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Exponential stability of density-velocity systems with boundary conditions and source term for the H^2 norm

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Abstract

In this paper, we address the problem of the exponential stability of density-velocity systems with boundary conditions. Density-velocity systems are omnipresent in physics as they encompass all systems that consist in a flux conservation and a momentum equation. In this paper we show that any such system can be stabilized exponentially quickly in the H^2 norm using simple local feedbacks, provided a condition on the source term which holds for most physical systems, even when it is not dissipative. Besides, the feedback laws obtained only depends on the target values at the boundaries, which implies that they do not depend on the expression of the source term or the force applied on the system and makes them very easy to implement in practice and robust to model errors. For instance, for a river modeled by Saint-Venant equations this means that the feedback laws do not require any information on the friction model, the slope or the shape of the channel considered. This feat is obtained by showing the existence of a basic H^2 Lyapunov functions and we apply it to numerous systems: the general Saint-Venant equations, the isentropic Euler equations, the motion of water in rigid-pipe, the osmosis phenomenon, etc.

1 Introduction

Density-velocity systems are important hyperbolic systems as they represent the physical phenomena where the flux is conserved, while the energy can be either increased or decreased. In physics they are found in fluid mechanics, electromegnetism, etc. The increase or decrease of the energy leads to nonuniform steady-states with sometimes large variations in space. In this paper, we address the exponential stability of such nonlinear systems for the H^2 norm, although the result is also true for the H^p norm for any $p \geq 2$. Mentioning the norm is not superfluous as, for nonlinear systems, the stability for different norms are not equivalent [11]. In particular it has been shown in [1] that the basic quadratic Lyapunov functions fail to ensure the stabilization in the L^2 norm for nonlinear hyperbolic systems systems and that one has to study the H^2 norm instead. Other attempt of basic Lyapunov functions have been constructed to ensure the stability of hyperbolic systems in the C^1 norm, for instance [7, 23, 24].

Physical density-velocity systems often have well-known conservative or dissipative energy or entropy functions when no source term occurs [12]. These dissipative energy or entropy functions are quite useful for the analysis of such system and enable to obtain stability results (see for instance [4, 8, 10] for the use of entropy as control Lyapunov function for Saint-Venant equations and [4] for the Euler equations). When source terms appear, however, no such function is usually known, especially when the source term is not dissipative. In the previous contribution [4], the authors also studied the stabilization of hyperbolic density-velocity equations, but with dissipative source terms only depending on the unknown functions. This is the case for Saint-Venant equations with no slope and with a constant friction, or for the isentropic Euler equations when the gas pressure is simply assumed to be a function of a gas density and a friction proportional to the square of the velocity. However, the source terms may also depend on the space variable in practice and may not be dissipative. This is the case for example for the Saint-Venant equations with both slope and arbitrary friction, or Euler equations with arbitrary friction slope, and general gas pressure, which are more realistic.

For general density-velocity systems, we find that for any H^2 steady-state, there always exists a basic quadratic Lyapunov function for the H^2 norm (or basic H^2 Lyapunov function) that guarantees the exponential stability of the steady-states for the H^2 norm provided suitable boundary conditions and a reasonable physical condition on the source term. Our result in this paper is quite generic and can be widely used in applications, we illustrate it by applying it to several physical systems: the general nonlinear Saint-Venant equations, the general isentropic Euler equations, the motion of water in a rigid pipe, a flow model under osmosis phenomenon. Moreover, our method has many advantages when applying in the real world. For example, to stabilize the Saint-Venant equations, we require some information on the section and the velocities only at the boundaries. No information on the internal section profile, on the slope or on the friction is required. This is very convenient in practice, as this feedback law can be applied without a clear information of the inner state of the channel (bathymetry, material, profile, etc.) since there may be no way to know properly the precise shape or material of the channel. Besides, while many friction models exist (see e.g. [6, Section 4.5]), it also completes the debate about which friction model to use as this feedback law works for any of them, without requiring to know it.

Another contribution of this paper is that we study the stabilization of general density-velocity systems with one single boundary control. This is for example in the regulation of navigable rivers, one usually applies only one boundary control at the downstream of the channel.
The organization of the paper is as follows. In Section 2, we first present the main results: the exponential stabilization of general density-velocity systems with two boundary controls in Theorem 2.1. Moreover in Theorem 2.2, the exponential stabilization result with a single boundary control is presented. Then we apply the result to several physical models. In Section 3, we give the proof of our main results, namely Theorem 2.1 and Theorem 2.2. In addition, we show the optimality of the conditions on the control in Section 4, Finally, some detailed computations are provided in the appendices.

2 Model considered and main result

A nonlinear hyperbolic density-velocity system is composed of a mass conservation law and a balance of momentum [4] and is thus given by

$$\partial_t H + \partial_x(HV) = 0, \tag{2.1}$$
$$\partial_t V + V\partial_x V + \partial_x(P(H,x)) + S(H,V,x) = 0, \tag{2.2}$$

where $t \in [0,+\infty)$, $x \in [0,L]$ with $L > 0$ any arbitrary constant. In many applications, $H : [0,+\infty)\times[0,L] \to (0,+\infty)$ denotes the density, $V : [0,+\infty) \times [0,L] \to (0,+\infty)$ denotes the propagation velocity, HV is the flow density and and $S(H,V,x)$ is a source term resulting of non-conservative forces acting on the system, such as slope or friction. The first equation expresses the flux conservation and is often known as continuity equation, while the second equation is usually referred as dynamical or momentum equation. In this second equation, $V\partial_x V$ represents the variation of the kinetic energy, while $\partial_x(P(H,x))$ represents the variation of the potential energy and corresponds to a conservative force (e.g. pressure, gravitation, etc.). As we are interested in physical systems, we assume that $S \in C^2((0,+\infty)^2 \times [0,L];\mathbb{R})$, $P \in C^2((0,+\infty) \times [0,L];\mathbb{R})$ and here and hereafter, we also assume that

$$H > 0, \quad V > 0, \quad \partial_H P(H,x) > 0. \tag{2.3}$$

The steady-states (H^*,V^*) of (2.1)–(2.2) are the solutions of

$$H^*_x = V^*_x \frac{H^*}{V^*}, \tag{2.4}$$

$$V^*_x = V^* \frac{S(H^*,V^*,\cdot) + \partial_x P(H^*,\cdot)}{\partial_H P(H^*,\cdot)H^* - V^{*2}}. \tag{2.5}$$

For each initial condition $(H^*(0),V^*(0)) \in (0,+\infty)^2$ satisfying $\partial_H P(H^*(0),0)H^*(0) - V(0)^{*2} > 0$, there exists a unique maximal solution to (2.4)-(2.5), and this maximal solution exists as soon as the condition $\partial_H P(H^*,\cdot)H^* > V^{*2}$ is satisfied. Besides, as hyperbolic systems with propagation velocities of the same sign can always be stabilized by the means of proportional boundary feedback (see e.g. [24]), we assume in the following that the propagation velocities of this system have opposite signs, which, from (2.1)–(2.2), means that $\partial_H P(H^*,\cdot)H^* > V^{*2}$. This holds for example in the case of the fluvial regime for Saint-Venant equations.

In the following, we give two strategies of boundary controls. As a first strategy, Theorem 2.1 relies on two boundary controls, i.e. the number of controls are equal to the number of the unknown functions. While in practice, one may control only one boundary. In the regulation of navigable rivers, for instance, one usually applies only one control at the downstream of the channel. Theorem 2.2 is thus concerned with the stabilization of general density-velocity systems with a single boundary control.

Two boundary controls We aim at stabilizing the steady-states of (2.1)–(2.2) with boundary feedback controls. We suppose that the boundary conditions have the form

$$\begin{aligned} V(t,0) &= \mathcal{B}_1(H(t,0)),\\ V(t,L) &= \mathcal{B}_2(H(t,L)), \end{aligned} \tag{2.6}$$

where the control function $\mathcal{B} = (\mathcal{B}_1,\mathcal{B}_2) : \mathbb{R}^2 \to \mathbb{R}^2$ is of class C^2. These kind of boundary conditions are imposed by physical devices in engineering system (e.g. sluice gates, feeding valves, pumps, etc.). This control function is one of the most simple potential feedback law as one does not need to know the full-state. Moreover, this control is local in the sense that one only needs to measure the value at the same end where the control acts.

As we study the stabilization in the H^2 norm, for any given initial condition

$$H(0,x) = H_0(x), \quad V(0,x) = V_0(x), \quad x \in [0,L], \tag{2.7}$$

with $(H_0,V_0) \in H^2((0,L);\mathbb{R}^2)$, the following first-order compatibility conditions are needed [2]

$$\begin{aligned} V_0(0) =& \mathcal{B}_1(H_0(0)),\\ V_0(L) =& \mathcal{B}_2(H_0(L)),\\ \left(V_0\partial_x V_0 + \partial_H P(H_0,\cdot)\partial_x H_0 + \partial_x P(H_0,\cdot) + S(H_0,V_0,\cdot)\right)(0) =& \mathcal{B}_1'(H_0(0))\partial_x(H_0V_0)(0),\\ \left(V_0\partial_x V_0 + \partial_H P(H_0,\cdot)\partial_x H_0 + \partial_x P(H_0,\cdot) + S(H_0,V_0,\cdot)\right)(L) =& \mathcal{B}_2'(H_0(L))\partial_x(H_0V_0)(L). \end{aligned} \tag{2.8}$$

We recall now the definition of the exponential stability for the H^2 norm.

Definition 2.1. A steady-state (H^*,V^*) of the system (2.1), (2.2), (2.6) is exponentially stable for the H^2 norm if there exist $\delta > 0$, $\gamma > 0$ and $C > 0$ such that, for any $(H_0,V_0) \in H^2((0,L);\mathbb{R}^2)$ satisfying

$$|H_0 - H^*|_{H^2} + |V_0 - V^*|_{H^2} < \delta \tag{2.9}$$

and the compatibility conditions (2.8), and for any $T > 0$, the Cauchy problem (2.1), (2.2), (2.6) and (2.7) has a unique solution $(H(t,\cdot),V(t,\cdot)) \in H^2((0,L);\mathbb{R}^2)$ satisfying

$$|H(t,\cdot) - H^*|_{H^2} + |V(t,\cdot) - V^*|_{H^2} \leq Ce^{-\gamma t}\left(|H_0 - H^*|_{H^2} + |V_0 - V^*|_{H^2}\right), \ \forall\, t \in [0,T). \tag{2.10}$$

Our main result is the following

Theorem 2.1. *Assume that* $S \in C^2((0,+\infty)^2 \times [0,L];\mathbb{R})$ *and* $P \in C^2((0,+\infty) \times [0,L];\mathbb{R})$*, let* (H^*,V^*) *be a steady-state of the nonlinear hyperbolic density-velocity system* (2.1)*,* (2.2)*,* (2.6) *satisfying*

$$\frac{\partial_V S(H^*,V^*,\cdot)}{V^*} - \frac{\partial_H S(H^*,V^*,\cdot)}{\partial_H P(H^*,\cdot)} \geq 0, \quad \forall x \in [0,L]. \tag{2.11}$$

If the boundary conditions satisfy:

$$\begin{aligned}
\mathcal{B}_1'(H^*(0)) &\in \left[-\frac{\partial_H P(H^*(0),0)}{V^*(0)}, -\frac{V^*(0)}{H^*(0)}\right], \\
\mathcal{B}_2'(H^*(L)) &\in \mathbb{R} \setminus \left[-\frac{\partial_H P(H^*(L),L)}{V^*(L)}, -\frac{V^*(L)}{H^*(L)}\right],
\end{aligned} \tag{2.12}$$

then the steady-state (H^*,V^*) *is exponentially stable for the* H^2 *norm.*

Remark 2.1. Note that condition (2.11) holds naturally for most physical systems with source terms, (e.g. friction, slope, electric field, external forces, etc.), as illustrated in the physical examples at the end of this section. Besides, note also that the source term is not necessarily dissipative, as S could be negative.

The proof of this result is given in Section 3. As announced in the introduction, this is done by showing the existence of a basic quadratic Lyapunov function for the H^2 norm (or basic H^2 Lyapunov function).

Single boundary control Suppose now that we have only a single feedback control, the other boundary condition being imposed, for instance by a constant but unknown upstream flow rate on which we cannot act. The boundary conditions are now

$$\begin{aligned}
&H(t,0)V(t,0) = Q_0, \\
&V(t,L) = \mathcal{B}_2(H(t,L)),
\end{aligned} \tag{2.13}$$

where Q_0 is the unknown constant inflow upstream, while $\mathcal{B}_2 : \mathbb{R} \to \mathbb{R}$ of class C^2 is still the control function. Using the same basic quadratic Lyapunov function for the H^2 norm we can still achieve the exponential stability which is a direct application of Theorem 2.1 by noticing now that $\mathcal{B}_1(H(t,0)) = Q_0/H(t,0)$ and that the steady-state satisfies $H^*(x)V^*(x) = Q_0$. We thus have

Theorem 2.2. *Assume that* $S \in C^2((0,+\infty)^2 \times [0,L];\mathbb{R})$ *and* $P \in C^2((0,+\infty) \times [0,L];\mathbb{R})$*, let* (H^*,V^*) *be a steady-state of the nonlinear hyperbolic density-velocity system* (2.1)*,* (2.2)*,* (2.13) *satisfying* (2.11)*. If the boundary control satisfies:*

$$\mathcal{B}_2'(H^*(L)) \in \mathbb{R} \setminus \left[-\frac{\partial_H P(H^*(L),L)}{V^*(L)}, -\frac{V^*(L)}{H^*(L)}\right], \tag{2.14}$$

then the steady-state (H^*,V^*) *is exponentially stable for the* H^2 *norm.*

Remark 2.2. Note that Q_0 is assumed to be constant otherwise no steady-state (H^*,V^*) exists. However, the stabilization of slowly-varying target-state when Q_0 can vary, possibly a lot, but slowly would hold under the same condition, adapting the control as in [25].

Remark 2.3. Theorem 2.2 still holds if the control is located on $x = 0$ instead, while the imposed flow is located on $x = L$, i.e.,

$$\begin{aligned}
&V(t,0) = \mathcal{B}_1(H(t,0)), \\
&H(t,L)V(t,L) = Q_L,
\end{aligned} \tag{2.15}$$

where Q_L is the unknown constant outflow downstream, while $\mathcal{B}_1 : \mathbb{R} \to \mathbb{R}$ of class C^2 is the control function. In this case, the condition on the control becomes

$$\mathcal{B}_1'(H^*(0)) \in \left(-\frac{\partial_H P(H^*(0),0)}{V^*(0)}, -\frac{V^*(0)}{H^*(0)}\right). \tag{2.16}$$

One can find a brief proof in Appendix C.

The abstract system (2.1), (2.2) covers many well-known systems and we give now a few examples.

General Saint-Venant equations The Saint-Venant equations are the basis model for the regulation of navigable rivers and irrigation networks in agriculture. The stabilization of the Saint-Venant equations by means of local boundary feedbacks has been widely studied [4, 5, 8, 9, 16, 28]. Recently in [26], the authors obtained the stabilization of the Saint-Venant equations with non-negligible friction and arbitrary slope. However, this result is obtained under the assumption of a rectangular cross section with a constant width and a known friction model. In the following, we show that our result applies to the most general 1D Saint-Venant equations with arbitrary varying slope, section profile and friction model [13]:

$$\partial_t A + \partial_x(AV) = 0, \tag{2.17}$$

$$\partial_t(AV) + \partial_x(AV^2) + gA(\partial_x H - S_b(x) + S_f(A,V,x)) = 0, \tag{2.18}$$

where A is the cross-sectional area of the section, V is the velocity, AV is consequently the flux, H is the height of the water, S_b is the slope, S_f is the friction and g is the gravity acceleration. Note that the friction logically depends on H and V but can also depend on x for external reasons, for instance if the material of the channel changes. Whatever is the section profile, A is strictly increasing with H, thus there exists a function G strictly increasing with A such that $H = G(A,x)$ and consequently (2.18) can be written as

$$\partial_t V + V\partial_x V + g\partial_A G(A,x)\partial_x A + g\partial_x G(A,x) + g(S_f(A,V,x) - S_b(x)) = 0. \tag{2.19}$$

Thus, system (2.17)-(2.18) has the form (2.1)–(2.2) with $P = gG(A,x)$ and $S = g(S_f - S_b)$. Besides, to be physically acceptable, the friction term has to be increasing with V and decreasing with A. Hence $\partial_V S = g\partial_V S_f > 0$ and $\partial_A S = g\partial_A S_f < 0$, noticing that $\partial_A P = g\partial_A G(A,x) > 0$, thus condition (2.11) is satisfied and we have the following theorem

Theorem 2.3. *Any steady-state (A^*, V^*) of the general Saint-Venant equations* (2.17), (2.19) *with boundary conditions* (2.6) *with A instead of H, is exponentially stable for the H^2 norm provided that*

$$\begin{aligned} \mathcal{B}_1'(A^*(0)) &\in \left[-\frac{g\partial_A G(A^*(0),0)}{V^*(0)}, -\frac{V^*(0)}{A^*(0)}\right], \\ \mathcal{B}_2'(A^*(L)) &\in \mathbb{R} \setminus \left[-\frac{g\partial_A G(A^*(L),L)}{V^*(L)}, -\frac{V^*(L)}{A^*(L)}\right]. \end{aligned} \tag{2.20}$$

Water motion in a rigid pipe The water motion in a rigid pipe is a common example for engineering system, whose equations are given in [3] as follows

$$\begin{aligned} \partial_t\left(\exp\left(\frac{g\mathcal{H}}{c^2}\right)\right) + \partial_x\left(V\exp\left(\frac{g\mathcal{H}}{c^2}\right)\right) &= 0, \\ \partial_t V + V\partial_x V + \partial_x(g\mathcal{H}) + S_f(V,x) &= 0, \end{aligned} \tag{2.21}$$

where $\mathcal{H}$ is the piezometric head, $V > 0$ is the water velocity, c is the sound velocity in water, g is the gravity acceleration, and S_f is the friction term. Denoting $H = \exp\left(g\mathcal{H}/c^2\right)$, this system has the form of (2.1)–(2.2) with $P = c^2 \ln H$. As previously, to be physically acceptable, the friction term S_f has to be nondecreasing with V, thus (2.11) holds and Theorem 2.1 applies again.

The isentropic Euler equations The isentropic Euler equations are used to model the gas transportation in pipelines. There are many literatures on the stabilization of the isentropic Euler equations [4, 14, 15, 17, 18, 19, 20]. But all those results are obtained without considering the pipeline slope and using the polytropic gas assumption or the isothermal assumption. The isentropic Euler equations with slope and friction have exactly the form (2.1)–(2.2) as (see e.g. [3][1.8.1] or [21])

$$\begin{aligned} &\partial_t\varrho + \partial_x(\varrho V) = 0, \\ &\partial_t V + V\partial_x V + \frac{\partial_x(\mathcal{P}(\varrho))}{\varrho} + \frac{1}{2}\theta V|V| + g\sin\alpha(x) = 0, \end{aligned} \tag{2.22}$$

where ϱ is the gas density, V is the velocity, $\alpha \in C^2(\mathbb{R})$ is the slope of the pipe, g is the acceleration gravity, $\mathcal{P}(\varrho)$ is the pressure (increasing with ϱ) with $\sqrt{\mathcal{P}'(\varrho)} > 0$ being the sound speed in the gas, $\theta = \lambda/D$ with $\lambda > 0$ is the friction coefficient and $D > 0$ the diameter of the pipe. In this case, $P := \int_0^{\varrho} \mathcal{P}'(s)/s\, ds$ and $S = \theta V|V|/2 + g \sin\alpha$. Thus, $\partial_\varrho P(\varrho) > 0$, $\partial_\varrho S = 0$, $\partial_V S > 0$ as long as $V > 0$, which implies that (2.11) holds and that Theorem 2.1 applies. Note that this holds in particular in the case where the gas is polytropic, i.e. $\mathcal{P} = a^2\varrho^\gamma$ with $\gamma > 1$ (as in [4]), and in the case of the isothermal Euler equation, i.e. $\mathcal{P} = a^2\varrho$ (as in [21]).

Flow under osmosis Osmosis is a spontaneous movement of solvant or solute through a semipermeable membrane in a solute/solvant mix. This phenomenon is extremely important in chemistry and biology as it is the main way by which water is transported out of cells in living organisms. Besides, biological membranes allow much faster filtration than any artificial mechanical membrane [], thus attempts have been recently made to design active membranes that would mimic this behavior and a mechanical model for this phenomenon can be found in [27].
Osmosis phenomenon through a membrane permeable to the solute but not to the solvant can be modeled by a potential barrier which acts on the solute. This creates, from Newton's law, a volume force on the fluid $-c(x)\partial_x\mathcal{U}$, where $\mathcal{U}$ is the profile of the potential barrier, compactly supported, c is the concentration and x is the space variable [27]. In an inviscid fluid modeled by the isentropic Euler equations (2.22), this reduces to adding an external compactly supported pressure term. Therefore, we still have $\partial_\varrho P(\varrho) > 0$, $\partial_\varrho S = 0$, $\partial_V S > 0$ as long as $V > 0$, and Theorem 2.1 applies. Note that any external potential acting on a fluid modeled by the isentropic Euler equations would fit in our framework, osmosis is only an example.

3 Exponential stability of density-velocity hyperbolic systems

In this section we prove Theorem 2.1. Let (H^*, V^*) be a steady-state of (2.1)–(2.2). We start by proving the exponential stability of the linearized system around this steady-state for the L^2 norm to give an idea of how the proof works and then, we show that the same type of Lyapunov function can be applied to ensure the exponential stability of the nonlinear system for the H^2 norm.

3.1 Exponential stability of the linearized system

Around the steady-state (H^*, V^*), the linearized system of (2.1)-(2.2) and (2.6) is given by:

$$\begin{aligned} &\partial_t h + V^*\partial_x h + H^*\partial_x v + V_x^* h + H_x^* v = 0, \\ &\partial_t v + V^*\partial_x v + V_x^* v + \partial_H P(H^*, x)\partial_x h + \partial^2_{HH} P(H^*, x) H_x^* h + \partial^2_{xH} P(H^*, x) h \\ &\qquad\qquad\qquad\qquad + \partial_H S(H^*, V^*, x) h + \partial_V S(H^*, V^*, x) v = 0. \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} v(t,0) &= c_1 h(t,0), \\ v(t,L) &= c_2 h(t,L), \end{aligned} \tag{3.2}$$

where $h = H - H^*$ and $v = V - V^*$ are the perturbations and $c_1 = \mathcal{B}_1'(H^*(0))$ and $c_2 = \mathcal{B}_2'(H^*(L))$. To simplify the notations, we denote from now on

$$\begin{gathered} \partial_H P(H^*, x) := f(H^*, x), \quad S_{H^*} := \partial_H S(H^*, V^*, x), \quad S_{V^*} := \partial_V S(H^*, V^*, x), \\ \mathscr{S}_{H^*} := \partial_x f(H^*, x) + S_{H^*}. \end{gathered} \tag{3.3}$$

Thus, the linearized system of (2.1)-(2.2) and (2.6) around the steady-state (H^*, V^*) given by (3.1) becomes

$$\begin{pmatrix} h \\ v \end{pmatrix}_t + \begin{pmatrix} V^* & H^* \\ f(H^*, x) & V^* \end{pmatrix} \begin{pmatrix} h \\ v \end{pmatrix}_x + \begin{pmatrix} V_x^* & H_x^* \\ \mathscr{S}_{H^*} + \partial_H f(H^*, x) H_x^* & S_{V^*} + V_x^* \end{pmatrix} \begin{pmatrix} h \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.4}$$

We recall that the steady-state $0 \in L^2((0,L);\mathbb{R}^2)$ (3.4) is said exponentially stable (for the norm of $L^2((0,L);\mathbb{R}^2)$) if there exist $\nu > 0$ and $C > 0$ such that, for every $(h_0(x), v_0(x)) \in L^2((0,L);\mathbb{R}^2)$, the Cauchy problem (3.4) with initial condition

$$h(0,x) = h_0(x), \quad v(0,x) = v_0(x) \tag{3.5}$$

is well-posed and its solution satisfies

$$|(h(t,\cdot), v(t,\cdot))|_{L^2((0,L);\mathbb{R}^2)} \leq Ce^{-\nu t}|(h_0, v_0)|_{L^2((0,L);\mathbb{R}^2)}, \quad \forall t \in [0,+\infty). \tag{3.6}$$

We prove the following proposition

Proposition 3.1. *Let (H^*, V^*) be any given steady-state such that (2.11) holds, if the boundary conditions satisfy:*

$$\begin{aligned} c_1 &\in \left[-\frac{f(H^*(0),0)}{V^*(0)}, -\frac{V^*(0)}{H^*(0)}\right], \\ c_2 &\in \mathbb{R} \setminus \left[-\frac{f(H^*(L),L)}{V^*(L)}, -\frac{V^*(L)}{H^*(L)}\right], \end{aligned} \tag{3.7}$$

then the null steady-state $h = 0$, $v = 0$ of the system (3.4) and (3.2) is exponentially stable for the L^2 norm.

Proof. Observe that the matrix $\begin{pmatrix} V^* & H^* \\ f(H^*,\cdot) & V^* \end{pmatrix}$ can be diagonalized, therefore the system can be put under the Riemann invariant form by the following change of variables

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{f(H^*,x)}{H^*}} & 1 \\ -\sqrt{\frac{f(H^*,x)}{H^*}} & 1 \end{pmatrix} \begin{pmatrix} h \\ v \end{pmatrix}. \tag{3.8}$$

Then (3.4) becomes (see Appendix A)

$$\begin{aligned} &\partial_t z_1 + \lambda_1 \partial_x z_1 + \gamma_1 z_1 + \delta_1 z_2 = 0, \\ &\partial_t z_2 - \lambda_2 \partial_x z_2 + \gamma_2 z_1 + \delta_2 z_2 = 0, \end{aligned} \tag{3.9}$$

where

$$\lambda_1 = V^* + \sqrt{f(H^*,x)H^*} > 0, \quad \lambda_2 = \sqrt{f(H^*,x)H^*} - V^* > 0 \tag{3.10}$$

and

$$\begin{aligned} \gamma_1 =& \frac{1}{4}\left(2S_{V^*} + 2\mathscr{S}_{H^*}\sqrt{\frac{H^*}{f(H^*,x)}} - 3\lambda_2 \frac{V_x^*}{V^*} - \lambda_1 \frac{\partial_x f(H^*,x)}{f(H^*,x)} + \lambda_2 \frac{\partial_H f(H^*,x) H_x^*}{f(H^*,x)}\right), \\ \gamma_2 =& \frac{1}{4}\left(2S_{V^*} + 2\mathscr{S}_{H^*}\sqrt{\frac{H^*}{f(H^*,x)}} + \lambda_1 \frac{V_x^*}{V^*} - \lambda_2 \frac{\partial_x f(H^*,x)}{f(H^*,x)} + \lambda_1 \frac{\partial_H f(H^*,x) H_x^*}{f(H^*,x)}\right), \\ \delta_1 =& \frac{1}{4}\left(2S_{V^*} - 2\mathscr{S}_{H^*}\sqrt{\frac{H^*}{f(H^*,x)}} - \lambda_2 \frac{V_x^*}{V^*} + \lambda_1 \frac{\partial_x f(H^*,x)}{f(H^*,x)} - \lambda_2 \frac{\partial_H f(H^*,x) H_x^*}{f(H^*,x)}\right), \\ \delta_2 =& \frac{1}{4}\left(2S_{V^*} - 2\mathscr{S}_{H^*}\sqrt{\frac{H^*}{f(H^*,x)}} + 3\lambda_1 \frac{V_x^*}{V^*} + \lambda_2 \frac{\partial_x f(H^*,x)}{f(H^*,x)} - \lambda_1 \frac{\partial_H f(H^*,x) H_x^*}{f(H^*,x)}\right). \end{aligned} \tag{3.11}$$

The boundary conditions (3.2) become

$$z_1(t,0) = k_1 z_2(t,0), \quad z_2(t,L) = k_2 z_1(t,L) \tag{3.12}$$

with

$$k_1 = \frac{c_1 + \sqrt{\frac{f(H^*(0),0)}{H^*(0)}}}{c_1 - \sqrt{\frac{f(H^*(0),0)}{H^*(0)}}}, \quad k_2 = \frac{c_2 - \sqrt{\frac{f(H^*(L),L)}{H^*(L)}}}{c_2 + \sqrt{\frac{f(H^*(L),L)}{H^*(L)}}}. \tag{3.13}$$

With these conditions, the Cauchy problem (3.9), (3.12) with any given initial condition $\mathbf{z}(0,x) = (z_{10}, z_{20}) \in L^2((0,L);\mathbb{R}^2)$ is well-posed (see [3, Appendix A]), which implies that the original system in physical coordinates is also well-posed.
We define the function ϕ by

$$\phi(x) = \exp\left(\int_0^x \frac{\gamma_1(s)}{\lambda_1(s)} + \frac{\delta_2(s)}{\lambda_2(s)} ds\right) \tag{3.14}$$

and we introduce the following lemma, that can also be found in [26] in the particular case of the Saint-Venant equations with constant rectangular section and friction given by $S_f = kV^2A^{-1}$ where $k > 0$ is a constant friction coefficient.

Lemma 3.1. *For any* $x \in [0,L]$,

$$\left(\frac{\lambda_2}{\lambda_1}\phi\right)'(x) = \frac{\phi\delta_1}{\lambda_1} + \frac{\phi^{-1}\gamma_2}{\lambda_2}\left(\frac{\lambda_2}{\lambda_1}\phi\right)^2. \tag{3.15}$$

The proof of this lemma is given in Appendix B.
We introduce now the following Lyapunov function candidate for the L^2 norm

$$V = \int_0^L \left(f_1(x)e^{2\int_0^x \frac{\gamma_1(s)}{\lambda_1(s)}ds} z_1^2(t,x) + f_2(x)e^{-2\int_0^x \frac{\delta_2(s)}{\lambda_2(s)}ds} z_2^2(t,x)\right) dx, \tag{3.16}$$

where $\mu > 0$ is a constant and f_1, f_2 are positive C^1 functions to be chosen later on. From the positivity of f_1 and f_2, there exist a_1 and a_2 positive constants such that

$$a_2\|(z_1,z_2)\|_{L^2((0,L);\mathbb{R}^2)} \leq V \leq a_1\|(z_1,z_2)\|_{L^2((0,L);\mathbb{R}^2)} \tag{3.17}$$

which means that V is equivalent to the L^2 norm of (z_1,z_2), thus is equivalent to the L^2 norm of (h,v) from the linear change of variables (3.8). Therefore, it suffices to show the exponential decay of V to obtain the exponential stability of (3.4) and (3.2) for the L^2 norm. Differentiating (3.16) with time along the C^1-solutions of (3.9) and (3.12), one has

$$\begin{aligned}\frac{dV}{dt} =& -\left[\lambda_1 f_1 e^{2\int_0^x \frac{\gamma_1(s)}{\lambda_1(s)}ds} z_1^2 - \lambda_2 f_2 e^{-2\int_0^x \frac{\delta_2(s)}{\lambda_2(s)}ds} z_2^2\right]_0^L \\ &- \int_0^L \Big[-(\lambda_1 f_1)' e^{2\int_0^x \frac{\gamma_1(s)}{\lambda_1(s)}ds} z_1^2 + (\lambda_2 f_2)' e^{-2\int_0^x \frac{\delta_2(s)}{\lambda_2(s)}ds} z_2^2 \\ &\qquad + 2(f_1\delta_1 e^{2\int_0^x \frac{\gamma_1(s)}{\lambda_1(s)}ds} + f_2\gamma_2 e^{-2\int_0^x \frac{\delta_2(s)}{\lambda_2(s)}ds}) z_1 z_2\Big] dx.\end{aligned} \tag{3.18}$$

Using the boundary conditions (3.12), we get

$$\begin{aligned}\frac{dV}{dt} =& -\left(\lambda_1(L)f_1(L)e^{2\int_0^L \frac{\gamma_1(s)}{\lambda_1(s)}ds} - k_2^2\lambda_2(L)f_2(L)e^{-2\int_0^L \frac{\delta_2(s)}{\lambda_2(s)}ds}\right) z_1^2(t,L) \\ &- \left(\lambda_2(0)f_2(0) - k_1^2\lambda_1(0)f_1(0)\right) z_2^2(t,0) \\ &- \int_0^L \begin{pmatrix} e^{\int_0^x \frac{\gamma_1(s)}{\lambda_1(s)}ds} z_1 \\ e^{-\int_0^x \frac{\delta_2(s)}{\lambda_2(s)}ds} z_2 \end{pmatrix}^T I \begin{pmatrix} e^{\int_0^x \frac{\gamma_1(s)}{\lambda_1(s)}ds} z_1 \\ e^{-\int_0^x \frac{\delta_2(s)}{\lambda_2(s)}ds} z_2 \end{pmatrix} dx,\end{aligned} \tag{3.19}$$

where

$$I = \begin{pmatrix} -(\lambda_1 f_1)' & f_1\delta_1\phi(x) + f_2\gamma_2\phi^{-1}(x) \\ f_1\delta_1\phi(x) + f_2\gamma_2\phi^{-1}(x) & (\lambda_2 f_2)' \end{pmatrix}. \tag{3.20}$$

Therefore, it suffices using the definition of ϕ given in (3.14) to show that there exist f_1 and f_2, such that the matrix I is positive definite and that

$$\begin{aligned} \lambda_1(L)f_1(L)\phi^2(L) - k_2^2\lambda_2(L)f_2(L) \geq 0, \\ \lambda_2(0)f_2(0) - k_1^2\lambda_1(0)f_1(0) \geq 0 \end{aligned} \tag{3.21}$$

to prove the exponential decay of V. Indeed, if I is positive definite, there exists a constant $\theta > 0$ and a small $\mu > 0$ such that for any $(z_1, z_2) \in L^2$, one has

$$\int_0^L \begin{pmatrix} e^{\int_0^x \frac{\gamma_1(s)}{\lambda_1(s)}\,ds} z_1 \\ e^{-\int_0^x \frac{\delta_2(s)}{\lambda_2(s)}\,ds} z_2 \end{pmatrix}^T I \begin{pmatrix} e^{\int_0^x \frac{\gamma_1(s)}{\lambda_1(s)}\,ds} z_1 \\ e^{-\int_0^x \frac{\delta_2(s)}{\lambda_2(s)}\,ds} z_2 \end{pmatrix} dx \geq \theta \int_0^L \left(e^{2\int_0^x \frac{\gamma_1(s)}{\lambda_1(s)}\,ds} z_1^2 + e^{-2\int_0^x \frac{\delta_2(s)}{\lambda_2(s)}\,ds} z_2^2\,dx \right) \geq \mu V. \tag{3.22}$$

Before going any further, observe that under the assumption (2.11), from (3.11), (3.10) and noticing the notations (3.3), one has

$$\begin{aligned} \frac{\phi\delta_1}{\lambda_1} + \frac{\phi^{-1}\gamma_2}{\lambda_2}\left(\frac{\lambda_2}{\lambda_1}\phi\right)^2 &= \frac{\phi}{\lambda_1^2}\left(\lambda_1\delta_1 + \lambda_2\gamma_2\right) \\ &= \frac{\phi}{\lambda_1^2}\left(\frac{(\lambda_1+\lambda_2)}{2}S_{V^*} + \frac{(\lambda_2-\lambda_1)}{2}\mathscr{S}_{H^*}\sqrt{\frac{H^*}{f(H^*,x)}} + \frac{(\lambda_1^2-\lambda_2^2)}{4}\frac{\partial_x f(H^*,x)}{f(H^*,x)}\right) \\ &= \frac{\phi}{\lambda_1^2}\left(\sqrt{f(H^*,x)H^*}S_{V^*} - V^*\sqrt{\frac{H^*}{f(H^*,x)}}S_{H^*}\right) \geq 0, \end{aligned} \tag{3.23}$$

which together with Lemma 3.1 implies that $\lambda_2\phi/\lambda_1$ is a solution to the differential equation

$$\eta' = \left|\frac{\delta_1}{\lambda_1}\phi + \frac{\gamma_2}{\lambda_2}\phi^{-1}\eta^2\right|, \quad \eta(0) = \frac{\lambda_2(0)}{\lambda_1(0)} \tag{3.24}$$

on $[0, L]$. Thus, there exists $\varepsilon_1 > 0$ such that for any $\varepsilon \in [0, \varepsilon_1)$, there exists a solution η_ε on $[0, L]$ to

$$\eta_\varepsilon' = \left|\frac{\delta_1}{\lambda_1}\phi + \frac{\gamma_2}{\lambda_2}\phi^{-1}\eta_\varepsilon^2\right| + \varepsilon, \quad \eta_\varepsilon(0) = \frac{\lambda_2(0)}{\lambda_1(0)} + \varepsilon \tag{3.25}$$

and such that we can define a map $\varepsilon \to \eta_\varepsilon$ which is C^0 on $[0, \varepsilon_1)$ (see [22]). Let us define

$$f_1 = (\lambda_1\eta_\varepsilon)^{-1} \text{ and } f_2 = \lambda_2^{-1}\eta_\varepsilon, \tag{3.26}$$

where $\varepsilon \in (0, \varepsilon_1)$ can be chosen later on. One has from (3.7) and (3.13) that

$$k_1^2 \leq \left(\frac{\lambda_2(0)}{\lambda_1(0)}\right)^2, \quad k_2^2 < \left(\frac{\lambda_1(L)}{\lambda_2(L)}\right)^2. \tag{3.27}$$

Therefore, from the continuity of $\varepsilon \to \eta_\varepsilon$, there exists $0 < \varepsilon_2 < \varepsilon_1$ such that for all $\varepsilon \in (0, \varepsilon_2)$

$$k_1^2 < \frac{\lambda_2(0)f_2(0)}{\lambda_1(0)f_1(0)}, \quad k_2^2 < \frac{\lambda_1(L)f_1(L)}{\lambda_2(L)f_2(L)}\phi^2(L), \tag{3.28}$$

which is exactly the same as condition (3.21) from the definition of ϕ in (3.14). We choose such $\varepsilon \in (0, \varepsilon_2)$, and we are left to prove that I defined by (3.20) is positive definite. We have from (3.20), (3.26) and (3.25) that

$$\begin{aligned}\det(I) =& -(\lambda_1 f_1)'(\lambda_2 f_2)' - \left(f_1\delta_1\phi + f_2\gamma_2\phi^{-1}\right)^2\\ =& \frac{1}{\eta_\varepsilon^2}\left((\eta_\varepsilon')^2 - \left(\frac{\delta_1}{\lambda_1}\phi + \frac{\gamma_2}{\lambda_2}\phi^{-1}\eta_\varepsilon^2\right)^2\right) > 0.\end{aligned} \tag{3.29}$$

Besides, from (3.25) and (3.26), one has $-(\lambda_1 f_1)' > 0$ and $(\lambda_2 f_2)' > 0$, hence I is positive definite. Thus, there exists $\mu > 0$ sufficiently small such that

$$\frac{dV}{dt} \leq -\mu V \tag{3.30}$$

along the C^1-solutions of the system (3.9) and (3.12) for any $\mu \in (0, \mu_1)$. Since the C^1-solutions are dense in the set of L^2-solutions, inequality (3.30) also holds in the sense of distributions for the L^2-solutions (see [3, Section 2.1]) for the details). Thus, the exponential stability of (3.4) and (3.2) in the L^2 norm is also guaranteed thanks to the linear change of variables (3.8). This ends the proof of Proposition 3.1. □

3.2 Exponential stability of the nonlinear system

For the exponential stability of nonlinear system, the proof will be similar to the linearized case. For a given steady-state (H^*, V^*) defined on $[0, L]$, we can still define $h = H - H^*$ and $v = V - V^*$ as previously and (z_1, z_2) using the same change of variables (3.8). Then, for (z_1, z_2) small enough, the system (2.1)–(2.2), (2.6) is equivalent to

$$\mathbf{z}_t + A(\mathbf{z}, x)\mathbf{z}_x + M(\mathbf{z}, x)\mathbf{z} = 0, \tag{3.31}$$

where

$$A(\mathbf{0}, x) = \begin{pmatrix} \lambda_1(x) & 0 \\ 0 & -\lambda_2(x) \end{pmatrix}, \quad M(\mathbf{0}, x) = \begin{pmatrix} \gamma_1(x) & \delta_1(x) \\ \gamma_2(x) & \delta_2(x) \end{pmatrix}, \tag{3.32}$$

and

$$\begin{aligned} z_1(t, 0) =& m_1(z_2(t, 0)),\\ z_2(t, L) =& m_2(z_1(t, L)),\end{aligned} \tag{3.33}$$

with

$$m_1'(0) = k_1, \quad m_2'(0) = k_2, \tag{3.34}$$

here, k_1 and k_2 are defined as (3.13). In (3.33), m_1 and m_2 are found by the implicit function theorem around $\mathbf{0}$, for z_1 and z_2 small enough (see [26, A.2] for more details in a similar case). Noticing that the exponential stability of the steady-state (H^*, V^*) of system (2.1)–(2.2) and (2.6) is therefore equivalent to the exponential stability of the null steady-state ($z_1 = 0$, $z_2 = 0$) of system (3.31)–(3.34), we use the following theorem, which is a direct application of [3, Theorem 6.10].

Theorem 3.2. *If there exists C^1 functions $g_1(x) > 0$ and $g_2(x) > 0$ such that, with $Q = diag(g_1(x), g_2(x))$, one has*

$$-(QA(\mathbf{0}, \cdot))' + QM(\mathbf{0}, x) + M^T(\mathbf{0}, x)Q \tag{3.35}$$

is positive definite on $[0, L]$ and the following inequalities hold

$$k_1^2 < \frac{\lambda_2(0)g_2(0)}{\lambda_1(0)g_1(0)}, \quad k_2^2 < \frac{\lambda_1(L)g_1(L)}{\lambda_2(L)g_2(L)}, \tag{3.36}$$

then the null steady-state of the system (3.31)–(3.34) is exponentially stable for the H^2 norm.

Remark 3.1. This theorem actually shows the existence of a Lyapunov function for the H^2 norm of the form

$$\begin{aligned} V = &\int_0^L (f_1(E(\mathbf{z},x)\mathbf{z})_1^2 + f_2(E(\mathbf{z},x)\mathbf{z})_2^2)dx + \int_0^L (f_1(E(\mathbf{z},x)\mathbf{z}_x)_1^2 + f_2(E(\mathbf{z},x)\mathbf{z}_x)_2^2)dx \\ &+ \int_0^L (f_1(E(\mathbf{z},x)\mathbf{z}_{xx})_1^2 + f_2(E(\mathbf{z},x)\mathbf{z}_{xx})_2^2)dx, \end{aligned} \tag{3.37}$$

where $E(\mathbf{0},\cdot) = Id$ (see [3, Chapter 6] for more details). This is the reason why we claim that this proof is actually the same as the proof of the exponential stability in the linearized case, and we will now see that we can use a similar Lyapunov function but for the H^2 norm.

Proof of Theorem 2.1. Let

$$g_1 := e^{2\int_0^x \frac{\gamma_1(s)}{\lambda_1(s)}\,ds} f_1, \quad g_2 := e^{-2\int_0^x \frac{\delta_2(s)}{\lambda_2(s)}\,ds} f_2,$$

where f_1 and f_2 are defined in (3.26). One can directly check that

$$-(QA(\mathbf{0},\cdot))' + QM(\mathbf{0},x) + M^T(\mathbf{0},x)Q = \begin{pmatrix} e^{\int_0^x \frac{\gamma_1(s)}{\lambda_1(s)}\,ds} & 0 \\ 0 & e^{-\int_0^x \frac{\delta_2(s)}{\lambda_2(s)}\,ds} \end{pmatrix} I \begin{pmatrix} e^{\int_0^x \frac{\gamma_1(s)}{\lambda_1(s)}\,ds} & 0 \\ 0 & e^{-\int_0^x \frac{\delta_2(s)}{\lambda_2(s)}\,ds} \end{pmatrix}$$

with I defined as (3.20), as I is positive definite from (3.29), condition (3.35) is thus satisfied. Condition (3.36) is satisfied from (3.28) by noticing the definition of ϕ given in (3.14). Thus, Theorem 3.2 applies and Theorem 2.1 holds. $\square$

4 Optimality of the conditions on the control

In this section, we will show the optimality of the conditions on the control in the sense that no basic quadratic Lyapunov function that would always exist for density-velocity systems satisfying (2.11) can give strictly less restrictive boundary conditions than (2.12) and (2.14) respectively, making these conditions quite sharp. These results are given by Theorem 4.1 and Theorem 4.2 respectively.

Theorem 4.1. *Assume that $S \in C^2((0,+\infty)^2 \times [0,L];\mathbb{R})$ and $P \in C^2((0,+\infty)\times[0,L];\mathbb{R})$. Let a steady-state $(H^*,V^*) \in C^1([0,L])$ of the nonlinear hyperbolic density-velocity system (2.1), (2.2), (2.6) satisfying (2.11) with $S(H^*,V^*,x) + \partial_x P(H^*,x) \leq 0$. For any $\varepsilon > 0$, there exists $L > 0$ such that, if there exists a basic quadratic Lyapunov function for the H^2 norm, then*

$$\begin{aligned} \mathcal{B}_1'(H^*(0)) &\in \left(-\varepsilon - \frac{\partial_H P(H^*(0),0)}{V^*(0)}, \varepsilon - \frac{V^*(0)}{H^*(0)}\right), \\ \mathcal{B}_2'(H^*(L)) &\in \mathbb{R} \setminus \left[\varepsilon - \frac{\partial_H P(H^*(L),L)}{V^*(L)}, -\varepsilon - \frac{V^*(L)}{H^*(L)}\right]. \end{aligned} \tag{4.1}$$

This theorem shows therefore that the condition (2.12) given in Theorem 2.1 is quite sharp and cannot be significantly improved. The situation is even clearer in the case of a single boundary control.

Theorem 4.2. *Assume that $S \in C^2((0,+\infty)^3;\mathbb{R})$ and $P \in C^2((0,+\infty)^2;\mathbb{R})$. Let a steady-state $(H^*,V^*) \in C^1([0,L])$ of the nonlinear hyperbolic density-velocity system (2.1), (2.2), (2.13) satisfying (2.11) with $S(H^*,V^*,x) + \partial_x P(H^*,x) \leq 0$. There exists a basic quadratic Lyapunov function for the H^2 norm if and only if*

$$\mathcal{B}_2'(H^*(L)) \in \mathbb{R} \setminus \left[-\frac{\partial_H P(H^*(L),L)}{V^*(L)}, -\frac{V^*(L)}{H^*(L)}\right]. \tag{4.2}$$

Remark 4.1. Note that the assumption $S(H^*,V^*,x)+\partial_x P(H^*,x)\leq 0$ in Theorems 4.1 and 4.2 is only to ensure that the steady-states exist and are C^1 on $[0,+\infty)$. Indeed, if $S(H^*,V^*,x)+\partial_x P(H^*,x)>0$, then from (2.1)–(2.2) the fluvial regime ends in finite length and thus regular steady-states exist only up to a finite length.

In the following, we give the proofs of Theorems 4.1 and 4.2.

Proof of Theorem 4.1. Let us assume that along (H^*,V^*), $S(H^*,V^*,x)+\partial_x P(H^*,V^*,x)\leq 0$. Then from (2.1)–(2.2), the steady-state (H^*,V^*) exists and is C^1 for any length $L>0$. Suppose that there exists $\varepsilon_1>0$ such that for any length $L>0$, there exists a basic quadratic Lyapunov function for the H^2 norm with

$$
\begin{aligned}
\mathcal{B}_1'(H^*(0)) &\in \mathbb{R}\setminus\left(-\varepsilon_1-\frac{\partial_H P(H^*(0),0)}{V^*(0)},\varepsilon_1-\frac{V^*(0)}{H^*(0)}\right),\\
\mathcal{B}_2'(H^*(L)) &\in \left[\varepsilon_1-\frac{\partial_H P(H^*(L),L)}{V^*(L)},-\varepsilon_1-\frac{V^*(L)}{H^*(L)}\right].
\end{aligned}
\tag{4.3}
$$

We can then use the same change of variables (3.8), as in Section 3. The system (2.1)–(2.2), (2.6) becomes (3.31) with boundary conditions (3.33). From (4.3), we have

$$
\begin{aligned}
&k_1^2>\eta^2(0)\\
\text{or } &k_2^2>\frac{\phi^2(L)}{\eta^2(L)},
\end{aligned}
\tag{4.4}
$$

where ϕ is defined by (3.14), k_1, k_2 are defined by (3.34) and $\eta=\lambda_2\phi/\lambda_1$. We define now

$$
a=\delta_1\phi,\qquad b=\gamma_2\phi^{-1}. \tag{4.5}
$$

As there exists a basic quadratic Lyapunov function for the H^2 norm, thus from [1] (see also [24, Theorem 3.5], and [24, (24),(40)–(43)]), there exists a function $\eta_2\in C^1([0,L])$ such that

$$
\eta_2'=\left|\frac{a}{\lambda_1}+\frac{b}{\lambda_2}\eta_2^2\right| \tag{4.6}
$$

on $[0,L]$ and there exists $\varepsilon_1>0$ depending only on ε such that

$$
\begin{aligned}
&\eta_2(L)\leq\eta(L)-\varepsilon_1,\\
\text{or } &\eta_2(0)\geq\eta(0)+\varepsilon_1.
\end{aligned}
\tag{4.7}
$$

Now, as L can be taken arbitrarily, η_2 exists for any L, and thus on $[0,+\infty)$. We claim now that

$$
\lim_{x\to+\infty}\eta_2(x)\in\mathbb{R}_+^*. \tag{4.8}
$$

Indeed, let us assume that $\lim_{x\to+\infty}\eta_2(x)=+\infty$. When x is large enough we have (see [24, Section 4])

$$
\gamma_2\leq 0,\ \frac{|b|\lambda_1}{a\lambda_2}\in\left(\frac{1}{2},2\right). \tag{4.9}
$$

Thus

$$
\eta_2'=\frac{|b|}{\lambda_2}\eta_2^2-\frac{a}{\lambda_1}, \tag{4.10}
$$

which implies that, using the estimates of [24, Section 4], there exist $C>0$ and $x_1>0$ such that for any $x\geq x_1$,

$$
\eta_2'\geq\frac{C}{x}\eta_2^2, \tag{4.11}
$$

hence

$$\eta_2 \geq \frac{1}{\frac{1}{\eta_2(x_1)} - C\ln(x/x_1)}. \tag{4.12}$$

And η_2 exists and is positive on $[x_1, \infty)$, hence the contradiction. Thus η_2 converges when x goes to ∞ to a limit $\eta_{2,\infty}$. Note that ϕ converges to $\phi_\infty > 0$ [24, Section 4]. Besides, using (4.9),

$$\eta_2' = \frac{|\delta_1|}{\lambda_1\phi(x)}\left|\phi^2(x) - \frac{\lambda_1|\gamma_2|}{\lambda_2\delta_1}\eta_2^2\right|. \tag{4.13}$$

As $\lambda_1|\gamma_2|/\lambda_2\delta_1$ goes to 1 when x goes to infinity [24, Section 4], assume by contradiction that $\eta_{2,\infty} \neq \phi_\infty$, there exists C_3 and x_3 such that for all $x > x_3$,

$$\eta_2' \geq \frac{C_3}{x}, \tag{4.14}$$

which implies that $\lim\limits_{x\to+\infty} \eta_2(x) = +\infty$, hence contradiction. Thus η_2 converges to ϕ_∞, just as $\eta(L)$, which implies that in any cases the condition at $x = L$ become arbitrarily close to the one we obtain with η when L goes to infinity and prove that the first inequality of (4.7) is impossible.
Now let us assume by contradiction that the second inequality of (4.7) is satisfied. Then $\eta_2(0) > \eta(0)$ and from (4.6),

$$\eta_2' \geq \left(\frac{a}{\lambda_1} + \frac{b}{\lambda_2}\eta_2^2\right), \tag{4.15}$$

which implies that

$$(\eta_2 - \eta)' \geq -2\frac{|b|}{\lambda_2}(\eta_2 - \eta)\phi_\infty. \tag{4.16}$$

Thus

$$(\eta_{2,\infty} - \eta_\infty) \geq (\eta_2(0) - \eta(0))\exp\left(-\phi_\infty\int_0^{+\infty} 2\frac{|b|}{\lambda_2}\,dx\right). \tag{4.17}$$

But, as seen in [24, Section 4], $\int_0^{+\infty} 2\frac{|b|}{\lambda_2}\,dx < +\infty$, which implies, using that $\eta_2(0) > \eta(0)$,

$$\eta_{2,\infty} > \eta_\infty, \tag{4.18}$$

while we know that $\eta_{2,\infty} = \eta_\infty$, hence the contradiction. This ends the proof of Theorem 4.1. □

We can now prove Theorem 4.2 in a very similar fashion.

Proof of Theorem 4.2. Let $(H^*, V^*) \in C^1([0,L])$ be a steady-state of (2.1)–(2.2). Let us assume by contradiction that there exists a basic quadratic Lyapunov function for the H^2 norm and that

$$\mathcal{B}_2'(H^*(L)) \in \left[-\frac{\partial_H P(H^*(L), L)}{V^*(L)}, -\frac{V^*(L)}{H^*(L)}\right]. \tag{4.19}$$

Then using again the change of variables (3.8), the system (2.1)–(2.2), (2.13) is again equivalent to (3.31) with boundary conditions (3.33). From (2.13), one has

$$k_1^2 := \eta^2(0), \tag{4.20}$$

where k_1 is again given by (3.34) and $\eta = \lambda_2\phi/\lambda_1$. As previously, as there exists a basic quadratic Lyapunov function for the H^2 norm, from [1] (see also [24, Theorem 3.5]) there exists a function $\eta_2 \in C^1([0,L])$ such that

$$\eta_2' = \left|\frac{a}{\lambda_1} + \frac{b}{\lambda_2}\eta_2^2\right| \tag{4.21}$$

on $[0, L]$, where a and b are defined by (4.5), and there exists $\varepsilon_1 > 0$ such that

$$\begin{aligned} &\eta_2(L) \leq \eta(L) - \varepsilon_1, \ \forall\ L > 0, \\ &\eta_2(0) \geq \eta(0). \end{aligned} \tag{4.22}$$

Using (4.22) and the same argument as (4.15)–(4.18), we get that $\eta_2(L) \geq \eta(L)$ thus

$$\eta_2(L) \geq \lambda_2(L)\phi(L)/\lambda_1(L) \tag{4.23}$$

which is in contradiction with (4.22). This ends the proof of Theorem 4.2. □

A Derivation of γ_1, γ_2, δ_1 and δ_2

Looking at 3.8, we denote by

$$\Delta = \begin{pmatrix} \sqrt{\frac{f(H^*,x)}{H^*}} & 1 \\ -\sqrt{\frac{f(H^*,x)}{H^*}} & 1 \end{pmatrix}$$

$$\Delta^{-1} = \frac{1}{2}\begin{pmatrix} \sqrt{\frac{H^*}{f(H^*,x)}} & -\sqrt{\frac{H^*}{f(H^*,x)}} \\ 1 & 1 \end{pmatrix}$$

Then, using the notations (3.3), (3.4) becomes

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}_t + \begin{pmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}_x + \begin{pmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix}\Delta\Delta_x^{-1}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \Delta\begin{pmatrix} V_x^* & H_x^* \\ \mathscr{S}_{H^*} + \partial_H f(H^*,x)H_x^* & S_{V^*} + V_x^* \end{pmatrix}\Delta^{-1}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{A.1}$$

where λ_1 and λ_2 are given by (3.10). Let us compute the coefficient of the first part of the source term,

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix}\Delta\Delta_x^{-1} = \frac{1}{4}\begin{pmatrix} \lambda_1\left(\frac{H_x^*}{H^*} - \frac{\frac{d}{dx}(f(H^*,x))}{f}\right) & -\lambda_1\left(\frac{H_x^*}{H^*} - \frac{\frac{d}{dx}(f(H^*,x))}{f}\right) \\ \lambda_2\left(\frac{H_x^*}{H^*} - \frac{\frac{d}{dx}(f(H^*,x))}{f}\right) & -\lambda_2\left(\frac{H_x^*}{H^*} - \frac{\frac{d}{dx}(f(H^*,x))}{f}\right) \end{pmatrix}. \tag{A.2}$$

The coefficient of the second part of the source term is

$$\begin{aligned} &\Delta\begin{pmatrix} V_x^* & H_x^* \\ \mathscr{S}_{H^*} + \partial_H f(H^*,x)H_x^* & S_{V^*} + V_x^* \end{pmatrix}\Delta^{-1} \\ &= \frac{1}{2}\begin{pmatrix} \partial_H f H_x^*\sqrt{\frac{H^*}{f}} + \mathscr{S}_{H^*}\sqrt{\frac{H^*}{f}} + H_x^*\sqrt{\frac{f}{H^*}} + 2V_x^* + S_{V^*} & -\partial_H f H_x^*\sqrt{\frac{H^*}{f}} - \mathscr{S}_{H^*}\sqrt{\frac{H^*}{f}} + H_x^*\sqrt{\frac{f}{H^*}} + S_{V^*} \\ \partial_H f H_x^*\sqrt{\frac{H^*}{f}} + \mathscr{S}_{H^*}\sqrt{\frac{H^*}{f}} - H_x^*\sqrt{\frac{f}{H^*}} + S_{V^*} & -\partial_H f H_x^*\sqrt{\frac{H^*}{f}} - \mathscr{S}_{H^*}\sqrt{\frac{H^*}{f}} - H_x^*\sqrt{\frac{f}{H^*}} + 2V_x^* + S_{V^*} \end{pmatrix}. \end{aligned} \tag{A.3}$$

Thus,

$$\begin{aligned} \gamma_1 =& \frac{1}{4}\left[\lambda_1\left(\frac{H_x^*}{H^*} - \frac{\partial_H f(H^*,x)H_x^* + \partial_x f(H^*,x)}{f}\right) + 2\left(\partial_H f H_x^*\sqrt{\frac{H^*}{f}} + \mathscr{S}_{H^*}\sqrt{\frac{H^*}{f}} + H_x^*\sqrt{\frac{f}{H^*}} + 2V_x^* + S_{V^*}\right)\right] \\ =& \frac{1}{4}\Big[2S_{V^*} + 2\mathscr{S}_{H^*}\sqrt{\frac{H^*}{f(H^*,x)}} - \lambda_1\frac{\partial_x f(H^*,x)}{f(H^*,x)} \\ &+ (V^* + \sqrt{fH^*})\left(\frac{H_x^*}{H^*} - \frac{\partial_H f(H^*,x)H_x^*}{f}\right) + 2\partial_H f H_x^*\sqrt{\frac{H^*}{f}} + 2H_x^*\sqrt{\frac{f}{H^*}} + 4V_x^*\Big] \\ =& \frac{1}{4}\left(2S_{V^*} + 2\mathscr{S}_{H^*}\sqrt{\frac{H^*}{f(H^*,x)}} - 3\lambda_2\frac{V_x^*}{V^*} - \lambda_1\frac{\partial_x f(H^*,x)}{f(H^*,x)} + \lambda_2\frac{\partial_H f(H^*,x)H_x^*}{f(H^*,x)}\right), \end{aligned}$$

and γ_2, δ_1 and δ_2 can be found similarly.

B Proof of Lemma 3.1

Differentiating $\lambda_2\phi/\lambda_1$ using (3.10), (3.11) and (3.14), we have

$$
\begin{aligned}
\left(\frac{\lambda_2}{\lambda_1}\phi\right)' &= \frac{\phi}{\lambda_1^2}\left(\lambda_1\lambda_2' - \lambda_1'\lambda_2 + (\lambda_2\gamma_1 + \lambda_1\delta_2)\right)\\
= \frac{\phi}{\lambda_1^2}\Bigg\{ & (V^* + \sqrt{f(H^*,x)H^*})\left(-V_x^* + \frac{(f(H^*,x)H^*)'}{2\sqrt{f(H^*,x)H^*}}\right)\\
& -(\sqrt{f(H^*,x)H^*} - V^*)\left(V_x^* + \frac{(f(H^*,x)H^*)'}{2\sqrt{f(H^*,x)H^*}}\right)\\
+\frac{1}{4}\Bigg[& (\lambda_1^2 - \lambda_2^2)\left(3\frac{V_x^*}{V^*} - \frac{\partial_H f(H^*,x)H_x^*}{f(H^*,x)}\right) + 2(\lambda_2-\lambda_1)\mathscr{S}_{H^*}\sqrt{\frac{H^*}{f(H^*,x)}} + 2(\lambda_2+\lambda_1)S_{V^*}\Bigg]\Bigg\}\\
= \frac{\phi}{\lambda_1^2}\Bigg(& V^*\sqrt{\frac{H^*}{f(H^*,x)}}\partial_x f(H^*,x) + H_x^*V^*\sqrt{\frac{f(H^*,x)}{H^*}} + V_x^*\sqrt{f(H^*,x)H^*}\\
& -V^*\mathscr{S}_{H^*}\sqrt{\frac{H^*}{f(H^*,x)}} + \sqrt{f(H^*,x)H^*}S_{V^*}\Bigg).
\end{aligned}
\tag{B.1}
$$

Noticing the notations (3.3) and from (2.4), (B.1) becomes

$$
\left(\frac{\lambda_2}{\lambda_1}\phi\right)' = \frac{\phi}{\lambda_1^2}\left(\sqrt{f(H^*,x)H^*}S_{V^*} - V\sqrt{\frac{H^*}{f(H^*,x)}}S_{H^*}\right), \tag{B.2}
$$

which, together with (3.23) gives

$$
\left(\frac{\lambda_2}{\lambda_1}\phi\right)' = \frac{\phi\delta_1}{\lambda_1} + \frac{\phi^{-1}\gamma_2}{\lambda_2}\left(\frac{\lambda_2}{\lambda_1}\phi\right)^2. \tag{B.3}
$$

C Adapting the proof of Theorem 2.2 when the control is imposed on $x = 0$

When the control is imposed on $x = 0$ and the flow is imposed on $x = L$ to an unknown constant, the proof is similar as in Theorem 2.1. First note that on L this condition implies

$$
H(t,L)V(t,L) = Q_L \tag{C.1}
$$

with Q_L the unknown constant inflow downstream, thus

$$
\mathcal{B}_1'(H^*(L)) = -\frac{V^*(L)}{H^*(L)}, \tag{C.2}
$$

which, after the change of variables (3.8) brings the system under the form (3.31)–(3.33), with

$$
k_2 = \left(\frac{\lambda_1(L)}{\lambda_2(L)}\right)^2, \tag{C.3}
$$

where k_2 is given by (3.34), and we have for k_1,

$$
k_1^2 < \left(\frac{\lambda_2(0)}{\lambda_1(0)}\right)^2. \tag{C.4}
$$

We can then do as in the proof of Theorem 2.1 but instead of defining η_ε by (3.25) with $\eta_\varepsilon(0)$ assigned, we define now η_ε as the solution of

$$\eta_\varepsilon' = \left|\frac{\delta_1}{\lambda_1}\phi + \frac{\gamma_2}{\lambda_2}\phi^{-1}\eta_\varepsilon^2\right| + \varepsilon, \quad \eta_\varepsilon(L) = \frac{\lambda_2(L)}{\lambda_1(L)}\phi(L)^2 - \varepsilon. \tag{C.5}$$

Still defining f_1 and f_2 as in (3.26), then one can check that there exists $\varepsilon_2 > 0$ such that for any $\varepsilon \in (0, \varepsilon_2)$, one still has (3.28) and I defined by (3.20) is still definite positive. The rest of the proof remains similar as the proof of Proposition 3.1.

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