



HAL
open science

AMPLENESS OF SCHUR POWERS OF COTANGENT BUNDLES AND p -HYPERBOLICITY

Antoine Etesse

► **To cite this version:**

Antoine Etesse. AMPLENESS OF SCHUR POWERS OF COTANGENT BUNDLES AND p -HYPERBOLICITY. 2019. hal-02190039v1

HAL Id: hal-02190039

<https://hal.science/hal-02190039v1>

Preprint submitted on 21 Jul 2019 (v1), last revised 8 Jan 2021 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

AMPLENESS OF SCHUR POWERS OF COTANGENT BUNDLES AND p -HYPERBOLICITY

ANTOINE ETESSE

INTRODUCTION

In [Deb05], Debarre proved that complete intersections of sufficiently ample general hypersurfaces in complex Abelian varieties whose codimensions are at least as large as their dimensions have ample cotangent bundles. By analogy, he suggested that this result should also hold when abelian varieties are replaced by complex projective spaces. This has been recently proved by Xie [Xie18] and Brotbek–Darondeau [BD18], independently, based on ideas and explicit methods developed in [Bro16]. It is well-known that complex compact manifolds with ample cotangent bundles are complex hyperbolic in the sense of Kobayashi (Cf. [Kob13]).

More generally, for $p \geq 1$, the notion of p -infinitesimal hyperbolicity (See Section 5) generalizes the usual notion of hyperbolicity. Indeed, Kobayashi hyperbolicity is equivalent to 1-infinitesimal hyperbolicity. For complex compact manifolds, 1-infinitesimal hyperbolicity implies p -infinitesimal hyperbolicity. Thus, the ampleness of the cotangent bundle implies p -infinitesimal hyperbolicity. However, to obtain this weaker property, it is sufficient to assume the ampleness of the p th exterior power of the cotangent bundle ([Dem, Prop. 3.4]). In this work, we will extend and generalize the results of [BD18] in order to obtain ampleness of Schur powers of cotangent bundles of generic complete intersections inside projective spaces.

Exterior powers are examples of Schur powers. These are obtained by applying to vector bundles certain Schur functors associated to partitions $\lambda = (\lambda_1 \geq \dots \geq \lambda_k > 0)$ (see Sect. 1 for more details). We can actually prove the following ampleness result for such general Schur powers.

Main Theorem. *Let M be a projective variety of dimension N over an algebraically closed field \mathbf{k} of characteristic zero, equipped with a fixed very ample line bundle $\mathcal{O}_M(1) \rightarrow M$. Let $c \leq N$ be a positive integer.*

For any partition $\lambda = (\lambda_1 \geq \dots \geq \lambda_k > 0)$ with k parts such that $(k+1)c \geq N$, the λ th Schur power of the cotangent bundle of a complete intersection $X := H_1 \cap \dots \cap H_c$ of c generic hypersurfaces $H_i \in |\mathcal{O}_M(d_i)|$ with respective degrees d_i such that

$$d_i \geq \left(1 + 2c \frac{2\lambda_1 + \lambda_2 + \dots + \lambda_k}{\gcd(\lambda_1, \dots, \lambda_k)} (N + k(N - k))^{c(k+1)}\right)^2$$

is ample.

Qualitatively, the λ th Schur power of the cotangent bundle of a complete intersection of at least $N/(k+1)$ generic hypersurfaces with large enough degrees will always be ample. This codimension condition is optimal. Indeed, let us recall the following vanishing

theorem of Brückmann and Rackwitz for $M = \mathbb{P}^N$. (For a partition λ , the *conjugate* partition λ^* is defined by $\lambda_j^* := \#\{i \mid \lambda_i \geq j\}$.)

Theorem ([BR90]). *Let X be a smooth complete intersection of codimension c in \mathbb{P}^N . Let λ be a partition such that $(\lambda_1^* + \cdots + \lambda_c^*) < (N - c)$. Then one has the following vanishing result:*

$$H^0(X, S^\lambda \Omega_X) = 0.$$

Now, assume λ satisfies $(k + 1)c < N$ (recall $k = \lambda_1^*$). On the one hand, the condition of the previous theorem for the partition $m\lambda$, $m \geq c$, becomes

$$\underbrace{k + \cdots + k}_{\times c} < N - c,$$

which is satisfied, hence $H^0(X, S^{m\lambda} \Omega_X) = 0$ for $m \geq c$. On the other hand, the ampleness of $S^\lambda \Omega_X$ would imply that $S^{m\lambda} \Omega_X$ is globally generated for m large enough, by Corollary 1.7 below and Bott's formula (cf. 1.2). Accordingly, $S^\lambda \Omega_X$ cannot be ample.

- Examples.**
- (1) For $k = 1$ (symmetric powers), one recovers Xie–Brotbek–Darondeau's theorem.
 - (2) For $\lambda = (1, 1)$ and $N = 5$, a complete intersection X of two generic hypersurfaces of degrees $\geq 6 \cdot 10^{16}$ has $\bigwedge^2 \Omega_X$ ample, whereas it cannot have Ω_X ample (since $2c = 4 < 5$).

Let us now briefly sketch the proof of our main theorem. The key ingredient in order to prove the ampleness of Schur powers $S^\lambda \Omega_X$ (for X generic complete intersection of codimension c) is to use explicit extrinsic equations to construct morphisms Ψ from (total spaces of) flag bundles of the relative tangent space of the universal family $\mathcal{X} \rightarrow \mathbf{S}$ of well-chosen particular complete intersections to simpler *generically finite* families $\mathcal{Y} \rightarrow \mathbf{G}$ of subschemes in some projective spaces (see Diagram 5). The parameter space of \mathcal{Y} has a natural ample line bundle $\mathcal{O}_{\mathbf{G}}(1)$. On the one hand, Nakayama's theorem allows us to control the augmented base locus of the pullback of $\mathcal{O}_{\mathbf{G}}(1)$ to \mathcal{Y} . On the other hand, by the very construction of the morphism Ψ , global sections of $\Psi^* \mathcal{O}_{\mathbf{G}}(m)$ are global sections of $S^{cm\lambda} \Omega_X$ (See 2.3). Combining carefully this two facts, one is able to prove the sought ampleness result.

The paper is organized as follows.

Section 1 is devoted to briefly introducing Schur bundles and flag bundles, and explaining how Hartshorne's point of view on ampleness can be generalized to this setting. In this natural framework, substantial difficulties (namely plethysm phenomena and obtaining extrinsic equations) that would occur following the most naive approach disappear.

Section 2 gives a more detailed overview of the proof sketched above.

Section 3 forms the technical core of the paper. We prove in full detail preliminary results leading to the construction of the morphisms Ψ in Section 4.

In Section 4, we implement the strategy described above in order to complete the proof.

In Section 5, as a corollary, an application in p -infinitesimal hyperbolicity is given, and we prove the following.

Corollary. *Let M be a projective variety of dimension N over \mathbb{C} equipped with fixed a very ample line bundle $\mathcal{O}_M(1)$. A generic hypersurface $X := H_1 \cap \dots \cap H_c$ of codimension c , $c(k+1) \geq N$, with $H_i \in |\mathcal{O}_M(d_i)|$ of degree $d_i \geq (1 + 2c(k+1)(N+k(N-k))^{c(k+1)+1})^2$ is k -infinitesimally hyperbolic.*

This property has a geometric interpretation in term of holomorphic maps $f : \mathbb{C} \times \mathbb{B}^{p-1} \rightarrow X$: it implies that every such map must be everywhere degenerate, in the sense that the Jacobian matrix of f is nowhere of maximal rank (Cf. Section 5). For $p = 1$, one recovers the notion of hyperbolicity in the sense of Brody.

1. AMPLENESS, SCHUR BUNDLES AND FLAG BUNDLES

1.1. Ampleness. For the theory of ample line bundles, we refer to [Laz04]. We briefly recall here Hartshorne’s point of view on ampleness of vector bundles. A vector bundle E over a variety M is said *ample* if the (dual) tautological line bundle $\mathcal{O}_{\mathbb{P}(E^\vee)}(1)$ is ample on $\mathbb{P}(E^\vee)$, the projective bundle of lines in E^\vee . Let $\pi : \mathbb{P}(E^\vee) \rightarrow M$ be the natural projection. By Bott’s formula [Bot57], one has that for $m \geq 1$

$$\pi_*(\mathcal{O}_{\mathbb{P}(E^\vee)}(m)) = S^m E, \tag{1}$$

where S^m is the m th symmetric power. Accordingly, the ampleness of E is equivalent to the ampleness of the symmetric powers $S^m E$, $m \geq 1$. As we will see in the sequel, this point of view can be generalized to Schur powers, looking at appropriate associated flag bundles.

1.2. Schur bundles and flag bundles. We introduce the terminology and some facts about partitions, flag varieties, flag bundles, and Schur bundles.

Any partition $\lambda = (\lambda_1 \geq \dots \geq \lambda_k > 0)$ with k parts is associated to a *Young diagram of shape λ* , which is a collection of cells arranged in left-justified rows: the first row contains λ_1 cells, the second λ_2 , and so on (cf. Figure 1 for an example).

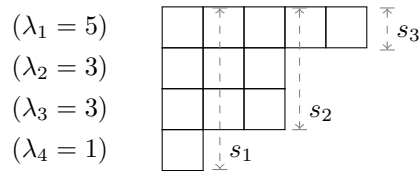


FIGURE 1. Young diagram of the partition $\lambda = (5, 3^2, 1)$, with associated jump sequence $s = (4, 3, 1)$.

By construction, if we read the Young diagram from top to bottom via the rows, we recover the partition λ ; observe that if instead we read it from left to right via the columns, we recover the conjugate partition λ^* mentioned in the introduction. There is a *jump sequence* $s = (s_1 > \dots > s_t)$ associated to a partition λ , defined by:

$$\lambda_i > \lambda_{i+1} \iff (i \in s)$$

(where by convention $\lambda_{k+1} = 0$). Equivalently, $s_1 > \dots > s_t$ are the (pairwise distinct) lengths of the parts of the conjugate partition λ^* (cf. Figure 1).

Let V be a \mathbf{k} -vector space of dimension n . For a decreasing sequence

$$s = (s_1 > s_2 > \cdots > s_t),$$

with $t \geq 1$, and $s_1 \leq n - 1$, the *partial flag variety* $\text{Flag}_s V$ of sequence s is defined as

$$\text{Flag}_s V := \{V \supseteq F_1 \supseteq \cdots \supseteq F_t \supseteq (0) \mid F_i \text{ is a } \mathbf{k}\text{-vector subspace of dimension } s_i\}.$$

On $\text{Flag}_s V$, there is a flag bundle filtration of the trivial bundle $\text{Flag}_s V \times V$ by universal subbundles $U_{s_t} \subseteq \cdots \subseteq U_{s_1}$, given over the point $\eta = (F_1 \supseteq \cdots \supseteq F_t) \in \text{Flag}_s V$ by $U_{s_i}(\eta) = F_i$, for $1 \leq i \leq s$.

Example 1.1. If we take the sequence $s = (k)$, where $k \geq 1$, then $\text{Flag}_s V$ is the Grassmannian of k -dimensional subspaces of V that we will denote $\text{Grass}(k, V)$, together with the tautological subbundle U_k .

Line bundles $\mathcal{L}_\lambda V$ on $\text{Flag}_s V$ are naturally associated to partitions λ with jump sequence s by setting:

$$\mathcal{L}_\lambda V := \bigotimes_{i=1}^t \det(U_i/U_{i+1})^{\lambda_{s_i}},$$

where by convention $U_{t+1} = 0$.

Example 1.2. For the partition $\lambda = (m^k)$, $\mathcal{L}_\lambda V$ is the m th power of the Plücker line bundle $\det(U_k)$ on $\text{Grass}(k, V)$, that allows to embed $\text{Grass}(k, V)$ in the projective space $\mathbb{P}(\bigwedge^k V)$.

Let us now take E a vector bundle of rank n over a variety M and λ a partition with jump sequence $s = (s_1 > s_2 > \cdots > s_t)$. The previous considerations make sense fiberwise, and we can glue them via a bundle structure to define:

- the projective bundle $\pi: \text{Flag}_s E^\vee \rightarrow X$, such that the fiber over $x \in M$ is $\text{Flag}_s E_x^\vee$.
- the line bundle $\mathcal{L}_\lambda E^\vee$ over $\text{Flag}_s E^\vee$, such that $(\mathcal{L}_\lambda E^\vee)|_{\text{Flag}_s E_x^\vee} = \mathcal{L}_\lambda E_x^\vee$.

Definition 1.3. The *Schur bundle* associated to a partition λ is the direct image

$$S^\lambda E := \pi_*(\mathcal{L}_\lambda E^\vee).$$

This, together with the vanishing of $R^i \pi_*(\mathcal{L}_\lambda E^\vee)$ for $i \geq 1$, is a proposition known as Bott's formulas in the literature (Cf. [Bot57]), but we take it as a definition.

Observe that for $\lambda = (m)$ one recovers (1), i.e. $S^{(m)} E = S^m E$. More generally, $S^\lambda E$ has a standard description as a quotient of products of symmetric powers, and we will now briefly sketch how to recover it from Definition 1.3.

Proposition 1.4 ([Wey03, Sect. 2.1, Theo. 4.1.8]). *The Schur power $S^\lambda E$ is a quotient of*

$$S^{\lambda_{s_1}} \left(\bigwedge^{s_1} E \right) \otimes S^{(\lambda_{s_2} - \lambda_{s_1})} \left(\bigwedge^{s_2} E \right) \otimes \cdots \otimes S^{(\lambda_{s_t} - \lambda_{s_{t-1}})} \left(\bigwedge^{s_t} E \right).$$

Proof. The partial flag bundle $\text{Flag}_s(E^\vee)$ has a natural structure of projective bundle: the product of *Plücker embeddings*

$$i: \text{Flag}_s E^\vee \longrightarrow \mathbb{P} \left(\bigwedge^{s_1} E^\vee \right) \times \cdots \times \mathbb{P} \left(\bigwedge^{s_t} E^\vee \right)$$

is fiberwise (say along $x \in M$) injective, with image a closed algebraic subset of $\mathbb{P}(\bigwedge^{s_1} E_x) \times \dots \times \mathbb{P}(\bigwedge^{s_t} E_x)$. This product of projective varieties is itself projective via the *Segre embedding*.

Observe that $\mathcal{L}_\lambda E^\vee$ can be rewritten:

$$\mathcal{L}_\lambda E^\vee = \det(U_{s_1})^{\lambda_{s_1}} \otimes \det(U_{s_2})^{\lambda_{s_2} - \lambda_{s_1}} \otimes \dots \otimes \det(U_{s_t})^{\lambda_{s_t} - \lambda_{s_{t-1}}},$$

where for $1 \leq i \leq t$, U_{s_i} is the relative version of the tautological subbundles introduced above. Now, by the very definition of the embedding i , this line bundle is actually nothing but the pullback bundle

$$\mathcal{L}_\lambda E^\vee = i^*(\mathcal{O}_{\mathbb{P}(\bigwedge^{s_1} E^\vee)}(\lambda_{s_1}) \boxtimes \mathcal{O}_{\mathbb{P}(\bigwedge^{s_2} E^\vee)}(\lambda_{s_2} - \lambda_{s_1}) \boxtimes \dots \boxtimes \mathcal{O}_{\mathbb{P}(\bigwedge^{s_t} E^\vee)}(\lambda_{s_t} - \lambda_{s_{t-1}})),$$

where \boxtimes denotes the *external tensor product*¹. Since i is a closed immersion, there is a natural surjective map

$$\mathcal{O}_{\mathbb{P}(\bigwedge^{s_1} E^\vee)}(\lambda_{s_1}) \boxtimes \mathcal{O}_{\mathbb{P}(\bigwedge^{s_2} E^\vee)}(\lambda_{s_2} - \lambda_{s_1}) \boxtimes \dots \boxtimes \mathcal{O}_{\mathbb{P}(\bigwedge^{s_t} E^\vee)}(\lambda_{s_t} - \lambda_{s_{t-1}}) \twoheadrightarrow i_* \mathcal{L}_\lambda E^\vee. \quad (2)$$

Let $p: \mathcal{O}_{\mathbb{P}(\bigwedge^{s_1} E^\vee)}(b_1) \boxtimes \dots \boxtimes \mathcal{O}_{\mathbb{P}(\bigwedge^{s_t} E^\vee)}(b_t) \rightarrow M$ be the natural projection. Noticing that $p_* i_* = \pi_*$, and applying p_* to (2) yields a map

$$S^{\lambda_{s_1}}(\bigwedge^{s_1} E) \otimes S^{(\lambda_{s_2} - \lambda_{s_1})}(\bigwedge^{s_2} E) \otimes \dots \otimes S^{(\lambda_{s_t} - \lambda_{s_{t-1}})}(\bigwedge^{s_t} E) \rightarrow S^\lambda E.$$

This map is still surjective (See e.g. the proof of Bott's theorem in [Wey03]), and its kernel constitutes the so-called *exchange relations*. One recovers the familiar description of the Schur power $S^\lambda E$ as a quotient of

$$S^{\lambda_{s_1}}(\bigwedge^{s_1} E) \otimes S^{(\lambda_{s_2} - \lambda_{s_1})}(\bigwedge^{s_2} E) \otimes \dots \otimes S^{(\lambda_{s_t} - \lambda_{s_{t-1}})}(\bigwedge^{s_t} E). \quad \square$$

1.3. Ampleness of Schur bundles. There is a natural generalization of the point of view of Hartshorne to study the ampleness of Schur powers of vector bundle. In [Dem88], Demailly proved that if E is ample on M , then $\mathcal{L}_\lambda E^\vee$ is ample on $\text{Flag}_s E^\vee$. The statement can be improved by weakening the hypothesis on E : it is enough to suppose $S^\lambda E$ ample to deduce the ampleness of $\mathcal{L}_\lambda E^\vee$. Darondeau asked whether the converse is true, namely, does the ampleness of the line bundle $\mathcal{L}_\lambda E^\vee$ imply the ampleness of the vector bundle $S^\lambda E$? In [LN18], the authors answered the question:

Proposition 1.5. ([LN18]) *Let λ be a partition with jump sequence s , and let E a vector bundle over M . The Schur bundle $S^\lambda E$ is ample on M if and only if the line bundle $\mathcal{L}_\lambda E^\vee$ is ample on $\text{Flag}_s E^\vee$.*

Example 1.6. In the case $\lambda = (1^k)$, the proposition states that $\bigwedge^k E$ is ample on M if and only if the Plücker line bundle is ample on $\text{Grass}(k, E^\vee)$.

As a corollary, one has the following:

¹Let X, Y be varieties, and let $p_1: X \times Y \rightarrow X$, $p_2: X \times Y \rightarrow Y$ be the first and second projection. If E_1 is a vector bundle over X , and E_2 is a vector bundle over Y , we denote $E_1 \boxtimes E_2 \rightarrow X \times Y$ the vector bundle

$$E_1 \boxtimes E_2 := pr_1^* E_1 \otimes pr_2^* E_2.$$

This definition generalizes to an arbitrary finite product of varieties.

Corollary 1.7. *Let λ be a partition with jump sequence s , and E a vector bundle over M . Then $S^\lambda E$ is ample if and only if $S^{m\lambda} E$ is very ample for some $m > 0$. Accordingly, $S^\lambda E$ is ample if and only if $S^{p\lambda} E$ is ample for any (some) $p \geq 1$.*

Proof. The previous proposition allows us to work with the line bundle $\mathcal{L}_\lambda E^\vee$. Noticing that $(\mathcal{L}_\lambda E^\vee)^{\otimes m} = \mathcal{L}_{m\lambda} E^\vee$, the very definition of $S^{m\lambda} E^\vee$ as the push-forward by π of $\mathcal{L}_{m\lambda} E^\vee$ yields the result. \square

1.4. Extrinsic equation of Schur powers of cotangent bundles of projective complete intersection.

The first obstacle one encounters in trying to generalize the method used in [BD18] to the case of Schur powers is in writing extrinsic equations defining the Schur power of the cotangent bundle of a complete intersection inside the Schur power of the cotangent bundle of the ambient space. To illustrate this, let $H := V(P)$ be a smooth hypersurface inside $M := \mathbb{P}^N$, $N \geq 1$, with P an homogenous polynomial of degree d . Let V be a trivializing open set of $O_M(1)$. The local equations of TH inside TM on V are as follows:

$$TH|_V = \{(x, v) \in TV \mid P(x) = 0, d(P)_V(x, v) = 0\},$$

where $(P)_V$ is the writing of P on the trivializing open set V . We see that the total space TH is of codimension 2 inside TM , and that we have two natural scalar equations defining it.

Let us continue this illustration with the case of the k th exterior power. There is a natural vectorial equation which, added to the equation $P = 0$, defines locally $\bigwedge^k TH$ inside $\bigwedge^k TM$. To see this, let us consider the linear map given on pure tensor by

$$dE: \begin{cases} \bigwedge^k TM|_V & \longrightarrow \bigwedge^{k-1} TM|_V \\ (x, v_1 \wedge \dots \wedge v_k) & \longmapsto \sum_{j=1}^k (-1)^j d(E)_V(x, v_j) v_1 \wedge \dots \wedge v_k. \end{cases}$$

This is well defined since by construction this is the map induced by a k -linear alternating form on $TM|_V^{\times k}$. If we look at the fiber of $x \in V$, one can show that $dE(x, \cdot)$ is surjective, and since its kernel contains $\bigwedge^k T_x H$, for dimension reason, it is actually equal to $\bigwedge^k T_x H$. If one restrict oneself to (non-zero) pure tensors $(x, v_1 \wedge \dots \wedge v_k)$, the vectorial equation splits into k scalar equations: $d(E)_V(x, v_1) = \dots = d(E)_V(x, v_k) = 0$. Restricting oneself to non-zero pure tensors is here equivalent to considering the grassmanian bundle $\text{Grass}(k, TM) = \text{Flag}_{(k)} TM$, and the previous scalar equations define locally $\text{Grass}(k, TH)$ inside $\text{Grass}(k, TM)$:

$$\text{Grass}(k, TH) \stackrel{loc}{=} \{(x, [v_1 \wedge \dots \wedge v_k]) \in \text{Grass}(k, TM) \mid E(x) = 0, d(E)_V(x, v_1) = \dots = d(E)_V(x, v_k) = 0\}.$$

Working with the appropriate flag variety, which is here the Grassmanian, is hence also advantageous in this perspective.

Let us finish the illustration with the case of the Schur power associated to a partition λ with jump sequence $s = (s_1 > \dots > s_t)$. With analogy to the case of exterior powers, one is interested in the equations defining locally $\text{Flag}_s(TH)$ inside $\text{Flag}_s(TM)$, which are as follows:

$$\text{Flag}_s(\mathcal{TH}) \stackrel{\text{loc}}{=} \{(x, F_1 \supseteq \dots \supseteq F_t) \in \text{Flag}_s(TM) \mid E(x) = 0, d(E)_V(x, v) = 0 \text{ for all } v \in F_1\}.$$

Following this point of view, we will generalize the method followed in [BD18], and obtain some ampleness results concerning the line bundle $\mathcal{L}_\lambda \mathcal{TH}$ over $\text{Flag}_s(\mathcal{TH})$, where $H := H_1 \cap \dots \cap H_c$ is a generic complete intersection of codimension c , with some specific constraints on the codimension and the largeness of the degrees of the hypersurfaces. With the Proposition 1.5, the ampleness of this line bundle is equivalent to the ampleness of the vector bundle $S^\lambda \Omega_H$ over H .

1.5. For the rest of the paper, we fix \mathbf{k} an algebraically closed field of characteristic 0, M a projective algebraic variety of dimension N , equipped with a fixed very ample line bundle $\mathcal{O}_M(1)$, and ξ_0, \dots, ξ_N $N + 1$ sections of $\mathcal{O}_M(1)$ in general position, i.e. each $D_i := \{\xi_i = 0\}$ is smooth, and the divisor $D = \sum_i D_i$ is simple normal crossing. In the case $M = \mathbb{P}^N$, it means that ξ_0, \dots, ξ_N are homogenous coordinates in \mathbb{P}^N . We will adopt the following multi-indexes notations. For a subset $J = (j_0, \dots, j_N) \subset \mathbb{N}^{N+1}$, the *support of J* is defined as

$$[J] := \{i \in [0, N] \mid j_i \neq 0\}$$

and the *length of J* is defined as $|J| := j_0 + \dots + j_N$. We will aswell denote

$$\boldsymbol{\xi}^J := \xi_0^{j_0} \dots \xi_N^{j_N} \in H^0(M, \mathcal{O}_M(|J|)).$$

Finally, we fix $\lambda = (\lambda_1 \geq \dots, \lambda_k > 0)$ a partition with k parts and jump sequence $s = (s_1 > s_2 > \dots > s_t)$, with $s_1 \leq (N - 1)$ and $t \geq 1$, whose dual partition writes $\lambda^* = (s_1^{b_1}, \dots, s_t^{b_t})$, $b_i \in \mathbb{N}_{\geq 1}$. As we are concerned with ampleness of certain vector bundles $S^\lambda E$, Corollary 1.7 allows us to suppose that $\gcd(\lambda_1, \dots, \lambda_k) = 1$. With the example 1.1 in mind, we will note $k = s_1$. Note the important role of k in the numerical hypothesis of our main theorem.

2. OUTLINE OF THE PROOF

We assume for simplicity until the end of this section that $M = \mathbb{P}^N$.

2.1. **Local equations of $\text{Flag}_s(\mathcal{TX}/\mathcal{S})$.** We consider complete intersections $H_1 \cap \dots \cap H_c$ of codimension c , where each H_p is a hypersurface defined by an homogenous polynomial of the form

$$E_p(\mathbf{a}^p, \cdot) := \sum_{|J|=\delta_p} a_J^p \boldsymbol{\xi}^{(r+1)J}, \tag{3}$$

where $J = (j_0, \dots, j_N) \in \mathbb{N}^{N+1}$, $\delta_p > 0$, $a_J^p \in H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(\varepsilon_p))$ with $\varepsilon_p > 0$, and $r \in \mathbb{N}$. Accordingly, $E_p(\mathbf{a}^p, \cdot) \in H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(\varepsilon_p + (r + 1)\delta_p))$. We denote, for $\delta \in \mathbb{N}$,

$$N_\delta = \dim H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(\delta)) = \#\{J \in \mathbb{N}^{N+1} \mid |J| = \delta\} = \binom{N + \delta}{N}$$

the number of parameters defining a degree δ homogenous polynomial in $N + 1$ variables, and consider

$$\mathbf{S} \subset H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(\varepsilon_1))^{N_{\delta_1}} \times \dots \times H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(\varepsilon_c))^{N_{\delta_c}}$$

the subspace of parameters $(\mathbf{a}^1, \dots, \mathbf{a}^c)$ of equations $(E_1(\mathbf{a}^1, \cdot), \dots, E_c(\mathbf{a}^c, \cdot))$ as above defining *smooth* complete intersections. We denote $\mathcal{X} \rightarrow S$ the universal family of such complete intersections, defined in $S \times \mathbb{P}^N$ by the universal equations (E_1, \dots, E_c) , and consider the flag bundle of the relative tangent bundle $T\mathcal{X}/S$

$$\pi_{\mathcal{X}} : \text{Flag}_s(T\mathcal{X}/S) \rightarrow \mathcal{X}.$$

As we saw in 1.4, local equations for $\text{Flag}_s(T\mathcal{X}/S) \subset \mathbf{S} \times \text{Flag}_s T\mathbb{P}^N$ are given via the universal equations (E_1, \dots, E_c) and their relative differentials $(d_{\mathbb{P}^N} E_1, \dots, d_{\mathbb{P}^N} E_c)$:

$$\text{Flag}_s(T\mathcal{X}/S) \stackrel{\text{loc}}{=} \{((\mathbf{a}^1, \dots, \mathbf{a}^c), (x, F_1 \supseteq \dots \supseteq F_t)) \in \mathbf{S} \times \text{Flag}_s(T\mathbb{P}^N) \mid E_1(\mathbf{a}^1, x) = \dots = E_c(\mathbf{a}^c, x) = 0, d_{\mathbb{P}^N} E_1(\mathbf{a}^1, x, \cdot)|_{F_1} = \dots = d_{\mathbb{P}^N} E_c(\mathbf{a}^c, x, \cdot)|_{F_1} = 0\}.$$

The equations (3) are such that their differentials can formally be written in the same form; locally on \mathbb{P}^N , for $1 \leq p \leq c$, one can write:

$$E_p(\mathbf{a}^p, \cdot) = \sum_{|J|=\delta_p} \alpha_J^p(\mathbf{a}^p, \cdot)(\xi^r)^J \quad \text{and} \quad d_{\mathbb{P}^N} E_p(\mathbf{a}^p, \cdot) \stackrel{\text{loc}}{=} \sum_{|J|=\delta_p} \theta_J^p(\mathbf{a}^p, \cdot)(\xi^r)^J, \quad (4)$$

where the notation $\stackrel{\text{loc}}{=}$ emphasizes that it makes senses only locally, after restricting oneself to an open subset V that trivializes $\mathcal{O}_{\mathbb{P}^N}(1)|_V$. Here, the coefficients α_J^p are homogenous polynomials of degree $\varepsilon_p + \delta_p$, and the coefficients θ_J^p are 1-form on V depending on the choice of the trivialization for $\mathcal{O}_{\mathbb{P}^N}(1)|_V$ (See subsection 3.1).

2.2. The morphism Ψ . After making the change of variable $z_i = \xi_i^r$ for $0 \leq i \leq N$, the equations (4) become

$$\sum_{|J|=\delta_p} \alpha_J^p \mathbf{z}^J \quad \text{and} \quad \sum_{|J|=\delta_p} \theta_J^p \mathbf{z}^J \quad \text{for } 1 \leq p \leq c.$$

We fix $1 \leq p \leq c$, and consider this time α_J^p and θ_J^p as variables in \mathbf{k} . As mentioned earlier, we implicitly assume that we restrict ourselves to a trivializing open set V of $\mathcal{O}_{\mathbb{P}^N}(1)$. One can then form, for $\mathbf{a}^p \in S_p := H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(\varepsilon_p))^{N_{\delta_p}}$, $x \in V$ and $v_1, \dots, v_k \in T_x \mathbb{P}^N$, the following matrix

$$A_p(\mathbf{a}^p; (x; v_1, \dots, v_k)) = \begin{pmatrix} ((\alpha_J^p)(\mathbf{a}^p, x))_J \\ (\theta_J^p(\mathbf{a}^p, x, v_1))_J \\ \cdot \\ \cdot \\ \cdot \\ (\theta_J^p(\mathbf{a}^p, x, v_k))_J \end{pmatrix},$$

which is of size $(k+1) \times N_{\delta_p}$. Let us suppose that $A_p(\mathbf{a}^p; (x; v_1, \dots, v_k))$ is of maximal rank, namely $k+1$ (this condition does not depend on the trivialization chosen to write the matrix). In this case, $\eta = (F_1 \supseteq \dots \supseteq F_t)$, where F_i is the subvector space of $\mathbf{k}^{N_{\delta_p}}$ spanned by the $s_i + 1$ first rows of the matrix, defines an element of $\text{Flag}_{s_+} \mathbf{k}^{N_{\delta_p}}$, where $s_+ = (s_1 + 1 > \dots > s_t + 1)$. One can then consider the coordinates of $\eta \in \text{Flag}_{s_+} \mathbf{k}^{N_{\delta_p}}$, which are given in term of some minors of the matrix. Now, one would like to be able to make the previous considerations on the whole space \mathbb{P}^N , for \mathbf{a}^p belonging to an open set of the space of parameters: one can show that it is possible as soon as δ_p is taken

large enough, and it is the object of 3.3. We denote $S_p^\circ \subset S_p$ such an open set. Let us mention that here lies the reason why $k = s_1$ plays a particular role: the condition so as to make the previous reasoning depends on k only, and not on the whole partition λ .

Suppose \mathbf{a}^p is in S_p° . One can show (cf. 3.2) that the right kind of product of coordinates of η can be interpreted as a *global* section of $\mathcal{L}_\lambda T\mathbb{P}^N$, tensorized by a power of $\pi_{\mathbb{P}^N}^* \mathcal{O}_{\mathbb{P}^N}(1)$, where $\pi_{\mathbb{P}^N} : \text{Flag}_s T\mathbb{P}^N \rightarrow \mathbb{P}^N$ is the natural projection. Considering every section arising in this way allows one to define a *morphism* $\varphi_p(\mathbf{a}^p, \cdot) : \text{Flag}_s T\mathbb{P}^N \rightarrow \mathbb{P}^M$. Here M is a certain natural number. This morphism depends algebraically on S_p° , so one can see it as a morphism

$$\varphi_p : S_p^\circ \times \text{Flag}_s T\mathbb{P}^N \rightarrow \mathbb{P}^M.$$

Let us be more specific on the image of the morphism: by construction, φ_p takes values in $\text{embed}_{\delta_p}(\text{Flag}_{s_+} \mathbf{k}^{N_{\delta_p}}) := G_{\delta_p}$, where

$$\text{embed}_{\delta_p} : \left| \begin{array}{ccc} \text{Flag}_{s_+} \mathbf{k}^{N_{\delta_p}} & \longrightarrow & \text{Grass}(s_1 + 1, \mathbf{k}^{N_{\delta_p}})^{b_1} \times \dots \times \text{Grass}(s_t + 1, \mathbf{k}^{N_{\delta_p}})^{b_t} \\ (\Delta_1 \supseteq \Delta_2 \supseteq \dots \supseteq \Delta_t) & \longmapsto & (\Delta_1, \dots, \Delta_1, \Delta_2, \dots, \Delta_2, \dots, \Delta_t, \dots, \Delta_t) \end{array} \right.$$

and accordingly:

$$\varphi_p : S_p^\circ \times \text{Flag}_s T\mathbb{P}^N \rightarrow G_{\delta_p}.$$

Define $\mathbf{S}^\circ := S_1^\circ \times \dots \times S_c^\circ$ and $\mathbf{G} := G_{\delta_1} \times \dots \times G_{\delta_c}$, and construct the morphism

$$\Psi : \left| \begin{array}{ccc} \mathbf{S}^\circ \times \text{Flag}_s T\mathbb{P}^N & \longrightarrow & \mathbf{G} \times \mathbb{P}^N \\ (\mathbf{a}^\cdot, \eta) & \longmapsto & ((\varphi_1(\mathbf{a}^1, \eta), \dots, \varphi_p(\mathbf{a}^c, \eta)), [\xi_0^r(\pi_{\mathbb{P}^N}(\eta)) : \dots : \xi_N^r(\pi_{\mathbb{P}^N}(\eta))]) \end{array} \right.$$

where $\mathbf{a}^\cdot = (\mathbf{a}^1, \dots, \mathbf{a}^c)$. By construction, $\Psi(\text{Flag}_s(T\mathcal{X}_{/\mathbf{S}^\circ}))$ factors through $\mathcal{Y} \subset \mathbf{G} \times \mathbb{P}^N$ defined as:

$$\mathcal{Y} := \{((\Delta_1^1, \dots, \Delta_t^1) \times \dots \times (\Delta_1^c, \dots, \Delta_t^c), z) \in \mathbf{G} \times \mathbb{P}^N \mid \forall 1 \leq i \leq c, \Delta_1^i(z) = 0\}$$

where $\langle \Delta_1^i(z) = 0 \rangle$ means that for every vector $v = (v_J)_{|J|=\delta_i} \in \Delta_1^i$, $\sum_{|J|=N_{\delta_i}} v_J z^J = 0$.

To sum up, we obtain the following diagram:

$$\begin{array}{ccc} \text{Flag}_s(T\mathcal{X}_{/\mathbf{S}^\circ}) & \xrightarrow{\Psi} & \mathcal{Y} \longrightarrow \mathbb{P}^N \\ \downarrow & & \downarrow \rho \\ \mathbf{S}^\circ & & \mathbf{G} \end{array} \quad (5)$$

where ρ is the first projection.

2.3. The model situation. The product $\mathbf{G} = G_{\delta_1} \times \dots \times G_{\delta_c}$ is naturally endowed with an ample line bundle $\mathcal{O}_{\mathbf{G}}((1, \dots, 1)) = \mathcal{O}_{G_{\delta_1}}(1) \boxtimes \dots \boxtimes \mathcal{O}_{G_{\delta_c}}(1)$. Denote $\mathcal{O}_{\mathcal{X}}(1) := (\mathcal{O}_{\mathbf{S}} \boxtimes \mathcal{O}_{\mathbb{P}^N}(1))|_{\mathcal{X}}$; by construction of the morphism Ψ , the pullback by Ψ of $(\mathcal{O}_{\mathbf{G}}(\mathbf{m}) \boxtimes \mathcal{O}_{\mathbb{P}^N}(-1))|_{\mathcal{Y}}$, where $\mathbf{m} = (m_1, \dots, m_c) \in \mathbb{N}^c$, is of the form

$$\Psi^*((\mathcal{O}_{\mathbf{G}}(\mathbf{m}) \boxtimes \mathcal{O}_{\mathbb{P}^N}(-1))|_{\mathcal{Y}}) = \mathcal{L}_\lambda T\mathcal{X}_{/\mathbf{S}^\circ}^{\otimes |\mathbf{m}|} \otimes \pi_{\mathcal{X}}^* \mathcal{O}_{\mathcal{X}}(-r + C(\lambda, \boldsymbol{\varepsilon}, \boldsymbol{\delta}, \mathbf{m})), \quad (6)$$

where $C(\lambda, \boldsymbol{\varepsilon}, \boldsymbol{\delta}, \mathbf{m})$ is a constant that depends on the partition λ , the c -uples $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_c)$, $\boldsymbol{\delta} = (\delta_1, \dots, \delta_c)$ and $\mathbf{m} = (m_1, \dots, m_c)$.

Suppose now that $c(k+1) \geq N$ so that the morphism ρ is generically finite. Accordingly, the pullback $\rho^* \mathcal{O}_{\mathbf{G}}((1, \dots, 1))$ of the ample line bundle $\mathcal{O}_{\mathbf{G}}((1, \dots, 1))$ is big

and nef. One can obtain a geometric control of the augmented base locus $\text{Bs}((\mathcal{O}_{\mathbf{G}}(\mathbf{m}) \boxtimes \mathcal{O}_{\mathbb{P}^N}(-1))|_{\mathbf{Y}})$ for \mathbf{m} with large enough components via Nakayama's theorem, but it does not provide an explicit bound. Instead, one can use an «effective Nakayama's theorem» ([Den16], and cf. 4.3.1) in order to obtain that for \mathbf{m} with large enough components (the bound depending on $\delta_1, \dots, \delta_c$):

$$\text{Bs}((\mathcal{O}_{\mathbf{G}}(\mathbf{m}) \boxtimes \mathcal{O}_{\mathbb{P}^N}(-1))|_{\mathbf{Y}}) \subset \text{Exc}(\rho) \quad (7)$$

where $\text{Exc}(\rho)$ is the reunion of all positive dimensional fibers of the morphism $\rho : \mathbf{Y} \rightarrow \mathbf{G}$.

2.4. Pulling back the positivity from \mathbf{Y} to $\text{Flag}_s(\mathbf{T}\mathcal{X}/S^\circ)$. Suppose $\mathbf{m} = (m_1, \dots, m_c)$ have large enough components so that (7) is valid. The next and final step is to investigate the base locus of the pull-back by Ψ of $\mathcal{O}_{\mathbf{G}}(\mathbf{m}) \boxtimes \mathcal{O}_{\mathbb{P}^N}(-1)|_{\mathbf{Y}}$. This is the object of 4.2, which shows in particular that for δ large enough and for a general \mathbf{a}^* in S° , any curve $\mathcal{C} \subset \text{Flag}_s TH_{\mathbf{a}^*}$ satisfies $\Psi(\mathcal{C}) \not\subset \text{Exc}(\rho)$, and a fortiori with (7) satisfies

$$\Psi(\mathcal{C}) \not\subset \text{Bs}((\mathcal{O}_{\mathbf{G}}(\mathbf{m}) \boxtimes \mathcal{O}_{\mathbb{P}^N}(-1))|_{\mathbf{Y}}). \quad (8)$$

Let us denote

$$\mathcal{Q}_{\mathbf{a}^*} := \mathcal{L}_\lambda TH_{\mathbf{a}^*}^{|\mathbf{m}|} \otimes \pi_{\mathbf{a}^*}^* \mathcal{O}_{H_{\mathbf{a}^*}}(-r + C(\lambda, \varepsilon, \delta, \mathbf{m})) \quad (9)$$

where \mathbf{a}^* belongs to S° , and $\pi_{\mathbf{a}^*} : \text{Flag}_s(TH_{\mathbf{a}^*}) \rightarrow H_{\mathbf{a}^*}$ is the natural projection. In view of (6) and (8), any curve $\mathcal{C} \subset \text{Flag}_s TH_{\mathbf{a}^*}$ satisfies

$$\mathcal{Q}_{\mathbf{a}^*} \cdot \mathcal{C} \geq 0.$$

This shows that $\mathcal{Q}_{\mathbf{a}^*}$ is nef. With (9), if one picks $r > C(\lambda, \varepsilon, \delta, \mathbf{m})$, $\mathcal{L}_\lambda TH_{\mathbf{a}^*}^{\otimes |\mathbf{m}|}$ writes as the sum of a nef line bundle with the pull-back by $\pi_{\mathbf{a}^*}^*$ of an ample line bundle. As $\mathcal{L}_\lambda TH_{\mathbf{a}^*}^{\otimes |\mathbf{m}|}$ is ample along every fiber of $\pi_{\mathbf{a}^*}$ (since it restricts to $\mathcal{L}_\lambda T_x H_{\mathbf{a}^*}^{\otimes |\mathbf{m}|}$ if we look at the fiber over $x \in M$), we deduce that $\mathcal{L}_\lambda TH_{\mathbf{a}^*}^{\otimes |\mathbf{m}|}$ is ample, and so is $\mathcal{L}_\lambda TH_{\mathbf{a}^*}$. By the openness property of ampleness, and the Proposition 1.5, it implies the main theorem.

However, as we will see in the sequel during the proof, technical difficulties prevent us from applying directly what was said above. Instead of working globally, we will be forced to work on the stratification of $\text{Flag}_s(\mathbb{P}^N)$ induced by the hyperplane coordinates.

3. NOTATIONS AND PRELIMINARY CONSTRUCTIONS

3.1. Notations. Let $\varepsilon \geq 1$, $\delta \geq 1$, $r \geq 1$, and $S := H^0(M, \mathcal{O}_M(\varepsilon))^{N_\delta}$, where $N_\delta = \binom{N+\delta}{N}$. For $\mathbf{a} = (a_J)_{J \subset \mathbb{N}^{N+1}, |J|=\delta} \in S$, let us form the following bihomogenous section of $\mathcal{O}_M(\varepsilon + (r+1)\delta)$

$$E(\mathbf{a}, \cdot) : x \mapsto \sum_{|J|=\delta} a_J(x) \boldsymbol{\xi}(x)^{(r+1)J},$$

where we recall the multi-indexes notations $\boldsymbol{\xi}(x)^J = \xi_0(x)^{j_0} \dots \xi_N(x)^{j_N}$ where $J = (j_0, \dots, j_N) \in \mathbb{N}^{N+1}$. Let us write $E(\mathbf{a}, \cdot)$ under the following form

$$\begin{aligned} E(\mathbf{a}, \cdot) &= \sum_{|J|=\delta} [a_J \boldsymbol{\xi}^J] \boldsymbol{\xi}^{rJ} \\ &= \sum_{|J|=\delta} \alpha_J(\mathbf{a}, \cdot) \boldsymbol{\xi}^{rJ} \end{aligned}$$

where $\alpha_J(\mathbf{a}, \cdot) = a_J \boldsymbol{\xi}^J \in H^0(M, \mathcal{O}_M(\varepsilon + \delta))$. We can then write, formally, that

$$d(E(\mathbf{a}, \cdot)) = \sum_{|J|=\delta} \theta_J(\mathbf{a}, \cdot) \boldsymbol{\xi}^{rJ},$$

where $\theta_J(\mathbf{a}, \cdot) = \boldsymbol{\xi}^J da_J + (r+1)a_J d\boldsymbol{\xi}^J$. Let us *insist* on the fact that the previous equality is formal: it makes sense only when writing the sections on a trivializing open set of $\mathcal{O}_M(1)$.

Recall that we denote $\pi_M : \text{Flag}_s(TM) \rightarrow M$ the projection on M . We will sometimes write, for sake of notation, $s(\eta)$ for $s(\pi_M(\eta))$, where $\eta \in \text{Flag}_s(TM)$ and s is a section of a line bundle on M .

3.2. Definition of a preliminary morphism. Let us denote, for \mathbf{a} in the space of parameters S , $H_{\mathbf{a}}$ the *smooth* hypersurface in M defined by the equation $(E(\mathbf{a}, \cdot) = 0)$. In the spirit of [BD18], the first step is to define a preliminary morphism $\Delta(\mathbf{a}, \cdot) : \text{Flag}_s(TM) \rightarrow G$ (G will be defined later), that depends algebraically on \mathbf{a} , which

- keeps track of the equations defining $\text{Flag}_s(TH_{\mathbf{a}})$ inside $\text{Flag}_s(TM)$ by its very definition.
- turns out to be a morphism for $\mathbf{a} \in S^\circ \subset S$, where S° is non-empty open set of S .
- is such that G possesses an ample line bundle whose pull-back by $\Delta(\mathbf{a}, \cdot)$ gives $\mathcal{L}_\lambda TM$ tensorized by a power of $\pi_M^* \mathcal{O}_M(1)$.

Let V be an open set that trivializes $\mathcal{O}_M(1)$. Let us consider, for $x \in V$, the following map from $T_x M$ to \mathbf{k}^{N_δ}

$$f_x : T_x M \rightarrow \mathbf{k}^{N_\delta}, \quad v \mapsto (\theta_J(\mathbf{a}, x, v))_J,$$

where we implicitly assume that θ_J is written in the trivialization chosen.

For $\mu = (\mu_1, \dots, \mu_s)$ a partition, we will denote $\mu_+ = (\mu_1 + 1, \dots, \mu_s + 1)$. Let us form the following map F_x from $\text{Flag}_s(T_x M)$ to $\text{Flag}_{s_+} \mathbf{k}^{N_\delta}$

$$F_x : \text{Flag}_s(T_x M) \rightarrow \text{Flag}_{s_+} \mathbf{k}^{N_\delta}, \quad (F_1 \supseteq \dots \supseteq F_t) \mapsto (f_x F_1 + \mathbf{k}v \supseteq \dots \supseteq f_x F_t + \mathbf{k}v),$$

where $v = (\alpha_J(\mathbf{a}, x))_J \in \mathbf{k}^{N_\delta}$. This is well defined only if for every free family (v_1, \dots, v_k) of $T_x M$, $((\alpha_J(\mathbf{a}, x))_J, f_x v_1, \dots, f_x v_k)$ remains a free family of \mathbf{k}^{N_δ} . Equivalently, it is well defined if the following $(k+1) \times N_\delta$ matrix

$$A(\mathbf{a}; (x; v_1, \dots, v_k)) = \begin{pmatrix} ((\alpha_J)(\mathbf{a}, x))_J \\ (\theta_J(\mathbf{a}, x, v_1))_J \\ \vdots \\ (\theta_J(\mathbf{a}, x, v_k))_J \end{pmatrix}$$

is of maximal rank for each free family $(v_1, \dots, v_k) \in T_x M$. Let us then consider the following condition

(*) $\forall x \in M, \forall (v_1, \dots, v_k)$ free family in $T_x M$, $A(\mathbf{a}; (x, v_1, \dots, v_k))$ is of maximal rank.

Observe that it makes sense since the rank does not depend on the trivialization chosen to form the matrix: any other matrix formed with another trivialization would have the

same rank. As we will be interested in coordinates in $\text{Flag}_{s_+} \mathbf{k}^{N_s}$, we make the following definition:

Definition 3.1. Let k, N be natural numbers, and let $A \in \mathcal{M}_{(k+1) \times N}(\mathbf{k})$ be a matrix of rank $k + 1$. We will call *Plücker coordinates of size $l \leq k + 1$* the determinant of a matrix obtained from A by keeping the l first rows, and keeping l distinct columns.

Let us now state a lemma that will allow us to define the wanted preliminary morphism:

Lemma 3.2. Let L be a line bundle on M . Let $l \geq 2$, and let $\omega_1, \dots, \omega_l$ be global sections of L . The following map:

$$\bigwedge^{l-1} TM \rightarrow \mathbf{k}, \quad (x, v_1 \wedge \dots \wedge v_{l-1}) \mapsto \det \begin{pmatrix} \omega_1(x) & \dots & \omega_l(x) \\ d\omega_1(x, v_1) & \dots & d\omega_l(x, v_1) \\ \vdots & & \vdots \\ d\omega_1(x, v_{l-1}) & \dots & d\omega_l(x, v_{l-1}) \end{pmatrix}$$

defines a global section of $(\bigwedge^{l-1} TM)^* \otimes L^l$.

Proof. For V an open set that trivializes the line bundle L , and ω a global section of L , we will denote $(\omega)_V$ the representation of the section ω on the trivializing open set V . Let us fix V and V' two open sets of M that trivializes the line bundle L , and let us denote $g_{VV'}$ the transition map. Let us fix $x \in V \cap V'$. We have that, on $V \cap V'$, for every $1 \leq i \leq l - 1$:

$$(\omega_i)_V = g_{VV'}(\omega_i)_{V'} \text{ and } d(\omega_i)_V = g_{VV'}d(\omega_i)_{V'} + (\omega_i)_{V'}dg_{VV'}.$$

Let us denote, for $v_1, \dots, v_{l-1} \in T_x M^{l-1}$,

$$B_V(x; v_1, \dots, v_{l-1}) = \begin{pmatrix} (\omega_1)_V(x) & \dots & (\omega_l)_V(x) \\ d(\omega_1)_V(x, v_1) & \dots & d(\omega_l)_V(x, v_1) \\ \vdots & & \vdots \\ d(\omega_1)_V(x, v_{l-1}) & \dots & d(\omega_l)_V(x, v_{l-1}) \end{pmatrix}.$$

We prove by induction on $l \geq 2$ that, for every $v_1, \dots, v_{l-1} \in T_x M^{l-1}$, the following equality is satisfied:

$$\det(B_V(x; v_1, \dots, v_{l-1})) = g_{VV'}^l \det(B_{V'}(x; v_1, \dots, v_{l-1})). \quad (10)$$

For $l = 2$, it follows from a straightforward computation. Fix $l > 2$, and suppose the equality (10) true for the previous ranks. Developing $\det(B_V(x; v_1, \dots, v_{l-1}))$ according to the last row and using the induction hypothesis, we get:

$$\det(B_V(x; v_1, \dots, v_{l-1})) = \sum_{i=1}^l (-1)^{l+i} d(\omega_i)_V(x, v_{l-1}) g_{VV'}^{l-1} \det(B_{V'}^{(l,i)}(x; v_1, \dots, v_{l-1})),$$

where $B_{V'}^{(l,i)}$ is the matrix obtained from $B_{V'}$ by suppressing the l th row and the i th column. The previous equality can be written as

$$\det(B_V(x; v_1, \dots, v_{l-1})) = g_{VV'}^{l-1} \det \begin{pmatrix} (\omega_1)_{V'}(x) & \dots & (\omega_l)_{V'}(x) \\ d(\omega_1)_{V'}(x, v_1) & \dots & d(\omega_l)_{V'}(x, v_1) \\ \vdots & & \vdots \\ d(\omega_1)_{V'}(x, v_{l-2}) & \dots & d(\omega_l)_{V'}(x, v_{l-2}) \\ d(\omega_1)_{V'}(x, v_{l-1}) & \dots & d(\omega_l)_{V'}(x, v_{l-1}) \end{pmatrix}.$$

Using the determinant by blocks formula, the right side of the equality writes:

$$\sum_{i < j} (-1)^{l+i+j} g_{VV'}^{l-1} \det \begin{pmatrix} (\omega_i)_{V'}(x) & (\omega_j)_{V'}(x) \\ d(\omega_i)_{V'}(x, v_{l-1}) & d(\omega_j)_{V'}(x, v_{l-1}) \end{pmatrix} \det(B_{V'}^{(1,i),(l,j)}(x; v_1, \dots, v_{l-1}))$$

where $B_{V'}^{(1,i),(l,j)}$ is the matrix obtained from $B_{V'}$ by suppressing the 1st and i th row as well as the i th and the j th column. It rewrites

$$\sum_{i < j} (-1)^{l+i+j} g_{VV'}^{l-2} \det \begin{pmatrix} (\omega_i)_V(x) & (\omega_j)_V(x) \\ d(\omega_i)_V(x, v_{l-1}) & d(\omega_j)_V(x, v_{l-1}) \end{pmatrix} \det(B_{V'}^{(1,i),(l,j)}(x; v_1, \dots, v_{l-1})),$$

Using the case $l = 2$, and the reverse of the previous determinant by blocks formula yields the equation (10), and concludes the induction.

Now,

$$T_x M^{l-1} \rightarrow \mathbf{k}, \quad (v_1, \dots, v_{l-1}) \mapsto \det(B_V(x; v_1, \dots, v_{l-1}))$$

is a $(l-1)$ -linear alternating form, so it induces a linear map from $\bigwedge^{l-1} T_x M$ to \mathbf{k} . From this, we infer that

$$\bigwedge^{l-1} TM|_V \rightarrow \mathbf{k}, \quad (x, v_1 \wedge \dots \wedge v_{l-1}) \mapsto \det(B_V(x; v_1, \dots, v_{l-1}))$$

defines a section of $(\bigwedge^{l-1} TM)|_V^*$. And the previous calculation shows exactly that those sections glue together to give a global section of $(\bigwedge^{l-1} TM)^* \otimes L^l$. \square

As an immediate application of this lemma, we have the following two corollaries:

Corollary 3.3. *A Plücker coordinate of size $l \leq k+1$ of the matrix $A(\mathbf{a}; (x; v_1, \dots, v_k))$ defines a global section of $(\bigwedge^{l-1} TM)^* \otimes \mathcal{O}_M(l(\varepsilon + \delta))$.*

Proof. Apply the previous lemma to l distinct sections $(a_{J_i} \xi^{(r+1)J_i})_{1 \leq i \leq l} \in H^0(M, \mathcal{O}_M(\varepsilon + (r+1)\delta))$, and write $(B)_V$ the matrix as in the proof of the previous lemma. If we denote $(P)_V$ the matrix obtained from $(A)_V(\mathbf{a}; \cdot)$ by keeping the columns corresponding to the indexes $(J_i)_{1 \leq i \leq l}$, observe that

$$(B)_V = (\xi^{rJ})_V (P)_V,$$

which implies that $\det(B)_V = (\xi^{lrJ})_V \det(P)_V$. From this, we deduce as in the previous lemma that the local sections $\det(P)_V$ of $\bigwedge^{l-1} TM|_V^*$ glue together to give a global section of $\bigwedge^{l-1} TM^* \otimes \mathcal{O}_M(l(\varepsilon + (r+1)\delta) - lr\delta) = \bigwedge^{l-1} TM^* \otimes \mathcal{O}_M(l(\varepsilon + \delta))$. \square

Corollary 3.4. *Let $p_1^1, \dots, p_1^{b_1}, \dots, p_t^1, \dots, p_t^{b_t}$ be a collection Plücker coordinates of size respectively $s_1 + 1, \dots, s_t + 1$ of the matrix*

$$A(\mathbf{a}; (x; v_1, \dots, v_k)) = \begin{pmatrix} ((\alpha_J)(\mathbf{a}, x))_J \\ (\theta_J(\mathbf{a}, x, v_1))_J \\ \vdots \\ \vdots \\ (\theta_J(\mathbf{a}, x, v_k))_J \end{pmatrix}.$$

Then $p_1^1 \otimes \dots \otimes p_1^{b_1} \otimes \dots \otimes p_t^1 \otimes \dots \otimes p_t^{b_t}$ defines a global section of $[(\wedge^{s_1} TM^*)^{\otimes b_1} \otimes \dots \otimes (\wedge^{s_t} TM^*)^{\otimes b_t}] \otimes \mathcal{O}_M(|\lambda_+^*|(\varepsilon + \delta))$.

Keeping the notation of the previous corollary, the section $p_1^1 \otimes \dots \otimes p_1^{b_1} \otimes \dots \otimes p_t^1 \otimes \dots \otimes p_t^{b_t}$ induces a section of $\mathcal{L}_\lambda TM \otimes \pi_M^* \mathcal{O}_M(|\lambda_+^*|(\varepsilon + \delta))$ (See e.g. Propostion 1.4).

Let us suppose that \mathbf{a} is such that the condition $(*)$ holds, and let us consider all the sections of $\mathcal{L}_\lambda TM \otimes \pi_M^* \mathcal{O}_M(|\lambda_+^*|(\varepsilon + \delta))$ we can create in the way just described. The fact that $(*)$ is satisfied implies that the sections cannot vanish simultaneously on a given point of $\text{Flag}_s(TM)$, and accordingly we can create with them in the usual way a morphism

$$\Delta(\mathbf{a}, \cdot): \text{Flag}_s(TM) \rightarrow \text{Grass}(s_1 + 1, \mathbf{k}^{N_\delta})^{b_1} \times \dots \times \text{Grass}(s_t + 1, \mathbf{k}^{N_\delta})^{b_t}.$$

Note here that we identify $\text{Grass}(s_1 + 1, \mathbf{k}^{N_\delta})^{b_1} \times \dots \times \text{Grass}(s_t + 1, \mathbf{k}^{N_\delta})^{b_t}$ with its image by the Segre embedding σ .

Observe that the image of $\Delta(\mathbf{a}, \cdot)$ can in fact factor through $G_\delta := \text{embed}_\delta(\text{Flag}_{s_+} \mathbf{k}^{N_\delta})$, where embed_δ is the embedding of $\text{Flag}_{s_+} \mathbf{k}^{N_\delta}$ inside $\text{Grass}(s_1 + 1, \mathbf{k}^{N_\delta})^{b_1} \times \dots \times \text{Grass}(s_t + 1, \mathbf{k}^{N_\delta})^{b_t}$ given by

$$\text{embed}_\delta: (\Delta_1 \supseteq \Delta_2 \supseteq \dots \supseteq \Delta_t) \mapsto (\Delta_1, \dots, \Delta_1, \Delta_2, \dots, \Delta_2, \dots, \Delta_t, \dots, \Delta_t).$$

Indeed, let us fix $\eta = (x, F_1 \supseteq \dots \supseteq F_t)$ an element of $\text{Flag}_s TM$, and let us consider the section s obtained via $p_1^1 \otimes \dots \otimes p_1^{b_1} \otimes \dots \otimes p_t^1 \otimes \dots \otimes p_t^{b_t}$. By construction, $s(\eta)$ is a coordinate of $\text{embed}_\delta(F_x(\eta))$ in $\text{Grass}(s_1 + 1, \mathbf{k}^{N_\delta})^{b_1} \times \dots \times \text{Grass}(s_t + 1, \mathbf{k}^{N_\delta})^{b_t}$. Considering all the sections of the previous form is equivalent to considering all the coordinates of $\text{embed}_\delta(F_x(\eta))$ in $\text{Grass}(s_1 + 1, \mathbf{k}^{N_\delta})^{b_1} \times \dots \times \text{Grass}(s_t + 1, \mathbf{k}^{N_\delta})^{b_t}$, hence the factorization.

For $\mathbf{u} = (u_1^1, \dots, u_1^{b_1}, \dots, u_t^1, \dots, u_t^{b_t}) \in \mathbb{N}^{b_1 + \dots + b_t}$, define the following line bundles over G_δ :

$$\mathcal{Q}_\delta(\mathbf{u}) := \mathcal{O}_{\text{Grass}(s_1+1, \mathbf{k}^{N_\delta})}(u_1^1) \boxtimes \dots \boxtimes \mathcal{O}_{\text{Grass}(s_1+1, \mathbf{k}^{N_\delta})}(u_1^{b_1}) \boxtimes \dots \boxtimes \mathcal{O}_{\text{Grass}(s_t+1, \mathbf{k}^{N_\delta})}(u_t^1) \boxtimes \dots \boxtimes \mathcal{O}_{\text{Grass}(s_t+1, \mathbf{k}^{N_\delta})}(u_t^{b_t})|_{G_\delta}.$$

Note that for $\mathbf{u} = (1, \dots, 1)$, $\mathcal{Q}_\delta(\mathbf{u})$ is the canonical line bundle on G_δ .

We can now state the following lemma, which follows immediately from the construction of $\Delta(\mathbf{a}, \cdot)$:

Lemma 3.5. *For \mathbf{a} that satisfies $(*)$, one has*

$$\Delta(\mathbf{a}, \cdot)^* \mathcal{Q}_\delta((1, \dots, 1)) = \mathcal{L}_\lambda TM \otimes \pi_M^* \mathcal{O}_M(|\lambda_+^*|(\varepsilon + \delta)).$$

To see how we keep track of the equations of $\text{Flag}_s(TH_{\mathbf{a}})$ inside $\text{Flag}_s(TM)$, let us first make the following definition:

Definition 3.6. Let Δ be a $s_1 + 1$ -space in $\text{Grass}(s_1 + 1, \mathbf{k}^{N_\delta})$. If $v = (v_J)_{|J|=N_\delta}$ is a vector inside this space, we form the polynomial $P_{\Delta,v}(z) := \sum_{|J|=N_\delta} v_J z^J$, $z \in \mathbb{P}^N$. We will write $\langle\langle \Delta(z) = 0 \rangle\rangle$ to mean that $P_{\Delta,v}(z) = 0$ for every $v \neq 0 \in \Delta$.

Let us then form the following morphism

$$\Psi(\mathbf{a}, \cdot): \text{Flag}_s(TM) \rightarrow G_\delta \times \mathbb{P}^N, \quad \eta \mapsto (\Delta(\mathbf{a}, \eta), [\xi(\eta)^r]),$$

where $[\xi(\eta)^r]$ stands for $[\xi_0(\eta)^r : \dots : \xi_N(\eta)^r]$. What was meant by keeping track of the equations of $\text{Flag}_s(TH_{\mathbf{a}})$ inside $\text{Flag}_s(TM)$ is that, by construction, $\Psi(\mathbf{a}, \text{Flag}_s TH_{\mathbf{a}})$ is included in the following set:

$$\mathcal{Y}_\delta := \{((\Delta_1, \dots, \Delta_1, \dots, \Delta_t, \dots, \Delta_t), z) \in G_\delta \times \mathbb{P}^N \mid \Delta_1(z) = 0\}.$$

Let

$$g_\delta: \left(\begin{array}{ccc} G_\delta & \longrightarrow & \text{Flag}_{s_+} \mathbf{k}^{N_\delta} \\ (\Delta_1, \dots, \Delta_1, \Delta_2, \dots, \Delta_2, \dots, \Delta_t, \dots, \Delta_t) & \longmapsto & (\Delta_1 \supseteq \Delta_2 \supseteq \dots \supseteq \Delta_t) \end{array} \right)$$

be the inverse of embed_δ , and let

$$\mathcal{Z}_\delta := (g_\delta \times \text{Id}_{\mathbb{P}^N})(\mathcal{Y}_\delta) = \{((\Delta_1 \supseteq \dots \supseteq \Delta_t), z) \in \text{Flag}_{s_+} \mathbf{k}^{N_\delta} \times \mathbb{P}^N \mid \Delta_1(z) = 0\}.$$

Let us define aswell

$$p_\delta: \text{Flag}_{s_+} \mathbf{k}^{N_\delta} \rightarrow \text{Grass}(s_1 + 1, \mathbf{k}^{N_\delta})$$

the natural projection onto $\text{Grass}(s_1 + 1, \mathbf{k}^{N_\delta})$, and

$$\mathcal{Y}_\delta^* := (p_\delta \times \text{Id}_{\mathbb{P}^N})(\mathcal{Z}_\delta) = \{(\Delta, z) \in \text{Grass}(s_1 + 1, \mathbf{k}^{N_\delta}) \times \mathbb{P}^N \mid \Delta(z) = 0\}.$$

We then have the following commutative diagram

$$\begin{array}{ccccc} \mathcal{Y}_\delta & \xrightarrow{g_\delta \times \text{Id}_{\mathbb{P}^N}} & \mathcal{Z}_\delta & \xrightarrow{p_\delta \times \text{Id}_{\mathbb{P}^N}} & \mathcal{Y}_\delta^* \\ \downarrow \rho & & \downarrow & & \downarrow \rho^* \\ G_\delta & \xrightarrow{g_\delta} & \text{Flag}_{s_+} \mathbf{k}^{N_\delta} & \xrightarrow{p_\delta} & \text{Grass}(s_1 + 1, \mathbf{k}^{N_\delta}), \end{array} \quad (11)$$

where the vertical arrows are the natural projections.

The next part will be devoted to the existence of an open subset $S^\circ \subset S$ on which $(*)$ is satisfied, allowing us to make all the previous considerations. Before going on, let us make an observation that we will need later (3.3 and 4.2.2). If we consider the particular case where $\lambda = (1^k)$, the condition $(*)$ is the same as when one considers a partition λ' with jump sequence $s = (s_1 = k > \dots > s_t)$, which is straightforward by definition. We denote for \mathbf{a} satisfying $(*)$

$$\Psi^*(\mathbf{a}, \cdot): \text{Grass}(k, TM) \rightarrow G_\delta^* \times \mathbb{P}^N$$

the morphism constructed in this particular case, where $G_\delta^* = \text{Grass}(k+1, TM)$. Observe that the following diagram is commutative

$$\begin{array}{ccccc}
 & & & \text{Grass}(k, TM), & \\
 & & & \downarrow \Psi^*(\mathbf{a}, \cdot) & \\
 \text{Flag}_s TM & \xrightarrow{q} & & & \\
 & \searrow & & & \\
 & & G_\delta \times \mathbb{P}^N & \xrightarrow{p_\delta \circ g_\delta \times Id_{\mathbb{P}^N}} & G_\delta^* \times \mathbb{P}^N \\
 & \xrightarrow{\Psi(\mathbf{a}, \cdot)} & & & \\
 & & & &
 \end{array} \tag{12}$$

where

$$q: \text{Flag}_s TM \rightarrow \text{Grass}(k, TM), (x, F_1 \supseteq \dots \supseteq F_t) \mapsto (x, F_1),$$

which is straightforward by the very construction of the morphisms Ψ and Ψ^* .

3.3. Existence of the open subset S° . Due to the specific form of the equations chosen, we need to slightly change the condition $(*)$. Let us define the following stratification of M .

Definition 3.7. For $I \subsetneq \{0, \dots, N\}$, denote $D_I = \bigcap_{i \in I} D_i$ where $D_i = \{\xi_i = 0\}$, as well as $M_I = D_I \setminus \bigcap_{i \notin I} D_i$.

If $|I| = N - k_0$, $\dim M_I = k_0$, and the M_I 's stratify M .

This stratification induces in a natural way a stratification on $\text{Flag}_s TM$ by considering

$$\text{Flag}_s TM|_{M_I} := \pi_M^{-1}(M_I).$$

Instead of considering $(*)$, one instead considers

$(*, I) \forall x \in M_I, \forall (v_1, \dots, v_k)$ free family in $T_x M, A(\mathbf{a}; (x, v_1, \dots, v_k))$ is of maximal rank.

We will prove that there exists an open set S° such that $(*, I)$ is satisfied for every $I \subsetneq \{0, \dots, N\}, |I| < N$.

3.3.1. Stratification of $\text{Grass}(k, TM)$. As we saw at the end of the last subsection, we can restrict ourselves to the particular case $\lambda = (1^k)$. We first describe a stratification of $\text{Grass}(k, TM)$ that we are going to need in the course of the proof.

Definition 3.8. Let $I \subsetneq \{0, \dots, N\}, |I| = N - k_0$, and let $I' \subset I, |I'| = N - k_1$, where $k_1 \geq k$. Define:

$$\Sigma(I, I') = \{\eta = (x, v_1 \wedge \dots \wedge v_k) \in \text{Grass}(k, TM|_{M_I}) \mid i \in I' \Leftrightarrow d\xi_i(x, v_1) = \dots = d\xi_i(x, v_k) = 0\}.$$

Note that since $I' \subset I$, the nullity condition given in the definition does not depend on the local coordinates chosen to write the differentials $d\xi_i$.

Lemma 3.9. $\Sigma(I, I')$ is of dimension $k_0 + k(k_1 - k)$.

Proof. With respect to M , $\Sigma(I, I')$ is locally defined by $|I| = N - k_0$ independent equations. Let us see what happens in the fiber over $x \in M$. By definition of $\Sigma(I, I')$, the fiber is $\text{Grass}(k, \bigcap_{i \in I'} T_x D_i)$, where we recall that $D_i = \{\xi_i = 0\}$. The computation of the dimension of $\text{Grass}(k, \bigcap_{i \in I'} T_x D_i)$ yields the result. \square

Denote, for $I \subsetneq \{0, \dots, N\}$

$$\mathbb{P}(\mathbf{k}_I) := \{(z_0, \dots, z_N) \in \mathbb{P}^N \mid \forall i \in I, z_i = 0\}.$$

The inclusion $\mathbb{P}(\mathbf{k}_I) \hookrightarrow \mathbb{P}^N$ induces a projection:

$$\text{res}_I: \begin{cases} \bigoplus_{|J|=\delta} \mathbf{k} \simeq H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(\delta)) & \longrightarrow & H^0(\mathbb{P}(\mathbf{k}_I), \mathcal{O}_{\mathbb{P}(\mathbf{k}_I)}(\delta)) \simeq \bigoplus_{|J|=\delta, [J] \cap I = \emptyset} \mathbf{k} \\ (t_J)_{|J|=\delta} & \longmapsto & (t_J)_{|J|=\delta, [J] \cap I = \emptyset} \end{cases}$$

For $J = (j_0, \dots, j_N) \in \mathbb{N}^{N+1}$, we say that j_i is the *weight* of J corresponding to the i th index.

3.3.2. Study of the rank of a linear map. Let us fix $\eta = (x, v_1 \wedge \dots \wedge v_k) \in \text{Grass}(k, TM)$, where (v_1, \dots, v_k) is a free family of $T_x M$, and let us study the rank of the following map

$$\varphi_\eta: S \rightarrow (\mathbf{k}^{k+1})^{N_\delta}, \quad \mathbf{a} = (a_J)_J \mapsto A(\mathbf{a}; (x, v_1, \dots, v_k)),$$

where we recall that $A(\mathbf{a}; (x, v_1, \dots, v_k)) = \begin{pmatrix} ((\alpha_J(\mathbf{a}, x))_J) \\ (\theta_J(\mathbf{a}, x, v_1))_J \\ \vdots \\ (\theta_J(\mathbf{a}, x, v_k))_J \end{pmatrix}$. Let us aswell recall that

when writing θ_J , we implicitly assumed that we fixed a trivialization of $\mathcal{O}_M(1)$ around $x \in M$, and wrote θ_J under this trivialization.

Proposition 3.10. *Let I, I' be like in the definitions B.II.3.a, with $|I| = N - k_0$ and $|I'| = N - k_1$, $k_1 \geq k$.*

(i) *For $\eta \in \Sigma(I, I')$, $\text{rank}(\varphi_\eta) = (k+1) \binom{k_0+\delta}{k_0} + (k_1 - k_0) \binom{k_0+\delta-1}{k_0}$.*

(ii) *Furthermore, $(\text{res}_I \times \dots \times \text{res}_{I'}) \circ \varphi_\eta$ is a surjective map from S to $H^0(\mathbb{P}(\mathbf{k}_I), \mathcal{O}_{\mathbb{P}(\mathbf{k}_I)}(\delta))^{\times(k+1)}$.*

Proof. The first thing to observe is that the map φ_η can be represented by blocks

$$\begin{pmatrix} \begin{pmatrix} * & * \\ * & * \end{pmatrix} & 0 & \dots & 0 \\ 0 & \begin{pmatrix} * & * \\ * & * \end{pmatrix} & \dots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \dots & \begin{pmatrix} * & * \\ * & * \end{pmatrix} \end{pmatrix}$$

where each block is a matrix with $k+1$ rows and $\dim(H^0(M, \mathcal{O}_M(\varepsilon)))$ columns, and the number of such blocks is equal to N_δ . To see this, one considers S as $(H^0(M, \mathcal{O}_M(\varepsilon)))_{J \subset \mathbb{N}^{N+1}, |J|=\delta}$ and \mathbf{k}^{N_δ} as $(\mathbf{k}^{k+1})_{J \subset \mathbb{N}^{N+1}, |J|=\delta}$; each vertical entity in the previous block representation corresponds to the image of $\{0\} \times \dots \times H^0(M, \mathcal{O}_M(\varepsilon)) \times \{0\} \times \dots \times \{0\}$, the non zero-term corresponding to a certain index J .

Now, let us study the rank of a block corresponding to a given index J .

1st case: J such that $\xi^J(\eta) = 0$. For every \mathbf{a} , one has that:

- (i) $\alpha_J(\mathbf{a}, \eta) = a_J(\eta)\boldsymbol{\xi}^J(\eta) = 0$.
- (ii) $\theta_J(\mathbf{a}, x, v_i) = (r + 1)a_J(\eta)d\boldsymbol{\xi}^J(x, v_i)$ for every $1 \leq i \leq k$.

Observe that $d\boldsymbol{\xi}^J(x, \cdot)$ is identically zero as soon as $|[J] \cap I| \geq 2$, where we recall that $[J]$ is the support of the multi-index J . If $[J] \cap I$ is a single index, observe that if the weight of J corresponding to this index is strictly greater than one, then $d\boldsymbol{\xi}^J(x, \cdot)$ is still identically zero. So, the only case where one can hope for a non-zero block on the J th spot is when $[J] \cap I$ is a singleton, with the weight of J at this singleton equal to one. In

this case, observe that the columns of the block are all proportionnal to $\begin{pmatrix} 0 \\ d\boldsymbol{\xi}^J(x, v_1) \\ \vdots \\ d\boldsymbol{\xi}^J(x, v_k) \end{pmatrix}$,

so the rank is at most one, and to get a rank equal to one, we need to have at least one i such that $d\boldsymbol{\xi}^J(x, v_i) \neq 0$. By hypothesis, we took $\eta \in \Sigma(I, I')$, so the only case where this is satisfied is when J is such that

- (i) $[J] \cap I$ is a singleton, with the weight of J at this singleton being one.
- (ii) The singleton $[J] \cap I$ is in $I \setminus I'$.

We will, for sake of notation, write abusively $\ll [J] \cap I = \{1\} \gg$ to mean these two conditions. One counts that there exists $(k_1 - k_0) \binom{k_0 + \delta - 1}{k_0}$ such indexes J .

2nd case: J such that $\boldsymbol{\xi}^J(\eta) \neq 0$. First, observe that under the hypothesis $\varepsilon \geq 1$, every element of $(T_x M)^*$ can be written under the form $da(x, \cdot)$ where $a \in H^0(M, \mathcal{O}_M(\varepsilon))$ satisfies $a(x) = 0$. Let \mathbf{a} such that $a_J(x) = 0$. We have that for every $1 \leq i \leq k$:

$$\theta_J(\mathbf{a}, x, v_i) = \boldsymbol{\xi}^J(\eta) da_J(x, v_i).$$

From the previous observation, for every $1 \leq i \leq k$, we can find \mathbf{a}_i such that

$$\theta_J(\mathbf{a}_i, x, v_j) = \delta_i^j$$

for every $1 \leq j \leq k$. And since we can always find $a \in H^0(M, \mathcal{O}_M(\varepsilon)) \setminus \text{Vect}((\mathbf{a}_1)_J, \dots, (\mathbf{a}_k)_J)$ such that $a(\eta) \neq 0$, we deduce that the rank of the J th block is at least equal to $k + 1$, and it is thus equal to $(k + 1)$, since the block has $(k + 1)$ rows. Since $\eta \in \Sigma(I, I')$, the number of indexes J such that $\boldsymbol{\xi}^J(\eta) \neq 0$ is equal to $\binom{k_0 + \delta}{k_0}$, and for those indexes, the rank of the corresponding block is thus maximal, i.e. equal to $k + 1$.

Combining the first and second case, due to the block decomposition, we deduce that:

$$\text{rank}(\varphi_\eta) = (k + 1) \binom{k_0 + \delta}{k_0} + (k_1 - k_0) \binom{k_0 + \delta - 1}{k_0}$$

which completes the proof of **(i)**.

As for the proof of **(ii)**, if we compose by the projection onto $H^0(\mathbb{P}(\mathbf{k}_I), \mathcal{O}_{\mathbb{P}(\mathbf{k}_I)}(\delta))^{\times k+1}$, we keep only the block of the second kind, and thus the rank is full. \square

3.3.3. Existence of the open subset S^0 . For $I \subset \{0, \dots, N\}$, $|I| < N$, let us denote

$$B(I) \stackrel{\text{loc}}{=} \{(\mathbf{a}, \eta) \in S \times \text{Grass}(k, TM|_{M_I}) \mid \text{rank } A(\mathbf{a}; \eta) \text{ is strictly less than } k + 1\}.$$

Let us observe that if we prove that $B(I)$ does not dominate S under the first projection

$$\text{proj}_1 : S \times \text{Grass}(k, TM) \rightarrow S,$$

then the condition $(*, \mathbf{I})$ is satisfied on a (Zariski) open set $S^{\circ, I} \subset S$, since $(*, \mathbf{I})$ is then an open condition satisfied by at least one point.

We denote

$$\pi_M^* : \text{Grass}(k, TM) \rightarrow M$$

the projection onto M , as well as

$$\text{proj}_2 : S \times \text{Grass}(k, TM) \rightarrow \text{Grass}(k, TM)$$

and

$$pr_1 : S \times M \rightarrow S.$$

Proposition 3.11. *For $\delta \geq k + 1$, for every $I \subset \{0, \dots, N\}$, $|I| < N$, $B(I)$ does not dominate S under proj_1 .*

Proof. Observe first that it is enough to prove that $B(I) \cap [S \times \Sigma(I, I')]$ does not dominate $S \times \Sigma(I, I')$ under proj_1 for every $I' \subset I \subsetneq \{0, \dots, N\}$ since $(\Sigma(I, I'))_{I'}$ stratifies $\text{Grass}(k, TM_{M_I})$.

Let us denote

$$Z(I, I') = B(I) \cap [S \times \Sigma(I, I')],$$

where $|I| = N - k_0$ and $|I'| = N - k_1$, for $1 \leq k_0 \leq k_1$, and let us decompose $Z(I, I')$ under the following form

$$Z(I, I') = Z^\alpha(I, I') \bigsqcup [Z(I, I') \setminus Z^\alpha(I, I')],$$

where

$$Z^\alpha(I, I') = \{(\mathbf{a}, \eta) \in S \times \Sigma(I, I') \mid \alpha_J(\mathbf{a}, \eta) = 0 \text{ for every } J\}.$$

Like before, it is enough to prove that $Z^\alpha(I, I')$ nor $Z(I, I') \setminus Z^\alpha(I, I')$ dominates S under proj_1 .

1st case: $Z^\alpha(I, I')$. Let us denote

$$Z^\alpha(I) = (id_S \times \pi_M^*)(Z^\alpha(I, I')) = \{(\mathbf{a}, \eta) \in S \times M_I \mid \alpha_J(\mathbf{a}, \eta) = 0 \forall J\}.$$

Observe that $\text{proj}_1(Z^\alpha(I, I')) = pr_1(Z^\alpha(I))$, thus $Z^\alpha(I, I')$ dominates S under proj_1 if and only if $Z^\alpha(I)$ dominates S under pr_1 . We fix $x \in M_I$. For any $\mathbf{a} = (a_J)_{|J|=\delta}$, one has

$$(\alpha_J(\mathbf{a}, x) = 0 \forall J) \iff (a_J(x) = 0 \forall J \text{ such that } [J] \cap I = \emptyset).$$

Indeed, $\alpha_J(\mathbf{a}, x) = a_J(x)\boldsymbol{\xi}^J(x)$, thus $\alpha_J(\mathbf{a}, x) = 0$ if and only if $a_J(x) = 0$ or $\boldsymbol{\xi}^J(x) = 0$. Since $x \in M_I$, $\xi_i(x) = 0$ if and only if $i \in I$. Thus we have that $\boldsymbol{\xi}^J(x) = 0$ if and only if $[J] \cap I \neq \emptyset$, and this gives the equivalence. Since there are $\binom{k_0 + \delta}{k_0}$ multi-indexes J satisfying the right-hand side condition, each multi-index imposing an independant condition, we have that $\dim(Z^\alpha(I) \cap pr_2^{-1}(x)) \leq \dim S - \binom{k_0 + \delta}{k_0}$. From this, we deduce that

$$\begin{aligned} \dim(Z^\alpha(I)) - \dim(S) &\leq \dim M_I + \max_{x \in M_I} \dim(Z^\alpha(I) \cap pr_2^{-1}(x)) - \dim S \\ &\leq k_0 - \binom{k_0 + \delta}{k_0} < 0 \end{aligned}$$

as soon as $\delta \geq 2$. Thus $Z^\alpha(I, I')$ does not dominate S .

2nd case: $Z(I, I') \setminus Z^\alpha(I, I')$. Let us fix $\eta = (x, [v_1 \wedge \dots \wedge v_k]) \in \Sigma(I, I')$, and let us fix $x \in V \subset M$ an open subset that trivializes $\mathcal{O}_M(1)$. Let us denote

$$Z_\eta := (\text{proj}_2)^{-1}(\eta) \cap [Z(I, I') \setminus Z^\alpha(I, I')],$$

and observe that Z_η is isomorphic to $\text{proj}_1(Z_\eta)$ under proj_1 . Let us denote, for $i \geq 1$, V_i the vectorial space that consist of the elements $(w_0, w_1, \dots, w_k) \in \mathbf{k}^{(k+1)N_\delta}$ that satisfy

$$(1.i) \quad w_i \in \text{Vect}(w_0, \dots, w_{i-1}, w_{i+1}, \dots, w_k),$$

$$(2) \quad \text{For } J \text{ such that } [J] \cap I \neq \emptyset, \quad \begin{pmatrix} (w_0)_J \\ \vdots \\ (w_k)_J \end{pmatrix} \in \mathbf{k} \cdot \begin{pmatrix} 0 \\ d\xi^J(x, v_1) \\ \vdots \\ d\xi^J(x, v_k) \end{pmatrix},$$

and observe that $\varphi_\eta(\text{proj}_1(Z_\eta))$ is included in $\bigcup_{i \geq 1} V_i$. Indeed, by definition of $Z(I, I') \setminus Z^\alpha(I, I')$, the first row of every element of $\varphi_\eta(\text{proj}_1(Z_\eta))$ is non-zero, and thus the conditions (1.i) $_{1 \leq i \leq k}$ expresses that the rank condition on $A(\mathbf{a}, \eta)$ is not satisfied; as for (2), it follows from the fact that $\eta \in \Sigma(I, I')$ (Cf. the discussion of the 1st case in the course of the proof of Proposition 3.10).

We will now evaluate the dimension of V_i . As we saw in the course of the proof of

proposition 3.10, there are $C_1 = (k_1 - k_0) \binom{k_0 + \delta - 1}{k_0}$ indexes J such that $\begin{pmatrix} 0 \\ d\xi^J(x, v_1) \\ \vdots \\ d\xi^J(x, v_k) \end{pmatrix}$ is

non-zero (they correspond to the indexes J such that $\ll [J] \cap I = \{1\} \gg$: cf. the proof of Proposition 3.10 for the notation) and $C_0 = \binom{k_0 + \delta}{k_0}$ indexes J such that $[J] \cap I = \emptyset$. Accordingly, the condition (2) allows one to see V_i as living in

$$\left(\bigoplus_{J, [J] \cap I = \emptyset} \mathbf{k} \right)^{\times k+1} \times \left(\bigoplus_{J, [J] \cap I = \{1\}} \mathbf{k} \right) \simeq \mathbf{k}^{(k+1)C_0 + C_1}.$$

Let us now use the conditions (1.i) and (2) to evaluate the dimension of V_i , and to do so, let us construct the (algebraic) morphism L_i :

$$\begin{aligned} \mathbf{k}^k \times \left(\bigoplus_{[J] \cap I = \emptyset} \mathbf{k} \right)^{\times k} \times \bigoplus_{[J] \cap I = \{1\}} \mathbf{k} &\rightarrow \left(\bigoplus_{[J] \cap I = \emptyset} \mathbf{k} \right)^{\times (k+1)} \times \bigoplus_{[J] \cap I = \{1\}} \mathbf{k} \\ \left((\omega_j)_{j \neq i}, (\lambda_j^j)_{[J] \cap I = \emptyset}^{j \neq i}, (\lambda_J)_{[J] \cap I = \{1\}} \right) &\mapsto \left((\lambda_j^j)_{[J] \cap I = \emptyset}^{j \neq i}, \left(\sum_{j \neq i} \omega_j \lambda_j^j \right)_{[J] \cap I = \emptyset}, (\lambda_J)_{[J] \cap I = \{1\}} \right) \end{aligned}$$

With the condition (1.i), $V_i \subset \text{Im}(L_i)$, and thus $V_i = L_i(L_i^{-1}(V_i))$. But one can say more with (1.i) combined with (2), namely that $L_i^{-1}(V_i)$ is included in

$$\left\{ \left((\omega_j)_{j \neq i}, ((\lambda_j^j)_{[J] \cap I = \emptyset})^{j \neq i}, (\lambda_J)_{[J] \cap I = \{1\}} \right) \mid \forall J, [J] \cap I = \{1\}, \lambda_J [d\xi^J(x, v_i) - \sum_{j \neq i} \omega_j d\xi^J(x, v_j)] = 0 \right\}$$

This set is a union of subvectorial spaces of codimension C_1 of $\mathbf{k}^k \times (\mathbf{k}^{C_0})^k \times \mathbf{k}^{C_1}$. Therefore, its dimension (as an algebraic set) is equal to $k + kC_0$. Since $V_i = L_i(L_i^{-1}(V_i))$, $\dim V_i \leq k + kC_0$, and accordingly one deduces that

$$\dim \varphi_\eta(\text{proj}_1(Z_\eta)) \leq k + k \binom{k_0 + \delta}{k_0}$$

since $\varphi_\eta(\text{proj}_1(Z_\eta)) \subset \bigcup_{i \geq 1} V_i$. Using this inequality, with proposition 3.10, one infers that

$$\begin{aligned} \dim(Z_\eta \simeq \text{proj}_1(Z_\eta)) &\leq k + k \binom{k_0 + \delta}{k_0} + \dim(\ker(\varphi_\eta)) \\ &= k + k \binom{k_0 + \delta}{k_0} + \dim S - \text{rank}(\varphi_\eta) \\ &\leq k - \binom{k_0 + \delta}{k_0} - (k_1 - k_0) \binom{k_0 + \delta - 1}{k_0} + \dim S. \end{aligned}$$

Since this is true for every $\eta \in \Sigma(I, I')$, one has that

$$\begin{aligned} &\dim(Z(I, I') \setminus Z^\alpha(I, I')) - \dim S \\ &\leq \dim \Sigma(I, I') + \max_{\eta \in \Sigma(I, I')} \dim Z_\eta - \dim S \\ &\leq k + k_0 + k(k_1 - k) - \binom{k_0 + \delta}{k_0} - (k_1 - k_0) \binom{k_0 + \delta - 1}{k_0} \\ &= (k + 1)k_0 - \binom{k_0 + \delta}{k_0} + k - k^2 + (k_1 - k_0) \left(k - \binom{k_0 + \delta - 1}{k_0} \right), \end{aligned}$$

which is strictly negative for any $k_1 \geq k_0 \geq 1$ as soon as $\delta \geq k + 1$ (Let us recall that by hypothesis on I , namely $|I| < N$, $k_0 \geq 1$). To see this, observe for instance that for $n, N \in \mathbb{N}_{\geq 1}$, $\binom{n+N}{n} \geq nN$, with a strict inequality for $n, N \geq 2$ (which can be proved by recurrence on $n + N \geq 2$). \square

With this proposition, we deduce that for every I , $|I| < N$, the condition $(*, I)$ is satisfied on an open set $S^{\circ, I} \subset S$. Since S is an irreducible variety (as product of irreducible varieties), the intersection of these open sets is a non-empty open set, that we will write S° . Following what was done in 3.2 in restriction to M_I , one can define for every $\mathbf{a} \in S^\circ$ a morphism

$$\Delta_I(\mathbf{a}, \cdot): \text{Flag}_s TM|_{M_I} \rightarrow G_\delta$$

such that

$$\Delta_I(\mathbf{a}, \cdot)^* \mathcal{Q}_\delta((1, \dots, 1)) = \mathcal{L}_\lambda TM|_{M_I} \otimes \pi_{M_I}^* \mathcal{O}_M(|\lambda_+^*|(\varepsilon + \delta)),$$

where $\pi_{M_I}: \text{Flag}_s TM|_{M_I} \rightarrow M_I$ is the projection. Then one constructs the morphism

$$\Psi_I(\mathbf{a}, \cdot): \text{Flag}_s TM|_{M_I} \rightarrow G_\delta \times \mathbb{P}(\mathbf{k}_I), \quad \eta \mapsto (\Delta(\mathbf{a}, \eta), [\xi(\eta)^r]),$$

which satisfies, for \mathbf{a} such that $H_{\mathbf{a}}$ is smooth

$$\Psi_I(\mathbf{a}, \text{Flag}_s (TH_{\mathbf{a}})_{M_I}) \subset \mathcal{Y}_\delta(I) := \{((\Delta_1, \dots, \Delta_1, \dots, \Delta_t, \dots, \Delta_t), z) \in G_\delta \times \mathbb{P}(\mathbf{k}_I) \mid \Delta_1(z) = 0\}.$$

Observe that the morphism $\Psi_I(\mathbf{a}, \cdot)$ depends algebraically on $\mathbf{a} \in S^\circ$; accordingly, we have a morphism

$$\Psi_I: S^\circ \times \text{Flag}_s TM|_{M_I} \rightarrow G_\delta \times \mathbb{P}(\mathbf{k}_I)$$

4. AMPLENESS OF SCHUR BUNDLES OF COTANGENT BUNDLES OF GENERAL COMPLETE INTERSECTIONS

4.1. The morphism Ψ . From now on, we consider not a single hypersurface $H_{\mathbf{a}}$, but a complete intersection $H_{\mathbf{a}^\bullet} := H_{\mathbf{a}^1} \cap \dots \cap H_{\mathbf{a}^c}$ of codimension c , where $\mathbf{a}^\bullet = (\mathbf{a}^1, \dots, \mathbf{a}^c)$ and \mathbf{a}^i is taken in the space of parameters $S_i := H^0(M, \mathcal{O}_M(\varepsilon_i))_{J \subset \mathbb{N}^{N+1}, |J|=\delta_i}$, where $\varepsilon_i \in \mathbb{N}_{\geq 1}$, and δ_i is large enough as in the Proposition 3.11. We denote

$$\overline{\mathcal{X}} := \{(\mathbf{a}^1, \dots, \mathbf{a}^c; x) \in S_1 \times \dots \times S_c \times M \mid E^1(\mathbf{a}^1, x) = \dots = E^c(\mathbf{a}^c, x) = 0\}$$

the universal family of such complete intersection, and we will denote, for each $1 \leq i \leq c$, S_i° the open set on which (\ast, \mathbf{I}) is valid for every $I \subset \{0, \dots, N\}, |I| < N$. According to the last section, for each $1 \leq i \leq c$, one can define

$$\Delta_I^i: S_i^\circ \times \text{Flag}_s TM|_{M_I} \rightarrow G_{\delta_i}.$$

The following lemma, which is for instance proved in [BD18]

Lemma 4.1. *A general fiber of the family $\overline{\mathcal{X}} \rightarrow S_1 \times \dots \times S_c$ is smooth.*

allows us to restrict ourselves to the open dense subset $\mathbf{S} \subset S_1 \times \dots \times S_c$ parametrizing smooth complete intersection varieties, and we denote $\mathcal{X} \rightarrow \mathbf{S}$ the corresponding universal family.

Denote $\mathbf{S}^\circ := \mathbf{S} \cap (S_1^\circ \times \dots \times S_c^\circ)$, and form the following morphism

$$\Psi_I: \mathbf{S}^\circ \times \text{Flag}_s TM|_{M_I} \rightarrow \mathbf{G} \times \mathbb{P}(\mathbf{k}_I), \quad (\mathbf{a}^\bullet, \eta) \mapsto \Delta_I^1(\mathbf{a}^1, \eta) \times \dots \times \Delta_I^c(\mathbf{a}^c, \eta) \times [\xi(\eta)^r]$$

where $\mathbf{G} := G_{\delta_1} \times \dots \times G_{\delta_c}$. As in the end of subsection 3.2, we define

$$\mathcal{Y} := \{((\Delta_1^1, \dots, \Delta_t^1) \times \dots \times (\Delta_1^c, \dots, \Delta_t^c), z) \in \mathbf{G} \times \mathbb{P}^N \mid \forall 1 \leq i \leq c, \Delta_1^i(z) = 0\}$$

as well as

$$g_\delta := g_{\delta_1} \times \dots \times g_{\delta_c} \text{ and } \mathcal{Z} := (g_\delta \times Id_{\mathbb{P}^N})(\mathcal{Y})$$

and

$$p_\delta := p_{\delta_1} \times \dots \times p_{\delta_c} \text{ and } \mathcal{Y}^\ast := (p_\delta \times Id_{\mathbb{P}^N})(\mathcal{Z}).$$

We have the following commutative diagram as in (11):

$$\begin{array}{ccccc} \mathcal{Y} & \xrightarrow{g_\delta \times Id_{\mathbb{P}^N}} & \mathcal{Z} & \xrightarrow{p_\delta \times Id_{\mathbb{P}^N}} & \mathcal{Y}^\ast \\ \downarrow \rho & & \downarrow & & \downarrow \rho^\ast \\ \mathbf{G} & \xrightarrow{g_\delta} & \prod_{i=1}^c \text{Flag}_{s_+} \mathbf{k}^{N_{\delta_i}} & \xrightarrow{p_\delta} & \prod_{i=1}^c \text{Grass}(k+1, \mathbf{k}^{N_{\delta_i}}) := \mathbf{G}^\ast \end{array} \quad (13)$$

where the vertical arrows are the natural projections. We denote for $I \subsetneq \{0, \dots, N\}$

$$\mathcal{Y}(I) := \mathcal{Y} \cap \mathbf{G} \times \mathbb{P}(\mathbf{k}_I),$$

and similarly for $\mathcal{Z}(I)$ and $\mathcal{Y}^\ast(I)$.

From the definition of the morphism, observe that for $\mathbf{a}^\bullet \in \mathbf{S}^\circ$

$$\Psi_I(\mathbf{a}^\bullet, \text{Flag}_s(TH_{\mathbf{a}^\bullet})|_{M_I}) \subset \mathcal{Y}(I).$$

Accordingly, we have the following commutative diagram as in (5):

$$\begin{array}{ccccc} \text{Flag}_s(T\mathcal{X}/\mathbf{S}^\circ) & \xrightarrow{\Psi_I} & \mathcal{Y}(I) & \longrightarrow & \mathbb{P}(\mathbf{k}_I) \\ \downarrow & & \downarrow \rho_I & & \\ \mathbf{S}^\circ & & \mathbf{G} & & \end{array} \quad (14)$$

where ρ_I is the restriction of the first projection ρ to $\mathcal{Y}(I)$. Suppose that $c(k+1) \geq N$ so that the morphism ρ_I is generically finite. As explained in 2.3 and 2.4, we would like $\Psi_I(\mathbf{a}^\bullet, \text{Flag}_s(TH_{\mathbf{a}^\bullet})|_{M_I})$ to avoid the exceptional locus of ρ_I for \mathbf{a}^\bullet belonging to an open set of \mathbf{S}° , for every $I \subsetneq \{0, \dots, N\}, |I| < N$.

4.2. Avoiding the exceptional locus. In this subsection, we prove the existence of an open subset $\mathbf{V} \subset \mathbf{S}^\circ$ such that for each $|I| < N$, and for all $\mathbf{a}^\bullet \in \mathbf{V}$, $\Psi_I(\mathbf{a}^\bullet, \text{Flag}_s(TH_{\mathbf{a}^\bullet})|_{M_I})$ avoids the exceptional locus of ρ_I , provided that the δ_i 's are taken large enough.

4.2.1. A particular case. Let us prove the existence of such an open set in the setting where $\lambda = (1^k)$. As we will see afterwards, it is enough to conclude the wanted result in full generality. In this setting, we denote (some notations were already introduced in (13)):

- $\text{Flag}_s(TM) = \text{Grass}(s_1, TM) = \text{Grass}(k, TM)$.
- $\Psi_I^*: \mathbf{S}^\circ \times \text{Grass}(k, TM) \rightarrow \text{Grass}(k+1, \mathbf{k}^{N_{\delta_1}}) \times \dots \times \text{Grass}(k+1, \mathbf{k}^{N_{\delta_c}})$.
- $\mathbf{G}^* = \text{Grass}(k+1, \mathbf{k}^{N_{\delta_1}}) \times \dots \times \text{Grass}(k+1, \mathbf{k}^{N_{\delta_c}})$.
- $\mathcal{Y}^* = \{(\Delta_1, \dots, \Delta_c, z) \in \mathbf{G}^* \times \mathbb{P}^N \mid \forall 1 \leq i \leq c, \Delta_i(z) = 0\}$.
- $\rho^*: \mathcal{Y}^* \rightarrow \mathbf{G}^*$ the first projection.

Observe that the set \mathbf{S}° in the previous setting and in this setting can be chosen to be the same (Cf. 3.3.3). Let us denote

$$\mathbf{G}_\infty^*(I) := \{\Delta = (\Delta_1, \dots, \Delta_c) \in \mathbf{G}^* \mid (\rho_I^*)^{-1}(\Delta) \text{ has positive dimension.}\}$$

as well as $\Phi_I^* = \rho_I^* \circ \Psi_I^*$. We prove the following lemma:

Lemma 4.2. *If $\delta_1, \dots, \delta_c$ are larger than $N + k(N - k)$, there exists a non-empty open subset $\mathbf{V} \subset \mathbf{S}^\circ$ such that for each $I \subsetneq \{0, \dots, N\}$ with cardinality $|I| < N$,*

$$(\mathbf{V} \times \text{Grass}(k, (TM)|_{M_I})) \cap \Phi_I^{*-1}(\mathbf{G}_\infty^*(I)) = \emptyset.$$

Proof. We first note that it is enough to prove that for each $I \subsetneq \{0, \dots, N\}$ with cardinality $N - k, k \geq 1$, there exists a non-empty open subset $\mathbf{V}(I) \subset \mathbf{S}^\circ$ such that

$$(\mathbf{V}(I) \times \text{Grass}(k, (TM)|_{M_I})) \cap \Phi_I^{*-1}(\mathbf{G}_\infty^*(I)) = \emptyset,$$

since taking their intersection will provide a non-empty open subset satisfying the required condition. We will prove that

$$\dim((\mathbf{S}^\circ \times \text{Grass}(k, (TM)|_{M_I})) \cap \Phi_I^{*-1}(\mathbf{G}_\infty^*(I))) < \dim \mathbf{S}^\circ,$$

which will imply the existence of the sought open subset. We fix $\eta \in \text{Grass}(k, (TM)_{|M_I})$, and we denote $\Phi_\eta^*: \mathbf{S}^\circ \rightarrow \mathbf{G}^*$ the map

$$\Phi_\eta^*(\mathbf{a}^\cdot) = \Phi_I^*(\mathbf{a}^\cdot, \eta).$$

We prove the following inequality

$$\dim(\Phi_\eta^{*-1}(\mathbf{G}_\infty^*(I))) < \dim \mathbf{S}^\circ - \dim \text{Grass}(k, TM_{|M_I}),$$

which will imply the lemma, since then

$$\begin{aligned} & \dim((\mathbf{S}^\circ \times \text{Grass}(k, (TM)_{|M_I})) \cap \Phi_\eta^{*-1}(\mathbf{G}_\infty^*(I))) \\ & \leq \dim \text{Grass}(k, TM_{|M_I}) + \max_\eta \dim(\Phi_\eta^{*-1}(\mathbf{G}_\infty^*(I))) \\ & < \dim \mathbf{S}^\circ. \end{aligned}$$

In order to study $\dim(\Phi_\eta^{*-1}(\mathbf{G}_\infty^*(I)))$, let us define:

- $\tilde{\mathbf{G}}^*(I) := H^0(\mathbb{P}(\mathbf{k}_I), \mathcal{O}_{\mathbb{P}(\mathbf{k}_I)}(\delta_1))^{\times(k+1)} \times \dots \times H^0(\mathbb{P}(\mathbf{k}_I), \mathcal{O}_{\mathbb{P}(\mathbf{k}_I)}(\delta_c))^{\times(k+1)}$.
- $\tilde{\mathcal{Y}}^*(I) := \{((\alpha_1, \theta_1^1, \dots, \theta_1^k, \dots, \alpha_c, \theta_c^1, \dots, \theta_c^k), z) \in \tilde{\mathbf{G}}^* \times \mathbb{P}(\mathbf{k}_I) \mid \forall 1 \leq i \leq c, \forall 1 \leq j \leq k, \alpha_i(z) = 0 \text{ and } \theta_i^j(z) = 0\}$, and $\tilde{\rho}_I^*$ the first projection associated.
- $\tilde{\Phi}_\eta^* := (\tilde{\varphi}_\eta^1 \times \dots \times \tilde{\varphi}_\eta^c): \mathbf{S} = S_1 \times \dots \times S_c \rightarrow \tilde{\mathbf{G}}^*(I)$, where we denote

$$\tilde{\varphi}_\eta^p = \underbrace{(\text{res}_I \times \dots \times \text{res}_I)}_{\times(k+1)} \circ \varphi_\eta^p.$$

Recall that φ_η was defined in 3.3.2, and that the Proposition 3.10, Φ_η^* is surjective.

- $\tilde{\mathbf{G}}_\infty^*(I) := \{\Delta \in \tilde{\mathbf{G}}^*(I) \mid (\tilde{\rho}_I^*)^{-1}(\Delta) \text{ has positive dimension}\}$.

By the very definition of the previous sets and maps, observe that for any $\mathbf{a}^\cdot \in \mathbf{S}^\circ$, one has

$$\rho_I^{*-1}(\Phi_\eta^*(\mathbf{a}^\cdot)) = \tilde{\rho}_I^{*-1}(\tilde{\Phi}_\eta^*(\mathbf{a}^\cdot)) \subset \mathbb{P}(\mathbf{k}_I).$$

We deduce that $\Phi_\eta^*(\mathbf{a}^\cdot) \in \mathbf{G}_\infty^*(I)$ if and only if $\tilde{\Phi}_\eta^*(\mathbf{a}^\cdot) \in \tilde{\mathbf{G}}_\infty^*(I)$, hence:

$$\Phi_\eta^{*-1}(\mathbf{G}_\infty^*(I)) = \tilde{\Phi}_\eta^{*-1}(\tilde{\mathbf{G}}_\infty^*(I)).$$

Since $\tilde{\Phi}_\eta^*$ is linear and surjective, its fibers are of dimension $\dim \mathbf{S} - \dim \tilde{\mathbf{G}}^*(I)$. Thus:

$$\begin{aligned} \dim(\Phi_\eta^{*-1}(\mathbf{G}_\infty^*(I))) &= \dim \mathbf{S} - \dim \tilde{\mathbf{G}}^*(I) + \dim \tilde{\mathbf{G}}_\infty^*(I) \\ &= \dim \mathbf{S} - \text{codim}(\tilde{\mathbf{G}}_\infty^*(I), \tilde{\mathbf{G}}^*(I)) \end{aligned}$$

It remains to evaluate $\text{codim}(\tilde{\mathbf{G}}_\infty^*(I), \tilde{\mathbf{G}}^*(I))$. It was already done in [BD18, Cor 3.3] as a corollary of a result in [Ben11], and one has that

$$\text{codim}(\tilde{\mathbf{G}}_\infty^*(I), \tilde{\mathbf{G}}^*(I)) > \min_{1 \leq p \leq c} (\delta_p).$$

Thus, if the δ_p 's are chosen greater than $N + k(N - k) = \dim \text{Grass}(k, TM) \geq \dim \text{Grass}(k, TM|_{M_I})$, we deduce that

$$\dim(\Phi_\eta^{*-1}(\mathbf{G}_\infty^*(I))) = \dim(\tilde{\Phi}_\eta^{*-1}(\tilde{\mathbf{G}}_\infty^*(I))) < \dim \mathbf{S} - \dim \text{Grass}(k, (TM)|_{M_I}),$$

which concludes the proof of the lemma. \square

4.2.2. *General case.* We now treat the original case, with $s = (s_1 = k > \dots > s_t)$ and the original partition λ . Let us denote

$$\mathbf{G}_\infty(I) := \{F = (\Delta_1^1, \dots, \Delta_1^t, \dots, \Delta_c^1, \dots, \Delta_c^t) \in \mathbf{G} \mid \rho_I^{-1}(F) \text{ has positive dimension}\}$$

and let us recall the commutative diagram (13)

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{(p_\delta \circ g_\delta) \times Id_{\mathbb{P}^N}} & \mathcal{Y}^* \\ \downarrow \rho & & \downarrow \rho^* \\ \mathbf{G} & \xrightarrow{p_\delta \circ g_\delta} & \mathbf{G}^* \end{array} \quad (15)$$

Fact 1: $\mathbf{G}_\infty(I) = (p_\delta \circ g_\delta)^{-1}(\mathbf{G}_\infty^*(I))$.

Indeed, if $(\rho_I^*)^{-1}(\Delta)$ has positive dimension, so does the fiber along \mathbb{P}^N of every element of $((p_\delta \circ g_\delta) \times Id_{\mathbb{P}^N})^{-1}((\rho_I^*)^{-1}(\Delta))$. By commutativity of the diagram, it implies that the reciprocal image by ρ_I of every element of $(p_\delta \circ g_\delta)^{-1}(\Delta)$ has positive dimension, and thus

$$\mathbf{G}_\infty(I) \supset (p_\delta \circ g_\delta)^{-1}(\mathbf{G}_\infty^*(I)).$$

In the other direction, let F such that $\rho_I^{-1}(F)$ has positive dimension. By commutativity of the diagram,

$$\rho_I^* \circ ((p_\delta \circ g_\delta) \times Id_{\mathbb{P}^N})(\rho_I^{-1}(F)) = p_\delta \circ g_\delta \circ \rho_I(\rho_I^{-1}(F)) = p_\delta \circ g_\delta(F),$$

which implies that

$$(\rho_I^*)^{-1}((p_\delta \circ g_\delta)(F)) \supset ((p_\delta \circ g_\delta) \times Id_{\mathbb{P}^N})(\rho_I^{-1}(F)).$$

Accordingly, $(\rho_I^*)^{-1}((p_\delta \circ g_\delta)(F))$ has positive dimension, and

$$\mathbf{G}_\infty(I) \subset (p_\delta \circ g_\delta)^{-1}(\mathbf{G}_\infty^*(I)).$$

Fact 2: The following diagram is commutative

$$\begin{array}{ccccc} & & & & (\Psi_I^*)^{-1}(\mathcal{Y}^*(I)) \\ & & & \nearrow Id \times q & \downarrow \Psi_I^* \\ \Psi_I^{-1}(\mathcal{Y}(I)) & \xrightarrow{\Psi_I} & \mathcal{Y}(I) & \xrightarrow{(p_\delta \circ g_\delta) \times Id_{\mathbb{P}^N}} & \mathcal{Y}^*(I) \\ & & \downarrow \rho_I & & \downarrow \rho_I^* \\ & & \mathbf{G} & \xrightarrow{p_\delta \circ g_\delta} & \mathbf{G}^* \end{array} \quad (16)$$

This follows from (12) for the top part of the diagram, and from the commutativity of (15) for the bottom part of the diagram.

Now, with the two previous facts, one sees that the open set \mathbf{V} given by the Lemma 4.2 satisfies aswell that for each $I \subsetneq \{0, \dots, N\}$ with cardinality $|I| < N$

$$(\mathbf{V} \times \text{Flag}_s TM|_{M_I}) \cap \Phi_I^{-1}(\mathbf{G}_\infty(I)) = \emptyset,$$

where $\Phi_I = \rho_I \circ \Psi_I$, since one has

$$\Phi_I^{-1}(\mathbf{G}_\infty(I)) = (p_\delta \circ g_\delta \circ \Phi_I)^{-1}(\mathbf{G}_\infty^*(I)) = (Id \times q)^{-1}(\Phi_I^{*-1}(\mathbf{G}_\infty^*(I))).$$

4.3. Pulling back the positivity. Suppose we have fixed $\delta = (\delta_1, \dots, \delta_c)$ large enough, so that the conclusion of the Proposition 3.11 and the Lemma 4.2 are satisfied. In particular, the open set $\mathbf{V} \subset \mathbf{S}$ on which the Lemma 4.2 is satisfied is fixed. Denote

$$pr_2^{[1]}: \mathbf{V} \times \text{Flag}_s TM \rightarrow \text{Flag}_s TM$$

the second projection aswell as $pr_{2,\mathcal{X}}^{[1]}$ its restriction to $\text{Flag}_s T\mathcal{X}/\mathbf{V}$. Recall that we defined

$$\pi_{\mathbf{a}^\bullet}: \text{Flag}_s(TH_{\mathbf{a}^\bullet}) \rightarrow H_{\mathbf{a}^\bullet}$$

the natural projection.

4.3.1. Effective version of Nakayame's theorem. Let us define, for $\mathbf{m} = (m_1, \dots, m_c) \in \mathbb{N}^c$

$$\mathcal{Q}^*(\mathbf{m}) := \mathcal{O}_{\text{Grass}(k+1, \mathbf{k}^{N_{\delta_1}})}(m_1) \boxtimes \dots \boxtimes \mathcal{O}_{\text{Grass}(k+1, \mathbf{k}^{N_{\delta_c}})}(m_c)$$

line bundle over \mathbf{G}^* . A theorem in [Den16, Theorem 3.1, Lemma 3.2] asserts that for

$\mathbf{m} = (m_1, \dots, m_c)$ such that $m_i \geq \frac{\prod_{j=1}^c \delta_j^{k+1}}{\delta_i}$, for every $I \subsetneq \{0, \dots, N\}$,

$$\text{Bs}(\mathcal{Q}^*(\mathbf{m}) \boxtimes \mathcal{O}_{\mathbb{P}^N}(-1)|_{\mathbf{y}^*(I)}) \subset (\rho_I^*)^{-1}(\mathbf{G}_\infty^*(I)).$$

Since the pullback of a globally generated line bundle by a morphism remains globally generated, one has

$$[(p_\delta \circ g_\delta) \times Id_{\mathbb{P}^N}]^{-1} \text{Bs}(\mathcal{Q}^*(\mathbf{m}) \boxtimes \mathcal{O}_{\mathbb{P}^N}(-1)) \supset \text{Bs}(g_\delta^* p_\delta^* \mathcal{Q}^*(\mathbf{m}) \boxtimes \mathcal{O}_{\mathbb{P}^N}(-1)).$$

What's more, observe that, if we note $\mathbf{e} = (1, 0, \dots, 0) \in \mathbb{N}^{b_1 + \dots + b_t}$,

$$g_\delta^* p_\delta^* \mathcal{Q}^*(\mathbf{m}) = \mathcal{Q}_{\delta_1}(m_1 \mathbf{e}) \boxtimes \dots \boxtimes \mathcal{Q}_{\delta_c}(m_c \mathbf{e})$$

(Cf. the paragraph before Lemma 3.5 for the definition of \mathcal{Q}_δ). Accordingly, with the commutativity of (15) and the fact that $\mathbf{G}_\infty(I) = (p_\delta \circ g_\delta)^{-1}(\mathbf{G}_\infty^*(I))$, one has that

$$\text{Bs}(\mathcal{Q}_{\delta_1}(m_1 \mathbf{e}) \boxtimes \dots \boxtimes \mathcal{Q}_{\delta_c}(m_c \mathbf{e}) \boxtimes \mathcal{O}_{\mathbb{P}^N}(-1)|_{\mathbf{y}(I)}) \subset \rho_I^{-1}(\mathbf{G}_\infty(I)).$$

Since for any $\mathbf{u} \in \mathbb{N}^{b_1 + \dots + b_t}$, $\mathcal{Q}_\delta(\mathbf{u})$ is generated by global sections, we deduce that for

$$m_i \geq \frac{\prod_{j=1}^c \delta_j^{k+1}}{\delta_i}:$$

$$\text{Bs}(\mathcal{Q}_{\delta_1}((1, \dots, 1))^{\otimes m_1} \boxtimes \dots \boxtimes \mathcal{Q}_{\delta_c}((1, \dots, 1))^{\otimes m_c} \boxtimes \mathcal{O}_{\mathbb{P}^N}(-1)|_{\mathbf{y}(I)}) \subset \rho_I^{-1}(\mathbf{G}_\infty(I)).$$

For sake of shortness, we denote, for $\mathbf{m} = (m_1, \dots, m_c)$:

$$\mathcal{O}_{\mathbf{G}}(\mathbf{m}) := \mathcal{Q}_{\delta_c}((m_1, \dots, m_1)) \boxtimes \dots \boxtimes \mathcal{Q}_{\delta_c}((m_c, \dots, m_c))$$

ample line bundle over $\mathbf{G} = G_{\delta_1} \times \dots \times G_{\delta_c}$, which accordingly to what was just said satisfies

$$\text{Bs}(\mathcal{O}_{\mathbf{G}}(\mathbf{m}) \boxtimes \mathcal{O}_{\mathbb{P}^N}(-1)|_{\mathcal{Y}(I)}) \subset \rho_I^{-1}(\mathbf{G}_{\infty}(I)) \quad (17)$$

for every $I \subsetneq \{0, \dots, N\}$, as soon as $m_i \geq \frac{\prod_{j=1}^c \delta_j^{k+1}}{\delta_i}$ for every $1 \leq i \leq c$.

4.3.2. *Proof of the main theorem.* Let us take $\mathbf{m}(\boldsymbol{\delta}) = (m_1, \dots, m_c)$ such that $m_i \geq \frac{\prod_{j=1}^c \delta_j^{k+1}}{\delta_i}$, so that (17) is satisfied for every $I \subsetneq \{0, \dots, N\}$. For a reason that we will see shortly, suppose ε, r are chosen such that:

$$u := r - |\lambda_+^*| \sum_{p=1}^c m_p(\varepsilon_p + \delta_p) > 0.$$

Observe that, as a direct consequence of Lemma 3.5, for $1 \leq p \leq c$ and $I \subsetneq \{0, \dots, N\}$, $|I| < N$,

$$(\Delta_I^p)^* \mathcal{O}_{G_p}(1) = (pr_2^{[1]})^* (\mathcal{L}_{\lambda} TM|_{M_I} \otimes \pi_{M_I}^* \mathcal{O}_M(|\lambda_+^*|(\varepsilon_p + \delta_p))),$$

which implies that

$$\Psi_I^* (\mathcal{O}_{\mathbf{G}}(\mathbf{m}(\boldsymbol{\delta})) \boxtimes \mathcal{O}_{\mathbb{P}(\mathbf{k}_I)}(-1)) = (pr_2^{[1]})^* (\mathcal{L}_{\lambda} TM|_{M_I}^{\otimes |\mathbf{m}(\boldsymbol{\delta})|} \otimes \pi_{M_I}^* \mathcal{O}_M(-u)). \quad (18)$$

Let us fix $\mathbf{a} \in \mathbf{V}$. We first prove that

$$\mathcal{L}_{\lambda} TH_{\mathbf{a}^{\bullet}}^{\otimes |\mathbf{m}(\boldsymbol{\delta})|} \otimes \pi_{H_{\mathbf{a}^{\bullet}}}^* \mathcal{O}_{H_{\mathbf{a}^{\bullet}}}(-u)$$

is nef. Let us take $\mathcal{C} \subset \text{Flag}_s TH_{\mathbf{a}^{\bullet}} \subset \text{Flag}_s TM$ any irreducible curve. As $(\text{Flag}_s TM|_{M_I})_{I \subsetneq \{0, \dots, N\}}$ stratifies $\text{Flag}_s(TM)$, there exists a unique set I such that $\text{Flag}_s TM|_{M_I}$ contains an open dense subset of \mathcal{C} . If $\dim(M_I) = 0$, then the irreducible curve \mathcal{C} is contained in a fiber of $\text{Flag}_s(TH_{\mathbf{a}^{\bullet}}) \rightarrow H_{\mathbf{a}^{\bullet}}$, say the fiber over x . As $\mathcal{L}_{\lambda} T_x H_{\mathbf{a}^{\bullet}}$ is ample, we indeed have that

$$\mathcal{C} \cdot \mathcal{L}_{\lambda} TH_{\mathbf{a}^{\bullet}}^{\otimes |\mathbf{m}(\boldsymbol{\delta})|} \otimes \pi_{H_{\mathbf{a}^{\bullet}}}^* \mathcal{O}_{H_{\mathbf{a}^{\bullet}}}(-u) \geq 0.$$

Suppose now that $\dim(M_I) = k \geq 1$, and denote $\mathcal{C}|_I$ the restriction of \mathcal{C} to $\text{Flag}_s TM|_{M_I}$. Since \mathbf{a}^{\bullet} is taken in \mathbf{V} , according to Lemma 4.2:

$$\mathcal{C}|_I \not\subset \Psi_I^{-1}(\rho_I^{-1}(\mathbf{G}_{\infty}(I))).$$

A fortiori, by what was seen in subsubsection 4.3.1,

$$\mathcal{C}|_I \not\subset \Psi_I^{-1}(\text{Bs}(\mathcal{O}_{\mathbf{G}}(\mathbf{m}(\boldsymbol{\delta})) \boxtimes \mathcal{O}_{\mathbb{P}^N}(-1)|_{\mathcal{Y}(I)})).$$

Therefore, there exists a section $\sigma \in H^0(\mathcal{Y}(I), \mathcal{O}_{\mathbf{G}}(\mathbf{m}(\boldsymbol{\delta})) \boxtimes \mathcal{O}_{\mathbb{P}^N}(-1)|_{\mathcal{Y}(I)})$ such that

$$\mathcal{C}|_I \not\subset (\Psi_I^* \sigma = 0). \quad (19)$$

Recall that we have the following diagram:

$$\begin{array}{ccccc} \text{Flag}_s(T\mathcal{X}/\mathbf{V}) & \xrightarrow{\Psi_I} & \mathcal{Y}(I) & \longrightarrow & \mathbb{P}(\mathbf{k}_I) \\ \downarrow & & \downarrow \rho_I & & \\ \mathbf{V} & & \mathbf{G} & & \end{array}$$

so that $\Psi_I^* \sigma$ can be seen as a section of $\Psi_I^*(\mathcal{O}_{\mathbf{G}}(\mathbf{m}(\delta)) \boxtimes \mathcal{O}_{\mathbb{P}(\mathbf{k}_I)}(-1))|_{\text{Flag}_s T\mathcal{X}/\mathbf{V}}$. Restricting (18) to $\text{Flag}_s T\mathcal{X}/\mathbf{V}$, we see that $\Psi_I^* \sigma$ is associated to a section of

$$(pr_{2,\mathcal{X}}^{[1]})^*(\mathcal{L}_\lambda TM|_{M_I}^{\otimes |\mathbf{m}(\delta)|} \otimes \pi_{M_I}^* \mathcal{O}_M(-u)).$$

With (19), we deduce that

$$\mathcal{C}|_I \cdot (pr_{2,\mathcal{X}}^{[1]})^*(\mathcal{L}_\lambda TM|_{M_I}^{\otimes |\mathbf{m}(\delta)|} \otimes \pi_{M_I}^* \mathcal{O}_M(-u)) \geq 0,$$

and since $\mathcal{C} \subset \text{Flag}_s TH_{\mathbf{a}^*}$, it implies in turn that

$$\mathcal{C} \cdot \mathcal{L}_\lambda TH_{\mathbf{a}^*}^{\otimes |\mathbf{m}(\delta)|} \otimes \pi_{H_{\mathbf{a}^*}}^* \mathcal{O}_{H_{\mathbf{a}^*}}(-u) \geq 0.$$

Now, $\mathcal{L}_\lambda TH_{\mathbf{a}^*}^{\otimes |\mathbf{m}(\delta)|}$ is ample on every fiber of $\text{Flag}_s TH_{\mathbf{a}^*} \xrightarrow{\pi_{H_{\mathbf{a}^*}}} H_{\mathbf{a}^*}$, and equal to the sum of a nef divisor and the pull-back by $\pi_{H_{\mathbf{a}^*}}$ of an ample divisor. Accordingly, $\mathcal{L}_\lambda TH_{\mathbf{a}^*}^{\otimes |\mathbf{m}(\delta)|}$ is ample, and so is $\mathcal{L}_\lambda TH_{\mathbf{a}^*}$.

Finally, according to Propostion 1.5, $S^\lambda \Omega_{H_{\mathbf{a}^*}}$ is ample on $H_{\mathbf{a}^*}$. With the openness property of ampleness, we have thus proved the following theorem:

Theorem 4.3. *Let λ a partition of jump sequence $s = (s_1 > \dots > s_t)$, and let M be a variety of dimension N equipped with a very ample line bundle $\mathcal{O}_M(1)$.*

Let us choose:

- c such that $c(s_1 + 1) \geq N \Leftrightarrow c(k + 1) \geq N$.
- $\delta = (\delta_1, \dots, \delta_c)$ such that for each $1 \leq i \leq c$, $\delta_i \geq N + k(N - k)$.
- $\mathbf{m}(\delta) = (m_1, \dots, m_c)$ such that for each $1 \leq i \leq c$, $m_i \geq \frac{\prod_{j=1}^c \delta_j^{k+1}}{\delta_i}$.
- $\varepsilon = (\varepsilon_1, \dots, \varepsilon_c)$ such that for each $1 \leq i \leq c$, $\varepsilon_i \geq 1$.
- r such that $r > |\lambda_+^*| \sum_{p=1}^c m_p(\varepsilon_p + \delta_p)$.

Let us denote for each $1 \leq i \leq c$, $d_i = \delta_i(r + 1) + \varepsilon_i$. Then a complete intersection $X := H_1 \cap \dots \cap H_c$ of c generic hypersurfaces $H_i \in |H^0(M, \mathcal{O}_M(d_i))|$ has $S^\lambda \Omega_X$ ample.

Notice the specific constraint on the degrees that prevents from embracing all large enough degrees. We will now see how, following the idea of «coup product» of Xie in [Xie18], it is possible to overcome this. Let us state the key observation in [Xie18, Observation 5.6]

Lemma 4.4. *For all positive integers $d \geq 1$, every integer $d_0 \geq d(d + 1)$ is a sum of non-negative multiples of $d + 1$ and $d + 2$.*

Denote $\delta = N + k(N - k)$, and fix $\delta_p = \delta$, $m_p = \delta^{c(k+1)-1}$ for $1 \leq p \leq c$. Fix aswell $r = (2c|\lambda_+^*| \delta^{c(k+1)} - 1)$ so that $r > |\lambda_+^*| \sum_{p=1}^c m_p(\varepsilon_p + \delta_p)$ for every choice of $\varepsilon_p \in \{1, 2\}$, and

denote $d = \delta(r + 1)$. We will prove that a complete intersection of c generic hypersurfaces $H_i \in |H^0(M, \mathcal{O}_M(d_i))|$ with $d_i \geq d(d + 1)$ has $S^\lambda \Omega_X$ ample.

Fix d_1, \dots, d_c with $d_i \geq d(d + 1)$, and, using Lemma 4.4, write $d_i = p_i(d + 1) + q_i(d + 2)$, with $p_i, q_i \in \mathbb{N}$. The idea is to consider, for $1 \leq i \leq c$, hypersurfaces of degree d_i defined by

$$H_i = h_{i,1} \dots h_{i,p_i} \cdot h_{i,p_i+1} \dots h_{i,p_i+q_i}$$

where $h_{i,1}, \dots, h_{i,p_i} \in H^0(M, \mathcal{O}_M(d+1))$ and $h_{i,p_i+1}, \dots, h_{i,p_i+q_i} \in H^0(M, \mathcal{O}_M(d+2))$. Let $(h_{i,j})_{\substack{1 \leq j \leq p_i+q_i \\ 1 \leq i \leq c}}$ be a generic choice of equations, and let $X = (H_1 = 0) \cap \dots \cap (H_c = 0)$.

Recall that $\text{Flag}_s TX$ is locally defined inside $\text{Flag}_s TM$ by:

$$\text{Flag}_s TX \stackrel{\text{loc}}{=} \{(x, (F_1 \supseteq \dots \supseteq F_t)) \mid \forall 1 \leq i \leq c, H_i(x) = 0, dH_i((x, v)) = 0 \forall v \in F_1\}.$$

If we restrict ourselves to regular points X_{reg} of X , we can decompose $\text{Flag}_s TX_{\text{reg}}$ as a union of subvarieties as follows. For $x \in X_{\text{reg}}$, each pair of equations

$$H_i(x) = 0, dH_i((x, \cdot)) = 0$$

decomposes as

$$h_{i,k}(x) = 0, dh_{i,k}((x, \cdot)) = 0$$

for a unique $1 \leq k \leq p_i + q_i$, since x cannot belong to $(h_{i,k} = 0) \cap (h_{i,k'} = 0)$ for $k \neq k'$ as this would imply that x is a singular point. Accordingly, we can decompose $\text{Flag}_s TX_{\text{reg}}$ as the disjoint union

$$\bigsqcup_{1 \leq k_i \leq p_i + q_i} \text{Flag}_s(T(X_{k_1, \dots, k_c} \cap X_{\text{reg}})) \quad (20)$$

where $X_{k_1, \dots, k_c} = (h_{1,k_1} = 0) \cap \dots \cap (h_{c,k_c} = 0)$.

Let us start with a lemma justifying that we can find $(h_{i,j})_{\substack{1 \leq j \leq p_i+q_i \\ 1 \leq i \leq c}}$ such that $\mathcal{L}_\lambda TX_{k_1, \dots, k_c}$ is ample for every k_1, \dots, k_c :

Lemma 4.5. *There exists a choice of $(h_{i,j})_{\substack{1 \leq j \leq p_i+q_i \\ 1 \leq i \leq c}}$ such that, for every $(1 \leq k_i \leq p_i + q_i)_{1 \leq i \leq c}$, X_{k_1, \dots, k_c} is smooth and the line bundle $\mathcal{L}_\lambda TX_{k_1, \dots, k_c} \rightarrow \text{Flag}_s TX_{k_1, \dots, k_c}$ is ample.*

Proof. Let us denote S_1 the space of parameters of $(d+1)$ -degree hypersurfaces, and S_2 the space of parameters of $(d+2)$ -degree hypersurfaces. For $1 \leq k_i \leq p_i + q_i$, with $1 \leq i \leq c$, denote aswell

$$\mathbf{S}_{k_1, \dots, k_c} = S_{\varepsilon_1(k_1)} \times \dots \times S_{\varepsilon_c(k_c)}$$

where $\varepsilon_i(k) = 1$ for $1 \leq k \leq p_i$ and $\varepsilon_i(k) = 2$ for $p_i + 1 \leq k \leq p_i + q_i$. By Theorem 4.3, there exists an open set $\mathbf{S}_{k_1, \dots, k_c}^\circ \subset \mathbf{S}_{k_1, \dots, k_c}$ such that every complete intersection $X_s \subset M$ defined by $s \in \mathbf{S}_{k_1, \dots, k_c}^\circ$ is smooth and has $\mathcal{L}_\lambda TX_s \rightarrow \text{Flag}_s TX_s$ ample.

Let $p_i^{k_1, \dots, k_c}$ be the projection of $\mathbf{S}_{k_1, \dots, k_c}$ onto the i th factor; note that $p_i^{k_1, \dots, k_c}(\mathbf{S}_{k_1, \dots, k_c}^\circ)$ is a dense open subset of $S_{\varepsilon_i(k_i)}$. First, for $1 \leq j \leq p_1 + q_1$, take $h_{1,j}$ in

$$\bigcap_{k_2, \dots, k_c} p_1^{j, k_2, \dots, k_c}(\mathbf{S}_{j, k_2, \dots, k_c}^\circ).$$

Second, for $1 \leq j \leq p_2 + q_2$, take $h_{2,j}$ in

$$\bigcap_{k_1, k_3, \dots, k_c} p_2^{k_1, j, \dots, k_c}((p_1^{k_1, j, \dots, k_c})^{-1}(\{h_{1, k_1}\}) \cap \mathbf{S}_{k_1, j, \dots, k_c}^\circ)$$

Keeping going in this fashion, one constructs a family $(h_{i,j})_{\substack{1 \leq j \leq p_i+q_i \\ 1 \leq i \leq c}}$ that has the required property. \square

Let $\mathcal{X}^1 \xrightarrow{\Pi} \mathbf{T}$ be the universal family of complete intersection of c hypersurfaces of dimension d_1, \dots, d_c . Denote $\mathcal{X}^{1,reg}$ the open subset whose fiber over $t \in \mathbf{T}$ is the regular points of the complete intersection X_t determined by t , and Π_{reg} the restriction of Π to $\mathcal{X}^{1,reg}$. Consider $\text{Flag}_s(T\mathcal{X}_{/\mathbf{T}}^{1,reg}) \subset \mathbf{T} \times \text{Flag}_s TM$ the flag bundle of the relative tangent bundle, and denote $Pr_1: \mathbf{T} \times \text{Flag}_s TM \rightarrow \mathbf{T}$ the first projection as well as $Pr_2: \mathbf{T} \times \text{Flag}_s TM \rightarrow \text{Flag}_s TM$ the second projection.

Consider $\overline{\text{Flag}_s(T\mathcal{X}_{/\mathbf{T}}^{1,reg})}$ the closure of $\text{Flag}_s(T\mathcal{X}_{/\mathbf{T}}^{1,reg})$ inside $\mathbf{T} \times \text{Flag}_s TM$, and denote $Pr_{1,\overline{\mathcal{X}^{1,reg}}}$ and $Pr_{2,\overline{\mathcal{X}^{1,reg}}}$ the restriction of Pr_1 and Pr_2 to this closed set.

Observation 1: If $t \in \mathbf{T}$ defines a smooth complete intersection X_t , then

$$(Pr_{1,\overline{\mathcal{X}^{1,reg}}})^{-1}(t) = \text{Flag}_s TX_t.$$

Indeed, as X_t is smooth, $(Pr_{1,\mathcal{X}^{1,reg}})^{-1}(t) = \text{Flag}_s TX_t$ is already closed.

Let us pick $(h_{i,j})_{\substack{1 \leq j \leq p_i + q_i \\ 1 \leq i \leq c}}$ as in the Lemma 4.5, and let $t_0 \in \mathbf{T}$ such that

$$X_{t_0} = X = (H_1 = 0) \cap \dots \cap (H_c = 0),$$

where we keep the notations introduced above.

Observation 2:

$$(Pr_{1,\overline{\mathcal{X}^{1,reg}}})^{-1}(t_0) = \bigcup_{1 \leq k_i \leq p_i + q_i} \text{Flag}_s(TX_{k_1, \dots, k_c}).$$

Indeed, by (20),

$$(Pr_{1,\mathcal{X}^{1,reg}})^{-1}(t_0) = \bigsqcup_{1 \leq k_i \leq p_i + q_i} \text{Flag}_s(T(X_{k_1, \dots, k_c} \cap X_{reg}))$$

Then, the Observation 2 follows from the following lemma:

Lemma 4.6. *Let $Y \subset M$ a smooth subvariety of M of dimension n , and $U \subset Y$ a dense open subset of Y . Then:*

$$\overline{\text{Flag}_s TU} = \text{Flag}_s TY$$

where the closure is taken in $\text{Flag}_s TM$.

Proof. Fix $y \in Y$, and $V \ni y$ a trivializing open set for $TY|_{V \cap Y}$ and $TM|_V$, so that

$$\text{Flag}_s(T(V \cap Y)) \simeq (V \cap Y) \times \text{Flag}_s \mathbf{k}^n \text{ and } \text{Flag}_s TV \simeq V \times \text{Flag}_s \mathbf{k}^N.$$

One has as well

$$\text{Flag}_s T(U \cap (V \cap Y)) \simeq (U \cap (V \cap Y)) \times \text{Flag}_s \mathbf{k}^n,$$

so that the closure of $\text{Flag}_s T(U \cap (V \cap Y))$ in $\text{Flag}_s TV$ is $\text{Flag}_s(T(Y \cap V))$, which yields the result. \square

Let us now conclude. Let $L = (Pr_{2,\overline{\mathcal{X}^{1,reg}}})^* \mathcal{L}_\lambda TM$, and denote $L_t := L|_{Pr_{2,\overline{\mathcal{X}^{1,reg}}}^{-1}(t)}$. With Observation 1, note that for $t \in \mathbf{T}$ such that X_t is smooth,

$$L_t = \mathcal{L}_\lambda TX_t,$$

and that for t_0 as fixed above

$$(L_{t_0})|_{\text{Flag}_s TX_{k_1, \dots, k_c}} = \mathcal{L}_\lambda TX_{k_1, \dots, k_c}.$$

The very choice of t_0 makes $\mathcal{L}_\lambda TX_{k_1, \dots, k_c}$ ample for every k_1, \dots, k_c , by the Lemma 4.5. By Observation 2, we deduce that L_{t_0} is ample as this is ample on every irreducible component of $(Pr_{1, \overline{\mathcal{X}^{1, reg}}})^{-1}(t_0)$. By openness of ampleness in family for $\overline{\text{Flag}_s(T\mathcal{X}_{\mathbf{T}}^{1, reg})} \rightarrow \mathbf{T}$, L_t remains ample for t in a neighborhood of t_0 , and since $\{t \in \mathbf{T} \mid X_t \text{ is smooth}\}$ is a dense open subset of \mathbf{T} , there exists $t \in \mathbf{T}$ such that $\mathcal{L}_\lambda TX_t$ is ample. Once again by openness of ampleness in family (this time for the family $\text{Flag}_s(T\mathcal{X}_{\mathbf{T}}^1) \rightarrow \mathbf{T}$), we deduce the following corollary:

Corollary 4.7. *Let λ a partition of jump sequence $s = (s_1 := k > \dots > s_t)$, and let M be a variety of dimension N equipped with a very ample line bundle $\mathcal{O}_M(1)$. Then a complete intersection $X := H_1 \cap \dots \cap H_c$ of c generic hypersurfaces $H_i \in |H^0(M, \mathcal{O}_M(d_i))|$ with $d_i \geq (1 + 2c|\lambda_+^*|(N + k(N - k))^{c(k+1)+1})^2$ has $S^\lambda \Omega_X$ ample.*

5. APPLICATIONS IN HYPERBOLICITY

A particular case of 4.7 for $\lambda = (1^k)$ implies that a generic complete intersection $X = H_1 \cap \dots \cap H_c$ of codimension $c \geq \frac{N}{k+1}$ with $H_i \in \mathcal{O}_M(d_i)$, $d_i \geq (1 + 2c(k+1)(N + k(N - k))^{c(k+1)+1})^2$, has $\bigwedge^k \Omega_X$ ample. We will explain in the sequel what can be deduced of X in term of k -infinitesimal hyperbolicity.

5.1. p -infinitesimal hyperbolicity. We briefly recall the notion of p -infinitesimal hyperbolicity, and we refer to [Dem] for more details on the subject. Let X be a compact complex manifold of dimension N , and let $1 \leq p \leq N$. Define the so-called *p -Kobayashi-Eisenman infinitesimal pseudometric* as the map defined on all decomposable p -vectors of $\bigwedge^p TX$ by:

$$e^p((x, v_1 \wedge \dots \wedge v_p)) = \inf \left\{ \frac{1}{R} > 0 \mid \exists f: \mathbb{B}_p \rightarrow X, f(0) = x, f_*(\partial/\partial t_1 \wedge \dots \wedge \partial/\partial t_p) = R \cdot v_1 \wedge \dots \wedge v_p \right\}$$

where \mathbb{B}_p is the unit ball in \mathbb{C}^p and $R \in \mathbb{R}_{>0}$. Consider $\text{Grass}(p, TX) \subset \mathbb{P}(\bigwedge^p TX)$, and denote $\mathcal{O}(-1)$ the restriction to $\text{Grass}(p, TX)$ of the tautological line bundle on $\mathbb{P}(\bigwedge^p TX)$. Define a function $H^p: \mathcal{O}(-1) \rightarrow \mathbb{R}_+$ by setting for $x \in X$ and $\xi = v_1 \wedge \dots \wedge v_p$:

$$H^p(((x, [\xi]), \xi)) = e^p((x, \xi)).$$

We say that X is *p -infinitesimally hyperbolic* if for any smooth metric ω on $\mathcal{O}(-1)$, there exists $\varepsilon > 0$ such that

$$H^p > \varepsilon \omega.$$

For the case $p = 1$, it is a theorem (cf. [Dem, Prop. 1.5] for instance) that being 1-infinitesimally hyperbolic is equivalent to the fact that every holomorphic map $f: \mathbb{C} \rightarrow X$ is constant, which, with a theorem of Brody ([Bro78]), is equivalent to being hyperbolic in the sense of Kobayashi.

Similarly, observe that for $p \geq 1$, being p -hyperbolic implies in particular that every holomorphic map $f: \mathbb{C} \times \mathbb{B}^{p-1} \rightarrow X$ must be degenerate in the sense that the jacobian

matrix of f is never of maximal rank. Indeed, otherwise, take z such that df_z is injective. Without loss of generality, suppose $z = 0$, and consider

$$f_R : B^p \rightarrow X, (z_1, z_2, \dots, z_p) \mapsto f(Rz_1, z_2, \dots, z_p).$$

Then, denoting $\xi = f_*(\partial/\partial t_1 \wedge \dots \wedge \partial/\partial t_p) \neq 0$, one has

$$(f_R)_*(\partial/\partial t_1 \wedge \dots \wedge \partial/\partial t_p) = R\xi$$

so that, letting $R \rightarrow \infty$, one sees that $(f(0), \xi)$ is a degenerate point for \mathbf{e}^p , which contradicts the p -infinitesimal hyperbolicity.

As a generalization of a theorem due to Kobayashi ([Kob13]) asserting that ampleness of the cotangent bundle implies hyperbolicity, we have the following (which is a particular case of [Dem, Prop. 3.4]):

Proposition 5.1 ([Dem]). *Let X be a compact complex manifold, with $\bigwedge^k \Omega_X$ ample. Then X is k -infinitesimal hyperbolic.*

This proposition combined with the Corollary 4.7 gives the following.

Corollary 5.2. *Let $X = H_1 \cap \dots \cap H_c$ be a generic complete intersection in \mathbb{P}^N of codimension $c \geq \frac{N}{k+1}$, with $H_i \in \mathcal{O}_{\mathbb{P}^N}(d_i)$, $d_i \geq (1 + 2c(k+1)(N+k(N-k))^{c(k+1)+1})^2$. Then X is k -infinitesimally hyperbolic.*

The Kobayashi conjecture for general hypersurfaces in projective spaces was recently proved by Brotbek in [Bro17]: a general hypersurface in \mathbb{P}^N of degree greater than a bound $d_N = (N+1)^{2N+6}$ is hyperbolic. Let us mention here that, as hyperbolic implies k -infinitesimally hyperbolic for $1 \leq k \leq N$, the previous result of Brotbek implies in particular this result (bounds included). However, using 4.3 instead of 4.7, it is possible to construct particular examples that are not covered by Brotbek's theorem. For instance, a generic complete intersection $H_1 \cap H_2$ of codimension 2, with $H_i \in H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d))$, $d \geq d'_N$, where $d'_N = 4(\lceil \frac{N}{2} - 1 \rceil + 1)(N + \lceil \frac{N}{2} - 1 \rceil(N - \lceil \frac{N}{2} - 1 \rceil))^{N+1}$, is $(N - \lceil \frac{N}{2} - 1 \rceil)$ -infinitesimally hyperbolic. Note that $d'_N \leq 2(N+1)(\frac{N-2}{2})^{2N+2}$ is indeed a better bound than d_N for large N .

ACKNOWLEDGEMENTS

I would like to thank my thesis supervisor Erwan Rousseau as well as my co-supervisor Lionel Darondeau for their help and support. This work is particularly in keeping of Lionel Darondeau and Damian Brotbek's work on the ampleness of the cotangent bundle of complete intersections in projective spaces, and Lionel's advices has substantially helped me carry it through.

REFERENCES

- [BD18] Damian Brotbek and Lionel Darondeau. Complete intersection varieties with ample cotangent bundles. *Inventiones mathematicae*, 212(3):913–940, 2018.
- [Ben11] Olivier Benoist. Le théorème de Bertini en famille. *Bull. Soc. Math. France*, 139(4):555–569, 2011.
- [Bot57] Raoul Bott. Homogeneous vector bundles. *Annals of mathematics*, pages 203–248, 1957.
- [BR90] P Brückmann and H-G Rackwitz. T-symmetrical tensor forms on complete intersections. *Mathematische Annalen*, 288(1):627–635, 1990.

- [Bro78] Robert Brody. Compact manifolds in hyperbolicity. *Transactions of the American Mathematical Society*, 235:213–219, 1978.
- [Bro16] Damian Brotbek. Symmetric differential forms on complete intersection varieties and applications. *Mathematische Annalen*, 366(1-2):417–446, 2016.
- [Bro17] Damian Brotbek. On the hyperbolicity of general hypersurfaces. *Publications mathématiques de l’IHÉS*, 126(1):1–34, 2017.
- [Deb05] Olivier Debarre. Varieties with ample cotangent bundle. *Compositio Mathematica*, 141(6):1445–1459, 2005.
- [Dem] JP Demailly. Algebraic criteria for Kobayashi hyperbolic projective varieties and jet differentials, algebraic geometry—santa cruz 1995, 285–360. 62(Part 2).
- [Dem88] Jean-Pierre Demailly. Vanishing theorems for tensor powers of an ample vector bundle. *Inventiones mathematicae*, 91(1):203–220, 1988.
- [Den16] Ya Deng. Effectivity in the hyperbolicity-related problems. *arXiv preprint arXiv:1606.03831*, 2016.
- [Kob13] Shoshichi Kobayashi. *Hyperbolic complex spaces*, volume 318. Springer Science & Business Media, 2013.
- [Laz04] Robert Lazarsfeld. *Positivity in algebraic geometry. A Series of Modern Surveys in Mathematics*. Springer Berlin, 2004.
- [LN18] Fatima Laytimi and Werner Nahm. Ampleness equivalence and dominance for vector bundles. *Geometriae Dedicata*, 2018. First Online: 19 May 2018.
- [Wey03] Jerzy Weyman. *Cohomology of vector bundles and syzygies*, volume 149. Cambridge University Press, 2003.
- [Xie18] Song-Yan Xie. On the ampleness of the cotangent bundles of complete intersections. *Inventiones mathematicae*, 212(3):941–996, 2018.

AIX MARSEILLE UNIV, CNRS, CENTRALE MARSEILLE, I2M, MARSEILLE, FRANCE
E-mail address: antoine.ettesse@univ-amu.fr