A study of the rate of convergence of generalized inverse Gaussian and Kummer distributions to the gamma distribution via Stein's method

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Essomanda Konzou*, Angelo Efoévi Koudou†, Kossi E. Gneyou ‡

Abstract

A sequence of random variables following the generalized inverse Gaussian or the Kummer distribution converges in law to the gamma distribution under certain conditions on the parameters. We provide explicit upper bounds for the distributional distance between such generalized inverse Gaussian (resp. Kummer) variables and their limiting gamma distribution applying Stein's approach.

Keywords: Generalized inverse Gaussian distribution, Kummer distribution, gamma distribution, Stein's method, Taylor expansions.

1 Introduction

The generalized inverse Gaussian (hereafter $\boldsymbol{GIG}$) distribution with parameters $\boldsymbol{p \in \mathbb{R}, a > 0, b > 0}$ has density

$$\boldsymbol{g_{p,a,b}(x) = \frac{(a/b)^{p/2}}{2K_p(\sqrt{ab})} x^{p-1} e^{-\frac{1}{2}(ax+b/x)}, \quad x > 0,} \tag{1}$$

where $\boldsymbol{K_p}$ is the modified Bessel function of the third kind.

For $\boldsymbol{a > 0}$, $\boldsymbol{b \in \mathbb{R}}$, $\boldsymbol{c > 0}$, the Kummer distribution $\boldsymbol{K(a, b, c)}$ has density function

$$\boldsymbol{k_{a,b,c}(x) = \frac{1}{\Gamma(a)\psi(a, 1-b; c)} x^{a-1}(1+x)^{-a-b}e^{-cx}, \ (x > 0)} \tag{2}$$

where $\boldsymbol{\psi}$ is the confluent hypergeometric function of the second kind and $\boldsymbol{\Gamma}$ is the gamma function.

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The following formulae, immediately obtained from (1) and (2), will be used in the proof of the main results. If $\boldsymbol{X} \sim \boldsymbol{GIG(p,a,b)}$ distribution, then

$$\mathbb{E}(\boldsymbol{X}) = \sqrt{\frac{\boldsymbol{b}}{\boldsymbol{a}}} \times \frac{\boldsymbol{K_{p+1}(\sqrt{ab})}}{\boldsymbol{K_p(\sqrt{ab})}}; \qquad \mathbb{E}(\boldsymbol{X^2}) = \frac{\boldsymbol{b}}{\boldsymbol{a}} \times \frac{\boldsymbol{K_{p+2}(\sqrt{ab})}}{\boldsymbol{K_p(\sqrt{ab})}}; \tag{3}$$

if $\boldsymbol{X} \sim \boldsymbol{K(a,b,c)}$ distribution, then

$$\mathbb{E}(\boldsymbol{X}) = \boldsymbol{a}\frac{\boldsymbol{\psi(a+1, 2-b; c)}}{\boldsymbol{\psi(a, 1-b; c)}}; \qquad \mathbb{E}(\boldsymbol{X^2}) = \boldsymbol{a(a+1)}\frac{\boldsymbol{\psi(a+2, 3-b; c)}}{\boldsymbol{\psi(a, 1-b; c)}}. \tag{4}$$

For more details on GIG and Kummer distributions see for instance Barndorff-Nielsen and Halgreen (1977), Koudou and Ley (2014), Koudou and Vallois (2012), Piliszek and Wesołowski (2018) and references therein.

For $\boldsymbol{\theta > 0}$, $\boldsymbol{\lambda > 0}$, the gamma distribution $\boldsymbol{\gamma(\theta, \lambda)}$ has density function

$$\boldsymbol{\gamma(\theta,\lambda)(x) = \frac{\lambda^\theta}{\Gamma(\theta)} x^{\theta-1} e^{-\lambda x} \mathbb{1}_{\{x>0\}}.}$$

We have the following well-known convolution properties which can be easily proved from the Laplace transforms of the considered distributions:

$$\boldsymbol{GIG(-p,a,b) * \gamma(p, a/2) = GIG(p,a,b);} \tag{5}$$

$$\boldsymbol{K(a,b,c) * \gamma(b,c) = K(b+a, -b, c).} \tag{6}$$

In this paper, the gamma distribution is considered in the context of Stein's method. This method, introduced by Stein (1972), is a powerful technique used in statistics and probability theory to characterize probability distributions and to bound the error in the approximation of the distribution of a random variable of interest, by another probability distribution. The main steps of the method are the determination of an operator (called Stein operator) characterizing the targeted distribution, the resolution of a differential equation called Stein equation, and the derivation of bounds of the solution obtained and its derivatives.

As a first example of the method, Stein (1972) showed that a random variable $\boldsymbol{X}$ has a standard normal distribution if and only if, for all real-valued absolutely continuous function $\boldsymbol{f}$ such that $\mathbb{E}\left|\boldsymbol{f'(Z)}\right| < \boldsymbol{\infty}$ for $\boldsymbol{Z \sim \mathcal{N}(0,1)}$,

$$\mathbb{E}\left[\boldsymbol{f'(X) - Xf(X)}\right] = \boldsymbol{0.}$$

The corresponding Stein equation is

$$\boldsymbol{f'(x) - xf(x) = h(x) - \mathbb{E}h(Z)} \tag{7}$$

where $\boldsymbol{h}$ is a bounded function and $\boldsymbol{Z}$ a random variable following the standard normal distribution.

If $\boldsymbol{f_h}$ solves equation (7), then for any random variable $\boldsymbol{X}$, we have

$$\left|\mathbb{E}\left[\boldsymbol{f_h'(X) - X f_h(X)}\right]\right| = \left|\mathbb{E}\boldsymbol{h(X)} - \mathbb{E}\boldsymbol{h(Z)}\right|.$$

Thus, we can bound $\left|\mathbb{E}\boldsymbol{h(X)} - \mathbb{E}\boldsymbol{h(Z)}\right|$ given $\boldsymbol{h}$, by finding a solution $\boldsymbol{f_h}$ of the Stein equation (7) and bounding the left-hand side of the previous equation. For instance, as shown in Ross (2011), if $\boldsymbol{X_1, \cdots, X_n}$

are independent mean zero random variables such that, $\mathbb{E}\boldsymbol{X_i^2} = \mathbf{1}$ and $\mathbb{E}|\boldsymbol{X_i}|^4 < \infty \ \boldsymbol{i} = \mathbf{1}, \cdots, \boldsymbol{n}$, then, by Stein's approach,

$$|\mathbb{E}\boldsymbol{h}(\boldsymbol{W_n}) - \mathbb{E}\boldsymbol{h}(\boldsymbol{Z})| \leq \frac{\mathbf{1}}{\boldsymbol{n}^{\frac{\mathbf{3}}{\mathbf{2}}}} \sum_{\boldsymbol{i}=\mathbf{1}}^{\boldsymbol{n}} \mathbb{E}|\boldsymbol{X_i}|^{\mathbf{3}} + \frac{\sqrt{\mathbf{2}}}{\boldsymbol{n}\sqrt{\boldsymbol{\pi}}} \sqrt{\sum_{\boldsymbol{i}=\mathbf{1}}^{\boldsymbol{n}} \mathbb{E}|\boldsymbol{X_i}|^{\mathbf{4}}} \tag{8}$$

where $\boldsymbol{W_n} = \dfrac{\mathbf{1}}{\sqrt{\boldsymbol{n}}} \displaystyle\sum_{\boldsymbol{i}=\mathbf{1}}^{\boldsymbol{n}} \boldsymbol{X_i}$ and $\boldsymbol{Z}$ has the standard normal distribution.
If the random variables $\boldsymbol{X_i}$ of (8) have common distribution, then the right-hand side is of order $\boldsymbol{n^{-1/2}}$. For more details on Stein's method, see Chen *et al* (2011), Ross (2011).

The aim of this paper is to provide a bound for the distance between a GIG (resp. a Kummer) random variable and its limiting gamma variable, and therefore to give a contribution to the study of the rate of convergence, via Stein's method. For this purpose we make use of bounds established in Luk (1994) and Gaunt *et al* (2017) in the framework of the Stein's method for the gamma distribution. These bounds are recalled in Section 2. Section 3 presents our results and section 4 contains the proofs.

2 Stein characterization for the gamma distribution

The following result (see Luk, 1994) gives the Stein characterization for the gamma distribution.

Theorem 2.1 *A random variable $\boldsymbol{X}$ follows the $\boldsymbol{\gamma(\theta, \lambda)}$ distribution if and only if, for any bounded and differentiable function $\boldsymbol{f}$,*

$$\mathbb{E}\left[\boldsymbol{X f'(X)} + (\boldsymbol{\theta} - \boldsymbol{\lambda X})\, \boldsymbol{f(X)}\right] = \mathbf{0}.$$

The corresponding differential equation is

$$\boldsymbol{x f'(x)} + (\boldsymbol{\theta} - \boldsymbol{\lambda x}) \boldsymbol{f(x)} = \boldsymbol{h(x)} - \mathbb{E}\boldsymbol{h}(\boldsymbol{\Lambda}),$$

where $\boldsymbol{h}$ is a bounded function and $\boldsymbol{\Lambda}$ a random variable following $\boldsymbol{\gamma(\theta, \lambda)}$ distribution. It is straightforward (see Gaunt, 2017, Schoutens, 2001) to prove that the solution of this Stein's equation is

$$\begin{aligned} \boldsymbol{f_h(x)} =& \frac{\mathbf{1}}{\boldsymbol{x}\gamma(\boldsymbol{\theta}, \boldsymbol{\lambda})(\boldsymbol{x})} \int_{\mathbf{0}}^{\boldsymbol{x}} \gamma(\boldsymbol{\theta}, \boldsymbol{\lambda})(\boldsymbol{t}) \left[\boldsymbol{h(t)} - \mathbb{E}\boldsymbol{h}(\boldsymbol{\Lambda})\right] \boldsymbol{dt} \\ =& -\frac{\mathbf{1}}{\boldsymbol{x}\gamma(\boldsymbol{\theta}, \boldsymbol{\lambda})(\boldsymbol{x})} \int_{\boldsymbol{x}}^{\infty} \gamma(\boldsymbol{\theta}, \boldsymbol{\lambda})(\boldsymbol{t}) \left[\boldsymbol{h(t)} - \mathbb{E}\boldsymbol{h}(\boldsymbol{\Lambda})\right] \boldsymbol{dt}. \end{aligned}$$

Luk (1994) used the generator approach of Barbour (1988) to derive the following Stein equation

$$\boldsymbol{x f''(x)} + (\boldsymbol{\theta} - \boldsymbol{\lambda x}) \boldsymbol{f'(x)} = \boldsymbol{h(x)} - \mathbb{E}\boldsymbol{h}(\boldsymbol{\Lambda}) \tag{9}$$

and obtained bounds for the derivatives of the solution. Let us recall this result:

Theorem 2.2 (Luk, 1994) *For $k \geq 1$ let $f^{(k)}$ denote the k-th derivative of a function f. Let*

$$C_{\lambda,k} = \{h : \mathbb{R}_+ \longrightarrow \mathbb{R} : \exists c > 0, a < \lambda \text{ such that } \forall x \in \mathbb{R}_+, \forall j = 0, 1, \cdots, k-1$$
$$|h^{(j)}(x)| \leq ce^{ax} \text{ and } h^{(k-1)} \text{ is absolutely continuous } \}.$$

If $h \in C_{\lambda,k}$, then the solution f_h of (9) is k times differentiable and

$$\left|\left|f_h^{(k)}\right|\right| \leq \frac{\left|\left|h^{(k)}\right|\right|}{k\lambda}, \qquad k \geq 1, \tag{10}$$

where $||f|| = \sup_{x>0} |f(x)|$ and $h^{(0)} = h$.

Alternative bounds were obtained by Picket (2004) and Gaunt (2013). Picket (2004) obtained for $h \in C_{\lambda,k}$

$$\left|\left|f_h^{(k)}\right|\right| \leq \left[\sqrt{\frac{2\pi}{\theta}} + \frac{2}{\theta}\right] \left|\left|h^{(k-1)}\right|\right|, \qquad k \geq 1$$

and Gaunt (2013) showed that for $h \in C_{\lambda,k}$,

$$\left|\left|f_h^{(k)}\right|\right| \leq \left[\frac{\sqrt{2\pi} + e^{-1}}{\sqrt{\theta + k - 1}} + \frac{2}{\theta + k - 1}\right] \left|\left|h^{(k-1)}\right|\right|, \qquad k \geq 1.$$

Recently, Gaunt *et al* (2017) provided new bounds.

Theorem 2.3 (Gaunt *et al* (2017)) *Let $k \geq 2$. Suppose $h \in C_{\lambda,k-1}$ and that $h^{(k-2)}$ is bounded. Then*

$$\left|\left|f_h^{(k)}\right|\right| \leq \frac{2}{\theta + k - 1} \left(3 \left|\left|h^{(k-1)}\right|\right| + 2\lambda \left|\left|h^{(k-2)}\right|\right|\right). \tag{11}$$

The bounds recalled in this section will be used in the proof of our results stated in next section.

3 Main results

3.1 About the rate of convergence of the generalized inverse Gaussian distribution to the gamma distribution

Before giving our first main result in Theorem 3.1 below, we recall in the following proposition the convergence of the GIG distribution to the gamma distribution as the third parameter tends to 0. We give the proof, even if it is elementary.

Proposition 3.1 *For $p > 0$, $a > 0$, let $(W_n)_{n\geq 1}$ be a sequence of random variables such that $W_n \sim GIG\left(p, a, \frac{1}{n}\right)$ for each $n \geq 1$. Then, as $n \to \infty$, the sequence $(W_n)_{n\geq 1}$ converges in distribution to a random variable following the $\gamma\left(p, \frac{a}{2}\right)$ distribution.*

Proof:
For all $x > 0$,

$$\mathbb{P}\left(W_n < x\right) = \frac{(an)^{p/2}}{2K_p\left(\sqrt{\frac{a}{n}}\right)} \int_0^x t^{p-1} e^{-\frac{1}{2}\left(at + \frac{1}{nt}\right)} dt.$$

We now use the well-known fact that, if $p \neq 0$,

$$K_p(z) \overset{z \to 0}{\sim} 2^{|p|-1}\Gamma(|p|) z^{-|p|} \tag{12}$$

and the Lebesgue's Dominated Convergence Theorem to see that

$$\lim_{n\to\infty} \mathbb{P}\left(W_n < x\right) = \int_0^x \frac{(a/2)^p}{\Gamma(p)} t^{p-1} e^{-\frac{1}{2}at} dt.$$

□

Unsing the convolution property (5) and Proposition 3.1 we have the following result :

Proposition 3.2 *Let $p > 0$, $a > 0$. Consider a random variable $Y \sim \gamma(p, a/2)$. Let (X_n) be a sequence of random variables such that, for each $n \geq 1$, X_n is independent of Y and $X_n \sim GIG\left(-p, a, \frac{1}{n}\right)$. Let $W_n = X_n + Y,\ n \geq 1$. Then*

1. $W_n \sim GIG\left(p, a, \frac{1}{n}\right)$.
2. *As $n \to \infty$, the sequence $(W_n)_{n\geq 1}$ converges in distribution to a random variable Λ following the $\gamma\left(p, \frac{a}{2}\right)$ distribution.*

In view of the previous proposition, one natural question is how close the $GIG\left(p, a, \frac{1}{n}\right)$ distribution of W_n is to the $\gamma\left(p, \frac{a}{2}\right)$ distribution of Λ for n big enough. The following theorem addresses this question by proposing a bound for $|\mathbb{E}h(W_n) - \mathbb{E}h(\Lambda)|$ given a regular function h.

Theorem 3.1 *With the notation of Proposition 3.2, if $h \in C_{\frac{a}{2},3}$ then*

$$|\mathbb{E}h(W_n) - \mathbb{E}h(\Lambda)| \leq \frac{A}{\sqrt{a}} \times \frac{1}{\sqrt{n}} \frac{K_{-p+1}\left(\sqrt{\frac{a}{n}}\right)}{K_{-p}\left(\sqrt{\frac{a}{n}}\right)} + \frac{B}{a} \times \frac{1}{n} \frac{K_{-p+2}\left(\sqrt{\frac{a}{n}}\right)}{K_{-p}\left(\sqrt{\frac{a}{n}}\right)},$$

where

$$A = 2a\left(1 + \frac{p}{1+p}\right)||h|| + \left(7 + \frac{6p}{1+p} + \frac{4p}{2+p}\right)||h'|| + \frac{12p}{a(2+p)}||h''||$$

and

$$B = \frac{4ap}{2+p}||h'|| + \left(\frac{12p}{2+p} + \frac{4p}{3+p}\right)||h''|| + \frac{12p}{a(3+p)}\left|\left|h^{(3)}\right|\right|.$$

Remark: Using the equivalence (12), for $p > 2$ we have

$$\frac{K_{-p+1}\left(\sqrt{\frac{a}{n}}\right)}{K_{-p}\left(\sqrt{\frac{a}{n}}\right)} \sim \frac{\sqrt{a}}{2p} \times \frac{1}{\sqrt{n}} \text{ and } \frac{K_{-p+2}\left(\sqrt{\frac{a}{n}}\right)}{K_{-p}\left(\sqrt{\frac{a}{n}}\right)} \sim \frac{a}{4p(p-1)} \times \frac{1}{n} \text{ as } n \to +\infty,$$

hence the upper bound provided by Theorem 3.1 is of order n^{-1} for all $p > 2$.

The proof of Theorem 3.1 will be given in Section 4.

3.2 About the rate of convergence of the Kummer distribution to the gamma distribution

The following theorem contains results corresponding to those given in the previous subsection, in the framework of the convergence of the Kummer distribution to the gamma distribution.

Theorem 3.2 *For any integer $\boldsymbol{n \geq 1}$, let $\boldsymbol{Z_n}$ and $\boldsymbol{Y_n}$ be independent random variables such that $\boldsymbol{Z_n \sim K(\frac{1}{n}, a - \frac{1}{n}, c)}$ and $\boldsymbol{Y_n \sim \gamma(a - \frac{1}{n}, c)}$ with $\boldsymbol{a > 0}$, $\boldsymbol{b \in \mathbb{R}}$, $\boldsymbol{c > 0}$.*
Let $\boldsymbol{V_n = Z_n + Y_n}$.

1. $\boldsymbol{V_n \sim K\left(a, -a + \frac{1}{n}, c\right)}$.
2. *As $\boldsymbol{n \to \infty}$, the sequence $\boldsymbol{(V_n)}$ converges in distribution to a random variable $\boldsymbol{\Lambda}$ following the $\boldsymbol{\gamma(a, c)}$ distribution.*
3. *For any $\boldsymbol{h \in C_{c,3}}$,*

$$|\mathbb{E}h(V_n) - \mathbb{E}h(\Lambda)| \leq \frac{1}{cn}||h'|| + \frac{F}{c} \times \frac{1}{n} \frac{\psi(a, a - \frac{1}{n}; c)}{\psi(a, 1 + a - \frac{1}{n}; c)}$$

$$+ \frac{H}{c^2} \times \frac{1}{n}\left(1 + \frac{1}{n}\right) \frac{\psi(a, -1 + a - \frac{1}{n}; c)}{\psi(a, 1 + a - \frac{1}{n}; c)}$$

where

$$F = \left(4c + \frac{4c}{1+c}\left|a - \frac{1}{n}\right|\right)||h|| + \left(7 + \left(\frac{6}{1+a} + \frac{4}{2+a}\right)\left|a - \frac{1}{n}\right|\right)||h'||$$

$$+ \frac{6}{c(2+a)}\left|a - \frac{1}{n}\right| ||h''||$$

and

$$H = \frac{4c}{2+a}\left(a + \left|a - \frac{1}{n}\right|\right)||h'|| + \frac{6}{c(3+a)}\left|a - \frac{1}{n}\right| ||h^{(3)}||$$

$$+ \left(\frac{6a}{2+a} + \left(\frac{6}{2+a} + \frac{4}{3+a}\right)\left|a - \frac{1}{n}\right|\right)||h''||.$$

Remark: Since ψ is a continuous function,

$$\frac{\psi(a, a - \frac{1}{n}; c)}{\psi(a, 1 + a - \frac{1}{n}; c)} \xrightarrow[n \to \infty]{} \frac{\psi(a, a; c)}{\psi(a, 1 + a; c)} < \infty;$$

$$\frac{\psi(a, -1 + a - \frac{1}{n}; c)}{\psi(a, 1 + a - \frac{1}{n}; c)} \xrightarrow[n \to \infty]{} \frac{\psi(a, -1 + a; c)}{\psi(a, 1 + a; c)} < \infty.$$

Theorem 3.2 shows an upper bound of order $\boldsymbol{n^{-1}}$.

4 Proofs

For convenience we denote by $f := f_h$ the solution of the Stein equation for the gamma distribution.

4.1 Proof of Theorem 3.1

The proof of Theorem 3.1 is based on the following lemmas.

Lemma 4.1 *X_n and W_n satisfy*

$$\left|\mathbb{E}\left[X_n f''(W_n) - \frac{a}{2} X_n f'(W_n)\right]\right| \leq \left[\frac{a}{2}||f'|| + ||f''||\right] \frac{1}{\sqrt{an}} \frac{K_{-p+1}\left(\sqrt{\frac{a}{n}}\right)}{K_{-p}\left(\sqrt{\frac{a}{n}}\right)}.$$

Proof

By the triangular inequality,

$$\begin{aligned}\left|\mathbb{E}\left[X_n f''(W_n) - \frac{a}{2} X_n f'(W_n)\right]\right| &\leq \left|\mathbb{E}\left[X_n f''(W_n)\right]\right| + \left|\frac{a}{2}\mathbb{E} X_n f'(W_n)\right| \\ &\leq \left(||f''|| + \frac{a}{2}||f'||\right)\mathbb{E}(X_n) \\ &= \left(||f''|| + \frac{a}{2}||f'||\right) \frac{1}{\sqrt{an}} \frac{K_{-p+1}\left(\sqrt{\frac{a}{n}}\right)}{K_{-p}\left(\sqrt{\frac{a}{n}}\right)} \text{ (by (3))}.\end{aligned}$$

□

Lemma 4.2 *Y and W_n satisfy*

$$\left|\mathbb{E}\left[\left(p - \frac{a}{2}Y\right)(W_n - Y) f''(Y)\right]\right| \leq 2p||f''|| \frac{1}{\sqrt{an}} \frac{K_{-p+1}\left(\sqrt{\frac{a}{n}}\right)}{K_{-p}\left(\sqrt{\frac{a}{n}}\right)}, \tag{13}$$

$$\left|\mathbb{E}\left[Y(W_n - Y) f^{(3)}(Y)\right]\right| \leq \frac{2p}{a}||f^{(3)}|| \frac{1}{\sqrt{an}} \frac{K_{-p+1}\left(\sqrt{\frac{a}{n}}\right)}{K_{-p}\left(\sqrt{\frac{a}{n}}\right)}, \tag{14}$$

$$\left|\mathbb{E}\left[\left(p - \frac{a}{2}Y\right)\int_Y^{W_n} (W_n - t) f^{(3)}(t) dt\right]\right| \leq 2p||f^{(3)}|| \frac{1}{an} \frac{K_{-p+2}\left(\sqrt{\frac{a}{n}}\right)}{K_{-p}\left(\sqrt{\frac{a}{n}}\right)}, \tag{15}$$

$$\left|\mathbb{E}\left[Y\int_Y^{W_n} (W_n - t) f^{(4)}(t) dt\right]\right| \leq \frac{2p}{a}||f^{(4)}|| \frac{1}{an} \frac{K_{-p+2}\left(\sqrt{\frac{a}{n}}\right)}{K_{-p}\left(\sqrt{\frac{a}{n}}\right)}. \tag{16}$$

Proof

We use the triangular inequality and the fact that the random variables X_n and Y are independent. We have

$$\begin{aligned}\left|\mathbb{E}\left[\left(p - \frac{a}{2}Y\right)(W_n - Y) f''(Y)\right]\right| &= \left|\mathbb{E}\left[\left(p - \frac{a}{2}Y\right) X_n f''(Y)\right]\right| \\ &\leq p\left|\mathbb{E}\left[X_n f''(Y)\right]\right| + \frac{a}{2}\left|\mathbb{E}\left[Y X_n f''(Y)\right]\right| \\ &\leq \left(p + \frac{a}{2}\mathbb{E}(Y)\right)\mathbb{E}(X_n)||f''|| \\ &= \left(p + \frac{a}{2} \times \frac{2p}{a}\right)||f''|| \times \frac{1}{\sqrt{an}} \frac{K_{-p+1}\left(\sqrt{\frac{a}{n}}\right)}{K_{-p}\left(\sqrt{\frac{a}{n}}\right)}\end{aligned}$$

by independance of $\boldsymbol{X_n}$ and $\boldsymbol{Y}$, which gives inequality (13). Noting that

$$\left|\mathbb{E}\left[\boldsymbol{Y}(\boldsymbol{W_n}-\boldsymbol{Y})\boldsymbol{f^{(3)}}(\boldsymbol{Y})\right]\right| = \left|\mathbb{E}\left[\boldsymbol{Y}\boldsymbol{X_n}\boldsymbol{f^{(3)}}(\boldsymbol{Y})\right]\right| \leq ||\boldsymbol{f^{(3)}}||\mathbb{E}(\boldsymbol{Y}\boldsymbol{X_n}) = ||\boldsymbol{f^{(3)}}||\mathbb{E}(\boldsymbol{Y})\mathbb{E}(\boldsymbol{X_n}),$$

we get inequality (14). For (15), we have

$$\begin{aligned}
&\left|\mathbb{E}\left[\left(\boldsymbol{p}-\frac{\boldsymbol{a}}{\boldsymbol{2}}\boldsymbol{Y}\right)\int_{\boldsymbol{Y}}^{\boldsymbol{W_n}}(\boldsymbol{W_n}-\boldsymbol{t})\boldsymbol{f^{(3)}}(\boldsymbol{t})\boldsymbol{dt}\right]\right| \\
&\qquad\leq ||\boldsymbol{f^{(3)}}||\mathbb{E}\left[\left|\left(\boldsymbol{p}-\frac{\boldsymbol{a}}{\boldsymbol{2}}\boldsymbol{Y}\right)\int_{\boldsymbol{Y}}^{\boldsymbol{W_n}}(\boldsymbol{W_n}-\boldsymbol{Y})\boldsymbol{dt}\right|\right] \\
&\qquad= ||\boldsymbol{f^{(3)}}||\mathbb{E}\left[\left|\left(\boldsymbol{p}-\frac{\boldsymbol{a}}{\boldsymbol{2}}\boldsymbol{Y}\right)(\boldsymbol{W_n}-\boldsymbol{Y})^{\boldsymbol{2}}\right|\right] \\
&\qquad= ||\boldsymbol{f^{(3)}}||\mathbb{E}\left[\left|\left(\boldsymbol{p}-\frac{\boldsymbol{a}}{\boldsymbol{2}}\boldsymbol{Y}\right)\boldsymbol{X_n^2}\right|\right] \\
&\qquad\leq ||\boldsymbol{f^{(3)}}||\left(\boldsymbol{p}+\frac{\boldsymbol{a}}{\boldsymbol{2}}\mathbb{E}(\boldsymbol{Y})\right)\mathbb{E}\boldsymbol{X_n^2} \\
&\qquad= \boldsymbol{2p}||\boldsymbol{f^{(3)}}|| \times \frac{\boldsymbol{1}}{\boldsymbol{an}}\frac{\boldsymbol{K_{-p+2}}\left(\sqrt{\frac{\boldsymbol{a}}{\boldsymbol{n}}}\right)}{\boldsymbol{K_{-p}}\left(\sqrt{\frac{\boldsymbol{a}}{\boldsymbol{n}}}\right)}.
\end{aligned}$$

Inequality (16) follows from the fact that

$$\begin{aligned}
\left|\mathbb{E}\left[\boldsymbol{Y}\int_{\boldsymbol{Y}}^{\boldsymbol{W_n}}(\boldsymbol{W_n}-\boldsymbol{t})\boldsymbol{f^{(4)}}(\boldsymbol{t})\boldsymbol{dt}\right]\right| &\leq ||\boldsymbol{f^{(4)}}||\mathbb{E}\left[\left|\boldsymbol{Y}(\boldsymbol{W_n}-\boldsymbol{Y})^{\boldsymbol{2}}\right|\right] \\
&= ||\boldsymbol{f^{(4)}}||\mathbb{E}\boldsymbol{Y}\mathbb{E}\boldsymbol{X_n^2}.
\end{aligned}$$

□

Proof of Theorem 3.1
We have

$$\begin{aligned}
&|\mathbb{E}\boldsymbol{h}(\boldsymbol{W_n})-\mathbb{E}\boldsymbol{h}(\boldsymbol{\Lambda})| \\
&\qquad= \left|\mathbb{E}\left[\boldsymbol{W_n}\boldsymbol{f''}(\boldsymbol{W_n})+\left(\boldsymbol{p}-\frac{\boldsymbol{a}}{\boldsymbol{2}}\boldsymbol{W_n}\right)\boldsymbol{f'}(\boldsymbol{W_n})\right]\right| \\
&\qquad= \left|\mathbb{E}\left[(\boldsymbol{X_n}+\boldsymbol{Y})\boldsymbol{f''}(\boldsymbol{W_n})+\left(\boldsymbol{p}-\frac{\boldsymbol{a}}{\boldsymbol{2}}(\boldsymbol{X_n}+\boldsymbol{Y})\right)\boldsymbol{f'}(\boldsymbol{W_n})\right]\right| \\
&\qquad= \left|\mathbb{E}\left[\boldsymbol{X_n}\boldsymbol{f''}(\boldsymbol{W_n})-\frac{\boldsymbol{a}}{\boldsymbol{2}}\boldsymbol{X_n}\boldsymbol{f'}(\boldsymbol{W_n})+\boldsymbol{Y}\boldsymbol{f''}(\boldsymbol{W_n})+\left(\boldsymbol{p}-\frac{\boldsymbol{a}}{\boldsymbol{2}}\boldsymbol{Y}\right)\boldsymbol{f'}(\boldsymbol{W_n})\right]\right| \\
&\qquad\leq \left|\mathbb{E}\left[\boldsymbol{X_n}\boldsymbol{f''}(\boldsymbol{W_n})-\frac{\boldsymbol{a}}{\boldsymbol{2}}\boldsymbol{X_n}\boldsymbol{f'}(\boldsymbol{W_n})\right]\right|+\left|\mathbb{E}\left[\boldsymbol{Y}\boldsymbol{f''}(\boldsymbol{W_n})+\left(\boldsymbol{p}-\frac{\boldsymbol{a}}{\boldsymbol{2}}\boldsymbol{Y}\right)\boldsymbol{f'}(\boldsymbol{W_n})\right]\right|.
\end{aligned}$$

The upper bound of the first term in the right hand side of the above inequality is given by Lemma 4.1 in terms of $||\boldsymbol{f'}||$ and $||\boldsymbol{f''}||$. By Taylor's expansion of $\boldsymbol{f''}(\boldsymbol{W_n})$ and $\boldsymbol{f'}(\boldsymbol{W_n})$ in the neighborhood of $\boldsymbol{Y}$, we have

$$\boldsymbol{Y}\boldsymbol{f''}(\boldsymbol{W_n}) = \boldsymbol{Y}\boldsymbol{f''}(\boldsymbol{Y})+\boldsymbol{Y}(\boldsymbol{W_n}-\boldsymbol{Y})\boldsymbol{f^{(3)}}(\boldsymbol{Y})+\boldsymbol{Y}\int_{\boldsymbol{Y}}^{\boldsymbol{W_n}}(\boldsymbol{W_n}-\boldsymbol{t})\boldsymbol{f^{(4)}}(\boldsymbol{t})\boldsymbol{dt}$$

and

$$\left(p-\tfrac{a}{2}Y\right)f'(W_n)=\left(p-\tfrac{a}{2}Y\right)f'(Y)+\left(p-\tfrac{a}{2}Y\right)(W_n-Y)f''(Y)$$

$$+\left(p-\frac{a}{2}Y\right)\int_Y^{W_n}(W_n-t)f^{(3)}(t)dt.$$

Hence

$$\left|\mathbb{E}\left[Yf''(W_n)+\left(p-\frac{a}{2}Y\right)f'(W_n)\right]\right|\leq I+II+III+IV+V$$

where

$$I=\left|\mathbb{E}\left[Yf''(Y)+\left(p-\frac{a}{2}Y\right)f'(Y)\right]\right|$$

$$II=\left|\mathbb{E}\left[Y(W_n-Y)f^{(3)}(Y)\right]\right|$$

$$III=\left|\mathbb{E}\left[\left(p-\frac{a}{2}Y\right)(W_n-Y)f''(Y)\right]\right|$$

$$IV=\left|\mathbb{E}\left[Y\int_Y^{W_n}(W_n-t)f^{(4)}(t)dt\right]\right|$$

$$V=\left|\mathbb{E}\left[\left(p-\frac{a}{2}Y\right)\int_Y^{W_n}(W_n-t)f^{(3)}(t)dt\right]\right|.$$

By Stein characterization of $\gamma(p,a/2)$ distribution deduced from (9), $I=0$. We obtain the upper bounds of II, III, IV and V by Lemma 4.2 in terms of $||f'||$, $||f''||$, $\left|\left|f^{(3)}\right|\right|$ and $\left|\left|f^{(4)}\right|\right|$. Inequality (10) gives

$$||f'||\leq\frac{2}{a}||h'|| \tag{17}$$

and inequality (11) gives

$$\begin{aligned}
||f''||&\leq\frac{2}{1+p}\left(3||h'||+a||h||\right)\\
\left|\left|f^{(3)}\right|\right|&\leq\frac{2}{2+p}\left(3||h''||+a||h'||\right)\\
\left|\left|f^{(4)}\right|\right|&\leq\frac{2}{3+p}\left(3||h^{(3)}||+a||h''||\right).
\end{aligned} \tag{18}$$

Theorem 3.1 follows by substituting, in (17) and (18, $||f'||$, $||f''||$, $\left|\left|f^{(3)}\right|\right|$ and $\left|\left|f^{(4)}\right|\right|$ for their upper bounds. □

4.2 Proof of Theorem 3.2

The first item comes from the convolution property (6).

The second item is easily proved as for Proposition 3.1.

In the sequel of the proof we need the following lemmas.

Lemma 4.3 *V_n and Z_n satisfy*

$$|\mathbb{E}\left[Z_nf''(V_n)-cZ_nf'(V_n)\right]|\leq\left[c||f'||+||f''||\right]\frac{1}{cn}\frac{\psi(a,a-\frac{1}{n};c)}{\psi(a,1+a-\frac{1}{n};c)}.$$

Proof

Using the triangle inequality, we have

$$|\mathbb{E}\left[\boldsymbol{Z_n f''(V_n) - cZ_n f'(V_n)}\right]| \leq (\boldsymbol{c}||\boldsymbol{f'}|| + ||\boldsymbol{f''}||)\,\mathbb{E}\boldsymbol{Z_n}$$

with

$$\mathbb{E}\boldsymbol{Z_n} = \frac{\boldsymbol{1}}{\boldsymbol{n}}\frac{\boldsymbol{\psi}\left(\boldsymbol{1}+\frac{\boldsymbol{1}}{\boldsymbol{n}}, \boldsymbol{2}-\boldsymbol{a}+\frac{\boldsymbol{1}}{\boldsymbol{n}}; \boldsymbol{c}\right)}{\boldsymbol{\psi}\left(\frac{\boldsymbol{1}}{\boldsymbol{n}}, \boldsymbol{1}-\boldsymbol{a}+\frac{\boldsymbol{1}}{\boldsymbol{n}}; \boldsymbol{c}\right)} = \frac{\boldsymbol{1}}{\boldsymbol{cn}}\frac{\boldsymbol{\psi(a, a-\frac{1}{n}; c)}}{\boldsymbol{\psi(a, 1+a-\frac{1}{n}; c)}}.$$

The last equality is obtained by using the fact that

$$\boldsymbol{\psi(a, b; x) = x^{1-b}\psi(a-b+1, 2-b; x).}$$

□

Lemma 4.4 *$\boldsymbol{V_n}$ and $\boldsymbol{Y_n}$ satisfy*

$$|\mathbb{E}\left[\boldsymbol{Y_n f''(Y_n) + (a - cY_n) f'(Y_n)}\right]| \leq \frac{\boldsymbol{||f'||}}{\boldsymbol{n}}, \tag{19}$$

$$|\mathbb{E}\left[\boldsymbol{(a - cY_n)(V_n - Y_n) f''(Y_n)}\right]| \leq \left(\boldsymbol{a} + \left|\boldsymbol{a} - \frac{\boldsymbol{1}}{\boldsymbol{n}}\right|\right)||\boldsymbol{f''}||\frac{\boldsymbol{1}}{\boldsymbol{cn}}\frac{\boldsymbol{\psi(a, a-\frac{1}{n}; c)}}{\boldsymbol{\psi(a, 1+a-\frac{1}{n}; c)}}, \tag{20}$$

$$\left|\mathbb{E}\left[\boldsymbol{Y_n(V_n - Y_n) f^{(3)}(Y_n)}\right]\right| \leq \frac{\boldsymbol{1}}{\boldsymbol{c}}\left|\boldsymbol{a} - \frac{\boldsymbol{1}}{\boldsymbol{n}}\right|||\boldsymbol{f^{(3)}}||\frac{\boldsymbol{1}}{\boldsymbol{cn}}\frac{\boldsymbol{\psi(a, a-\frac{1}{n}; c)}}{\boldsymbol{\psi(a, 1+a-\frac{1}{n}; c)}}, \tag{21}$$

$$\left|\mathbb{E}\left[\boldsymbol{(a - cY_n)}\int_{\boldsymbol{Y_n}}^{\boldsymbol{V_n}}\boldsymbol{(V_n - t) f^{(3)}(t) dt}\right]\right| \leq \left(\boldsymbol{a} + \left|\boldsymbol{a} - \frac{\boldsymbol{1}}{\boldsymbol{n}}\right|\right)||\boldsymbol{f^{(3)}}||\frac{\boldsymbol{1}}{\boldsymbol{c^2 n}}\left(\boldsymbol{1} + \frac{\boldsymbol{1}}{\boldsymbol{n}}\right) \times \frac{\boldsymbol{\psi(a, -1+a-\frac{1}{n}; c)}}{\boldsymbol{\psi(a, 1+a-\frac{1}{n}; c)}}, \tag{22}$$

$$\left|\mathbb{E}\left[\boldsymbol{Y_n}\int_{\boldsymbol{Y_n}}^{\boldsymbol{V_n}}\boldsymbol{(V_n - t) f^{(4)}(t) dt}\right]\right| \leq \frac{\boldsymbol{1}}{\boldsymbol{c}}\left|\boldsymbol{a} - \frac{\boldsymbol{1}}{\boldsymbol{n}}\right|||\boldsymbol{f^{(4)}}||\frac{\boldsymbol{1}}{\boldsymbol{c^2 n}}\left(\boldsymbol{1} + \frac{\boldsymbol{1}}{\boldsymbol{n}}\right) \times \frac{\boldsymbol{\psi(a, -1+a-\frac{1}{n}; c)}}{\boldsymbol{\psi(a, 1+a-\frac{1}{n}; c)}}. \tag{23}$$

Proof

For inequality (19) we have

$$\begin{aligned}
&|\mathbb{E}\left[\boldsymbol{Y_n f''(Y_n) + (a - cY_n) f'(Y_n)}\right]| \\
&\qquad = \left|\mathbb{E}\left[\boldsymbol{Y_n f''(Y_n) + (a - \frac{1}{n} - cY_n) f'(Y_n) + \frac{1}{n} f'(Y_n)}\right]\right| \\
&\qquad \leq \left|\mathbb{E}\left[\boldsymbol{Y_n f''(Y_n) + (a - \frac{1}{n} - cY_n) f'(Y_n)}\right]\right| + \\
&\qquad \boldsymbol{\frac{1}{n}}\,|\mathbb{E}\left[\boldsymbol{f'(Y_n)}\right]| \\
&\qquad \leq \frac{\boldsymbol{||f'||}}{\boldsymbol{n}},
\end{aligned}$$

noting that $\mathbb{E}\left[\boldsymbol{Y_n f''(Y_n)} + (\boldsymbol{a} - \frac{\boldsymbol{1}}{\boldsymbol{n}} - \boldsymbol{cY_n})\boldsymbol{f'(Y_n)}\right] = \boldsymbol{0}$ by the Stein characterization for $\boldsymbol{\gamma(a - \frac{1}{n}, c)}$ distribution.

For inequality (20), we have

$$\begin{aligned}|\mathbb{E}\left[(\boldsymbol{a} - \boldsymbol{cY_n})(\boldsymbol{V_n} - \boldsymbol{Y_n})\boldsymbol{f''(Y_n)}\right]| &= |\mathbb{E}\left[(\boldsymbol{a} - \boldsymbol{cY_n})(\boldsymbol{Z_n})\boldsymbol{f''(Y_n)}\right]| \\ &\leq ||\boldsymbol{f''}||\left(\boldsymbol{a} + \boldsymbol{c}|\mathbb{E}(\boldsymbol{Y_n})|\right)\mathbb{E}(\boldsymbol{Z_n}) \\ &= \left(\boldsymbol{a} + \left|\boldsymbol{a} - \frac{\boldsymbol{1}}{\boldsymbol{n}}\right|\right)||\boldsymbol{f''}||\frac{\boldsymbol{1}}{\boldsymbol{cn}}\frac{\boldsymbol{\psi(a, a - \frac{1}{n}; c)}}{\boldsymbol{\psi(a, 1 + a - \frac{1}{n}; c)}}.\end{aligned}$$

We obtain inequality (21) by

$$\begin{aligned}\left|\mathbb{E}\left[\boldsymbol{Y_n}(\boldsymbol{V_n} - \boldsymbol{Y_n})\boldsymbol{f^{(3)}}(\boldsymbol{Y_n})\right]\right| &= \left|\mathbb{E}\left[\boldsymbol{Y_n}(\boldsymbol{Z_n})\boldsymbol{f^{(3)}}(\boldsymbol{Y_n})\right]\right| \\ &\leq ||\boldsymbol{f^{(3)}}|||\mathbb{E}\boldsymbol{Y_n}\mathbb{E}\boldsymbol{Z_n}| \quad (\boldsymbol{Y_n} \text{ and } \boldsymbol{Z_n} \text{ are independent}) \\ &= \frac{\boldsymbol{1}}{\boldsymbol{c}}\left|\boldsymbol{a} - \frac{\boldsymbol{1}}{\boldsymbol{n}}\right|||\boldsymbol{f^{(3)}}||\frac{\boldsymbol{1}}{\boldsymbol{cn}}\frac{\boldsymbol{\psi(a, a - \frac{1}{n}; c)}}{\boldsymbol{\psi(a, 1 + a - \frac{1}{n}; c)}}.\end{aligned}$$

For inequality (22), we have

$$\begin{aligned}&\left|\mathbb{E}\left[(\boldsymbol{a} - \boldsymbol{cY_n})\int_{\boldsymbol{Y_n}}^{\boldsymbol{V_n}}(\boldsymbol{V_n} - \boldsymbol{t})\boldsymbol{f^{(3)}}(\boldsymbol{t})\boldsymbol{dt}\right]\right| \\ &\qquad \leq (\boldsymbol{a} + \boldsymbol{c}|\mathbb{E}\boldsymbol{Y_n}|)||\boldsymbol{f^{(3)}}||\mathbb{E}\left(\int_{\boldsymbol{Y_n}}^{\boldsymbol{V_n}}|\boldsymbol{V_n} - \boldsymbol{Y_n}|\boldsymbol{dt}\right) \\ &\qquad = (\boldsymbol{a} + \boldsymbol{c}|\mathbb{E}\boldsymbol{Y_n}|)||\boldsymbol{f^{(3)}}||\mathbb{E}(\boldsymbol{Z_n^2}) \\ &\qquad = \left(\boldsymbol{a} + \left|\boldsymbol{a} - \frac{\boldsymbol{1}}{\boldsymbol{n}}\right|\right)||\boldsymbol{f^{(3)}}||\frac{\boldsymbol{1}}{\boldsymbol{c^2 n}}\left(\boldsymbol{1} + \frac{\boldsymbol{1}}{\boldsymbol{n}}\right) \times \\ &\qquad \frac{\boldsymbol{\psi(a, -1 + a - \frac{1}{n}; c)}}{\boldsymbol{\psi(a, 1 + a - \frac{1}{n}; c)}},\end{aligned}$$

since

$$\begin{aligned}\mathbb{E}(\boldsymbol{Z_n^2}) &= \frac{\boldsymbol{1}}{\boldsymbol{n}}\left(\boldsymbol{1} + \frac{\boldsymbol{1}}{\boldsymbol{n}}\right)\frac{\boldsymbol{\psi\left(2 + \frac{1}{n}, 3 - a + \frac{1}{n}; c\right)}}{\boldsymbol{\psi\left(\frac{1}{n}, 1 - a + \frac{1}{n}; c\right)}} \\ &= \frac{\boldsymbol{1}}{\boldsymbol{c^2 n}}\left(\boldsymbol{1} + \frac{\boldsymbol{1}}{\boldsymbol{n}}\right)\frac{\boldsymbol{\psi(a, -1 + a - \frac{1}{n}; c)}}{\boldsymbol{\psi(a, 1 + a - \frac{1}{n}; c)}}.\end{aligned}$$

The last inequality of Lemma 4.4 is obtained by

$$\begin{aligned}\left|\mathbb{E}\left[\boldsymbol{Y_n}\int_{\boldsymbol{Y_n}}^{\boldsymbol{V_n}}(\boldsymbol{V_n} - \boldsymbol{t})\boldsymbol{f^{(4)}}(\boldsymbol{t})\boldsymbol{dt}\right]\right| &\leq ||\boldsymbol{f^{(4)}}|||\mathbb{E}\boldsymbol{Y_n}|\mathbb{E}\boldsymbol{Z_n^2} \\ &= \frac{\boldsymbol{1}}{\boldsymbol{c}}\left|\boldsymbol{a} - \frac{\boldsymbol{1}}{\boldsymbol{n}}\right|||\boldsymbol{f^{(4)}}||\frac{\boldsymbol{1}}{\boldsymbol{c^2 n}}\left(\boldsymbol{1} + \frac{\boldsymbol{1}}{\boldsymbol{n}}\right) \times \\ &\frac{\boldsymbol{\psi(a, -1 + a - \frac{1}{n}; c)}}{\boldsymbol{\psi(a, 1 + a - \frac{1}{n}; c)}}.\end{aligned}$$

We are now able to prove the last item of Theorem 3.2.

Proof of Theorem 3.2

We check that

$$\begin{aligned}
&|\mathbb{E}\boldsymbol{h}(\boldsymbol{V_n}) - \mathbb{E}\boldsymbol{h}(\boldsymbol{\Lambda})| \\
&\qquad = |\mathbb{E}\left[\boldsymbol{V_n f''(V_n)} + (\boldsymbol{a} - \boldsymbol{cV_n})\, \boldsymbol{f'(V_n)}\right]| \\
&\qquad = |\mathbb{E}\left[(\boldsymbol{Z_n} + \boldsymbol{Y_n})\boldsymbol{f''(V_n)} + (\boldsymbol{a} - \boldsymbol{c}(\boldsymbol{Z_n} + \boldsymbol{Y_n}))\, \boldsymbol{f'(V_n)}\right]| \\
&\qquad = |\mathbb{E}\left[\boldsymbol{Z_n f''(V_n)} - \boldsymbol{cZ_n f'(V_n)} + \boldsymbol{Y_n f''(V_n)} + (\boldsymbol{a} - \boldsymbol{cY_n})\, \boldsymbol{f'(V_n)}\right]| \\
&\qquad \leq |\mathbb{E}\left[\boldsymbol{Z_n f''(V_n)} - \boldsymbol{cZ_n f'(V_n)}\right]| + |\mathbb{E}\left[\boldsymbol{Y_n f''(V_n)} + (\boldsymbol{a} - \boldsymbol{cY_n})\, \boldsymbol{f'(V_n)}\right]|\,.
\end{aligned}$$

The upper bound of $|\mathbb{E}\left[\boldsymbol{Z_n f''(V_n)} - \boldsymbol{cZ_n f'(V_n)}\right]|$ is given by Lemma 4.4 in terms of $||\boldsymbol{f'}||$ and $||\boldsymbol{f''}||$. By Taylor's expansion of $\boldsymbol{f''(V_n)}$ and $\boldsymbol{f'(V_n)}$ in the neighborhood of $\boldsymbol{Y_n}$, we have

$$\boldsymbol{Y_n f''(V_n)} = \boldsymbol{Y_n f''(Y_n)} + \boldsymbol{Y_n(V_n - Y_n)f^{(3)}(Y_n)} + \boldsymbol{Y_n}\int_{\boldsymbol{Y_n}}^{\boldsymbol{V_n}} \boldsymbol{(V_n - t)f^{(4)}(t)dt}$$

and

$$\begin{aligned}
(\boldsymbol{a} - \boldsymbol{cY_n})\, \boldsymbol{f'(V_n)} &= (\boldsymbol{a} - \boldsymbol{cY_n})\, \boldsymbol{f'(Y_n)} + (\boldsymbol{a} - \boldsymbol{cY_n})\, \boldsymbol{(V_n - Y_n)f''(Y_n)} \\
&\quad + (\boldsymbol{a} - \boldsymbol{cY_n}) \int_{\boldsymbol{Y_n}}^{\boldsymbol{V_n}} \boldsymbol{(V_n - t)f^{(3)}(t)dt.}
\end{aligned}$$

Hence

$$|\mathbb{E}\left[\boldsymbol{Y_n f''(V_n)} + (\boldsymbol{a} - \boldsymbol{cY_n})\, \boldsymbol{f'(V_n)}\right]| \leq |\mathbb{E}\left[\boldsymbol{Y_n f''(Y_n)} + (\boldsymbol{a} - \boldsymbol{cY_n})\, \boldsymbol{f'(Y_n)}\right]| +$$

$$\left|\mathbb{E}\left[\boldsymbol{Y_n(V_n - Y_n)f^{(3)}(Y_n)}\right]\right| + |\mathbb{E}\left[(\boldsymbol{a} - \boldsymbol{cY_n})\, \boldsymbol{(V_n - Y_n)f''(Y_n)}\right]| +$$

$$\left|\mathbb{E}\left[\boldsymbol{Y_n}\int_{\boldsymbol{Y_n}}^{\boldsymbol{V_n}} \boldsymbol{(V_n - t)f^{(4)}(t)dt}\right]\right| + \left|\mathbb{E}\left[(\boldsymbol{a} - \boldsymbol{cY_n}) \int_{\boldsymbol{Y_n}}^{\boldsymbol{V_n}} \boldsymbol{(V_n - t)f^{(3)}(t)dt}\right]\right|. \quad (24)$$

We obtain the upper bounds of the expressions of the right-hand side of the inequality (24) by Lemma 4.3 in terms of $||\boldsymbol{f'}||$, $||\boldsymbol{f''}||$, $\left|\left|\boldsymbol{f^{(3)}}\right|\right|$ and $\left|\left|\boldsymbol{f^{(4)}}\right|\right|$.

Inequality (10) gives $||\boldsymbol{f'}|| \leq \frac{\boldsymbol{1}}{\boldsymbol{c}}||\boldsymbol{h'}||$ and inequality (11) gives

$$\begin{aligned}
||\boldsymbol{f''}|| &\leq \frac{\boldsymbol{2}}{\boldsymbol{1+a}}\left(\boldsymbol{3}||\boldsymbol{h'}|| + \boldsymbol{2c}||\boldsymbol{h}||\right) \\
\left|\left|\boldsymbol{f^{(3)}}\right|\right| &\leq \frac{\boldsymbol{2}}{\boldsymbol{2+a}}\left(\boldsymbol{3}||\boldsymbol{h''}|| + \boldsymbol{2c}||\boldsymbol{h'}||\right) \\
\left|\left|\boldsymbol{f^{(4)}}\right|\right| &\leq \frac{\boldsymbol{2}}{\boldsymbol{3+a}}\left(\boldsymbol{3}||\boldsymbol{h^{(3)}}|| + \boldsymbol{2c}||\boldsymbol{h''}||\right),
\end{aligned}$$

which ends the proof of Theorem 3.2.

□

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References

[1] A. D. Barbour. Stein's method and Poisson process approximations. *J.Appl. Probab.* **25A** (1988) 175-184.

[2] Barndorff-Nielsen, O. E. and Halgreen, C. (1977). Infinite divisibility of the Hyperbolic and generalized inverse Gaussian distribution. $Z.\ Wahrscheinlichkeitstheorie\ und\ Verw.\ Gebiete$, **38**, 309-312.

[3] L. H. Y. Chen, L. Goldstein and Q.-M. Shao. $Normal\ approximation\ by\ Stein's\ method$. Probability and its Applications (New York). Springer, Heidelberg, 2011.

[4] R. E. Gaunt. Rates of Convergence of Variance-Gamma Approximations via Stein's Method. DPhil thesis, University of Oxford, (2013).

[5] R. E.Gaunt. A Stein characterisation of the generalized hyperbolic distribution. *ESAIM: Probability and Statistics*. **21** (2017) 303-316.

[6] R.E. Gaunt, A. Pickett and G. Reinert. Chi-square approximation by Stein's method with application to Pearson's statistic.To appear in *Ann. Appl. Probab.* (2017).

[7] A. E. Koudou and C. Ley. Characterizations of GIG laws: a survey complemented with two new results *Proba. Surv.* **11** (2014) 161–176.

[8] A. E. Koudou and P. Vallois. Independence properties of the Matsumoto-Yor type. $Bernoulli$ **18** (2012) 119-136.

[9] H. M. Luk. Stein's method for the gamma distribution and related statistical applications. Ph.D. thesis. University of Southern California. Los Angeles, USA (1994).

[10] A. Picket. Rates of Convergence of $\boldsymbol{\chi^2}$ Approximations via Stein's Method. DPhil thesis, University of Oxford, (2004).

[11] A. Piliszek and J. Wesołowski. Change of measure technique in characterizations of the gamma and Kummer distributions. J. Math. Anal. Appl. 458 (2018) 967-979.

[12] N. Ross. Fundamentals of Stein's method. *Probab. Surv.* **8** (2011) 210–293.

[13] W. Schoutens. Orthogonal Polynomials in Steins Method. *J. Math. Anal. Appl.* **253** (2001) 515–531.

[14] C. Stein. A bound for the error in the normal approximation to the distribution of a sum of dependent random variables. In *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability* (Vol 2, pp. 583–602) (1972). Berkeley: University of California Press.