

Some ideas about constructive tensor field theory

Vincent Rivasseau¹ Fabien Vignes-Tourneret²

¹Université Paris-Saclay

²CNRS & Université de Lyon



Outline

Random tensors, random spaces

Loop Vertex Expansion

A constructive result for tensors

Random tensors, random spaces

Random tensors, random spaces

Why?

How?

Loop Vertex Expansion

A constructive result for tensors

Why tensor fields?

1. Generalize matrix models to higher dimensions
 - w.r.t. their symmetry properties,
 - provide a theory of random spaces.
2. Define a canonical way of summing over spaces
3. Implement a geometrogenesis scenario
 - spacetime from scratch,
 - background independent.

Invariant actions

Symmetry

Consider $T, \bar{T} : \mathbb{Z}^D \rightarrow \mathbb{C}$, complex rank D tensors with *no symmetry*.

Invariant actions

Symmetry

Consider $T, \bar{T} : \mathbb{Z}^D \rightarrow \mathbb{C}$, complex rank D tensors with *no symmetry*.

- Matrix models: invariant under (at most) two copies of $U(N)$.
Tensor models (rank D): invariant under D copies of $U(N)$.

$$T_{n_1 n_2 \dots n_D} \longrightarrow \sum_m U_{n_1 m_1}^{(1)} U_{n_2 m_2}^{(2)} \dots U_{n_D m_D}^{(D)} T_{m_1 m_2 \dots m_D}$$

Invariant actions


Symmetry

Consider $T, \bar{T} : \mathbb{Z}^D \rightarrow \mathbb{C}$, complex rank D tensors with *no symmetry*.

- Matrix models: invariant under (at most) two copies of $U(N)$.
Tensor models (rank D): invariant under D copies of $U(N)$.

$$T_{n_1 n_2 \dots n_D} \longrightarrow \sum_m U_{n_1 m_1}^{(1)} U_{n_2 m_2}^{(2)} \dots U_{n_D m_D}^{(D)} T_{m_1 m_2 \dots m_D}$$

- Invariants as D -coloured graphs



The diagram shows two vertices, labeled T on the left and \bar{T} on the right. There are four edges connecting them, labeled 1, 2, 3, and 4 from bottom to top. Edge 1 is the bottom-most curve, edge 2 is the second curve from the bottom, edge 3 is the third curve, and edge 4 is the top-most curve.

$$\sum_{n_i} T_{n_1 n_2 n_3 n_4} \bar{T}_{n_1 n_2 n_3 n_4} =: T \cdot \bar{T}$$

Invariant actions


Symmetry

Consider $T, \bar{T} : \mathbb{Z}^D \rightarrow \mathbb{C}$, complex rank D tensors with *no symmetry*.

- Matrix models: invariant under (at most) two copies of $U(N)$.
Tensor models (rank D): invariant under D copies of $U(N)$.

$$T_{n_1 n_2 \dots n_D} \longrightarrow \sum_m U_{n_1 m_1}^{(1)} U_{n_2 m_2}^{(2)} \dots U_{n_D m_D}^{(D)} T_{m_1 m_2 \dots m_D}$$

- Invariants as D -coloured graphs (bipartite D -reg. properly edge-coloured)



The diagram shows two nodes, T and \bar{T} , connected by four edges labeled 1, 2, 3, and 4. The edges are arranged in a symmetric, overlapping pattern between the two nodes.

$$\sum_{n_i} T_{n_1 n_2 n_3 n_4} \bar{T}_{n_1 n_2 n_3 n_4} =: T \cdot \bar{T}$$

Invariant actions

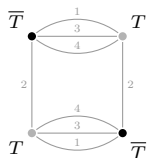
Symmetry

Consider $T, \bar{T} : \mathbb{Z}^D \rightarrow \mathbb{C}$, complex rank D tensors with *no symmetry*.

- Matrix models: invariant under (at most) two copies of $U(N)$.
Tensor models (rank D): invariant under D copies of $U(N)$.

$$T_{n_1 n_2 \dots n_D} \longrightarrow \sum_m U_{n_1 m_1}^{(1)} U_{n_2 m_2}^{(2)} \dots U_{n_D m_D}^{(D)} T_{m_1 m_2 \dots m_D}$$

- Invariants as D -coloured graphs (bipartite D -reg. properly edge-coloured)



$$\sum_{m_i, n_j} T_{m_1 m_2 m_3 m_4} \bar{T}_{m_1 n_2 m_3 m_4} T_{n_1 n_2 n_3 n_4} \bar{T}_{n_1 m_2 n_3 n_4} =: \text{Tr}_{\text{cylinder}} [T, \bar{T}]$$

Invariant actions

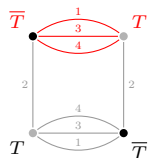
Symmetry

Consider $T, \bar{T} : \mathbb{Z}^D \rightarrow \mathbb{C}$, complex rank D tensors with *no symmetry*.

- Matrix models: invariant under (at most) two copies of $U(N)$.
Tensor models (rank D): invariant under D copies of $U(N)$.

$$T_{n_1 n_2 \dots n_D} \longrightarrow \sum_m U_{n_1 m_1}^{(1)} U_{n_2 m_2}^{(2)} \dots U_{n_D m_D}^{(D)} T_{m_1 m_2 \dots m_D}$$

- Invariants as D -coloured graphs (bipartite D -reg. properly edge-coloured)



$$\sum_{m_i, n_j} T_{m_1 m_2 m_3 m_4} \bar{T}_{m_1 n_2 m_3 m_4} T_{n_1 n_2 n_3 n_4} \bar{T}_{n_1 m_2 n_3 n_4} =: \text{Tr}_{\text{cylinder}} [T, \bar{T}]$$

Invariant actions

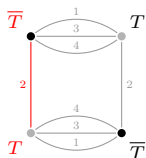
Symmetry

Consider $T, \bar{T} : \mathbb{Z}^D \rightarrow \mathbb{C}$, complex rank D tensors with *no symmetry*.

- Matrix models: invariant under (at most) two copies of $U(N)$.
Tensor models (rank D): invariant under D copies of $U(N)$.

$$T_{n_1 n_2 \dots n_D} \longrightarrow \sum_m U_{n_1 m_1}^{(1)} U_{n_2 m_2}^{(2)} \dots U_{n_D m_D}^{(D)} T_{m_1 m_2 \dots m_D}$$

- Invariants as D -coloured graphs (bipartite D -reg. properly edge-coloured)



$$\sum_{m_i, n_j} T_{m_1 m_2 m_3 m_4} \bar{T}_{m_1 n_2 m_3 m_4} T_{n_1 n_2 n_3 n_4} \bar{T}_{n_1 m_2 n_3 n_4} =: \text{Tr}_{\text{cylinder}} [T, \bar{T}]$$

Invariant actions

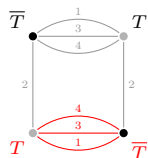
Symmetry

Consider $T, \bar{T} : \mathbb{Z}^D \rightarrow \mathbb{C}$, complex rank D tensors with *no symmetry*.

- Matrix models: invariant under (at most) two copies of $U(N)$.
Tensor models (rank D): invariant under D copies of $U(N)$.

$$T_{n_1 n_2 \dots n_D} \longrightarrow \sum_m U_{n_1 m_1}^{(1)} U_{n_2 m_2}^{(2)} \dots U_{n_D m_D}^{(D)} T_{m_1 m_2 \dots m_D}$$

- Invariants as D -coloured graphs (bipartite D -reg. properly edge-coloured)



$$\sum_{m_i, n_j} T_{m_1 m_2 m_3 m_4} \bar{T}_{m_1 n_2 m_3 m_4} T_{n_1 n_2 n_3 n_4} \bar{T}_{n_1 m_2 n_3 n_4} =: \text{Tr}_{\text{cylinder}} [T, \bar{T}]$$

Invariant actions

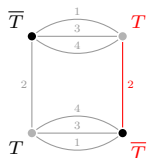
Symmetry

Consider $T, \bar{T} : \mathbb{Z}^D \rightarrow \mathbb{C}$, complex rank D tensors with *no symmetry*.

- Matrix models: invariant under (at most) two copies of $U(N)$.
Tensor models (rank D): invariant under D copies of $U(N)$.

$$T_{n_1 n_2 \dots n_D} \longrightarrow \sum_m U_{n_1 m_1}^{(1)} U_{n_2 m_2}^{(2)} \dots U_{n_D m_D}^{(D)} T_{m_1 m_2 \dots m_D}$$

- Invariants as D -coloured graphs (bipartite D -reg. properly edge-coloured)



$$\sum_{m_i, n_j} T_{m_1 m_2 m_3 m_4} \bar{T}_{m_1 n_2 m_3 m_4} T_{n_1 n_2 n_3 n_4} \bar{T}_{n_1 m_2 n_3 n_4} =: \text{Tr}_{\text{cylinder}} [T, \bar{T}]$$

Invariant actions

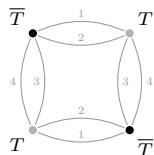
Symmetry

Consider $T, \bar{T} : \mathbb{Z}^D \rightarrow \mathbb{C}$, complex rank D tensors with *no symmetry*.

- Matrix models: invariant under (at most) two copies of $U(N)$.
Tensor models (rank D): invariant under D copies of $U(N)$.

$$T_{n_1 n_2 \dots n_D} \longrightarrow \sum_m U_{n_1 m_1}^{(1)} U_{n_2 m_2}^{(2)} \dots U_{n_D m_D}^{(D)} T_{m_1 m_2 \dots m_D}$$

- Invariants as D -coloured graphs (bipartite D -reg. properly edge-coloured)



$$\sum_{m_i, n_j} T_{m_1 m_2 m_3 m_4} \bar{T}_{m_1 m_2 n_3 n_4} T_{n_1 n_2 n_3 n_4} \bar{T}_{n_1 n_2 m_3 m_4} =: \text{Tr}_{\text{cube}} [T, \bar{T}]$$

Invariant actions

Feynman graphs

- Action of a tensor model

$$S(T, \bar{T}) = T \cdot \bar{T} + \sum_{B \in \mathfrak{J}} g_B \text{Tr}_B[T, \bar{T}],$$

$$\mathfrak{J} \subset \{D\text{-coloured graphs of order } \geq 4\}$$

interaction vertices

Invariant actions

Feynman graphs

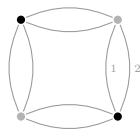
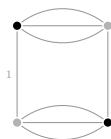
- Action of a tensor model

$$S(T, \bar{T}) = T \cdot \bar{T} + \sum_{B \in \mathfrak{J}} g_B \text{Tr}_B[T, \bar{T}],$$

$$\mathfrak{J} \subset \{D\text{-coloured graphs of order } \geq 4\}$$

interaction vertices

- Feynman graphs



Invariant actions

Feynman graphs

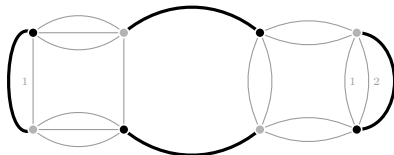
- Action of a tensor model

$$S(T, \bar{T}) = T \cdot \bar{T} + \sum_{B \in \mathfrak{J}} g_B \text{Tr}_B[T, \bar{T}],$$

$$\mathfrak{J} \subset \{D\text{-coloured graphs of order } \geq 4\}$$

interaction vertices

- Feynman graphs



Invariant actions

Feynman graphs

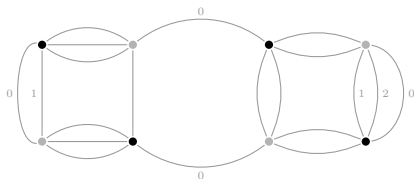
- Action of a tensor model

$$S(T, \bar{T}) = T \cdot \bar{T} + \sum_{B \in \mathfrak{J}} g_B \text{Tr}_B[T, \bar{T}],$$

$$\mathfrak{J} \subset \{D\text{-coloured graphs of order } \geq 4\}$$

interaction vertices

- Feynman graphs = $(D + 1)$ -coloured graphs



Invariant actions

Feynman graphs

- Action of a tensor model

$$S(T, \bar{T}) = T \cdot \bar{T} + \sum_{B \in \mathfrak{J}} g_B \text{Tr}_B[T, \bar{T}],$$

$$\mathfrak{J} \subset \{D\text{-coloured graphs of order } \geq 4\}$$

interaction vertices

- Feynman graphs = $(D + 1)$ -coloured graphs = D -Triang. spaces

Invariant actions

Feynman graphs

- Action of a tensor model

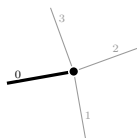
$$S(T, \bar{T}) = T \cdot \bar{T} + \sum_{B \in \mathfrak{J}} g_B \text{Tr}_B[T, \bar{T}],$$

$\mathfrak{J} \subset \{D\text{-coloured graphs of order } \geq 4\}$

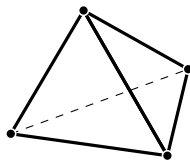
interaction vertices

- Feynman graphs = $(D + 1)$ -coloured graphs = D -Triang. spaces

vertex



D -simplex



Invariant actions

Feynman graphs

- Action of a tensor model

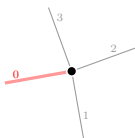
$$S(T, \bar{T}) = T \cdot \bar{T} + \sum_{B \in \mathfrak{J}} g_B \text{Tr}_B[T, \bar{T}],$$

$\mathfrak{J} \subset \{D\text{-coloured graphs of order } \geq 4\}$

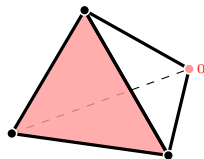
interaction vertices

- Feynman graphs = $(D + 1)$ -coloured graphs = D -Triang. spaces

half-edge



$(D - 1)$ -face



Invariant actions

Feynman graphs

- Action of a tensor model

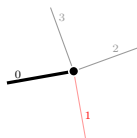
$$S(T, \bar{T}) = T \cdot \bar{T} + \sum_{B \in \mathcal{I}} g_B \text{Tr}_B[T, \bar{T}],$$

$\mathcal{I} \subset \{D\text{-coloured graphs of order } \geq 4\}$

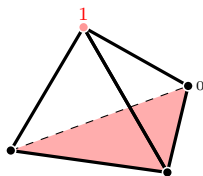
interaction vertices

- Feynman graphs = $(D + 1)$ -coloured graphs = D -Triang. spaces

half-edge



$(D - 1)$ -face



Invariant actions

Feynman graphs

- Action of a tensor model

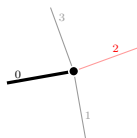
$$S(T, \bar{T}) = T \cdot \bar{T} + \sum_{B \in \mathfrak{J}} g_B \text{Tr}_B[T, \bar{T}],$$

$\mathfrak{J} \subset \{D\text{-coloured graphs of order } \geq 4\}$

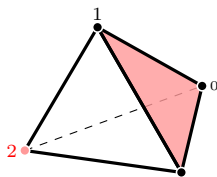
interaction vertices

- Feynman graphs = $(D + 1)$ -coloured graphs = D -Triang. spaces

half-edge



$(D - 1)$ -face



Invariant actions

Feynman graphs

- Action of a tensor model

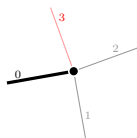
$$S(T, \bar{T}) = T \cdot \bar{T} + \sum_{B \in \mathcal{I}} g_B \text{Tr}_B[T, \bar{T}],$$

$\mathcal{I} \subset \{D\text{-coloured graphs of order } \geq 4\}$

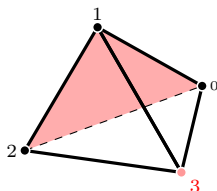
interaction vertices

- Feynman graphs = $(D + 1)$ -coloured graphs = D -Triang. spaces

half-edge



$(D - 1)$ -face



Invariant actions

Feynman graphs

- Action of a tensor model

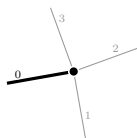
$$S(T, \bar{T}) = T \cdot \bar{T} + \sum_{B \in \mathfrak{J}} g_B \text{Tr}_B[T, \bar{T}],$$

$\mathfrak{J} \subset \{D\text{-coloured graphs of order } \geq 4\}$

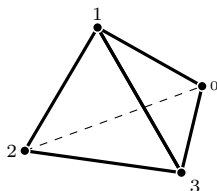
interaction vertices

- Feynman graphs = $(D + 1)$ -coloured graphs = D -Triang. spaces

vertex



D -simplex



Invariant actions

Feynman graphs

- Action of a tensor model

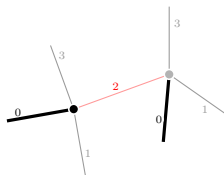
$$S(T, \bar{T}) = T \cdot \bar{T} + \sum_{B \in \mathcal{I}} g_B \text{Tr}_B[T, \bar{T}],$$

$\mathcal{I} \subset \{D\text{-coloured graphs of order } \geq 4\}$

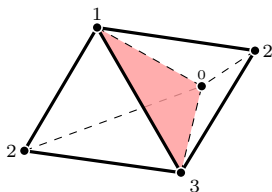
interaction vertices

- Feynman graphs = $(D + 1)$ -coloured graphs = D -Triang. spaces

edge



gluing



Loop Vertex Expansion

Random tensors, random spaces

Loop Vertex Expansion

Why?

How?

A constructive result for tensors

Constructive field theory

A functional integral point of view

- Aim: get some control on connected quantities via the derivation of tractable formulas for the *logarithm* of correlation functions (say).

Constructive field theory

The classical approach

- Aim: get some control on connected quantities via the derivation of tractable formulas for the *logarithm* of correlation functions (say).
- How? By finding an expansion which interpolates between the functional integral and the perturbative series.

Constructive field theory

The classical approach

- Aim: get some control on connected quantities via the derivation of tractable formulas for the *logarithm* of correlation functions (say).
- How? By finding an expansion which interpolates between the functional integral and the perturbative series.
- Tools: With cluster and Mayer expansions which are both a clever application of the forest formula.

The BKAR forest formula

- Fix an integer $n \geq 2$.
- f a function of $\frac{n(n-1)}{2}$ variables x_ℓ , sufficiently differentiable.
- K_n , complete graph on $\{1, 2, \dots, n\}$. $\#E(K_n) = \frac{n(n-1)}{2}$

Then,

$$f(1, 1, \dots, 1) = \sum_{\mathcal{F}} \int dw_{\mathcal{F}} \partial_{\mathcal{F}} f(X^{\mathcal{F}}(w_{\mathcal{F}}))$$

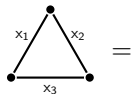
where

- the sum is over spanning forests of K_n ,
- $\int dw_{\mathcal{F}} := \prod_{\ell \in E(\mathcal{F})} \int_0^1 dw_\ell$,
- $\partial_{\mathcal{F}} := \prod_{\ell \in E(\mathcal{F})} \frac{\partial}{\partial x_\ell}$,
- $X^{\mathcal{F}} = (x_\ell^{\mathcal{F}})_{\ell \in E(K_n)}$ – evaluation point of $\partial_{\mathcal{F}} f$.

The forest formula

An example

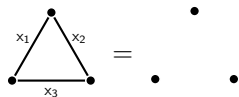
$$f(1, 1, 1) =$$



The forest formula

An example

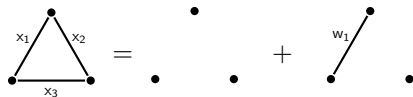
$$f(1, 1, 1) = f(0, 0, 0)$$



The forest formula

An example

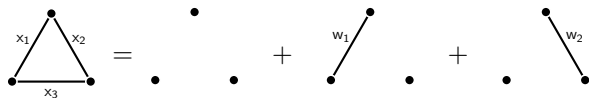
$$f(1, 1, 1) = f(0, 0, 0) + \int_0^1 \partial_{x_1} f(w_1, 0, 0) dw_1$$



The forest formula

An example

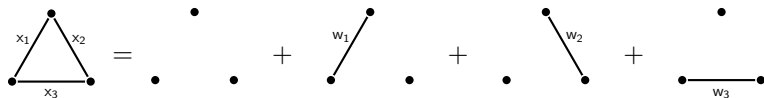
$$f(1, 1, 1) = f(0, 0, 0) + \int_0^1 \partial_{x_1} f(w_1, 0, 0) dw_1 + \int_0^1 \partial_{x_2} f(0, w_2, 0) dw_2$$



The forest formula

An example

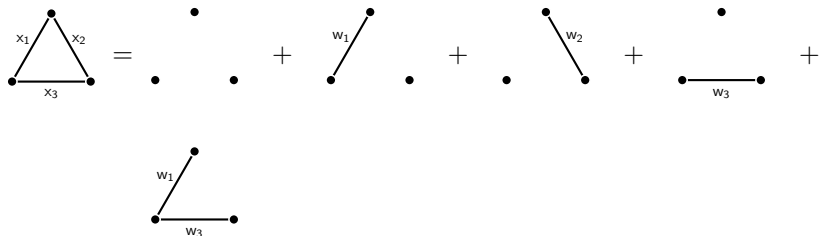
$$f(1, 1, 1) = f(0, 0, 0) + \int_0^1 \partial_{x_1} f(w_1, 0, 0) dw_1 + \int_0^1 \partial_{x_2} f(0, w_2, 0) dw_2 + \int_0^1 \partial_{x_3} f(0, 0, w_3) dw_3$$



The forest formula

An example

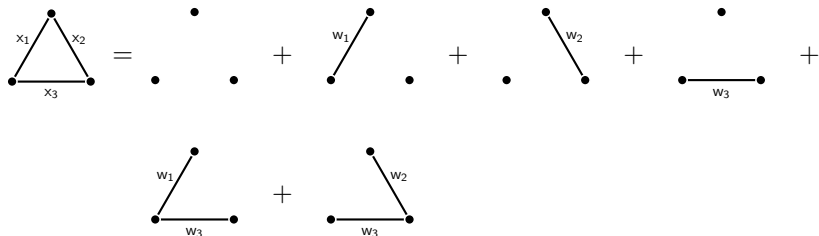
$$f(1, 1, 1) = f(0, 0, 0) + \int_0^1 \partial_{x_1} f(w_1, 0, 0) dw_1 + \int_0^1 \partial_{x_2} f(0, w_2, 0) dw_2 + \int_0^1 \partial_{x_3} f(0, 0, w_3) dw_3 + \iint \partial_{x_1, x_3}^2 f(w_1, \min\{w_1, w_3\}, w_3)$$



The forest formula

An example

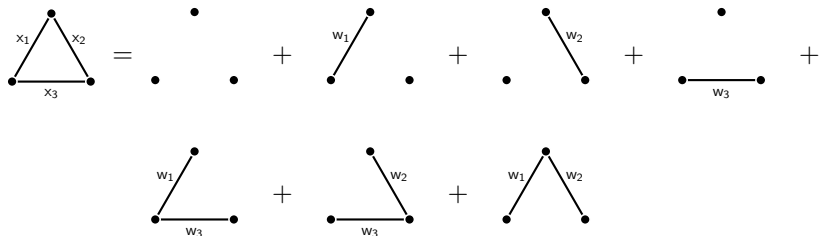
$$f(1, 1, 1) = f(0, 0, 0) + \int_0^1 \partial_{x_1} f(w_1, 0, 0) dw_1 + \int_0^1 \partial_{x_2} f(0, w_2, 0) dw_2 + \\ \int_0^1 \partial_{x_3} f(0, 0, w_3) dw_3 + \iint \partial_{x_1, x_3}^2 f(w_1, \min\{w_1, w_3\}, w_3) + \\ \iint \partial_{x_2, x_3}^2 f(\min\{w_2, w_3\}, w_2, w_3)$$



The forest formula

An example

$$f(1, 1, 1) = f(0, 0, 0) + \int_0^1 \partial_{x_1} f(w_1, 0, 0) dw_1 + \int_0^1 \partial_{x_2} f(0, w_2, 0) dw_2 + \\ \int_0^1 \partial_{x_3} f(0, 0, w_3) dw_3 + \iint \partial_{x_1, x_3}^2 f(w_1, \min\{w_1, w_3\}, w_3) + \\ \iint \partial_{x_2, x_3}^2 f(\min\{w_2, w_3\}, w_2, w_3) + \iint \partial_{x_1, x_2}^2 f(w_1, w_2, \min\{w_1, w_2\}).$$



Constructive field theory

The classical approach

- Aim: get some control on connected quantities via the derivation of tractable formulas for the *logarithm* of correlation functions (say).
- How? By finding an expansion which interpolates between the functional integral and the perturbative series.
- Tools: With cluster and Mayer expansions which are both a clever application of the forest formula.
- **But** classical constructive techniques are unsuited to matrices.

Loop Vertex Expansion

Motivations

LVE = main constructive tool for matrices and tensors

-
- Originally designed for random matrices. [Rivasseau 2007]
 - Initial goals:
 1. Constructive ϕ_4^{*4} ,
 2. Simplify Bosonic constructive theory.
-

Loop Vertex Expansion

Motivations

LVE = main constructive tool for matrices and tensors

-
- Originally designed for random matrices. [Rivasseau 2007]
 - Initial goals:
 1. Constructive ϕ_4^{*4} ,
 2. Simplify Bosonic constructive theory.
-
2. Bosonic perturbative series cannot be resummed as easily as Fermionic ones.

$$S(\lambda) = \sum_{n \geq 0} \frac{\lambda^n}{n!} \sum_{G, n(G)=n} A_G$$

Loop Vertex Expansion

Motivations

LVE = main constructive tool for matrices and tensors

-
- Originally designed for random matrices. [Rivasseau 2007]
 - Initial goals:
 1. Constructive ϕ_4^{*4} ,
 2. Simplify Bosonic constructive theory.
-
2. Bosonic perturbative series cannot be resummed as easily as Fermionic ones.

$$S(\lambda) = \sum_{n \geq 0} \frac{\lambda^n}{n!} \sum_{G, n(G)=n} A_G = \sum_{n \geq 0} \frac{\lambda^n}{n!} \sum_{G, n(G)=n} \sum_{TCG} w(G, T) A_G$$

Loop Vertex Expansion

Motivations

LVE = main constructive tool for matrices and tensors

-
- Originally designed for random matrices. [Rivasseau 2007]
 - Initial goals:
 1. Constructive ϕ_4^{*4} ,
 2. Simplify Bosonic constructive theory.
-
2. Bosonic perturbative series cannot be resummed as easily as Fermionic ones.

$$\begin{aligned} S(\lambda) &= \sum_{n \geq 0} \frac{\lambda^n}{n!} \sum_{G, n(G)=n} A_G = \sum_{n \geq 0} \frac{\lambda^n}{n!} \sum_{G, n(G)=n} \sum_{T \subset G} w(G, T) A_G \\ &= \sum_{n \geq 0} \frac{\lambda^n}{n!} \sum_{T, n(T)=n} A_T, \quad A_T = \sum_{G \supset T} w(G, T) A_G \end{aligned}$$

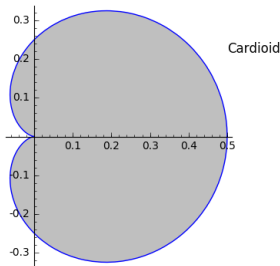
Loop Vertex Expansion

Analyticity of the free energy of ϕ_0^4

$$Z(g) = \int_{\mathbb{R}} e^{-\frac{g}{2}\phi^4} d\mu(\phi), \quad d\mu(\phi) = \frac{d\phi}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2}$$

Theorem

$\log Z$ is analytic in the cardioid domain $\{g \in \mathbb{C} : |g| < \frac{1}{2} \cos^2(\frac{1}{2} \arg g)\}$.



Loop Vertex Expansion

Analyticity of the free energy of ϕ_0^4

$$Z(g) = \int_{\mathbb{R}} e^{-\frac{g}{2}\phi^4} d\mu(\phi), \quad d\mu(\phi) = \frac{d\phi}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2}$$

Theorem

$\log Z$ is analytic in the cardioid domain $\{g \in \mathbb{C} : |g| < \frac{1}{2} \cos^2(\frac{1}{2} \arg g)\}$.

Proof. LVE is made of 2 ingredients:

1. Intermediate field representation,
2. Forest formula.

Loop Vertex Expansion

Analyticity of the free energy of ϕ_0^4

1. Intermediate field representation: $e^{-\frac{g}{2}\phi^4} = \int_{\mathbb{R}} e^{\lambda\sigma\phi^2} d\mu(\sigma), \lambda := \sqrt{g}$

$$\begin{aligned} Z(g) &= \int_{\mathbb{R}} e^{-\frac{g}{2}\phi^4} d\mu(\phi) \\ &= \int_{\mathbb{R}} e^{V(\sigma)} d\mu(\sigma), \quad V(\sigma) = -\frac{1}{2} \log(1 - \lambda\sigma). \end{aligned}$$

Loop Vertex Expansion

Analyticity of the free energy of ϕ_0^4

1. Intermediate field representation: $e^{-\frac{g}{2}\phi^4} = \int_{\mathbb{R}} e^{i\lambda\sigma\phi^2} d\mu(\sigma), \lambda := \sqrt{g}$

$$Z(g) = \int_{\mathbb{R}} e^{V(\sigma)} d\mu(\sigma), \quad V(\sigma) = -\frac{1}{2} \log(1 - i\lambda\sigma).$$

$\frac{3}{2}$. Replication of fields:

$$Z(g) = \sum_{n \geq 0} \frac{1}{n!} \int_{\mathbb{R}} V(\sigma)^n d\mu(\sigma) = \sum_{n \geq 0} \frac{1}{n!} \int_{\mathbb{R}^n} \left(\prod_{i=1}^n V(\sigma_i) \right) d\mu_{\mathbb{1}_n}(\vec{\sigma}).$$

Loop Vertex Expansion

Analyticity of the free energy of ϕ_0^4

1. Intermediate field representation: $e^{-\frac{g}{2}\phi^4} = \int_{\mathbb{R}} e^{i\lambda\sigma\phi^2} d\mu(\sigma)$, $\lambda := \sqrt{g}$

$$Z(g) = \int_{\mathbb{R}} e^{V(\sigma)} d\mu(\sigma), \quad V(\sigma) = -\frac{1}{2} \log(1 - i\lambda\sigma).$$

$\frac{3}{2}$. Replication of fields:

$$Z(g) = \sum_{n \geq 0} \frac{1}{n!} \int_{\mathbb{R}} V(\sigma)^n d\mu(\sigma) = \sum_{n \geq 0} \frac{1}{n!} \int_{\mathbb{R}^n} \left(\prod_{i=1}^n V(\sigma_i) \right) d\mu_{\mathbb{1}_n}(\vec{\sigma}).$$

2. Forest formula:

$$Z = \sum_{n \geq 0} \frac{1}{n!} \sum_{\mathcal{F} \subset K_n} \int d\omega_{\mathcal{F}} \int \left[\prod_{\ell \in E(\mathcal{F})} \frac{\partial}{\partial \sigma_{i(\ell)}} \frac{\partial}{\partial \sigma_{j(\ell)}} \prod_{i=1}^n V(\sigma_i) \right] d\mu_{C^{\mathcal{F}}(\omega)}(\vec{\sigma})$$

Loop Vertex Expansion

Analyticity of the free energy of ϕ_0^4

1. Intermediate field representation: $e^{-\frac{g}{2}\phi^4} = \int_{\mathbb{R}} e^{i\lambda\sigma\phi^2} d\mu(\sigma), \lambda := \sqrt{g}$

$$Z(g) = \int_{\mathbb{R}} e^{V(\sigma)} d\mu(\sigma), \quad V(\sigma) = -\frac{1}{2} \log(1 - i\lambda\sigma).$$

$\frac{3}{2}$. Replication of fields:

$$Z(g) = \sum_{n \geq 0} \frac{1}{n!} \int_{\mathbb{R}} V(\sigma)^n d\mu(\sigma) = \sum_{n \geq 0} \frac{1}{n!} \int_{\mathbb{R}^n} \left(\prod_{i=1}^n V(\sigma_i) \right) d\mu_{\mathbb{1}_n}(\vec{\sigma}).$$

2. Forest formula:

$$Z = \sum_{n \geq 0} \frac{1}{n!} \sum_{\mathcal{F} \subset K_n} \int dw_{\mathcal{F}} \int \left[\prod_{\ell \in E(\mathcal{F})} \frac{\partial}{\partial \sigma_{i(\ell)}} \frac{\partial}{\partial \sigma_{j(\ell)}} \prod_{i=1}^n V(\sigma_i) \right] d\mu_{C^{\mathcal{F}}(w)}(\vec{\sigma})$$

$$\log Z = \sum_{n \geq 1} \frac{1}{n!} \sum_{\mathcal{T} \subset K_n} \int dw_{\mathcal{T}} \int \left[\prod_{\ell \in E(\mathcal{T})} \frac{\partial}{\partial \sigma_{i(\ell)}} \frac{\partial}{\partial \sigma_{j(\ell)}} \prod_{i=1}^n V(\sigma_i) \right] d\mu_{C^{\mathcal{T}}(w)}(\vec{\sigma}).$$

Loop Vertex Expansion

Analyticity of the free energy of ϕ_0^4

$$\begin{aligned}\log Z &= \sum_{n \geq 1} \frac{1}{n!} \sum_{\mathcal{T} \subset K_n} \int dw_{\mathcal{T}} \int \left[\prod_{\ell \in E(\mathcal{T})} \frac{\partial}{\partial \sigma_{i(\ell)}} \frac{\partial}{\partial \sigma_{j(\ell)}} \prod_{i=1}^n V(\sigma_i) \right] d\mu_{C^{\mathcal{T}}(w)}(\vec{\sigma}) \\ &= \frac{1}{2} \sum_{n \geq 1} \frac{(-g/2)^{n-1}}{n!} \sum_{\mathcal{T} \subset K_n} \int dw_{\mathcal{T}} \int \left[\prod_{i=1}^n \frac{(d_i - 1)!}{(1 - \lambda \sigma_i)^{d_i}} \right] d\mu_{C^{\mathcal{T}}(w)}(\vec{\sigma}).\end{aligned}$$

Loop Vertex Expansion

Analyticity of the free energy of ϕ_0^4

$$\begin{aligned}\log Z &= \sum_{n \geq 1} \frac{1}{n!} \sum_{\mathcal{T} \subset K_n} \int dw_{\mathcal{T}} \int \left[\prod_{\ell \in E(\mathcal{T})} \frac{\partial}{\partial \sigma_{i(\ell)}} \frac{\partial}{\partial \sigma_{j(\ell)}} \prod_{i=1}^n V(\sigma_i) \right] d\mu_{C^{\mathcal{T}}(w)}(\vec{\sigma}) \\ &= \frac{1}{2} \sum_{n \geq 1} \frac{(-g/2)^{n-1}}{n!} \sum_{\mathcal{T} \subset K_n} \int dw_{\mathcal{T}} \int \left[\prod_{i=1}^n \frac{(d_i - 1)!}{(1 - i\lambda\sigma_i)^{d_i}} \right] d\mu_{C^{\mathcal{T}}(w)}(\vec{\sigma}).\end{aligned}$$

Using $|1 - i\lambda\sigma| \geq \cos(\frac{1}{2} \arg g)$, we get

$$\begin{aligned}|\log Z| &\leq \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{|g|}{2 \cos^2(\frac{1}{2} \arg g)} \right)^{n-1} \sum_{\mathcal{T} \subset K_n} \prod_{i=1}^n (d_i - 1)! \\ &\leq 2 \sum_{n=1}^{\infty} \left(\frac{2|g|}{\cos^2(\frac{1}{2} \arg g)} \right)^{n-1}\end{aligned}$$

which is convergent for all $g \in \mathbb{C}$ such that $|g| < \frac{1}{2} \cos^2(\frac{1}{2} \arg g)$. \square

A constructive result for tensors

Random tensors, random spaces

Loop Vertex Expansion

A constructive result for tensors

The T_4^4 field theory

- Tensors:

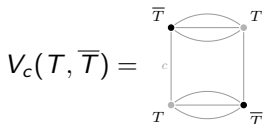
$$T : \mathbb{Z}^4 \rightarrow \mathbb{C}, \quad T_{\mathbf{n}}, \bar{T}_{\bar{\mathbf{n}}} \text{ with } \mathbf{n}, \bar{\mathbf{n}} \in \mathbb{Z}^4.$$

- Free action:

$$C_{\mathbf{n}, \bar{\mathbf{n}}} = \delta_{\mathbf{n}, \bar{\mathbf{n}}} \frac{(\mathbf{1}_{\leq j_{\max}})_{\mathbf{n}\bar{\mathbf{n}}}}{\mathbf{n}^2 + 1}, \quad \mathbf{n}^2 := n_1^2 + n_2^2 + n_3^2 + n_4^2.$$

- Interactions:

$$V(T, \bar{T}) = \frac{g}{2} \sum_{c=1}^4 V_c(T, \bar{T}),$$



Lemma

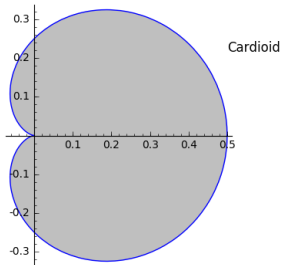
T_4^4 is (super-)renormalizable to all orders of perturbation with a power-counting similar to ϕ_3^4 .

The T_4^4 field theory

Analyticity

Theorem (Rivasseau, V.-T. 2017)

There exists $\rho > 0$ such that $\lim_{j_{\max} \rightarrow \infty} \log Z_{j_{\max}}(g)$ is analytic in the cardioid domain defined by $|\arg g| < \pi$ and $|g| < \rho \cos^2(\frac{1}{2} \arg g)$.



The T_4^4 field theory

Analyticity

Theorem (Rivasseau, V.-T. 2017)

There exists $\rho > 0$ such that $\lim_{j_{\max} \rightarrow \infty} \log Z_{j_{\max}}(g)$ is analytic in the cardioid domain defined by $|\arg g| < \pi$ and $|g| < \rho \cos^2(\frac{1}{2} \arg g)$.

Remarks

- *Intermediate field is a matrix,*
- *Renormalization needed,*
- *Multiscale analysis required,*
- *Forests turn into jungles,*
- *Non-perturbative bounds necessary.*

Conclusion and perspectives

- Regarding T_4^4 , one could also prove Borel summability of the connected correlation functions.
- LVE makes Bosons as convergent as Fermions (in dimension 0).

- T_5^4 (just renormalisable, asymptotically free)
- New Loop Vertex Representation [Rivasseau 2017]
- Inductive approach to LVE [Fang-Jie Zhao 2019]
- Simplify Bosonic constructive theory?

Thank you for your attention