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p-value presheaves

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p-value presheaves

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1. Introduction

This report presents a preliminary study of the sheaf-ness properties of the p-value.

2. Notation

$\mathcal{B}(\mathbb{R}^n)$ is the Borel set (the smallest σ -algebra containing all open sets of \mathbb{R}^n).

3. P-values

Definition 3.1 (Size). A *size* is a function $\alpha : \mathcal{B}(\mathbb{R}^n) \rightarrow [0, 1]$.

Definition 3.2 (Retract). Let $\alpha : \mathcal{B}(\mathbb{R}^n) \rightarrow [0, 1]$ be a size.

A *retract* is a function $R : [0, 1] \rightarrow \mathcal{B}(\mathbb{R}^n)$ such that $\alpha \circ R = \text{id}_{[0,1]}$.

In other words, a retract is a right inverse for a size.

Definition 3.3 (Test). A *family of tests*, or simply *test*, is a retract $T : [0, 1] \rightarrow \mathcal{B}(\mathbb{R}^n)$ such that $T(0) = \emptyset$, $T(1) = \mathbb{R}^n$ and $a \leq a' \Rightarrow T(a) \leq T(a')$.

Remark 3.4. If $T : [0, 1] \rightarrow \mathcal{B}(\mathbb{R}^n)$ is a test, then for $a \in [0, 1]$, $T(a)$ is the rejection region of the null hypothesis with size a .

Definition 3.5 (p-value). Let T be a test.

For $x \in \mathbb{R}^n$, the p-value of x is defined as:

$$\text{pval}_T(x) = \inf(\{a \in [0, 1] \mid x \in T(a)\})$$

The p-value of x is the minimum size of the test that puts x into the rejection region.

Definition 3.6 (Topological space \mathcal{T}). For all $a \in [0, 1]$, we define $U_a = [0, a[$ and $\mathcal{T} = \{U_a \mid a \in [0, 1]\}$.

It is easy to see that:

Proposition 3.7. $([0, 1[, \mathcal{T})$ is a topological space.

Remark 3.8. Note that $U_a \cap U_b = U_{\min(a,b)}$ and $\bigcup_{a \in A} U_a = U_{\sup(A)}$.

Definition 3.9 (P-value sheaf). Let $T : [0, 1] \rightarrow \mathcal{B}(\mathbb{R}^n)$ be a test.

The p-value sheaf for T is the following functor:

$$S_T : \begin{cases} \mathcal{T}^{\text{op}} & \longrightarrow & \mathbf{Sets} \\ U_a & \longmapsto & \{x \in \mathbb{R}^n \mid \text{pval}_T(x) \geq a\} \\ U_a \subseteq U_b & \longmapsto & \begin{cases} S_T(U_b) & \longrightarrow & S_T(U_a) \\ x & \longmapsto & x \end{cases} \end{cases}$$

Proposition 3.10. Let $T : [0, 1] \rightarrow \mathcal{B}(\mathbb{R}^n)$ be a test and let S_T be its p-value sheaf.

Then, S_T is an actual sheaf.

Proof. It's easy to see that S_T is a presheaf: it's a functor $\mathcal{T}^{\text{op}} \rightarrow \mathbf{Sets}$. We now have to check the sheaf condition.

Let $A \subset [0, 1[$ and $(U_a)_{a \in A} \in \mathcal{T}^A$ be a cover of $U \in \mathcal{T}$. We have $U = \bigcup_{a \in A} U_a$. Let $(s_a)_{a \in A}$ be a matching family of S_T -sections over $(U_a)_{a \in A}$: $s_a \in S_T(U_a)$ and $s_a|_{U_a \cap U_b} = s_b|_{U_a \cap U_b}$.

By definition of S_T , we have:

$$s_a|_{U_a \cap U_b} = S_T(U_a \cap U_b \subset U_a)(s_a) = s_a$$

which yields, for all $a, b \in A$, $s_a = s_b$. Let $s = s_a$; then s satisfies $\forall a \in A$, $\text{pval}_T(s) \geq a$, so $\text{pval}_T(s) \geq \sup(A)$ and $s \in S_T(U_{\sup(A)}) = S_T\left(\bigcup_{a \in A} U_a\right)$. Consequently, s is the unique gluing of $(s_a)_{a \in A}$, and S_T satisfies the sheaf condition. \square

Consider the sheaf topos based on \mathcal{T} : $\mathbf{Shv}([0, 1[, \mathcal{T})$. Its subobject classifier is:

$$\Omega : \begin{cases} \mathcal{T}^{\text{op}} & \longrightarrow & \mathbf{Sets} \\ U_a & \longmapsto & \{U_{a'} \mid a' \leq a\} \\ U_a \subset U_b & \longmapsto & \begin{cases} \Omega(U_b) & \longrightarrow & \Omega(U_a) \\ U_c & \longmapsto & U_a \cap U_c = U_{\min(a,c)} \end{cases} \end{cases}$$

Let T be a test and S_T its associated p-value sheaf. A sheaf morphism $p : S_T \rightarrow \Omega$ makes the following diagram commute for all $a \leq b$:

$$\begin{array}{ccc} U_a & & S_T(U_b) \xrightarrow{p_{U_b}} \Omega(U_b) \\ \downarrow u & \rightsquigarrow & \downarrow S_T(u) \quad \checkmark \quad \downarrow \Omega(u) \\ U_b & & S_T(U_a) \xrightarrow{p_{U_a}} \Omega(U_a) \end{array}$$

For $x \in S_T(U_b)$, we have:

$$\begin{aligned} p_{U_a} \circ S_T(u)(x) &= \Omega(u) \circ p_{U_b}(x) \\ p_{U_a}(x) &= p_{U_b}(x) \cap U_a \end{aligned}$$

The canonical example is the following natural transformation:

$$p_{U_a} : \begin{cases} S_T(U_a) & \longrightarrow & \Omega(U_a) \\ x & \longmapsto & U_a \end{cases}$$

4. RDT and sheaves

Consider the set-theoretic integer $n \in \mathbb{N}$, $n = \{0, \dots, n-1\}$. Endow n with the discrete topology $(n, \mathcal{P}(n))$. For $I \subset n$, \mathbb{R}^I is the set of functions $I \rightarrow \mathbb{R}$; equivalently, it is the set of I -indexed sets of real numbers.

Let Π_n be the following functor:

$$\Pi_n : \begin{cases} \mathcal{P}(n) & \longrightarrow & \mathbf{Sets} \\ I & \longmapsto & \mathbb{R}^I \\ I \subset J & \longmapsto & \begin{cases} \Pi_n(J) & \longrightarrow & \Pi_n(I) \\ f & \longmapsto & f|_I \end{cases} \end{cases}$$

Proposition 4.1. Π_n is a sheaf.

For $I \subset n$, $\|\cdot\|_{\mathbb{R}^I}$ is the Euclidean norm on \mathbb{R}^I .

Definition 4.2 (τ functor). We define the functor τ by setting:

$$\tau : \begin{cases} \mathcal{P}(n) & \longrightarrow & \mathcal{T} \\ I & \longmapsto & \tau(I) \\ I \subset J & \longmapsto & \tau(I) \subset \tau(J) \end{cases}$$

where \mathcal{T} is the topology defined in Definition 3.6. For $I \in \mathcal{P}(n)$, the unique a such that $\tau(I) = [0, a[$ is denoted τ_I , so that: $\tau(I) = [0, \tau_I[$.

The interest of this definition will tentatively be given later.

Let $f, \theta \in \mathbb{R}^n$. For all $I \in \mathcal{P}(n)$, the proposition: $\|\theta|_I - f|_I\|_{\mathbb{R}^I} \leq \tau_I$ is a proposition the truth of which can be known by simple inspection, provided that θ is known. In practice, we don't have access to θ but to a modified, noisy version $Y = Y(\theta)$ of it, and the question is then, for any given $I \subset n$, whether the proposition $\|\theta|_I - f|_I\|_{\mathbb{R}^I} \leq \tau_I$ is plausible in a certain sense or not, when we observe $Y|_I$.

Definition 4.3 (Optimal pair). Let $f \in \mathbb{R}^n$, we call it model. Let Θ be a random vector whose distribution we don't know, and $X \sim \mathcal{N}(0, C^2)$ is a n -dimensional Gaussian vector. We observe $Y = \Theta + X$.

For all $I \subset n$, there exists a pair $(\alpha_I^{f,\tau}, T_I^{f,\tau})$ [1], called *optimal pair restricted to I* , where:

1. $\alpha_I^{f,\tau} : \mathcal{B}(\mathbb{R}^I) \rightarrow [0, 1]$ is a size
2. $T_I^{f,\tau} : [0, 1] \rightarrow \mathcal{B}(\mathbb{R}^I)$ is a monotone retract of $\alpha_I^{f,\tau}$
3. $T_I^{f,\tau}$ is optimal for testing $\|\Theta|_I - f|_I\|_{\mathbb{R}^I} \leq \tau_I$ against its alternative $\|\Theta|_I - f|_I\|_{\mathbb{R}^I} \geq \tau_I$ when we observe $Y(\Theta)$.

Definition 4.4 (Restricted p-value). We consider the τ functor described above. Let $I \subset n$ and let $(\alpha_I^{f,\tau}, T_I^{f,\tau})$ be an optimal pair restricted to I .

The *I -restricted p-value of x* is the following value:

$$\text{pval}_I^{f,\tau}(y) = \inf \left(\left\{ a \in [0, 1] \mid y \in T_I^{f,\tau}(a) \right\} \right)$$

The *I -restricted p-value sheaf* is the following sheaf:

$$S_I^{f,\tau} : \begin{cases} [0, 1] & \longrightarrow & \mathbb{R}^I \\ U_a & \longmapsto & \left\{ y \in \mathbb{R}^n \mid \text{pval}_I^{f,\tau}(y) \geq a \right\} \\ U_a \subseteq U_b & \longmapsto & \begin{cases} S_T(U_b) & \longrightarrow & S_T(U_a) \\ x & \longmapsto & x \end{cases} \end{cases}$$

The p-value $\text{pval}_I^{f,\tau}(y)$ measures the plausability of proposition $\|\Theta|_I - f|_I\|_{\mathbb{R}^I} \leq \tau_I$ when we observe $y \sim Y = \Theta + X$. The higher $\text{pval}_I^{f,\tau}(y)$, the more plausible the proposition $\|\Theta|_I - f|_I\|_{\mathbb{R}^I} \leq \tau_I$.

5. First results

5.1. About p-value sheaves

It is easy to see that:

Proposition 5.1. (\mathcal{T}, \subset) is a total order with a minimal and maximal element (resp. \emptyset and \mathbb{R}^d).

It is also easy, but less easy, to see that:

Proposition 5.2. \mathcal{T}^{op} has all small limits and colimits.

Proof. Let $D : \mathcal{I} \rightarrow \mathcal{T}^{\text{op}}$ be a diagram in \mathcal{T}^{op} . For all $i \in \mathcal{I}$, define $U_{d_i} = D(i)$. Then:

$$\begin{aligned} \text{Colim}(D) &= \sup_{i \in \text{Ob}_{\mathcal{I}}} D_i = \bigcap_{i \in \text{Ob}_{\mathcal{I}}} U_{d_i} = U_{\inf(\mathcal{I})} \\ \text{Lim}(D) &= \inf_{i \in \text{Ob}_{\mathcal{I}}} D_i = \bigcup_{i \in \text{Ob}_{\mathcal{I}}} U_{d_i} = U_{\sup(\mathcal{I})} \end{aligned}$$

Note that the sup becomes an intersection, and the inf becomes a union, because we are considering \mathcal{T}^{op} and not \mathcal{T} . Also note that the infima and suprema always exist because these are infima and suprema of $\{d_i \mid i \in \text{Ob}_{\mathcal{I}}\}$, which is a subset of \mathbb{R} . \square

Proposition 5.3. For any test T , its p-value sheaf S_T is continuous and cocontinuous.

Proof. Let $D : \mathcal{I} \rightarrow \mathcal{T}^{\text{op}}$ be any (small) diagram in \mathcal{T}^{op} . For all $i \in \mathcal{I}$, define $U_{d_i} = D(i)$.

We only consider the case of a limit; the proof is very similar for colimits.

$$\begin{aligned}
 S_T(\text{Lim}(D)) &= S_T\left(\bigcup_{i \in \mathcal{I}} U_{d_i}\right) \\
 &= S_T(U_{\sup(d_i)}) \\
 &= \left\{x \in \mathbb{R}^d \mid \text{pval}_T(x) \geq \sup_{i \in \mathcal{I}}(d_i)\right\} \\
 &= \left\{x \in \mathbb{R}^d \mid \forall i \in \mathcal{I}, \text{pval}_T(x) \geq d_i\right\} \\
 &= \bigcap_{i \in \mathcal{I}} \left\{x \in \mathbb{R}^d \mid \text{pval}_T(x) \geq d_i\right\} \\
 &= \bigcap_{i \in \mathcal{I}} S_T(U_{d_i})
 \end{aligned}$$

We now have to check that $\bigcap_{i \in \mathcal{I}} S_T(U_{d_i}) \cong \text{Lim}(S_T \circ D)$.

For all $i \rightarrow j \in \mathcal{I}$, we have $S_T(U_{d_i}) \subset S_T(U_{d_j})$. We denote by $\iota_{i,j} = S_T(U_{d_i} \subset U_{d_j}) : S_T(U_{d_i}) \rightarrow S_T(U_{d_j})$ the inclusion mapping between $S_T(U_{d_i})$'s. We also denote by $\iota_i : \bigcap_{i \in \mathcal{I}} S_T(U_{d_i}) \rightarrow S_T(U_{d_i})$ the inclusion mapping of the intersection. For all $i \rightarrow j \in \mathcal{I}$, we have $\iota_{i,j} \circ \iota_i = \iota_j$, so that $\iota = \left(\iota_i : \bigcap_{i \in \mathcal{I}} S_T(U_{d_i}) \rightarrow S_T(U_{d_i})\right)_{i \in \mathcal{I}}$ is a cone to $S_T \circ D$.

Let (A, α) be any cone to $S_T \circ D$. We denote the $S_T(U_{d_i})$ -components of α by $\alpha_i : A \rightarrow S_T(U_{d_i})$. For all $i \rightarrow j \in \mathcal{I}$, and for all $x \in X$, we have $\iota_{i,j} \circ \alpha_i(x) = \alpha_j(x) = \alpha_i(x)$.

$$\begin{array}{ccc}
 S_T(U_{d_i}) & \xleftarrow{\alpha_i} & A \\
 \downarrow \iota_{i,j} & \swarrow \alpha_j & \\
 S_T(U_{d_j}) & &
 \end{array}$$

So in fact, $\alpha_i(X) = \alpha_j(X) \subset S_T(U_{d_i})$, which yields that, for all $i \in \mathcal{I}$, $\alpha_i(X) \subset \bigcap_{i \in \mathcal{I}} S_T(U_{d_i})$.

Let u be such that:

$$u : \begin{cases} X & \longrightarrow \bigcap_{i \in \mathcal{I}} S_T(U_{d_i}) \\ x & \longmapsto \alpha_i(x) \end{cases}$$

for any of the $i \in \mathcal{I}$, because $\alpha_j(x) = \alpha_i(x)$. Then for all $i \in \mathcal{I}$, we have $\iota_i \circ u = \alpha_i$.

It is also easy to check the unicity of that u : suppose $u, u' : X \rightarrow \bigcap_{i \in \mathcal{I}} S_T(U_{d_i})$, then for all $x \in X$, we have:

$$\begin{aligned}
 \iota_i \circ u(x) &= \alpha_i(x) = \iota_i \circ u'(x) \\
 u(x) &= u'(x)
 \end{aligned}$$

which leads to $u = u'$.

Consequently, $\bigcap_{i \in \mathcal{I}} S_T(U_{d_i}) \cong \text{Lim}(S_T \circ D)$. □

5.2. About sheaf morphisms

Let us first study the sheaf morphisms between p-value sheaves $p : S_R \rightarrow S_T$.

Proposition 5.4 (Nesting property). *Let $R, T : [0, 1] \rightarrow \mathcal{B}(\mathbb{R}^n)$ be two tests (of the same size or not). Let $p : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a function.*

The function p defines a natural transformation $\bar{p} : S_R \rightarrow S_T = (p|_{S_R(U_a)})_{a \in [0,1]} \Leftrightarrow$ for all $a \in [0, 1]$, $p(S_R(U_a)) \subset S_T(U_a)$.

In other words, p defines a function between nested open sets, as in Figure 1.

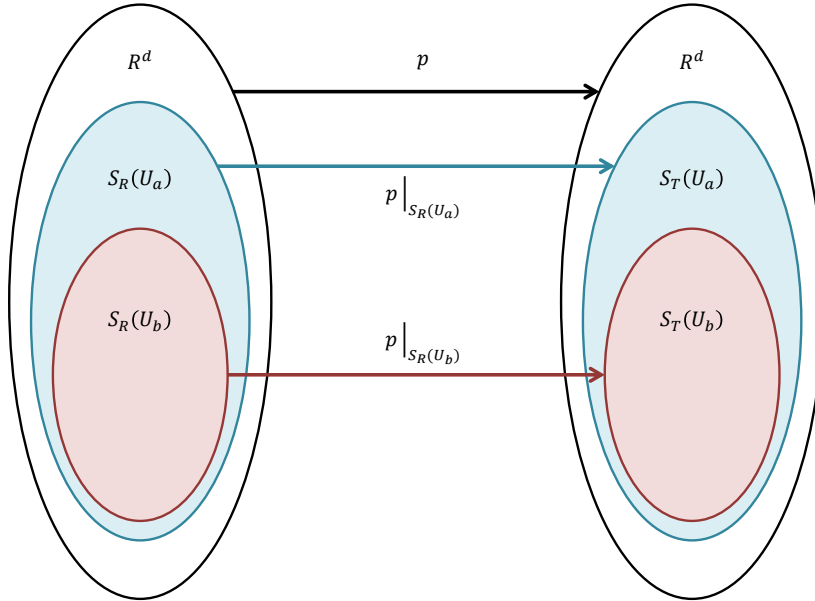


Figure 1: Illustration of the nesting property of p . If $p : \mathbb{R}^d \rightarrow \mathbb{R}^d$ verifies this nesting property, then it defines a natural transformation between two p-value sheaves.

Proof. Easy deduction from the following natural transformation diagram:

$$\begin{array}{ccc}
 U_a & & S_R(U_b) \xrightarrow{p|_{S_R(U_b)}} S_T(U_b) \\
 \downarrow \subset & \rightsquigarrow & \downarrow \subset \quad \checkmark \quad \downarrow \subset \\
 U_b & & S_R(U_a) \xrightarrow{p|_{S_R(U_a)}} S_T(U_a)
 \end{array}$$

□

Corollary 5.5. *If $\bar{p} = (p_{U_a})_{a \in [0,1]}$ is a natural transformation $S_R \rightarrow S_T$, then $p = \bigcup_{a \in A} p_{U_a}$ is a function that verifies the nesting property.*

Proposition 5.6. *For all $a \in [0, 1]$, $S_T(U_a) = \mathbb{R}^d \setminus \left(\bigcup_{b < a} T(b) \right)$.*

Proof. By computation:

$$\begin{aligned}
 S_T(U_a) &= \{x \in \mathbb{R}^d \mid \text{pval}_T(x) \geq a\} \\
 &= \left\{x \in \mathbb{R}^d \mid \inf_{x \in T(b)} (b \in [0, 1]) \geq a\right\} \\
 &= \{x \in \mathbb{R}^d \mid \forall b < a, x \notin T(b)\} \\
 &= \bigcap_{b < a} \mathbb{R}^d \setminus T(b) \\
 &= \mathbb{R}^d \setminus \left(\bigcup_{b < a} T(b)\right)
 \end{aligned}$$

□

Corollary 5.7. For all $a \in [0, 1]$, if $\bigcup_{b < a} T(b) = T(a)$ then $S_T(U_a) = \mathbb{R}^d \setminus T(a)$.

Example 5.8. Note that we are considering very general tests $T : [0, 1] \rightarrow \mathcal{B}(\mathbb{R}^n)$, so $\left(\bigcup_{b < a} T(b)\right)$ has no reason to be equal to $T(a)$. Let us give a counterexample.

Consider the following functions:

$$\begin{aligned}
 f : \begin{cases} [0, 1] & \longrightarrow & [0, 1] \\ x & \longmapsto & \begin{cases} x & \text{if } x < \frac{1}{2} \\ \frac{1}{2}x + \frac{1}{2} & \text{if } x \geq \frac{1}{2} \end{cases} \end{cases} \\
 g : \begin{cases}]0, 1[& \longrightarrow & \mathbb{R} \\ x & \longmapsto & \tan\left(\pi x - \frac{\pi}{2}\right) \end{cases} \\
 T : \begin{cases} [0, 1] & \longrightarrow & \mathcal{B}(\mathbb{R}^n) \\ x & \longmapsto & \begin{cases}]-g \circ f(x), g \circ f(x)[& \text{if } a < 1 \\ \mathbb{R}^d & \text{if } a = 1 \end{cases} \end{cases}
 \end{aligned}$$

The function f is strictly increasing and establishes a bijection $[0, 1] \rightarrow [0, \frac{1}{2}[\cup [\frac{3}{4}, 1]$. Then g is bijective and strictly increasing, so $g \circ f$ is strictly increasing and injective, and finally T is strictly increasing in the Borel set, so it is injective, and has a left inverse $\mathcal{B}(\mathbb{R}^n) \rightarrow [0, 1]$ (which is a size). In other words, T is a test. However:

$$\begin{aligned}
 \bigcup_{a < \frac{1}{2}} T(a) &= \bigcup_{a < \frac{1}{2}}]-g \circ f(a), g \circ f(a)[\\
 &= \left] -\lim_{a < \frac{1}{2}} g \circ f(a), \lim_{a < \frac{1}{2}} g \circ f(a) \right[\\
 &= \left] -g\left(\lim_{a < \frac{1}{2}} f(a)\right), g\left(\lim_{a < \frac{1}{2}} f(a)\right) \right[\\
 &= \left] -g\left(\frac{1}{2}\right), g\left(\frac{1}{2}\right) \right[\\
 &\subsetneq T\left(\frac{1}{2}\right) = \left] -g\left(\frac{3}{4}\right), g\left(\frac{3}{4}\right) \right[
 \end{aligned}$$

Definition 5.9 (Continuous test). Let T be a test. We call T a *continuous test* when, for all $a \in [0, 1]$, we have $\bigcup_{b < a} T(b) = T(a)$.

In the following, we give an example of such a continuous test.

Definition 5.10 (Likelihood ratio). Let $f_0, f_1 : \mathbb{R}^d \rightarrow \mathbb{R}^+$ be two probability density functions, and let $x \in \mathbb{R}^d$.

The *likelihood ratio of x being from $X \sim f_0$ instead of $X \sim f_1$* , written $\Lambda^{f_0, f_1}(x)$ or simply $\Lambda(x)$ when there is no ambiguity, is the following function:

$$\Lambda : \begin{cases} \mathbb{R}^d & \longrightarrow & \mathbb{R}^+ \cup \{\infty\} \\ x & \longmapsto & \begin{cases} \frac{f_0(x)}{f_1(x)} & \text{if } f_1(x) \neq 0 \\ \infty & \text{if } f_1(x) = 0 \text{ and } f_0(x) > 0 \\ 0 & \text{if } f_1(x) = 0 \text{ and } f_0(x) = 0 \end{cases} \end{cases} \quad (1)$$

In the practical case where f_1 is always strictly positive on its domain, Λ has values only in \mathbb{R}^+ , and Λ becomes a measurable function.

Consider a random variable $X \sim f_0$. One can compute $\mathbb{P}[\Lambda(X) \leq k | X \sim f_0] = a$. When $k = 0$, we have $a = 1$. When $k \rightarrow \infty$, we have $a \rightarrow 0$.

Definition 5.11 (NP-threshold function). Let $f_0, f_1 : \mathbb{R}^d \rightarrow \mathbb{R}^+$ be two probability density functions so that $\Lambda^{f_0, f_1} = \Lambda$ is continuous.

The *NP-threshold function*, written k_{NP} or simply k , is the function that assigns to each $a \in [0, 1]$, threshold $k(a)$ such that:

$$a = \int_{\Lambda(x) \leq k(a)} f_0(x) dx = \mathbb{P}[\Lambda(X) \leq k | X \sim f_0]$$

Proposition 5.12. *The threshold function is continuous and decreasing.*

Proof. To be completed. □

Definition 5.13 (Neyman-Pearson test). Let $f_0, f_1 : \mathbb{R}^d \rightarrow \mathbb{R}^+$ be two probability density functions so that $\Lambda^{f_0, f_1} = \Lambda$ is continuous.

The *Neyman-Pearson test* is the following test:

$$\text{NP} : \begin{cases} [0, 1] & \longrightarrow & \mathcal{B}(\mathbb{R}^n) \\ a & \longmapsto & \{x \in \mathbb{R}^d \mid \Lambda(x) \leq k(a)\} \end{cases}$$

Proposition 5.14. *For all $a \in]0, 1[$, we have $\bigcup_{b < a} \text{NP}(b) = \text{NP}(a)$. In other words, NP is a continuous test.*

Proof. This is due to the continuity of the threshold function k , which remains to be proved:

$$\begin{aligned} \bigcup_{b < a} \text{NP}(b) &= \bigcup_{b < a} \{x \in \mathbb{R}^d \mid \Lambda(x) \leq k(b)\} \\ &= \{x \in \mathbb{R}^d \mid \exists b < a, \Lambda(x) \leq k(b)\} \\ &= \left\{x \in \mathbb{R}^d \mid \Lambda(x) \leq \sup_{b < a} (k(b))\right\} \\ &= \{x \in \mathbb{R}^d \mid \Lambda(x) \leq k(a)\} \\ &= \text{NP}(a) \end{aligned}$$

□

Corollary 5.15. $S_{\text{NP}}(U_a) = \mathbb{R} \setminus \text{NP}(a)$.

5.3. About predicates

Definition 5.16 (Predicate). Let S be a sheaf in $\mathbf{Shv}([0, 1[, \mathcal{T})$.

A *predicate* is a sheaf morphism $p : S \rightarrow \Omega$.

Computing predicates

Let $S : \mathcal{T}^{\text{op}} \rightarrow \mathbf{Sets}$ be a sheaf, and let $p_1, p_2 : S \rightarrow \Omega$ be predicates. Let $a \in [0, 1]$. For $x \in S(U_a)$, define $U_{a_1} = p_{1,U_a}(x)$ and $U_{a_2} = p_{2,U_a}(x)$. We have the trivial results:

$$\begin{aligned} (p_1 \wedge p_2)_{U_a}(x) &= U_{\min(a_1, a_2)} \\ (p_1 \vee p_2)_{U_a}(x) &= U_{\max(a_1, a_2)} \end{aligned}$$

Also:

$$\begin{aligned} (p_1 \rightarrow p_2)_{U_a}(x) &= p_{1,U_a}(x) \rightarrow p_{2,U_a}(x) \\ &= \bigcup_{R \cap U_{a_1} \subset U_{a_2}} R \\ &= \bigcup_{\substack{b \in [0, a] \\ U_b \cap U_{a_1} \subset U_{a_2}}} U_b \\ &= \bigcup_{\substack{b \in [0, a] \\ U_{\min(a_1, b)} \subset U_{a_2}}} U_b \\ &= \bigcup_{\substack{b \in [0, a] \\ \min(a_1, b) \leq a_2}} U_b \\ &= U_c \end{aligned}$$

for some $c \in [0, a]$. Let us take a look at that c . We have:

$$c = \sup_{\substack{b \in [0, a] \\ \min(a_1, b) \leq a_2}} (b) = \begin{cases} a & \text{if } a_1 \leq a_2 \\ a_2 & \text{otherwise} \end{cases}$$

The negation $\neg p_1$ corresponds to the special case where $U_{a_2} = U_0 = \emptyset$:

$$\begin{aligned} (\neg p_1)_{U_a}(x) &= p_{1,U_a}(x) \rightarrow \emptyset \\ &= \bigcup_{\substack{b \in [0, a] \\ \min(a_1, b) \leq 0}} U_b \\ &= U_c \end{aligned}$$

With c being:

$$c = \sup_{\substack{b \in [0, a] \\ \min(a_1, b) \leq 0}} (b) = \begin{cases} a & \text{if } a_1 = 0 \\ 0 & \text{if } a_1 > 0 \end{cases}$$

In summary:

Formula	Condition	Result
$p_{1,U_a}(x)$		U_{a_1}
$p_{2,U_a}(x)$		U_{a_2}
$(p_1 \wedge p_2)_{U_a}(x)$		$U_{\min(a_1, a_2)}$
$(p_1 \vee p_2)_{U_a}(x)$		$U_{\max(a_1, a_2)}$
$(p_1 \rightarrow p_2)_{U_a}(x)$	if $a_1 \leq a_2$ if $a_1 > a_2$	U_a U_{a_2}
$(\neg p_1)_{U_a}(x)$	if $a_1 = 0$ if $a_1 > 0$	U_a $U_0 = \emptyset$

References

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