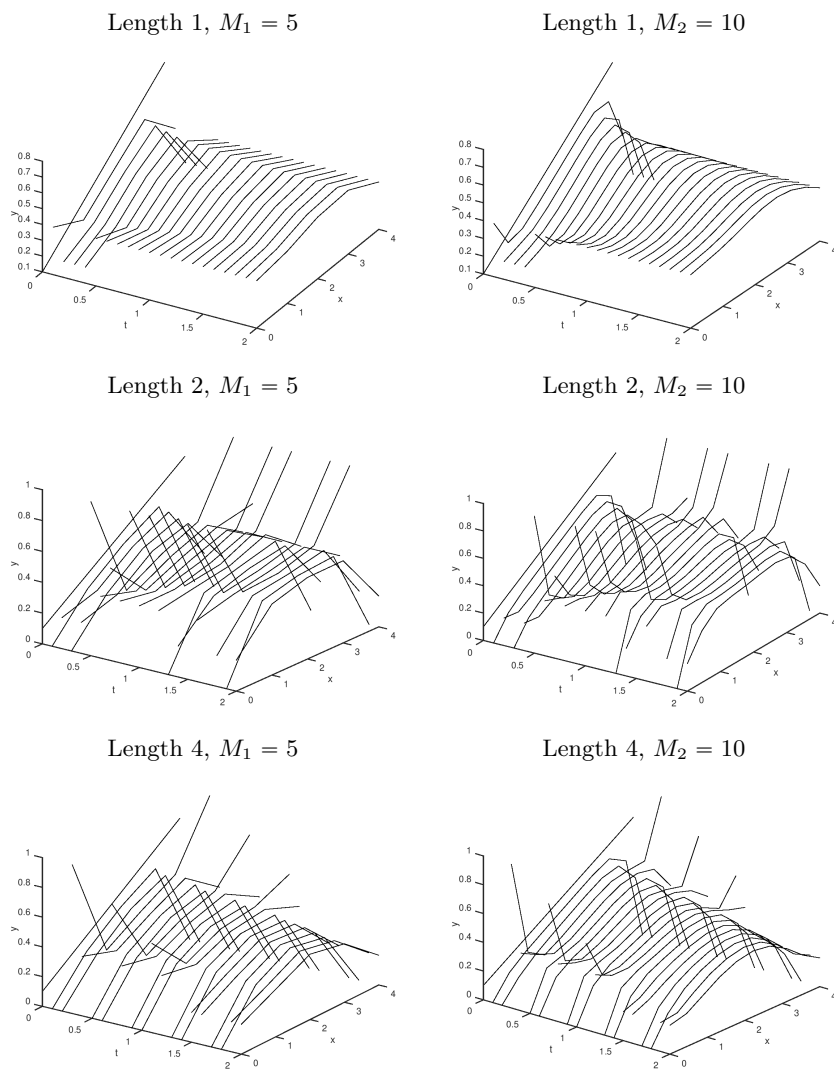
**Table 1.** Value  $\|y_i^{\pi^\varepsilon}(T) - y_i^f\|$  for  $\sigma = 1$  and  $\sigma = 0.5$  ( $T = 2$ ,  $i = 1, 2$ ).

Extended mode length	$\ Py_2^{\pi^\varepsilon}(T) - y_1^f\ $ for $\sigma = 1$	$\ Py_2^{\pi^\varepsilon}(T) - y_1^f\ $ for $\sigma = 0.5$
1	0.67429	0.77322
2	0.27501	0.72322
4	0.31385	0.21481

**Table 2.** Projection value  $\|Py_2^{\pi^\varepsilon}(T) - y_1^f\|$  for  $\sigma = 1$ ,  $\sigma = 0.5$  ( $T = 2$ ).



**Fig. 5.** Simulation of the controllers for  $\sigma = 1$ .



**Fig. 6.** Simulation of the controllers for  $\sigma = 0.5$ .