# Uniqueness of entropy solutions to fractional conservation laws with ”fully infinite” speed of propagation

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# Uniqueness of entropy solutions to fractional conservation laws with "fully infinite" speed of propagation

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**Abstract**

Our goal is to study the uniqueness of bounded entropy solutions for a multidimensional conservation law including a non-Lipschitz convection term and a diffusion term of nonlocal porous medium type. The nonlocality is given by a fractional power of the Laplace operator. For a wide class of nonlinearities, the $L^1$-contraction principle is established, despite the fact that the "finite-infinite" speed of propagation [Alibaud, JEE 2007] cannot be exploited in our framework; existence is deduced with perturbation arguments. The method of proof, adapted from [Andreianov, Maliki, NoDEA 2010], requires a careful analysis of the action of the fractional laplacian on truncations of radial powers.

**Keywords:** Fractional laplacian, Radial powers, Nonlocal conservation law, Entropy formulation, Kato inequality, $L^1$-contraction principle, Differential inequalities.

## Contents



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# 1 Introduction

We study the uniqueness of solutions for a multidimensional fractional conservation law

$$\begin{cases} \partial_t u + \mathrm{div}_x\Big(f(u)\Big) + A[\varphi(u)] = g & \text{in } I \times \mathbb{R}^N, \\ u(t=0) = u_0 \in L^\infty(\mathbb{R}^N) & \text{in } \mathbb{R}^N, \end{cases} \tag{1}$$

associated with a non-local operator $A = c(-\Delta_x)^{\alpha/2}$ taken as a positive multiple ($c \geq 0$) of a "fractional laplacian". The time interval is here $I = \mathbb{R}_+$ and the space domain is the whole $\mathbb{R}^N$. The nonlinearities $f : \mathbb{R} \to \mathbb{R}^N$ and $\varphi : \mathbb{R} \to \mathbb{R}$ are assumed at least continuous. In addition, $\varphi$ is assumed nondecreasing, so that (1) is the Cauchy problem for a non-local (fractional) convection-diffusion equation.

## 1.1 Finite speed of propagation and uniqueness for local or fractional conservation laws.

The best-known case is $A = 0$, which corresponds to a pure hyperbolic convection equation without diffusion. In this setting, it is well-known that regular solutions do not exist in general; that weak solutions fail to be unique; and that the appropriate notion of solution relies on the so-called entropy inequalities introduced by Kruzhkov [32]. Via the method of doubling variables, these entropy inequalities imply Kato's inequality: for any couple $(u, v)$ of entropy solutions

$$\partial_t |u - v| + \mathrm{div}_x\Big(\mathrm{sgn}(u - v)(f(u) - f(v))\Big) \leq 0 \tag{2}$$

in the sense of distributions (i.e. with compactly supported smooth test functions), where sgn(.) denotes the sign function. The *finite speed of propagation* (for bounded $f'$) is used in [32] to construct appropriate sequences of test functions to be inserted into Kato's inequality (2) in order to infer uniqueness. As a matter of fact, the *speed of propagation* is the key heuristic issue in the discussion below.

The general problem with $c \geq 0$ and $\alpha \in (0, 1)$ inherits many key features of the purely hyperbolic case ($A = 0$). Indeed, for $\alpha < 1$ it has been shown:

- by Alibaud, Droniou and Vovelle in [5], that jump singularities may develop from smooth data;
- by Alibaud in [1], that a well-posedness theory for entropy solutions can be derived in the spirit of [32] for locally Lipschitz $f$ and $\varphi = \mathrm{Id}$; this was later extended by Cifani, Jakobsen [20] to include nonlinear $\varphi$;
- by Alibaud and the first author in [2], that weak solutions are ill-posed at least for a subclass of equations (1) including the fractional Burgers equation.

Although solutions do not develop singularities for $\alpha \in [1, 2)$ (see Droniou, Gallouët, Vovelle [26] for $\alpha > 1$ and Constantin, Vicol [21] for $\alpha = 1$), it should be stressed that the entropy solutions concept yields well-posedness for the whole range $\alpha \in (0, 2]$ at least when $f$,$\varphi$ are locally Lipschitz (see Alibaud [1] for the basic choice $\varphi = \mathrm{Id}$, and Cifani, Jakobsen [20] for $\varphi \neq \mathrm{Id}$ in the non-local case; cf. [19, 39] for the local case $\alpha = 2$, and Karlsen, Ulusoy [31] for a mixture of local and non-local diffusion). Note in passing that the setting of [1, 20] is appropriate for deriving optimal continuous dependence estimates with respect to the nonlinearities (see Alibaud, Cifani and Jakobsen [4] and references therein); we stress that these estimates rely explicitly on $L^\infty$-bounds for $f'$,$\varphi'$. Finally, let us mention the alternative approach to well-posedness of (1) using an adaptation of the kinetic formulation (see [3]) which has the advantage to extend to $L^1$ data. Also in the kinetic context $f, \varphi$ should be at least locally Lipshitz continuous.

Going into details, note that the presence of a local or non-local diffusion operator naturally leads to the infinite speed of propagation, i.e., compactly supported data generically lead to solutions supported in the whole space. In this context, the method used in [1, 31, 20] (see also [27] which contains the local convection-diffusion case) permits to deduce $L^1$-contraction inequalities based on the *finite-infinite speed of propagation* approach. The latter relies upon the Lipschitz continuity of the nonlinearities in (1). Indeed, the *finite-infinite speed of propagation* approach of Alibaud exploits, via a kind of splitting argument, the combination of the rapid decay at infinity for the fractional heat kernel and of the finite speed of propagation proper to the hyperbolic conservation law with Lipschitz flux. A recent deep improvement of these results is due to Endal and Jakobsen [27], who introduced localized contraction estimates in order to provide subtler information on local stability of solutions to (1). These estimates are obtained by inserting, as test functions in (2), super-solutions of an auxiliary Hamilton-Jacobi equation defined for locally Lipschitz $f, \varphi$. The ideas and thechniques of [27] are further pushed forward in [6], where a kind of duality is established between (1) in the local case and the auxiliary Hamilton-Jacobi equation.

### 1.2 Infinite speed of propagation, associated uniqueness techniques and possible non-uniqueness.

This paper aims at going, in the entropy solution framework, beyond the case of locally Lipschitz nonlinearities. In contrast to the setting of [1, 31, 20, 27, 6], we will focus on the case of what we call *fully infinite speed of propagation*: indeed, under our assumptions, the local convection and the non-local diffusion operators both feature infinite speed of propagation.

This issue was first addressed in the pure hyperbolic local case ($A = 0$) by Bénilan [12] and by Kruzhkov and Hildebrandt [35], where a Hölder continuity of order $1 - 1/N$ on $f$ was shown to be a sufficient condition for uniqueness (when $N = 1$ no condition is needed). Kruzhkov, Panov and Bénilan in [33, 36, 37, 16] pushed the theory of the hyperbolic case further, by elaborating counterexamples to uniqueness and by giving sufficient conditions for uniqueness thanks to the product of the moduli of continuity of the components of the flux vector $f$. The possible non-uniqueness demonstrated by counterexamples due to Panov may be explained by the *infinite speed of propagation* caused by the unboundedness of $f'$. Heuristically, one can say that information may come not only from the initial data but also from infinity, as if $\mathbb{R}^N$ had a boundary on which different boundary conditions could be applied. Further sufficient conditions based on the monotonicity of $N - 1$ merely continuous components of the flux were put forward by Bénilan, Kruzhkov and the first author in [8] under the additional assumption of integrability of data. This shows that sharp conditions combining the irregularity of $f$ and the decay of solutions at infinity are still not well-identified (see also Szepessy [45], Bendahmane and Karlsen [11] for related results based on very different techniques). The fundamental idea of [8] does not permit its extension to cases with diffusion, therefore it is not relevant for our study of (1). Note in passing that the program of research on infinite speed of propagation in hyperbolic conservation laws put forward by Kruzhkov ([34]) yet contains further unsolved issues such as the (non?)uniqueness of $L^1 \cap L^\infty$ entropy solutions.

The line of research starting from [37, 16] can be continued in the case $c > 0$. In the local diffusion case $\alpha = 2$, the corresponding generalizations were obtained by Maliki and Touré in [39], relying on the fundamental work of Carrillo [19] that established the entropy formulation and the technique of doubling variables for local quasilinear convection-diffusion equations. In [39], a Hölder restriction was imposed on $\varphi$. Extensions to anisotropic diffusion with the same method were obtained by Ouédraogo et al. in [38, 42, 41] with *ad hoc* conditions on products of the moduli of continuity of $f_i$'s and of the components of the diffusion matrix.

In absence of counterexamples to uniqueness in this hyperbolic-parabolic case, Maliki and the first author investigated the optimality of some conditions obtained in [39], and discovered in [9] that, under the basic isotropic Hölder of $f$, mere continuity of $\varphi$ is sufficient to imply uniqueness. The work [9] (see also the related investigation [10] of the pure diffusion case) provided a new method of proof, using a well-chosen test function obtained by appropriate truncation of the fundamental solution of the Laplace operator. Roughly speaking, in [9], a sequence of (almost) super-harmonic functions with sufficient regularity is built thanks to explicit calculations. As in all the other works on the subject, these test functions are exploited in the context of Kato's inequality (2), see Section 3. It should be stressed that the method of [9], which is essentially isotropic, does not permit to recover the fine (and sometimes sharp) anisotropic conditions for uniqueness established by Kruzhkov, Panov, Bénilan for the hyperbolic case and by Maliki, Touré, Ouédraogo for the degenerate parabolic case. In contrast, we show in the present contribution that this method turns out to be suitable for generalization to fractional conservation laws (1). We obtain this generalization at the price of a painstaking investigation of the image, by the fractional laplacian, of the family of radial functions suggested in [9].

### 1.3 Comments on the (fractional) porous medium case

Our framework includes as an important particular case the situation where $f = 0$, so that equation (1) becomes a nonlinear non-local diffusion equation. Although the investigation of this case was not our main motivation, the results we prove and the techniques we use should be confronted to the state-of-the-art for porous medium / fast diffusion equations, in both local and non-local settings.

The classical local case is by far the most studied one; we refer to the books of Vázquez [47] and Daskalopoulos, Kenig [23] and references therein for an extensive account on this case. One important point of the theory here is that in general, entropy solutions are not the most relevant notion (the very weak solutions can be considered). Solutions with $L^2_{loc}(I;\mathbb{R}^N)$ gradient $\nabla u$ often exist, and these turn out to satisfy entropy inequalities (cf. [19]). The study of uniqueness of very weak solutions cannot rely on the Kato inequalities, therefore it is beyond our scope. Further, in relation with our focus on low regularity of nonlinearities, let us mention that under the general continuity assumption on the nonlinearity $\varphi$, Brézis and Crandall [18] studied bounded very weak (distributional) solutions of (1) with $f = 0$ and $\alpha = 2$. They proved that two solutions $u, v$ in that case coincide a.e. in $(0,T) \times \mathbb{R}^N$ as long as $u - v \in L^1((0,T) \times \mathbb{R}^N)$. The latter condition was removed in e.g. [13, 22] and [29] for classical homogeneous nonlinearities $\varphi(u) = |u|^{m-1}u$ with $m > 1$ and $0 < m < 1$, respectively. Some new information on the case of local diffusion with general continuous $\varphi$ is provided in [10].

In the non-local case, the theory is more recent and not so extensive. The analogue of [18] was obtained in [24, 25] for very general operators $A$, and the integrability condition was very recently removed by Grillo, Muratori and Punzo in [28] when $A$ is the fractional Laplacian and the nonlinearity $\varphi$ is locally Lipschitz. Compared to the result of [28], one can interpret our result in this paper (for the case $f = 0$) as the removal of the mentioned $W^{1,\infty}_{loc}$ regularity condition on $\varphi$; however, we do so in the somewhat restrictive context of entropy solutions.

Finally, a historical perspective on techniques we use should be provided. First, note that the choice of the test function in Section 4 has appeared – in combination with properties reminiscent of our Lemma 4.9 below – in the uniqueness proofs of e.g. [13, 22, 29] (for the local case); after this work was completed, we learned of its use in [28] (for the non-local case). Second, the technique of Section 3, inspired by [9], is motivated by the difficulties of the hyperbolic-parabolic case; the way the fundamental solution of the equation is exploited is different from the preceding literature, to the best of our knowledge. However, generally speaking, instrumentalization of the fundamental solution of the laplacian is not new in uniqueness proofs for nonlinear diffusion equations (see, e.g., Pierre [43] for the local case and Bonforte, Vázquez [15] for the non-local counterpart).

### 1.4 Outline of the paper.

The main goal of the present paper is to provide sufficient conditions on $f$ and $\varphi$ for uniqueness in (1).

Since we shall follow the line of [9], the essential role will be played by truncations of the fundamental solution of the diffusion operator. However, the non-local nature of the fractional laplacian makes it very delicate to control the effect of truncations on the values of the operator. As a matter of fact, a large part of the paper is concerned with a careful study of the action of the fractional laplacian operator $A$ on a class of radial functions $(E_r^R)$ encompassing the desired truncations $(e_r)$ of the fundamental solution, see Definition (6) in Section 2. The simple explicit calculations of the local case [9] are here replaced by carefully chosen upper and lower pointwise bounds derived from symmetry and scaling properties. Particularly, we study the positivity properties of $A[E_r^R]$, its decay at infinity and its continuity w.r.t. parameters, i.e., w.r.t. to the power $\beta$ in $|x|^{-\beta}$ and w.r.t. to the truncation parameters $r$ and $R$. Several phenomena are observed, such as the lack of $L^1$ continuity as $R \to \infty$. The latter makes it particularly delicate to adapt to the non-local equation (1) the uniqueness method developed for the local case in [9].

The results obtained in Section 2 are exploited in Section 3 to derive uniqueness and stability for entropy solutions of equation (1) for general $f$ and $\varphi$. For the sake of completeness, an existence proof is also given at the end of Section 3. If we decide to treat in a unified way the nonlinearities $f$ and $\varphi$, then a much simpler uniqueness and stability proof becomes available at the price of an additional regularity restriction on $\varphi$. This is the purpose of Section 4: in this case, the analysis does not rely on the monotonicity of $\varphi$ but on appropriate Hölder conditions (namely $f$ is $1-1/N$ Hölder and $\varphi$ is $1-\alpha/N$ Hölder; cf. [39] for the case $\alpha = 2$). Due to this assumption on $\varphi$, Section 4 yields a sub-optimal result but it is self-contained, in the sense that it does not depend on the critical study of action of $A$ on truncated radial powers developed in Section 2.

### 1.5 Assumptions and well-posedness results for (1)

The precise assumptions under which uniqueness of solutions to (1) and its generalizations will be established are the following. We assume that

$$\left|\begin{array}{ll} \text{if } N=1, & \text{then } f \text{ is continuous;} \\ \text{if } N>1, & \text{then } f \text{ is locally Hölder-continuous of exponent } \sigma := 1-1/N. \end{array}\right. \tag{$H_f$}$$

The latter assumption reads

$$\forall M \in \mathbb{R}_+ \;\; \exists L_f^M \in \mathbb{R}_+ \; : \; \forall u,v \in [-M,+M] \qquad |f(u)-f(v)| \le L_f^M |u-v|^\sigma,$$

while for $N=1$ this condition degenerates into a trivial one. Besides, we assume that

$$\text{the nonlinearity } \varphi : \mathbb{R} \to \mathbb{R} \text{ is continuous and nondecreasing.} \tag{$H_\varphi$}$$

For any $0 < \alpha < 2$, uniqueness will be established within the class of bounded entropy solutions introduced by Alibaud [1], Cifani and Jakobsen [20], as a consequence of the classical $L^1$-contraction principle which is our main result:

**Theorem 1.1.** *Assume $(H_f)$ and $(H_\varphi)$. Let $u$ be an entropy solution of (1) in the sense of the entropy formulation (see Definition 3.1 in Section 3) with initial datum $u_0 \in L^\infty(\mathbb{R}^N)$ and source term*[i] $g \in L^1_{loc}(I; L^\infty(\mathbb{R}^N))$. *Let $v$ be an entropy solution with the respective data $v_0$ and $h$. Then we have (with values in $[0,+\infty]$)*

$$\|u(t)-v(t)\|_{L^1(\mathbb{R}^N)} \le \|u_0-v_0\|_{L^1(\mathbb{R}^N)} + \int_0^t \|g-h\|_{L^1(\mathbb{R}^N)}(\tau)\,d\tau \qquad \textit{for a.e. } t \in I. \tag{3}$$

*Moreover, a similar estimate holds with $u-v$, $u_0-v_0$, $g-h$ replaced therein by their positive parts $(u-v)^+$, $(u_0-v_0)^+$, $(g-h)^+$, respectively.*

[i] Our notation is slightly abusive: as usual (cf., e.g., [8]) in the context, throughout the paper the space $L^1_{loc}(I)$ of $L^\infty(\mathbb{R}^N)$-valued functions is considered under the weak-$*$ measurability assumption.

It turns out that a simpler proof of (3) is available (see Section 4) under the additional assumption

$$\varphi \text{ is locally Hölder-continuous of exponent } s := 1 - \alpha/N > 0 \qquad (N > \alpha) \qquad\qquad (H_\varphi^{\text{bis}})$$

The paper eventually covers any dimension $N \geq 1$ and any power $0 < \alpha < 2$. But the case $\alpha \geq N$ is special in many respects and deserves a distinctive argument (well-detailed in Remark 4.6), indeed, in this case we manage to work with merely continuous $f$ and $\varphi$. The reader may find it worth while to leave this marginal case aside.

Notice that beyond uniqueness, the result of Theorem 1.1 also leads to well-known a priori estimates and comparison inequalities on entropy solutions (for a.e. $t \in I$)

$$\begin{cases} u(t) \leq v(t) \ \text{ whenever } u_0 \leq v_0 \text{ and } g \leq h, \\ \|u(t)\|_{L^1(\mathbb{R}^N)} \leq \|u_0\|_{L^1(\mathbb{R}^N)} + \int_0^t \|g(\tau, \cdot)\|_{L^1(\mathbb{R}^N)} \, d\tau, \\ \|u(t)\|_{L^\infty(\mathbb{R}^N)} \leq \|u_0\|_{L^\infty(\mathbb{R}^N)} + \int_0^t \|g(\tau, \cdot)\|_{L^\infty(\mathbb{R}^N)} \, d\tau. \end{cases} \tag{4}$$

The two last properties follow upon comparison of $u$ with explicit space-independent solutions $v$ of (1). The comparison principle is also a cornerstone for the simple existence result contained in our well-posedness claim:

**Theorem 1.2.** *Assume that $f$ is merely continuous. Assume that $\varphi$ is continuous and non-decreasing (i.e. $(H_\varphi)$ holds). Then there exists an entropy solution to* (1) *for all initial datum $u_0 \in L^\infty(\mathbb{R}^N)$ and source term $g \in L^1_{loc}(I; L^\infty(\mathbb{R}^N))$.*
*If we assume in addition that $f$ satisfies the Hölder continuity $(H_f)$ then there exists a unique entropy solution to* (1) *for all data $(u_0, g) \in L^\infty(\mathbb{R}^N) \times L^1_{loc}(I; L^\infty(\mathbb{R}^N))$; the solution is stable w.r.t. data in the sense of* (3).

### 1.6 Interpretation of the non-local operator and conventions on notation

Everywhere in the sequel, the non-local operator $A = c(-\Delta)^{\alpha/2}$ on $\mathbb{R}^N$ given by the (fractional) power of the laplacian will be viewed either as a Fourier multiplier $A = c\,\mathcal{F}^{-1} \circ |\cdot|^\alpha \circ \mathcal{F}$ or as a (pseudo)convolution operation $A = * \ \mathrm{PF}\left(\dfrac{c}{|\cdot|^{N+\alpha}}\right)$ (see (5) below), in the sense of the dual formulations

$$A[\phi](x) = c\,\mathcal{F}^{-1}\Big(|\xi|^\alpha \mathcal{F}\phi(\xi)\Big)(x) = -c \lim_{H \to 0^+} \int_{\mathbb{R}^N} \mathbb{I}_{|h|>H} \ \Big(\phi(x+h) - \phi(x)\Big) \frac{dh}{|h|^{N+\alpha}}.$$

For this kind of formulae to make sense, $\phi$ should vary in a space of locally regular functions with some prescribed decay at infinity, for instance in the Schwartz class $\mathcal{S}(\mathbb{R}^N)$ as explained in Ch.VII of the classical book [44] on distribution theory.

Finally, we take the following conventions of notation. From now on, all constants ($c$,$C$,$\gamma$) are positive and may change from place to place. As a rule $c$ is used for a positive constant coming from the non-local operator. Everywhere $s_N$ stands for the surface of the unit sphere in $\mathbb{R}^N$ appearing in polar coordinates. The symbols $\mathbb{I}_{<R}$ and $\mathbb{I}_{>R}$ are sometimes used for the indicator of the ball of radius $\mathfrak{r}$ ($\mathfrak{r} = r$ or $\mathfrak{r} = R$) and of its exterior in $\mathbb{R}^N$, respectively. Similarly $\int_{>R}$ and $\int_{<R}$ refer to integration over these exterior and interior domains. On a few occasions when $\alpha \geq 1$, we shall make use of test functions in the space

$$W^{2-,1}(\mathbb{R}^N) := \bigcap_{\varepsilon>0} W^{2-\varepsilon,1}(\mathbb{R}^N),$$

where $W^{s,1}(\mathbb{R}^N)$ for $s \in \mathbb{R}$ refer to the standard (fractional order) Sobolev spaces based on integrable decay (with the convention $W^{0,1} := L^1$). Whenever a function space is given the subscript $_c$ this indicates 'compact support' as for instance in the $W^{1,1}_c(\mathbb{R}^N)$ space of (8) or in the $C^\infty_c(I \times \mathbb{R}^N)$ space of test functions of Definition 3.1.

# 2 Action of $A$ on radial powers

Since $A$ may be viewed as a Fourier multiplier, the starting point of the study of $A$ is to recall how the Fourier transform $\mathcal{F}$ itself acts on radial powers. In distribution theory, this problem is best explained and best solved by means of pseudofunctions (PF), in the sense that a nice formula shows that the class of pseudofunctions associated with radial powers is invariant through $\mathcal{F}$, up to some exceptional values of the power:

$$\mathcal{F}\left(\mathrm{PF}\frac{1}{|\cdot|^{N-m}}\right) = \pi^{\frac{N}{2}-m}\frac{\Gamma(\frac{m}{2})}{\Gamma(\frac{N-m}{2})}\mathrm{PF}\frac{1}{|\cdot|^{m}} \qquad \forall m \in \mathbb{C} : m \notin (-2\mathbb{N}) \cup (N+2\mathbb{N}). \tag{5}$$

Here, the symbol PF may be dropped whenever it is followed by an $L^1_{loc}(\mathbb{R}^N)$-function, otherwise it represents an essential change of the power as a function of $L^1_{loc}(\mathbb{R}^N - \{0\})$ into a distribution of $\mathcal{D}'(\mathbb{R}^N)$ origin included.

**Remark 2.1.** *Particularly, in cases when PF cannot be dropped, the resulting distribution on $\mathbb{R}^N$ (which is not even a measure) has no sign in the vicinity of the origin, although it is the extension of a positive function on $\mathbb{R}^N - \{0\}$.*

All this material on pseudofunctions is masterly detailed for instance in [44, Ch.II §3 Ex.2 ; Ch.VII §7 Ex.5]. In slightly different notations, formula (5) is just [44, formula (VII,7;13)]. Concerning the constant involving the standard $\Gamma$ function, its explicit expression[ii] will only be used in the sequel for real values of $m$, to identify what its sign is.

The radial power

$$e := 1/|\cdot|^{N-\alpha} \in L^1_{loc}(\mathbb{R}^N)$$

is readily seen to be a fundamental solution of $A$ i.e. $A[e] = c_0\delta_0$ for some positive constant $c_0$. This results from (5) by realizing $A$ as a Fourier multiplier:

$$A[e] = c\,\mathcal{F}^{-1}(|\cdot|^{\alpha} \times \mathcal{F}e) = c_0\,\mathcal{F}^{-1}\left(|\cdot|^{\alpha} \times \frac{1}{|\cdot|^{\alpha}}\right) = c_0\,\mathcal{F}^{-1}(\mathbb{I}) = c_0\delta_0.$$

Since unfortunately $e$ does not always belong to $W^{1,1}_{loc}(\mathbb{R}^N)$ due to a lack of local integrability in the gradient, we approximate it by

$$e_r := \min\{e, e(r)\} \in W^{1,1}_{loc}(\mathbb{R}^N) \;\text{ for fixed } r > 0, \tag{6}$$

the singularity at the origin being replaced by a step value. The fact that $A[e_r]$ is a nonnegative function on $\mathbb{R}^N$ will be checked later in Lemma 2.4 in a slightly more general context. It is also interesting (and a bit surprising) to notice that $A[e_r]$ has always the same finite integral:

**Lemma 2.1.** *$A[e_r]$ is an integrable and nonnegative function on $\mathbb{R}^N$ whose integral is*

$$\int_{\mathbb{R}^N} A[e_r] = \int_{\mathbb{R}^N} A[e] = c_0 < \infty. \tag{7}$$

*Proof.* Remark first that the tempered $\mathcal{F}$-transform of $A[e_r] \in \mathcal{S}'(\mathbb{R}^N)$ is a continuous function on $\mathbb{R}^N$ whose value at the origin is $c_0$, since in the decomposition

$$\mathcal{F}\Big(A[e_r]\Big) = \mathcal{F}\Big(A[e]\Big) - \mathcal{F}\Big(A[e - e_r]\Big) = c_0 - c|\cdot|^{\alpha}\mathcal{F}(e - e_r)$$

[ii] The computation is relative to the definition of $\mathcal{F}$ as $\mathcal{F}\phi(\xi) := \int_{\mathbb{R}^N} \phi(x)e^{-2i\pi x.\xi}dx$.

the term $e - e_r \in L^1(\mathbb{R}^N)$ transforms into $\mathcal{F}L^1 \subset C^0$. Next, let $G = e^{-|\cdot|^2}$ denote a Gauss-type function on $\mathbb{R}^N$, so that $\mathcal{F}^{-1}G$ is also a Gauss-type function on $\mathbb{R}^N$. The nonnegativity[iii] of $A[e_r] \geq 0$ allows to pass to the limit as $\varepsilon \downarrow 0^+$ by monotonic convergence in the relation

$$\int_{\mathbb{R}^N} A[e_r]\, G(\varepsilon\cdot) = \int_{\mathbb{R}^N} \mathcal{F}\Big(A[e_r]\Big)\, \mathcal{F}^{-1}\Big(G(\varepsilon\cdot)\Big) = \int_{\mathbb{R}^N} \mathcal{F}\Big(A[e_r]\Big)(\varepsilon\cdot)\, \mathcal{F}^{-1}G,$$

and consequently to obtain

$$\int_{\mathbb{R}^N} A[e_r]\, G(0) = \int_{\mathbb{R}^N} \mathcal{F}\Big(A[e_r]\Big)(0)\, \mathcal{F}^{-1}G = \mathcal{F}\Big(A[e_r]\Big)(0)\, G(0)$$

as a relation in $[0, \infty]$. Since $\mathcal{F}\Big(A[e_r]\Big)(0) = c_0 < \infty$ it follows that the nonnegative function $A[e_r]$ has a finite integral equal to $c_0$. $\square$

Once the local regularity has been fixed up by turning $e$ into $e_r$, following [9] we would like to compactify the support of $e_r$ by a simple subtraction-truncation procedure, and for this purpose we introduce

$$e_r^R := \Big[e_r - e(R)\Big]^+ \in W_c^{1,1}(\mathbb{R}^N)$$

for $R > r$ intended to go to infinity. Unfortunately, this second procedure is a real problem at the level of $A$, inasmuch as it destroys many properties, for instance:

- *(i)* $A[e_r^R] \in L^1(\mathbb{R}^N)$ is a function whose sign changes on $\mathbb{R}^N$;
- *(ii)* there is a lack of $L^1$ continuity w.r.t. $R$ in the sense that $\int_{\mathbb{R}^N} A[e_r^R] = 0 < c_0 = \int_{\mathbb{R}^N} A[e_r]$, a collapse showing that the approximation $A[e_r^R] \to A[e_r]$ as $R \to \infty$ cannot hold in $L^1(\mathbb{R}^N)$.

In complete analogy, all this may also be applied to other powers than $\alpha$, by setting for $\beta \in ]0, \alpha[$

$$\begin{cases} E := 1 \,/\, |\cdot|^{N-\beta} \in L^1_{loc}(\mathbb{R}^N), \\[2ex] E_r := \min\{E, E(r)\} \in W^{1,1}_{loc}(\mathbb{R}^N), \\[2ex] E_r^R := \Big[E_r - E(R)\Big]^+ \in W_c^{1,1}(\mathbb{R}^N). \end{cases} \tag{8}$$

In order to keep the notation readable, we will omit the dependence of these functions on $\beta$, since the value $\beta \leq \alpha$ will be fixed through all calculations. The case $\beta = \alpha$ will be referred to as the *critical case*; it corresponds to $E = e$, $E_r = e_r$, $E_r^R = e_r^R$. In the statements below, we focus on the subcritical case $\beta < \alpha$; the discussion at the end of this section highlights the pecularities of the critical case.

We will rely upon the following obvious properties:

$$E_r^R \leq E_r, \qquad E_r^R \longrightarrow E_r \text{ pointwise as } R \to \infty.$$

**Lemma 2.2.** *The distribution* $A[E] = -(\alpha - \beta)c\ PF\dfrac{1}{|\cdot|^{N+\alpha-\beta}} \in \mathcal{D}'(\mathbb{R}^N)$ *is negative on* $\mathbb{R}^N - \{0\}$*. Moreover, it has no sign at all in the vicinity of the origin.*

[iii] Postponed till Lemma 2.4 without any vicious circle.

*Proof.* The same kind of computation as in checking $A[e] = c_0\delta_0$ together with formula (5) now yields

$$A[E] = c\,\mathcal{F}^{-1}(|\cdot|^\alpha \times \mathcal{F}E) = c\,\mathcal{F}^{-1}\left(|\cdot|^\alpha \times \pi^{\frac{N}{2}-\beta}\frac{\Gamma(\frac{\beta}{2})}{\Gamma(\frac{N-\beta}{2})}\frac{1}{|\cdot|^\beta}\right) = \frac{c}{\pi^\alpha}\frac{\Gamma(\frac{\beta}{2})\Gamma(\frac{N+\alpha-\beta}{2})}{\Gamma(\frac{N-\beta}{2})\Gamma(\frac{\beta-\alpha}{2})}\mathrm{PF}\frac{1}{|\cdot|^{N+\alpha-\beta}}.$$

In order to exhibit a positive constant, we insert the identity

$$\Gamma(1+\frac{\beta-\alpha}{2}) = -\frac{\alpha-\beta}{2}\Gamma(\frac{\beta-\alpha}{2}),$$

in which the $\Gamma$-term on the l.h.s. is now positive ($\alpha-\beta<2$). This provides the negative factor $-(\alpha-\beta)$ followed by a new positive constant. In view of Remark 2.1, this completes the proof. $\square$

**Lemma 2.3.** *$A[E_r^R]$ is an integrable function on $\mathbb{R}^N$ (which is moreover continuous and negligible at infinity when $\alpha<1$).*

*Proof.* Let us distinguish between two cases.

(*i*) Case $\alpha<1$. Since $E_r^R \in W^{1,1}(\mathbb{R}^N)\cap W^{1,\infty}(\mathbb{R}^N)$, the lemma is a direct consequence of the fact that $A$ maps $W^{1,1}(\mathbb{R}^N)$ into $L^1(\mathbb{R}^N)$ while it maps $W^{1,\infty}(\mathbb{R}^N)$ into the space $C_b^0(\mathbb{R}^N)$ of bounded continuous functions. Indeed, the operator $A : W^{1,p}(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$ for $p=1$ or $p=\infty$ may be seen to be well-defined through the interpolation estimate $\|A[\phi]\| \le c\|\phi\|^{1-\alpha}\|\nabla\phi\|^\alpha$ relatively to the $L^p$-norm. To check it, notice that for any translation-invariant norm $\|\cdot\|$ on functions defined in $\mathbb{R}^N$, we may write

$$\begin{aligned}\|A[\phi]\| &= c\Big\|\int_{|h|>H}\Big(\phi(\cdot+h)-\phi(\cdot)\Big)\frac{dh}{|h|^{N+\alpha}} + \int_{|h|<H}\frac{h}{|h|}\int_0^1\nabla\phi(\cdot+sh)ds\frac{dh}{|h|^{N-(1-\alpha)}}\Big\| \\ &\le c\left(2\|\phi\|\int_{|h|>H}\frac{dh}{|h|^{N+\alpha}} + \|\nabla\phi\|\int_{|h|<H}\frac{dh}{|h|^{N-(1-\alpha)}}\right),\end{aligned}$$

and that this relation optimizes into $\|A[\phi]\| \le c\|\phi\|^{1-\alpha}\|\nabla\phi\|^\alpha$ when $H$ varies over $(0,\infty)$. So, the only remaining point to be established is the continuity of $A[\phi]$ for any bounded Lipschitz-continuous $\phi$, an easy fact that can be proved by the same idea as before ($H:=1$). Indeed, the parameterized integrals

$$A[\phi] = -c\int_{|h|>1}\Big(\phi(\cdot+h)-\phi(\cdot)\Big)\frac{dh}{|h|^{N+\alpha}} - c\int_{|h|<1}\frac{\phi(\cdot+h)-\phi(\cdot)}{|h|}\frac{dh}{|h|^{N-(1-\alpha)}}$$

both inherit the continuity of their integrands, because of some integrable dominations (w.r.t. $h$) ensured by the pointwise estimate

$$\frac{|\phi(\cdot+h)-\phi(\cdot)|}{|h|^{N+\alpha}} \le \|\phi\|_{W^{1,\infty}(\mathbb{R}^N)}\left(2\frac{\mathbb{I}_{|h|>1}}{|h|^{N+\alpha}} + \frac{\mathbb{I}_{|h|<1}}{|h|^{N-(1-\alpha)}}\right).$$

Note also that this integrable domination legitimates the equality

$$\lim_{|x|\to\infty}\int_{\mathbb{R}^N}\Big(\phi(x+h)-\phi(x)\Big)\frac{dh}{|h|^{N+\alpha}} = \int_{\mathbb{R}^N}\lim_{|x|\to\infty}\Big(\phi(x+h)-\phi(x)\Big)\frac{dh}{|h|^{N+\alpha}} = 0$$

as well, whenever $\phi\in W^{1,\infty}(\mathbb{R}^N)$ satisfies $\phi(x+h)-\phi(x)\to 0$ as $|x|\to\infty$ for every fixed $h\in\mathbb{R}^N$. Such is the case when typically

$$\phi\in W^{1,\infty}(\mathbb{R}^N) \text{ with } \lim_\infty|\phi| = 0 \text{ or } \lim_\infty|\nabla\phi| = 0.$$

As a consequence, $A[E_r^R]$ and $A[E_r]$ tend to zero at infinity, since $\phi = E_r^R$ and $\phi = E_r$ are two such functions.

(*ii*) Case $\alpha \geq 1$. A convenient representation formula for $A[\phi]$ (which takes into account some higher regularity of $\phi$) is now

$$A[\phi] = c\int_{|h|>H}\Big(\phi(\cdot) - \phi(\cdot + h)\Big)\frac{dh}{|h|^{N+\alpha}} - c\int_{|h|<H}\frac{h}{|h|}\int_0^1 \frac{\nabla\phi(\cdot + sh) - \nabla\phi(\cdot)}{|h|}ds\frac{dh}{|h|^{N-(2-\alpha)}}, \tag{9}$$

so that (again for any translation-invariant norm)

$$\begin{aligned}
\|A[\phi]\| &\leq c\left(2\|\phi\|\int_{|h|>H}\frac{dh}{|h|^{N+\alpha}} + \int_0^1 s^{\alpha-1}ds\int_{|h|<H}\frac{\|\nabla\phi(\cdot + sh) - \nabla\phi(\cdot)\|}{|sh|^{N+\alpha-1}}s^N dh\right)\\
&= c\left(2\|\phi\|\int_{|h|>H}\frac{dh}{|h|^{N+\alpha}} + \int_0^1 s^{\alpha-1}ds\int_{|h|<sH}\frac{\|\nabla\phi(\cdot + h) - \nabla\phi(\cdot)\|}{|h|^{N+\alpha-1}}dh\right).
\end{aligned}$$

The use of the $L^1$-norm leads to the classical double integral ($dxdy$) defining the so-called *intrinsic norm* of $W^{\alpha-1,1}(\mathbb{R}^N)$, namely

$$\begin{aligned}
\|A[\phi]\|_{L^1(\mathbb{R}^N)} &\leq c\Big(2\|\phi\|_{L^1(\mathbb{R}^N)}\int_{|h|>H}\frac{dh}{|h|^{N+\alpha}} + H^\theta\int_0^1 s^{\theta+\alpha-1}ds\int\int_{|x-y|<sH}\frac{|\nabla\phi(x) - \nabla\phi(y)|}{|x-y|^{N+\theta+\alpha-1}}dxdy\Big)\\
&\leq c\|\phi\|_{W^{\theta+\alpha,1}(\mathbb{R}^N)},
\end{aligned}$$

where $\theta > 0$ is arbitrarily small[iv] when $\alpha = 1$ while $\theta$ may be set to zero otherwise. As a conclusion $A$ maps $W^{2-,1}(\mathbb{R}^N)$ into $L^1(\mathbb{R}^N)$. The assertion of the lemma for $\alpha \geq 1$ is then a consequence of the fact that $E_r^R$ belongs to $W^{2-,1}(\mathbb{R}^N)$ (see Remark 2.2 below). $\square$

**Remark 2.2.** *The observation $E_r^R \in W^{2-,1}(\mathbb{R}^N)$ is a special case of the more general relation $\nabla E_r^R \in W^{s,p}(\mathbb{R}^N)$ valid for $0 \leq s < 1/p \leq 1$. The latter property can be checked by mimicking a classical exercise on fractional order Sobolev spaces, according to which the characteristic function $\mathbb{I}_\Omega$ of a bounded domain $\Omega \subset \mathbb{R}^N$ with Lipschitz boundary belongs to $W^{s,p}(\mathbb{R}^N)$ for any $s < 1/p$. See for instance [40] or [46]. Obviously, the presence here of an extra $C^\infty$ non-constant function within $\Omega := \{r < |\cdot| < R\}$ makes no problem.*

The following lower bound on $A[E_r]$ is fundamental in the sequel: it contains both some nonnegativity within the ball of radius $r$ (which is no mystery since $E_r$ is maximal there) and some information on its decay at infinity (roughly speaking an improvement of a power $\alpha$). Recall that $\beta < \alpha$ is a parameter of $E_r$.

**Lemma 2.4.** *One has* $A[E_r] \geq -(\alpha - \beta)c\dfrac{E_r}{|\cdot|^\alpha}\,\mathbb{I}_{>r}$.

*Proof.* The function $\theta_r : s \in \mathbb{R} \mapsto \min\{s, E(r)\} \in \mathbb{R}$ used in the construction of $E_r = \theta_r \circ E$ is concave on $\mathbb{R}$. So, the pointwise inequality $\theta_r(a) - \theta_r(b) \geq \theta_r'(a)(a-b)$ leads to a convexity property for the operator $A$, namely

$$\begin{aligned}
A[E_r](x) &= -c\lim_{H\to 0^+}\int_{\mathbb{R}^N}\mathbb{I}_{|h|>H}\,\Big(E_r(x+h) - E_r(x)\Big)\frac{dh}{|h|^{N+\alpha}}\\
&\geq -c\lim_{H\to 0^+}\int_{\mathbb{R}^N}\mathbb{I}_{|h|>H}\,(\theta_r' \circ E)(x)\Big(E(x+h) - E(x)\Big)\frac{dh}{|h|^{N+\alpha}}\\
&= (\theta_r' \circ E)(x)\ A[E](x) = \mathbb{I}_{|x|>r}\ A[E](x).
\end{aligned}$$

It remains to insert the expression of $A[E]$ just found in Lemma 2.2 to conclude. $\square$

[iv]Note that for $\alpha = 1$, use of $\theta > 0$ permits to avoid introducing Besov spaces in this calculation.

We will also need for $A[E_r^R]$ some alternative expression of its main part ($|x| < R$) and tail ($|x| > R$):

**Lemma 2.5.** *$A[E_r^R](x)$ is equal to*

$$\begin{cases} A[E_r](x) - c\displaystyle\int_{\mathbb{R}^N} \left[\frac{1}{R^{N-\beta}} - \frac{1}{|x+h|^{N-\beta}}\right]^+ \frac{dh}{|h|^{N+\alpha}} & \textit{for } |x| < R, \\ -\dfrac{c}{r^{N-\beta}}\left(1 - \dfrac{r^{N-\beta}}{R^{N-\beta}}\right)\displaystyle\int_{B(x,r)} \frac{1}{|\cdot|^{N+\alpha}} - c\int_{|x+h|>r} \left[\frac{1}{|x+h|^{N-\beta}} - \frac{1}{R^{N-\beta}}\right]^+ \frac{dh}{|h|^{N+\alpha}} & \textit{for } |x| > R. \end{cases}$$

*Proof.* For $|x| < R$ fixed, the integral in $h$ defining $A[E_r^R]$ may be cut into two parts according to whether $|x+h|$ is situated below or above the threshold $R$, i.e.,

$$\begin{aligned} A[E_r^R](x) &= -c\int_{|x+h|<R} \Big(E_r^R(x+h) - E_r^R(x)\Big)\frac{dh}{|h|^{N+\alpha}} - c\int_{|x+h|>R} \Big(E_r^R(x+h) - E_r^R(x)\Big)\frac{dh}{|h|^{N+\alpha}} \\ &= -c\int_{|x+h|<R} \Big(E_r(x+h) - E_r(x)\Big)\frac{dh}{|h|^{N+\alpha}} + c\int_{|x+h|>R} E_r^R(x)\frac{dh}{|h|^{N+\alpha}} \\ &= A[E_r](x) + c\int_{|x+h|>R} \Big(E_r(x+h) - E_r(x) + E_r^R(x)\Big)\frac{dh}{|h|^{N+\alpha}} \\ &= A[E_r](x) - c\int_{|x+h|>R} \left(\frac{1}{R^{N-\beta}} - \frac{1}{|x+h|^{N-\beta}}\right)\frac{dh}{|h|^{N+\alpha}}, \end{aligned}$$

whence the result. For $|x| > R$ fixed, the computation is even more natural:

$$\begin{aligned} A[E_r^R](x) &= -c\int_{\mathbb{R}^N} E_r^R(x+h)\frac{dh}{|h|^{N+\alpha}} \\ &= -c\int_{|x+h|<r} \left(\frac{1}{r^{N-\beta}} - \frac{1}{R^{N-\beta}}\right)\frac{dh}{|h|^{N+\alpha}} - c\int_{r<|x+h|<R} \left(\frac{1}{|x+h|^{N-\beta}} - \frac{1}{R^{N-\beta}}\right)\frac{dh}{|h|^{N+\alpha}}, \end{aligned}$$

which is the stated formula. □

We are now in a position to state and prove the main result of this section:

**Lemma 2.6.** *One has $\|A[E_r^R] - \mathbb{I}_{<R}\, A[E_r]\|_{L^1(\mathbb{R}^N)} \to 0$ as $R \to \infty$.*

*Proof.* By the previous lemma, $\|A[E_r^R] - \mathbb{I}_{<R}\, A[E_r]\|_{L^1(\mathbb{R}^N)} = c(I_r^R + J_r^R + K_r^R)$, where

$$\begin{aligned} I_r^R &:= \int_{|x|<R}\Big(\int_{|x+h|>R} \left[\frac{1}{R^{N-\beta}} - \frac{1}{|x+h|^{N-\beta}}\right]\frac{dh}{|h|^{N+\alpha}}\Big)dx, \\ J_r^R &:= \int_{|x|>R}\Big(\int_{R>|x+h|>r} \left[\frac{1}{|x+h|^{N-\beta}} - \frac{1}{R^{N-\beta}}\right]\frac{dh}{|h|^{N+\alpha}}\Big)dx, \\ K_r^R &:= \Big(\frac{1}{r^{N-\beta}} - \frac{1}{R^{N-\beta}}\Big)\int_{|x|>R}\Big(\int_{|x+h|<r} \frac{dh}{|h|^{N+\alpha}}\Big)dx. \end{aligned}$$

We shall start with the estimate of $K_r^R$ which turns out to be a mild term, in the sense that $K_r^R$ remains asymptotically small whatever the choice of $\beta$ be. Next, we shall deal with $J_r^R$ in detail, leaving the similar study of $I_r^R$ to the reader, since $I_r^R$ and $J_r^R$ have been designed in order to be in complete analogy from a computational point of view.

(*i*) Study of $K_r^R$. In the double integral defining $K_r^R$, introduce the change of variables $h = -x + |x|y$ for every fixed $x$ (so that $dh = |x|^N dy$), then make the polar change of variables w.r.t. $x$ for every fixed $h$ (i.e., set $\rho := |x|$). By the Fubini-Tonelli theorem, the integrals can be freely interchanged. Recall that $s_N$ is the measure of the unit sphere. This leads to the following chain of equalities:

$$\int_{|x|>R}\Big(\int_{|x+h|<r}\frac{dh}{|h|^{N+\alpha}}\Big)dx = \int_{|x|>R}\frac{dx}{|x|^\alpha}\int_{|y|<r/|x|}\frac{dy}{\left|\frac{x}{|x|}-y\right|^{N+\alpha}}$$

$$= s_N\int_{\rho>R}\rho^{N-\alpha}\frac{d\rho}{\rho}\int_{|\cdot|<r/\rho}\frac{1}{|\epsilon-\cdot|^{N+\alpha}} = s_N\int_{|\cdot|<r/R}\frac{1}{|\epsilon-\cdot|^{N+\alpha}}\int_R^{r/|\cdot|}\rho^{N-\alpha}\frac{d\rho}{\rho},$$

where the choice of the reference point $\epsilon := (1, 0, \dots, 0)$ on the unit sphere of $\mathbb{R}^N$ is immaterial here due to the angular invariance of the integral involved. The conclusion of this computation is the identity

$$\int_{|x|>R}\Big(\int_{|x+h|<r}\frac{dh}{|h|^{N+\alpha}}\Big)dx = s_N\int_{|\cdot|<r/R}\frac{1}{|\epsilon-\cdot|^{N+\alpha}}\frac{1}{N-\alpha}\Big(\frac{r^{N-\alpha}}{|\cdot|^{N-\alpha}}-R^{N-\alpha}\Big),$$

in which an extra $r/R$-dilation can be made to get finally the more transparent expression

$$\int_{|x|>R}\Big(\int_{|x+h|<r}\frac{dh}{|h|^{N+\alpha}}\Big)dx = \frac{r^N}{R^\alpha}\frac{s_N}{N-\alpha}\int_{|\cdot|<1}\frac{1}{\left|\epsilon-\frac{r}{R}\cdot\right|^{N+\alpha}}\Big(\frac{1}{|\cdot|^{N-\alpha}}-1\Big).$$

The asymptotic behavior as $R\to\infty$ is readily seen on this formula: we find $K_r^R \sim Cr^\beta/R^\alpha$, in particular there is a constant $C$ for which $K_r^R \le Cr^\beta/R^\alpha \to 0$ as $R\to\infty$.

(*ii*) Study of $J_r^R$. In the double integral defining $J_r^R$, consider first a translation of $h$ in the $x$-variable for every fixed $h$, then rename $h$ as $h = x - |x|y$ for every fixed $x$ (so that $dh = |x|^N dy$), and finally use polar coordinates to take advantage of the angular invariance (as before $\epsilon$ is a reference point on the unit sphere). This amounts to successively turning $J_r^R$ into

$$\begin{aligned}
J_r^R &= \int_{|x|>R}\Big(\int_{R>|x+h|>r}\Big[\frac{1}{|x+h|^{N-\beta}}-\frac{1}{R^{N-\beta}}\Big]\frac{dh}{|h|^{N+\alpha}}\Big)dx\\
&= \int\int_{r<|x|<R<|x-h|}\Big(\frac{1}{|x|^{N-\beta}}-\frac{1}{R^{N-\beta}}\Big)dx\frac{dh}{|h|^{N+\alpha}}\\
&= \int_{r<|x|<R}\Big(\frac{1}{|x|^{N-\beta}}-\frac{1}{R^{N-\beta}}\Big)\frac{dx}{|x|^\alpha}\int_{|y|>R/|x|}\frac{dy}{\left|\frac{x}{|x|}-y\right|^{N+\alpha}}\\
&= s_N\int_r^R\frac{1}{\rho^{\alpha-\beta}}\Big(1-\frac{1}{(R/\rho)^{N-\beta}}\Big)\frac{d\rho}{\rho}\int_{|\cdot|>R/\rho}\frac{1}{|\epsilon-\cdot|^{N+\alpha}}.
\end{aligned}$$

A last change of $\rho$ into $R/\rho$ (so that $d\rho/\rho$ is unchanged) leads to an expression of $J_r^R$ as a monotonic quantity of $R/r$, i.e.,

$$J_r^R = \frac{s_N}{R^{\alpha-\beta}}\int_1^{R/r}\rho^{\alpha-\beta}\Big(1-\frac{1}{\rho^{N-\beta}}\Big)\frac{d\rho}{\rho}\int_{|\cdot|>\rho}\frac{1}{|\epsilon-\cdot|^{N+\alpha}},$$

so that $J_r^R \leq \gamma_J / R^{\alpha-\beta}$ with

$$\gamma_J := s_N \int_1^\infty \rho^{\alpha-\beta}\Big(1 - \frac{1}{\rho^{N-\beta}}\Big)\frac{d\rho}{\rho}\int_{|\cdot|>\rho} \frac{1}{|\epsilon - \cdot|^{N+\alpha}}$$

$$= s_N \int_{|\cdot|>1} \frac{1}{|\epsilon - \cdot|^{N-(2-\alpha)}} \times \frac{\frac{|\cdot|^{\alpha-\beta}-1}{\alpha-\beta} - \frac{1}{N-\alpha}\left(1 - \frac{1}{|\cdot|^{N-\alpha}}\right)}{|\epsilon - \cdot|^2} < \infty,$$

the integrability being ensured by the boundedness of the last ratio in the last integrand. Specifically,

$$\frac{\frac{|\cdot|^{\alpha-\beta}-1}{\alpha-\beta} - \frac{1}{N-\alpha}\left(1 - \frac{1}{|\cdot|^{N-\alpha}}\right)}{|\epsilon - \cdot|^2} \sim \frac{1}{2}(N-\beta)\left(\frac{|\cdot| - 1}{|\epsilon - \cdot|}\right)^2$$

remains bounded in a vicinity of $\epsilon$.

(*iii*) Study of $I_r^R$. Similar transformations of the double integral defining $I_r^R$ show that $I_r^R$ is in fact of the form $I_r^R = \gamma_I / R^{\alpha-\beta}$ with

$$\gamma_I := s_N \int_{|\cdot|<1} \frac{1}{|\epsilon - \cdot|^{N-(2-\alpha)}} \times \frac{\frac{|\cdot|^{\alpha-\beta}-1}{\alpha-\beta} + \frac{1}{N-\alpha}\left(\frac{1}{|\cdot|^{N-\alpha}} - 1\right)}{|\epsilon - \cdot|^2} < \infty,$$

the integrability being ensured by the same argument as before. □

We shall end this section with a natural question concerning the border case of powers: what still holds when $\beta = \alpha$? The discussion here will focus on the case $\alpha < 1$ only to avoid some lengthy digression. Obviously, Lemma 2.2 has to be re-interpreted as a relation ($A[e] = c_0\delta_0$) expressing that $e (= E)$ is then an elementary solution. All other lemmata and proofs remain unchanged, with the outstanding exception that Lemma 2.6 should now state an $L^1$-bound instead of an $L^1$-convergence. Precisely

$$\|A[e_r^R] - \mathbb{I}_{<R}\, A[e_r]\|_{L^1(\mathbb{R}^N)} \to (\hat\gamma_I + \hat\gamma_J)c \quad \text{as } R \to \infty,$$

or equivalently (since $A[e_r]$ is integrable by Lemma 2.1)

$$\|A[e_r^R] - A[e_r]\|_{L^1(\mathbb{R}^N)} \to (\hat\gamma_I + \hat\gamma_J)c \quad \text{as } R \to \infty,$$

where $\hat\gamma_I + \hat\gamma_J > 0$ is the constant appearing in (*ii*)-(*iii*) of the proof of Lemma 2.6 when $\beta := \alpha$. Note that the expression we would find in this case

$$\hat\gamma_I + \hat\gamma_J = s_N \int_{\mathbb{R}^N} \frac{1}{|\epsilon - \cdot|^{N-(2-\alpha)}} \times \frac{\frac{|\cdot|^{0}-1}{0} + \frac{1}{N-\alpha}\left(\frac{1}{|\cdot|^{N-\alpha}} - 1\right)}{|\epsilon - \cdot|^2} < \infty$$

involves the logarithm through the usual convention for the zero power $\dfrac{|\cdot|^0 - 1}{0} := \log|\cdot|$.

To complete the picture for $\alpha < 1$, let us mention that the convergence $A[e_r^R] \to A[e_r]$ as $R \to \infty$ may be seen to hold in $L^p(\mathbb{R}^N)$ for any $1 < p \leq \infty$, but not in $L^1$ (nor even in $L^1$ weak, see the comment at the end of the section). In the sequel, this lack of duality is a serious obstacle to get rid of 'non-local' terms of the type

$$\liminf_{R\to\infty} \int_{\mathbb{R}^N} W\, A[e_r^R],$$

where $W = |\varphi(u) - \varphi(v)|$. These terms are required to be nonnegative in the proof of Section 3 when $W \in L^\infty_+(\mathbb{R}^N)$ is only known to be bounded (and not in $L^{p'}$ for some $1 < p \leq \infty$). In this respect, all we can apparently do in an $L^\infty$-framework with the aforementioned material for the critical case $\beta = \alpha$, is the following claim.

**Lemma 2.7.** *Assume that $W \in L^\infty(\mathbb{R}^N)$ tends to some limit $l$ at infinity. Then*

$$\int_{\mathbb{R}^N} W\ A[e_r^R] \ \longrightarrow\ -c_0 l + \int_{\mathbb{R}^N} W\ A[e_r] \quad \text{as } R \to \infty$$

*where $c_0$ is the constant of Lemma 2.1.*

*Proof.* Consider the decomposition

$$\begin{aligned}\int_{\mathbb{R}^N} W\ A[e_r^R] - \int_{\mathbb{R}^N} W\ A[e_r] = &\int_{\mathbb{R}^N} (W-l)\ \Big(A[e_r^R] - A[e_r]\Big) - l\int_{>R} A[e_r] \\ &+ l\int_{>R} A[e_r^R] + l\int_{<R}\Big(A[e_r^R] - A[e_r]\Big).\end{aligned}$$

As $R \to \infty$, it is easy to see

- $(i)$ that the first term tends to zero because $A[e_r^R] - A[e_r] \to 0$ uniformly on $\mathbb{R}^N$ with an $L^1(\mathbb{R}^N)$-bound;
- $(ii)$ that the second term tends to zero because $A[e_r] \in L^1(\mathbb{R}^N)$ (see Lemma 2.1);
- $(iii)$ and that the sum of the two last terms tends to $-(\hat{\gamma}_I + \hat{\gamma}_J)c$ as the proof of Lemma 2.6 shows when $\beta = \alpha$.

The case of a constant $W$ permits to calculate $(\hat{\gamma}_I + \hat{\gamma}_J)c = c_0$ as the value (7) of the constant in the fundamental equation $A[e] = c_0\delta_0$. In other terms $\hat{\gamma}_I + \hat{\gamma}_J = \pi^{\frac{N}{2}-\alpha}\Gamma(\frac{\alpha}{2})/\Gamma(\frac{N-\alpha}{2})$. □

The case of a general $W \in L^\infty(\mathbb{R}^N)$ is not covered by Lemma 2.7. As a conclusion to this section, we could therefore say that introducing a strictly smaller power $\beta < \alpha$ in (8) can be viewed as a technicality to get some replacement of Lemma 2.7 valid for any bounded $W$. In the context of Section 3 this will allow the study of solutions without any prescribed behavior at infinity.

# 3 Entropy formulation and proofs of Theorems 1.1, 1.2

## 3.1 Entropy formulation recalled

Following [1] (see also [31, 20, 27] for variants of the definition), we define the entropy formulation of the nonlocal conservation law (1) as follows. Note that in order to make most apparent the link to (10) below we chose to state it with Kruzhkov entropies and not with general smooth convex entropies; we also include the initial datum into entropy inequalities instead of using the original $\operatorname{ess\,lim}_{t\to 0^+}$ formulation of [32].

**Definition 3.1.** *Let $u_0 \in L^\infty(\mathbb{R}^N)$ and $g \in L^1_{loc}(I; L^\infty(\mathbb{R}^N))$. A function $u \in L^\infty(I \times \mathbb{R}^N)$ is called an entropy solution of (1) if for any nonnegative test function $\phi \in C_c^\infty(I \times \mathbb{R}^N)$ and any cut value $\mathfrak{r} \in \mathbb{R}^*_+$ the following* *entropy inequalities hold:*

$$\int_{I\times\mathbb{R}^N} |u-k|\partial_t\phi + \Phi(u,k).\nabla_x\phi + \mathrm{sgn}(u-k)\,g\phi + \int_{\mathbb{R}^N} |u_0-k|\phi(0,\cdot)$$
$$\geq c\int_{I\times\mathbb{R}^N} \mathrm{sgn}(u-k)\phi \int_{|h|>\mathfrak{r}} \frac{\varphi(u(\cdot))-\varphi(u(\cdot+h))}{|h|^{N+\alpha}}dh$$
$$- c\int_{I\times\mathbb{R}^N} |\varphi(u)-\varphi(k)| \int_{|h|<\mathfrak{r}} \frac{\phi(\cdot+h)-\phi(\cdot)-h\nabla\phi(\cdot)}{|h|^{N+\alpha}}dh,$$

*where $\Phi(u,k) = sgn(u-k)(f(u)-f(k))$ is the associated entropy flux ([32]) and $c$ is the normalization constant of the non-local diffusion operator.*

## 3.2 Uniqueness and contraction(-comparison) proof

Let $u$ and $v$ be two bounded entropy solutions of (1) associated with the initial data $u_0$ and $v_0$ and the source terms $g$ and $h$. Kruzhkov's method of doubling variables applied to the entropy inequality of Definition 3.1 allows to derive the Kato-like inequality

$$\int_I \left(\int_{\mathbb{R}^N} |u-v|\phi\,dx\right)\psi'(t)dt + \int_I \left(\int_{\mathbb{R}^N} \mathrm{sgn}(u-v)\Big(f(u)-f(v)\Big).\nabla_x\phi\,dx\right)\psi(t)dt \tag{10}$$
$$+ \int_I \left(\int_{\mathbb{R}^N} \mathrm{sgn}(u-v)(g-h)\phi\,dx\right)\psi(t)dt + \psi(0)\int_{\mathbb{R}^N} |u_0-v_0|\phi\,dx \geq \int_I \left(\int_{\mathbb{R}^N} |\varphi(u)-\varphi(v)|\; A[\phi]\,dx\right)\psi(t)dt,$$

in which $\phi=\phi(x)\in W_+^{2-,1}(\mathbb{R}^N)$ and $\psi=\psi(t)\in W_+^{1,1}(I)$ are arbitrary (regular[v]) functions of space and time. Although this point is of crucial importance in the global argument, we shall not discuss it further here, since this has already been done several times elsewhere (see e.g. [1, 27]).

Note also that the monotonicity assumption on the nonlinearity $\varphi$ is fundamental to end up with a nonnegative r.h.s. of the type $\mathrm{sgn}(u-v)\big(\varphi(u)-\varphi(v)\big) = |\varphi(u)-\varphi(v)|$; but the usually assumed Lipschitz regularity of $f$,$\varphi$ plays no role in the argument as soon as $f(u)$,$\varphi(u)$ are $L^1_{loc}$ functions.

The remaining part of this section aims at deducing from (10) the uniqueness and stability of entropy solutions through the classical $L^1$-contraction principle

$$\|w(t)\|_{L^1(\mathbb{R}^N)} \leq \|w_0\|_{L^1(\mathbb{R}^N)} + \int_0^t \|g-h\|_{L^1(\mathbb{R}^N)}(\tau)d\tau \qquad (\text{ for a.e. } t\in I)$$

for

$$w := |u-v| \in L^\infty_+(I\times\mathbb{R}^N)$$

and $w_0 := |u_0-v_0| \in L^\infty_+(\mathbb{R}^N)$. Of course, the underlying assumption is that the data differ here by integrable terms $u_0-v_0\in L^1(\mathbb{R}^N)$ and $g-h\in L^1_{loc}(I;L^1(\mathbb{R}^N))$, otherwise we have nothing to prove.

In an entirely analogous way, the choice $w := (u-v)^+$ leads to the contraction-comparison principle also stated in Theorem 1.1. For this sake, it is enough to start with the version of the Kato inequality (2) where the Kruzhkov entropy $|\cdot -k|$ and its entropy flux are replaced by the semi-entropies $(\cdot - k)^\pm$ and the associated fluxes, while $|\varphi(u)-\varphi(k)|$ is replaced by $(\varphi(u)-\varphi(k))^\pm$.

For notational convenience, let us set once for all

$$M := \max\{\|u\|_{L^\infty(I\times\mathbb{R}^N)}, \|v\|_{L^\infty(I\times\mathbb{R}^N)}, \|f(u)-f(v)\|_{L^\infty(I\times\mathbb{R}^N)}, \|\varphi(u)-\varphi(v)\|_{L^\infty(I\times\mathbb{R}^N)}\} < \infty.$$

[v] The extension from $C_c^\infty$ to $W^{2-,1}$ follows from an easy density argument.

### 3.2.1 The local term

Besides the trivial bound of source terms by $\operatorname{sgn}(u-v)(g-h)E_r^R \le |g-h|E_r$, we shall also plug into (10) the following estimate of the local term (or div-term)

$$\operatorname{sgn}(u-v)\Big(f(u)-f(v)\Big).\nabla_x E_r^R \le L_f^M w^\sigma |\nabla E_r^R|, \tag{11}$$

where

$$|\nabla E_r^R| = |\nabla E_r|\, \mathbb{I}_{r<|\cdot|<R} = (N-\beta)\frac{E_r}{|\cdot|}\, \mathbb{I}_{r<|\cdot|<R}. \tag{12}$$

Now, assume $N>1$. Fix $\delta>0$ small and set $\delta' := \delta^{-\frac{1}{N-1}}$ (as well as $N' := \dfrac{N}{N-1}$), in order to write

$$\frac{w^\sigma}{|\cdot|} \le \frac{\delta}{N}\frac{1}{|\cdot|^N} + \frac{\delta'}{N'}w. \tag{13}$$

This estimate results from the Young inequality for nonnegative numbers

$$ab \le \frac{a^p}{p} + \frac{b^{p'}}{p'} \text{ applied with } a := \frac{\delta^{1/N}}{|\cdot|},\ b := \frac{w^\sigma}{\delta^{1/N}},\ p := N.$$

Now, (11),(12),(13) give

$$\operatorname{sgn}(u-v)\Big(f(u)-f(v)\Big).\nabla_x E_r^R \ \le\ L_f^M (N-\beta)\left(\frac{\delta}{N}\frac{E_r}{|\cdot|^N} + \frac{\delta'}{N'}wE_r\right)\mathbb{I}_{r<|\cdot|<R},$$

so that after integration in space

$$\begin{aligned}\int_{\mathbb{R}^N} \operatorname{sgn}(u-v)\Big(f(u)-f(v)\Big).\nabla_x E_r^R \ &\le\ L_f^M (N-\beta)\left(\frac{\delta}{N}\int_{>r}\frac{E_r}{|\cdot|^N} + \frac{\delta'}{N'}\int_{>r} wE_r\right)\\ &=\ L_f^M \frac{\delta}{N} s_N \frac{1}{r^{N-\beta}} + L_f^M (N-\beta)\frac{\delta'}{N'}\int_{>r} wE_r. \end{aligned}\tag{14}$$

**Remark 3.3.** *In the monodimensional case $N=1$, the modulus of continuity of $f$ defined by*

$$\omega_f : \delta \in \mathbb{R}_+^* \mapsto \sup_{u,v\in[-M,+M]:0<|u-v|\le\delta} |f(u)-f(v)| \in \mathbb{R}_+ \tag{15}$$

*allows to avoid the use of $N'$ via the estimate*

$$\begin{aligned}\operatorname{sgn}(u-v)\Big(f(u)-f(v)\Big).\nabla_x E_r^R \ &\le\ (1-\beta)\Big(\omega_f(\delta) + \frac{M}{\delta}w\Big)\frac{E_r}{|\cdot|}\mathbb{I}_{r<|\cdot|<R}\\ &\le\ (1-\beta)\Big(\omega_f(\delta)\frac{E_r}{|\cdot|} + \frac{M}{\delta r}wE_r\Big)\mathbb{I}_{r<|\cdot|<R}.\end{aligned}$$

*In other words, the constants $(\frac{\delta}{N}, \frac{\delta'}{N'})$ of the multidimensional case are to be re-interpreted as $(\omega_f(\delta), \frac{M}{\delta r})$ when $N=1$.*

### 3.2.2 The non-local term

In the splitting of the non-local term (or $A$-term) in the r.h.s. of (10) as

$$\int_{\mathbb{R}^N} |\varphi(u)-\varphi(v)|\, A[E_r^R] = \int_{\mathbb{R}^N} |\varphi(u)-\varphi(v)| \Big(A[E_r^R] - \mathbb{I}_{<R} A[E_r]\Big) \\ + \int_{r<|\cdot|<R} |\varphi(u)-\varphi(v)|\, A[E_r] + \int_{|\cdot|<r} |\varphi(u)-\varphi(v)|\, A[E_r],$$

we exploit the nonnegativity of $A[E_r]$ for $|\cdot|<r$ and the lower bound of Lemma 2.4 for the case $r<|\cdot|<R$ to get

$$\int_{\mathbb{R}^N} |\varphi(u)-\varphi(v)|\, A[E_r^R] \geq -M\|A[E_r^R] - \mathbb{I}_{<R}A[E_r]\|_{L^1(\mathbb{R}^N)} - (\alpha-\beta)c \int_{>r} |\varphi(u)-\varphi(v)|\, \frac{E_r}{|\cdot|^\alpha}. \tag{16}$$

Next, we make a classical use of the modulus of continuity

$$\omega_\varphi : \varepsilon \in \mathbb{R}_+^* \mapsto \sup_{u,v\in[-M,+M]:0<|u-v|\leq\varepsilon} |\varphi(u)-\varphi(v)| \in \mathbb{R}_+. \tag{17}$$

by distinguishing between small and large values of $w$, i.e.

$$\int_{>r} |\varphi(u)-\varphi(v)|\, \frac{E_r}{|\cdot|^\alpha} = \int_{>r} \mathbb{I}_{(w\leq\varepsilon)}\, |\varphi(u)-\varphi(v)|\, \frac{E_r}{|\cdot|^\alpha} + \int_{>r} \mathbb{I}_{(w>\varepsilon)}\, |\varphi(u)-\varphi(v)|\, \frac{E_r}{|\cdot|^\alpha} \\ \leq \int_{>r} \omega_\varphi(\varepsilon)\frac{E_r}{|\cdot|^\alpha} + \int_{>r} \frac{M}{\varepsilon} w \frac{E_r}{r^\alpha} = \omega_\varphi(\varepsilon) s_N \frac{1}{(\alpha-\beta)r^{\alpha-\beta}} + \frac{M}{\varepsilon}\frac{1}{r^\alpha}\int_{>r} wE_r. \tag{18}$$

### 3.2.3 Estimates derived from the evolution in time

The various estimates (14),(16),(18) assembled together yield

$$\int_I\Big(\int_{\mathbb{R}^N} wE_r^R\Big)\psi' + L_f^M \int_I\Big(\frac{\delta}{N} s_N \frac{1}{r^{N-\beta}} + (N-\beta)\frac{\delta'}{N'}\int_{>r} wE_r\Big)\psi \\ + \int_I\Big(\int_{\mathbb{R}^N} |g-h|E_r\Big)\psi + \psi(0)\int_{\mathbb{R}^N} w_0 E_r^R \\ \geq \int_I\Big(-M\|A[E_r^R] - \mathbb{I}_{<R}A[E_r]\|_{L^1(\mathbb{R}^N)} - \omega_\varphi(\varepsilon) c s_N \frac{1}{r^{\alpha-\beta}} - \frac{M}{\varepsilon}\frac{\alpha-\beta}{r^\alpha} c \int_{>r} wE_r\Big)\psi. \tag{19}$$

In order to focus on time dependence only, let us set

$$p := \int_{<r} wE_r^R \leq \int_{<r} wE_r =: \hat{p} \qquad \text{and} \qquad q := \int_{>r} wE_r^R \leq \int_{>r} wE_r =: \hat{q},$$

$$d := \int_{\mathbb{R}^N} |g-h|E_r,$$

as functions of time $t$; they are defined for a.e. $t \in I$. Let us stress that at this stage, $\hat{q}$ is not known to be finite-valued. Introduce the three constants

$$\nu := M\|A[E_r^R] - \mathbb{I}_{<R}A[E_r]\|_{L^1(\mathbb{R}^N)},$$

$$\rho := L_f^M \frac{\delta}{N} s_N \frac{1}{r^{N-\beta}} + \omega_\varphi(\varepsilon) c s_N \frac{1}{r^{\alpha-\beta}} \quad \text{and} \quad \hat{l} := L_f^M (N-\beta)\frac{\delta'}{N'} + \frac{M}{\varepsilon}\frac{\alpha-\beta}{r^\alpha}c,$$

the dependence upon $r$ (and also upon $R$ for $p, q, \nu$) being dropped for the sake of readability. Notice that $\nu$ vanishes as $R \to \infty$ thanks to Lemma 2.6. With these notations, the inequality obtained in (19) reads

$$\int_I (p+q)\psi' + \hat{l} \int_I \hat{q}\psi + \int_I (d+\rho+\nu)\psi \geq -\psi(0) \int_{\mathbb{R}^N} w_0 E_r^R, \tag{20}$$

*provided that* the $\hat{q}$-term coming from (18) has been proved *finite*. Such is the purpose of the following result, which states an intermediate integrability on $w$. In the sequel this claim will be improved into global integrability through the $L^1$-contraction principle itself. Note that the border case $\beta := \alpha$ is allowed in the statement below.

**Lemma 3.8.** *For all $\beta \leq \alpha$, one has $\hat{q} \in L^\infty_{loc}(I)$.*

*Proof.* Coming back to (11),(12) and continuing the chain of inequalities (16) by

$$\int_{\mathbb{R}^N} |\varphi(u) - \varphi(v)| \, A[E_r^R] \geq -M \|A[E_r^R] - \mathbb{I}_{<R} A[E_r]\|_{L^1(\mathbb{R}^N)} - M(\alpha-\beta)c \int_{>r} \frac{E_r}{|\cdot|^\alpha} = -\nu - \frac{Mcs_N}{r^{\alpha-\beta}},$$

from (10) we obtain in the same spirit as before

$$\int_I (p+q)\psi' + L_f^M (N-\beta) \int_I \Big( \int_{>r} w^\sigma \, \frac{E_r}{|\cdot|} \Big) \psi + \int_I (d + \nu + \frac{Mcs_N}{r^{\alpha-\beta}}) \psi \geq -\psi(0) \int_{\mathbb{R}^N} w_0 E_r^R$$

and consequently (after integration in time)

$$(p+q)(t) \leq \int_{\mathbb{R}^N} w_0 E_r + \int_0^t d + t\nu + t\frac{Mcs_N}{r^{\alpha-\beta}} + L_f^M N \int_0^t \Big( \int_{>r} w^\sigma \, \frac{E_r}{|\cdot|} \Big).$$

Fatou's lemma as $R \to \infty$ then leads to

$$(\hat{p}+\hat{q})(t) \leq \int_{\mathbb{R}^N} w_0 E_r + \int_0^t d + t\frac{Mcs_N}{r^{\alpha-\beta}} + L_f^M N \int_0^t \Big( \int_{>r} w^\sigma \, \frac{E_r}{|\cdot|} \Big). \tag{21}$$

When $\alpha < 1$, this relation shows as expected that $\hat{p}$ and $\hat{q}$ remain bounded locally uniformly in time, since the integrability

$$\int_0^t \Big( \int_{>r} w^\sigma \, \frac{E_r}{|\cdot|} \Big) \leq tM^\sigma \int_{>r} \frac{E_r}{|\cdot|} < \infty$$

is obvious in this case. It is also of interest to remark that the limiting procedure $\beta \to \alpha^-$ is allowed in this argument, since taking $\beta := \alpha$ in the final bound (21) makes no problem.

When $\alpha \geq 1$ an extra argument is needed. Let $(\alpha_k)_{k\in\mathbb{N}}$ denote the decreasing sequence of powers defined from $\alpha_0 := \alpha$ by the inductive formula

$$\alpha_k := \alpha - \sum_{0 \leq j < k} \frac{N - \alpha_j}{N-1}.$$

Since its limit as $k \to \infty$ is obviously $\lim \alpha_k = -\infty$ (since the general term of the series does not tend to zero), we may restrict $k$ to the smallest integer $K$ s.t. $\alpha_K \leq 1$. For $0 \leq k \leq K$, let $\mathcal{E}_{r,k}$ denote the same quantity as $E_r$, but with $\beta$ changed into $\beta - \alpha + \alpha_k$ inside. Instead of (13), we shall now use the similar estimate

$$\frac{w^\sigma}{|\cdot|} \mathcal{E}_{r,k} \leq \frac{1}{N} \frac{1}{|\cdot|^{\alpha_k}} \mathcal{E}_{r,k} + \frac{1}{N'} \frac{w}{|\cdot|^{(N-\alpha_k)/(N-1)}} \mathcal{E}_{r,k},$$

in which the first term of the r.h.s. has been designed to be integrable at infinity or more precisely, on the set $(>r)$. So, repeating (21) for any $0 \leq k \leq K$ with $\mathcal{E}_{r,k}$ (instead of $E_r$) produces a system of $K+1$ inequalities

$$\begin{aligned}\int_{\mathbb{R}^N} w(t)\mathcal{E}_{r,k} &\leq \int_{\mathbb{R}^N} w_0\mathcal{E}_{r,k} + \int_0^t d + t\frac{Mcs_N}{r^{\alpha-\beta}} + L_f^M N \int_0^t \Big(\int_{>r} w^\sigma \, \frac{\mathcal{E}_{r,k}}{|\cdot|}\Big) \\ &\leq \int_{\mathbb{R}^N} w_0\mathcal{E}_{r,k} + \int_0^t d + t\frac{Mcs_N}{r^{\alpha-\beta}} + tL_f^M \int_{>r} \frac{\mathcal{E}_{r,k}}{|\cdot|^{\alpha_k}} + L_f^M \frac{N}{N'} \int_0^t \Big(\int_{>r} w \, \frac{\mathcal{E}_{r,k}}{|\cdot|^{(N-\alpha_k)/(N-1)}}\Big),\end{aligned}$$

ending (in line $K$) with a finite r.h.s. due to the bound

$$\int_0^t \Big(\int_{>r} w^\sigma \, \frac{\mathcal{E}_{r,K}}{|\cdot|}\Big) \leq tM^\sigma \int_{>r} \frac{\mathcal{E}_{r,K}}{|\cdot|} < \infty.$$

A careful inspection of powers shows that this system is essentially inductive, in the sense that the r.h.s. of any line is the l.h.s. of the next line up to some easy terms. Finally, this system exhibits a control from above of the l.h.s. $(\hat{p}+\hat{q})(t)$ of its first line $(k=0)$ by the finite r.h.s. of its last line $(k=K)$, showing as expected that $\hat{p}$ and $\hat{q}$ remain bounded locally uniformly in time. □

The (spatial) integrability expressed by Lemma 3.8 allows to pass to the limit in (20) as $R \to \infty$, with the aim to recover a similar differential inequation for $\hat{p}$ and $\hat{q}$, i.e.

$$\int_I (\hat{p}+\hat{q})\psi' + \hat{l}\int_I \hat{q}\psi + \int_I (d+\rho)\psi \geq -\psi(0)\int_{\mathbb{R}^N} w_0 E_r.$$

Therein, it is now possible to let $\beta$ tend towards $\alpha$, thanks again to the integrability contained in Lemma 3.8 for the border case. The resulting relation reads

$$\int_I (P+Q)\psi' + l\int_I Q\psi + \int_I \left(D + \overline{\rho} + \omega_\varphi(\varepsilon)cs_N\right)\psi \geq -\psi(0)\int_{\mathbb{R}^N} w_0 e_r, \tag{22}$$

where

$$P := \int_{<r} we_r \ \text{ and } \ Q := \int_{>r} we_r \text{ and } D := \int_{\mathbb{R}^N} |g-h|e_r$$

are the quantities corresponding to the border case $\beta = \alpha$, $E_r = e_r$, while

$$\overline{\rho} := L_f^M \frac{\delta}{N} s_N \frac{1}{r^{N-\alpha}} \text{ and } l := L_f^M (N-\alpha)\frac{\delta'}{N'}.$$

Finally $\varepsilon \to 0^+$ in (22) yields $-(P+Q)' + lQ + D + \overline{\rho} \geq 0$ in the sense of distributions on $I$. We shall now integrate this evolution inequation in two slightly different ways, viewing it, on the one hand like

$$(P+Q)' \leq D + \overline{\rho} + lQ,$$

and on the other hand like

$$\frac{d}{dt}\left(P(t)e^{-lt} + Q(t)e^{-lt}\right) \leq (D + \overline{\rho} - lP)e^{-lt}.$$

From the first standpoint, we get

$$(P+Q)(t) \leq \int_{\mathbb{R}^N} w_0 e_r + \int_0^t D + t\overline{\rho} + l\int_0^t Q,$$

so that (after multiplying by $r^{N-\alpha}$)

$$
\begin{aligned}
\int_{\mathbb{R}^N} w(t) r^{N-\alpha} e_r &\le \int_{\mathbb{R}^N} w_0 r^{N-\alpha} e_r + \int_0^t \Big( \int_{\mathbb{R}^N} |g-h| r^{N-\alpha} e_r \Big) + t L_f^M \frac{\delta}{N} s_N + l \int_0^t \Big( \int_{>r} w r^{N-\alpha} e_r \Big) \\
&\le \|w_0\|_{L^1(\mathbb{R}^N)} + \int_0^t \|g-h\|_{L^1(\mathbb{R}^N)} + t L_f^M \frac{\delta}{N} s_N + l \int_0^t \|w \mathbb{I}_{>r}\|_{L^1(\mathbb{R}^N)}. \qquad (23)
\end{aligned}
$$

By Fatou's lemma as $r \to \infty$, this relation leads to the $L^1$-contraction principle

$$
\|w(t)\|_{L^1(\mathbb{R}^N)} \le \|w_0\|_{L^1(\mathbb{R}^N)} + \int_0^t \|g-h\|_{L^1(\mathbb{R}^N)} + t L_f^M \frac{\delta}{N} s_N
$$

up to some arbitrarily small $\delta$-term, provided that the function $w$ occuring in the r.h.s. of (23) is known to be integrable on $\mathbb{R}^N$. This crucial point is now to be checked.

From the second standpoint, we get

$$
(P+Q)(t) \le e^{+lt} \Big( \int_{\mathbb{R}^N} w_0 e_r + \int_0^t (D + \overline{\rho} - lP)(\tau) e^{-l\tau} d\tau \Big) \le e^{+lt} \Big( \int_{\mathbb{R}^N} w_0 e_r + \int_0^t D + t\overline{\rho} \Big),
$$

so that (after multiplying by $r^{N-\alpha}$)

$$
\begin{aligned}
\int_{\mathbb{R}^N} w(t) r^{N-\alpha} e_r &\le e^{+lt} \Big( \int_{\mathbb{R}^N} w_0 r^{N-\alpha} e_r + \int_0^t \big( \int_{\mathbb{R}^N} |g-h| r^{N-\alpha} e_r \big) + t L_f^M \frac{\delta}{N} s_N \Big) \\
&\le e^{+lt} \Big( \|w_0\|_{L^1(\mathbb{R}^N)} + \int_0^t \|g-h\|_{L^1(\mathbb{R}^N)} + t L_f^M \frac{\delta}{N} s_N \Big).
\end{aligned}
$$

By Fatou's lemma as $r \to \infty$, this relation leads to

$$
\|w(t)\|_{L^1(\mathbb{R}^N)} \le e^{+lt} \Big( \|w_0\|_{L^1(\mathbb{R}^N)} + \int_0^t \|g-h\|_{L^1(\mathbb{R}^N)} + t L_f^M \frac{\delta}{N} s_N \Big) < \infty,
$$

establishing the expected integrability of $w$. This concludes the proof of Theorem 1.1.

### 3.3 The existence claim (sketched)

For the sake of completeness, let us briefly indicate how the existence for our case of irregular $f, \varphi$ can be deduced with the known tools and existence results designed for the case of locally Lipschitz nonlinearities. For the sake of conciseness, we skip the mention of the space domain $\mathbb{R}^N$ in notations like $L^\infty(\mathbb{R}^N)$, $BV(\mathbb{R}^N)$, etc..

At the first step of the argument, approximate $f, \varphi$ uniformly on compact sets by locally Lipschitz nonlinearities $f^l, \varphi^l$ converging as $l \to \infty$, keeping the monotonicity of $\varphi$: this can be done by the standard mollification argument. Consider a countable family $\mathcal{U}_0 \subset BV \cap L^\infty$ of compactly supported data $u_0$ such that $\mathcal{U}_0$ is dense in $L^1$ for the $L^1$ topology. Analogously, consider a countable family $\mathcal{G} \subset L^1(I; BV \cap L^\infty)$ of compactly supported source terms such that $\mathcal{G}$ is dense in $L^1_{loc}(I; L^1)$ for its topology.

The existence results of Cifani, Jakobsen [20] (case without source) and Endal, Jakobsen [27] (case with source) provide for each fixed $(u_0, g) \in \mathcal{U}_0 \times \mathcal{G}$ the sequence of respective entropy solutions $(u^l_{u_0,g})_l$ of (1) with nonlinearities $f^l, \varphi^l$. Moreover, from the proofs of e.g. [20], using in particular the translation invariance of the underlying PDE and the $L^1$ contraction, one infers bounds on space and time translates of $u^l_{u_0,g}$ *independently* of $l$, which allow the application of the $C^0(0,T;L^1)$-compactness result of [30, Th. A.8]. So, a diagonal extraction permits to deduce, for a subsequence $l_k \to \infty$, the simultaneous – for all $(u_0, g) \in \mathcal{U}_0 \times \mathcal{G}$ – a.e. convergence of $u^{l_k}_{u_0,g}$ to some limits $u_{u_0,g}$. The entropy formulation of Definition 3.1 is stable under the a.e. convergence $u^{l_k}_{u_0,g} \to u_{u_0,g}$, having in mind the locally uniform in time bound given by the last line of (4)

$$\|u^l_{u_0,g}(t,\cdot)\|_\infty \le \|u_0\|_\infty + \int_0^t \|g(\tau,\cdot)\|_\infty \, d\tau.$$

By the stability results of [20, 27], the solvers $(u_0, g) \mapsto u^l_{u_0,g}$ are $L^1$-contractive and order-preserving with respect to data in $\mathcal{U}_0 \times \mathcal{G}$; this structure is preserved at the limit $l_k \to \infty$.

At the second step of the argument, the above constructed solver is extended by density to general $L^1 \cap L^\infty$ data $u_0$ and $L^1_{loc}(I; L^1 \cap L^\infty)$ sources $g$. Here again, extension by density preserves the entropy formulation of Definition 3.1, the extended solver $(u_0, g) \mapsto u_{u_0,g}$ on $L^1 \cap L^\infty$ is $L^1$-contractive and order-preserving, and the $L^\infty$ norm of the solution is controlled by $\|u_0\|_\infty + \int_0^t \|g(\tau,\cdot)\|_\infty \, d\tau$ due to the last line of (4). We now denote this solver by $S$.

At the final step of the argument, we attain general initial data $u_0 \in L^\infty$ and source terms $g \in L^1_{loc}(I; L^\infty)$ by means of the bimonotone approximation due to Ammar, Wittbold [7] by setting

$$u_0^{m,n} := (u_0)^+ \mathbb{I}_{<n} - (u_0)^- \mathbb{I}_{<m} \in L^1 \cap L^\infty,$$

$$g^{m,n} := (g)^+ \mathbb{I}_{<n} - (g)^- \mathbb{I}_{<m} \in L^1_{loc}(I; L^1 \cap L^\infty).$$

By construction, $(u_0^{m,n})_{m,n}$ is bi-monotone in the sense that it is non-decreasing in $n$ and non-increasing in $m$. Also the sequence $(g^{m,n})_{m,n}$ is constructed to be bi-monotone. The order-preservation of $S$ makes $\big(S(u_0^{m,n}, g^{m,n})\big)_{m,n}$ bi-monotone as well. Then we apply twice the monotone convergence theorem on compact subsets of $I \times \mathbb{R}^N$, in the context of the locally uniform in time $L^\infty$ control on $\big(S(u_0^{m,n}, g^{m,n})\big)_{m,n}$ (the latter is inherited from the previous steps, being understood that the truncation procedure via $\mathbb{I}_{<n}$ and $\mathbb{I}_{<m}$ does not increase the $L^\infty$ norm). We infer a.e. convergence of $S(u_0^{m,n}, g^{m,n})$ to an a.e. finite limit we call $S(u_0, g)$. As previously, we pass to the limit in the entropy formulation of Definition 3.1 and infer the existence claim.

# 4 Simpler proof of $L^1$-contraction under Hölder regularity of $\varphi$

In this section we take the assumption $(H_\varphi^{\text{bis}})$ in addition to $(H_f)$ and $(H_\varphi)$ . More precisely, we suppose that $\varphi$ is locally Hölder-continuous of exponent $s := 1 - \dfrac{\alpha}{N}$, i.e.[vi],

$$\forall M \in \mathbb{R}_+ \;\; \exists L^M_\varphi \in \mathbb{R}_+ \; : \; \forall u, v \in [-M, +M] \qquad |\varphi(u) - \varphi(v)| \le L^M_\varphi |u - v|^s.$$

We provide a simpler proof of the result (3) in Theorem 1.1. This alternative proof does not rely on the study of radial powers developped in Section 2. Instead, the idea is to use the dilations $\phi(\varepsilon\cdot)$ of a special positive test function $\phi \in W^{2,1}(\mathbb{R}^N) \cap W^{2,\infty}(\mathbb{R}^N) : 0 < \phi \le 1$ satisfying $\phi(x) = 1$ for $|x| < 1$ and $\phi(x) = 1/|x|^{N+\alpha}$ for $|x| > 2$. Note that in the case $f = 0$ (pure diffusion case) this line of analysis can be compared to [29] (in the local case) and [14] (in the fractional case).

**Lemma 4.9.** *There is a constant $\gamma$ for which $|\nabla\phi| \le \gamma\phi$ and $|A[\phi]| \le \gamma\phi$ on $\mathbb{R}^N$.*

Note that related statements can be found in [14].

[vi]In case $s \le 0$, the condition degenerates and must be understood as the mere continuity (see Remark 4.6 below).

*Proof.* For $|x| > 2$ fixed, we treat the singular part of the integral defining $A[\phi](x)$ in the spirit of Lemma 2.3 (specifically through formula (9) with $H := 1$), to get

$$|A[\phi]|(x) \leq c\int_{|h|<1}\int_0^1 \frac{|\nabla\phi(x+sh)-\nabla\phi(x)|}{|h|}ds\frac{dh}{|h|^{N-(2-\alpha)}}$$
$$+ c\int_{|h|>1}\phi(x+h)\,\mathbb{I}_{|x+h|<1}\,\frac{dh}{|h|^{N+\alpha}} + c\int_{|h|>1}\phi(x+h)\,\mathbb{I}_{|x+h|>1}\,\frac{dh}{|h|^{N+\alpha}} + c\phi(x)\int_{|h|>1}\frac{dh}{|h|^{N+\alpha}}. \quad (24)$$

The last term in the r.h.s. of (24) will lead us to the desired conclusion provided we are able to control the other terms. Since in the first integral of the r.h.s. of (24) we have

$$\frac{|\nabla\phi(x+sh)-\nabla\phi(x)|}{|h|} \leq \sup_{0<\tau<1}|\nabla^2\phi|(x+s\tau h) \leq \sup_{0<\tau<1}\frac{C}{|x+s\tau h|^{N+\alpha+2}} \quad (25)$$
$$\leq \frac{C}{(|x|-1)^{N+\alpha+2}} \leq \frac{C}{(|x|/2)^{N+\alpha+2}} \leq 2^{N+\alpha}C\phi(x), \quad (26)$$

it remains to study the second and third terms only. In the second term of the r.h.s. of (24), note that $|x+h| < 1 < |h|$ implies $|h| \geq |x| - |x+h| > |x|/2$, so

$$\int_{|h|>1}\phi(x+h)\,\mathbb{I}_{|x+h|<1}\,\frac{dh}{|h|^{N+\alpha}} = \int_{|x+h|<1}\mathbb{I}_{|h|>1}\,\frac{dh}{|h|^{N+\alpha}} \leq \int_{|x+h|<1}\frac{dh}{(|x|/2)^{N+\alpha}} = C\phi(x).$$

In the third term of the r.h.s. of (24), note that at least one of the following cases occurs: either $|x+h| \geq |x|/2$ or $|h| \geq |x|/2$. Whence

$$\int_{|h|>1}\phi(x+h)\,\mathbb{I}_{|x+h|>1}\,\frac{dh}{|h|^{N+\alpha}} \leq \Gamma\int_{\{|x+h|>1 \text{ and } |h|>1\}}\frac{dh}{|x+h|^{N+\alpha}|h|^{N+\alpha}}$$
$$\leq \Gamma\int_{|x+h|>|x|/2}\mathbb{I}_{|h|>1}\,\frac{dh}{|x+h|^{N+\alpha}|h|^{N+\alpha}} + \Gamma\int_{|h|>|x|/2}\mathbb{I}_{|x+h|>1}\,\frac{dh}{|x+h|^{N+\alpha}|h|^{N+\alpha}}$$
$$\leq \Gamma\frac{2}{(|x|/2)^{N+\alpha}}\int_{>1}\frac{1}{|\cdot|^{N+\alpha}} = C\phi(x) \qquad \text{with } \Gamma := \sup_{\mathbb{R}^N}|\cdot|^{N+\alpha}\phi.$$

All this proves the expected estimate $|A[\phi]| \leq \gamma\phi$ outside the ball of radius 2 for some constant $\gamma$. Inside the ball, in order to conclude it is enough to say that $A[\phi]$ is bounded as a consequence of the proof of Lemma 2.3.

This concludes the proof, indeed, the claim of the lemma concerning the bound on $\nabla\phi$ is obvious. □

**Remark 4.4.** *When $\alpha < 1$, the choice $\phi := \min\{1, 1/|\cdot|^{N+\alpha}\} \in W^{1,1}(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$ would be enough, since the regularization up to the second order is needless in this case.*

**Remark 4.5.** *Even if the argument just given seems simple, the result is sharp, inasmuch as the decay at infinity of $A[\phi]$ for a nonnegative (regular) $\phi$ cannot be made too strong: for $\alpha < 1$ (to simplify), it is indeed impossible to find out a non trivial element $\phi \in W^{1,1}_+(\mathbb{R}^N) - \{0\}$ for which $|\cdot|^{N+\alpha}A[\phi]$ would tend to zero at infinity ... This is in sharp contrast with the local case when $A$ is a differential operator. Thus, regarding its asymptotic behavior, $\gamma\phi$ turns out to be the optimal pointwise bound.*

For notational convenience, set

$$n := N/\alpha \quad (\text{so that } s = 1/n' \text{ in } (H_\varphi^{\text{bis}}))$$

and

$$l := \gamma \max\{L_f^M \|\phi\|_{L^1(\mathbb{R}^N)}^{1/N}, L_\varphi^M \|\phi\|_{L^1(\mathbb{R}^N)}^{1/n}\}.$$

In (10) applied with $\phi(\varepsilon\cdot)$ instead of $\phi$, let us estimate the source term and the non-local term by, respectively,

$$\begin{aligned}
\operatorname{sgn}(u-v)(g-h)\phi(\varepsilon\cdot) \ &\leq \ |g-h|, \\
|\varphi(u)-\varphi(v)|\ A[\phi(\varepsilon\cdot)] = \varepsilon^\alpha |\varphi(u)-\varphi(v)|\ A[\phi](\varepsilon\cdot) \ &\geq \ 0 \ \text{ for } |\cdot| < 1/\varepsilon,
\end{aligned}$$

the nonnegativity of $A[\phi]$ within the unit ball of $\mathbb{R}^N$ being derived from the maximality of $\phi$ there.

Now, given that $f$ and $\varphi$ are supposed Hölder-continuous of respective exponents $\sigma = 1/N'$ and $s = 1/n'$, the div-term and $A$-term at infinity may be dealt with in a completely similar way, via

$$\begin{aligned}
\varepsilon \int_{\mathbb{R}^N} \operatorname{sgn}(u-v)\Big(f(u)-f(v)\Big).\nabla_x \phi(\varepsilon\cdot) &\leq \varepsilon L_f^M \int_{\mathbb{R}^N} w^\sigma |\nabla\phi|(\varepsilon\cdot) \leq \varepsilon\gamma L_f^M \int_{>1/\varepsilon} w^\sigma \phi(\varepsilon\cdot) \\
&= \varepsilon\gamma L_f^M \int_{>1/\varepsilon} \Big(\phi(\varepsilon\cdot)\Big)^{1/N} \Big(w\phi(\varepsilon\cdot)\Big)^{1/N'} \\
&\leq \varepsilon\gamma L_f^M \|\phi(\varepsilon\cdot)\|_{L^1(\mathbb{R}^N)}^{1/N} \|\mathbb{I}_{>1/\varepsilon}\ w\phi(\varepsilon\cdot)\|_{L^1(\mathbb{R}^N)}^{1/N'} \\
&= \gamma L_f^M \|\phi\|_{L^1(\mathbb{R}^N)}^{1/N} \Big(\int_{>1/\varepsilon} w\phi(\varepsilon\cdot)\Big)^\sigma \leq l\Big(\int_{>1/\varepsilon} w\phi(\varepsilon\cdot)\Big)^\sigma, \qquad (27)
\end{aligned}$$

and likewise

$$\begin{aligned}
\varepsilon^\alpha \int_{>1/\varepsilon} |\varphi(u)-\varphi(v)|\ A[\phi](\varepsilon\cdot) &\geq -\varepsilon^\alpha \gamma L_\varphi^M \int_{>1/\varepsilon} w^s \phi(\varepsilon\cdot) = -\varepsilon^\alpha\gamma L_\varphi^M \int_{>1/\varepsilon} \Big(\phi(\varepsilon\cdot)\Big)^{1/n} \Big(w\phi(\varepsilon\cdot)\Big)^{1/n'} \\
&\geq -\varepsilon^\alpha \gamma L_\varphi^M \|\phi(\varepsilon\cdot)\|_{L^1(\mathbb{R}^N)}^{1/n} \|\mathbb{I}_{>1/\varepsilon}\ w\phi(\varepsilon\cdot)\|_{L^1(\mathbb{R}^N)}^{1/n'} \\
&= -\gamma L_\varphi^M \|\phi\|_{L^1(\mathbb{R}^N)}^{1/n} \Big(\int_{>1/\varepsilon} w\phi(\varepsilon\cdot)\Big)^s \geq -l\Big(\int_{>1/\varepsilon} w\phi(\varepsilon\cdot)\Big)^s, \qquad (28)
\end{aligned}$$

thanks to Lemma 4.9 (and to a few obvious properties of $\phi$). Basically, (27) and (28) are just two simple Hölder inequalities for integrals, w.r.t. the exponents $N$ and $n$. Altogether, this turns (10) into

$$\begin{aligned}
\int_I \Big(\int_{\mathbb{R}^N} w\phi(\varepsilon\cdot)\Big)\psi' + l\int_I \Big(\int_{>1/\varepsilon} w\phi(\varepsilon\cdot)\Big)^\sigma \psi + \int_I \Big(\int_{\mathbb{R}^N} |g-h|\phi(\varepsilon\cdot)\Big)\psi(t)dt + \psi(0)\int_{\mathbb{R}^N} w_0\phi(\varepsilon\cdot) \\
\geq \int_I \Big(\int_{\mathbb{R}^N} |\varphi(u)-\varphi(v)|\ A[\phi(\varepsilon\cdot)]\Big)\psi \\
= \varepsilon^\alpha \int_I \Big(\int_{<1/\varepsilon} |\varphi(u)-\varphi(v)|\ A[\phi](\varepsilon\cdot)\Big)\psi + \varepsilon^\alpha \int_I \Big(\int_{>1/\varepsilon} |\varphi(u)-\varphi(v)|\ A[\phi](\varepsilon\cdot)\Big)\psi \\
\geq -l\int_I \Big(\int_{>1/\varepsilon} w\phi(\varepsilon\cdot)\Big)^s \psi. \qquad (29)
\end{aligned}$$

**Remark 4.6.** *The monodimensional case $N=1$ requires here a special argument because (27) makes no sense for $N=1$. Specifically, it will be enough in this case to use the modulus of continuity (15) of $f$ (for the value*

*$M := \|f(u)-f(v)\|_{L^\infty(\mathbb{R})}$) as follows:*

$$
\begin{aligned}
\varepsilon \int_{\mathbb{R}} sgn(u-v)\Big(f(u)-f(v)\Big).\nabla_x\phi(\varepsilon\cdot) &\leq \varepsilon \int_{\mathbb{R}} |f(u)-f(v)|\,|\phi'|(\varepsilon\cdot) \\
&= \int_{\mathbb{R}} \mathbb{I}_{(w\leq\delta)}\,|f(u)-f(v)|\,\varepsilon|\phi'|(\varepsilon\cdot) + \varepsilon\int_{>1/\varepsilon} \mathbb{I}_{(w>\delta)}\,|f(u)-f(v)|\,|\phi'|(\varepsilon\cdot) \\
&\leq \omega_f(\delta)\int_{\mathbb{R}} \varepsilon|\phi'|(\varepsilon\cdot) + M\frac{\gamma}{\delta}\varepsilon\int_{>1/\varepsilon} w\phi(\varepsilon\cdot) = \omega_f(\delta)\|\phi'\|_{L^1(\mathbb{R})} + M\frac{\gamma}{\delta}\varepsilon\int_{>1/\varepsilon} w\phi(\varepsilon\cdot). \qquad (30)
\end{aligned}
$$

*The multidimensional proof developped below may then be adapted to cover the case $N=1$ as well, through replacing (27) by (30) everywhere. Once $\varepsilon\to0^+$ has been sent to zero, this only creates an extra $\delta$-term, which can easily be made small in the end (since $\omega_f(\delta)\|\phi'\|_{L^1(\mathbb{R})}\to0$ as $\delta\to0^+$).*

*In a completely similar fashion, (28) makes no sense when $s=1-\alpha/N\leq0$ i.e. when $N=1\leq\alpha$, but in this exceptionally favourable case the non-local term may be treated as previously in (30), namely*

$$
\begin{aligned}
\varepsilon^{\alpha-1}\times\varepsilon\int_{>1/\varepsilon}|\varphi(u)-\varphi(v)|\,A[\phi](\varepsilon\cdot) &\geq -\varepsilon\int_{>1/\varepsilon}|\varphi(u)-\varphi(v)|\,|A[\phi]|(\varepsilon\cdot) \\
&= -\int_{>1/\varepsilon}\mathbb{I}_{(w\leq\delta)}\,|\varphi(u)-\varphi(v)|\,\varepsilon|A[\phi]|(\varepsilon\cdot) - \varepsilon\int_{>1/\varepsilon}\mathbb{I}_{(w>\delta)}\,|\varphi(u)-\varphi(v)|\,|A[\phi]|(\varepsilon\cdot) \\
&\geq -\omega_\varphi(\delta)\int_{\mathbb{R}}\varepsilon|A[\phi]|(\varepsilon\cdot) - M\frac{\gamma}{\delta}\varepsilon\int_{>1/\varepsilon} w\phi(\varepsilon\cdot) = -\omega_\varphi(\delta)\|A[\phi]\|_{L^1(\mathbb{R})} - M\frac{\gamma}{\delta}\varepsilon\int_{>1/\varepsilon} w\phi(\varepsilon\cdot),
\end{aligned}
$$

*where $\omega_\varphi$ stands for the modulus of continuity of $\varphi$. We arrive to the same conclusion as after (30).*

In order to focus on time dependence only, let us set

$$
p := \int_{<1/\varepsilon} w\phi(\varepsilon\cdot) = \int_{<1/\varepsilon} w, \quad \text{then} \;\; q := \int_{>1/\varepsilon} w\phi(\varepsilon\cdot) \;\; \text{and finally} \;\; L := \int_{\mathbb{R}^N}|g-h|
$$

as functions of time ($t\in I$). With these notations, the inequality obtained in (29) reads

$$
\int_I (p+q)\psi' + l\int_I q^\sigma\psi + l\int_I q^s\psi + \int_I L\psi \geq -\psi(0)\int_{\mathbb{R}^N} w_0\phi(\varepsilon\cdot) \qquad \forall\psi\in W^{1,1}_+(I),
$$

and essentially expresses the differential inequation $-(p+q)'+lq^\sigma+lq^s+L\geq0$ in the sense of distributions on $I$. We shall integrate it with respect to time in two slightly different ways, viewing it, on the one hand like $(p+q)'\leq L+lq^\sigma+lq^s$, and on the other hand like

$$
\frac{d}{dt}\Big((p+q)(t)e^{-l(\sigma+s)t}\Big) \leq \Big(L + l(q^\sigma-\sigma q) + l(q^s - sq) - l(\sigma+s)p\Big)e^{-l(\sigma+s)t} \qquad (31)
$$

In the first case, we get

$$
\begin{aligned}
\int_{\mathbb{R}^N} w(t)\phi(\varepsilon\cdot) = (p+q)(t) &\leq \int_{\mathbb{R}^N} w_0\phi(\varepsilon\cdot) + \int_0^t (L+lq^\sigma+lq^s) \\
&\leq \|w_0\|_{L^1(\mathbb{R}^N)} + \int_0^t\|g-h\|_{L^1(\mathbb{R}^N)} + l\int_0^t\Big(\int_{>1/\varepsilon} w\Big)^\sigma + l\int_0^t\Big(\int_{>1/\varepsilon} w\Big)^s. \qquad (32)
\end{aligned}
$$

By Fatou's lemma as $\varepsilon \to 0$, this relation leads to the $L^1$-contraction principle (3), provided that the function $w$ occuring in the r.h.s. of (32) is known to be integrable on $\mathbb{R}^N$. This crucial point is now to be checked.

In the second case, to avoid unnecessary technicalities regarding the exact integration of (31), we choose first to convert it into a *linear* differential inequation by estimating its r.h.s. thanks to two standard inequalities of the type (13), namely $q^\sigma \le 1 + \sigma q$ and $q^s \le 1 + sq$. Consequently, we get

$$\int_{\mathbb{R}^N} w(t)\phi(\varepsilon\cdot) = (p+q)(t) \le e^{+l(\sigma+s)t}\left(\int_{\mathbb{R}^N} w_0\phi(\varepsilon\cdot) + \int_0^t (L+2l)(\tau)e^{-l(\sigma+s)\tau}d\tau\right)$$
$$\le e^{+l(\sigma+s)t}\left(\|w_0\|_{L^1(\mathbb{R}^N)} + \int_0^t \|g-h\|_{L^1(\mathbb{R}^N)} + 2tl\right),$$

and also (letting $\varepsilon \to 0$)

$$\|w(t)\|_{L^1(\mathbb{R}^N)} \le e^{+l(\sigma+s)t}\left(\|w_0\|_{L^1(\mathbb{R}^N)} + \int_0^t \|g-h\|_{L^1(\mathbb{R}^N)} + 2tl\right) < \infty,$$

which proves the desired integrability of $w$.

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