p-value presheaves

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1. Introduction

This report presents a preliminary study of the sheaf-ness properties of the p-value.

2. Notation

$\mathcal{B}(\mathbb{R}^n)$ is the Borel set (the smallest σ-algeba containing all open sets of $\mathbb{R}^n$).

3. P-values

Definition 3.1 (Size)**.** A *size* is a function $\alpha : \mathcal{B}(\mathbb{R}^n) \to [0,1]$.

Definition 3.2 (Retract)**.** Let $\alpha : \mathcal{B}(\mathbb{R}^n) \to [0,1]$ be a size.
A *retract* is a function $R : [0,1] \to \mathcal{B}(\mathbb{R}^n)$ such that $\alpha \circ R = \mathrm{id}_{[0,1]}$.

In other words, a retract is a right inverse for a size.

Definition 3.3 (Test)**.** A *family of tests*, or simply *test*, is a retract $T : [0,1] \to \mathcal{B}(\mathbb{R}^n)$ such that $T(0) = \emptyset$, $T(1) = \mathbb{R}^n$ and $a \leqslant a' \Rightarrow T(a) \leqslant T(a')$.

Remark 3.4. If $T : [0,1] \to \mathcal{B}(\mathbb{R}^n)$ is a test, then for $a \in [0,1]$, $T(a)$ is the rejection region of the null hypothesis with size a.

Definition 3.5 (p-value)**.** Let T be a test.
For $x \in \mathbb{R}^n$, the p-value of x is defined as:

$$\mathrm{pval}_T(x) = \inf(\{a \in [0,1] \mid x \in T(a)\})$$

The p-value of x is the minimum size of the test that puts x into the rejection region.

Definition 3.6 (Topological space $\mathscr{T}$)**.** For all $a \in [0,1]$, we define $U_a = [0,a[$ and $\mathscr{T} = \{U_a \mid a \in [0,1]\}$.

It is easy to see that:

Proposition 3.7. *$([0,1[, \mathscr{T})$ is a topological space.*

Remark 3.8. Note that $U_a \cap U_b = U_{\min(a,b)}$ and $\bigcup_{a \in A} U_a = U_{\sup(A)}$.

Definition 3.9 (P-value sheaf)**.** Let $T : [0,1] \to \mathcal{B}(\mathbb{R}^n)$ be a test.
The p-value sheaf for T is the following functor:

$$S_T : \begin{cases} \mathscr{T}^{\mathrm{op}} & \longrightarrow & \textbf{Sets} \\ U_a & \longmapsto & \{x \in \mathbb{R}^n \mid \mathrm{pval}_T(x) \geqslant a\} \\ U_a \subseteq U_b & \longmapsto & \begin{cases} S_T(U_b) & \longrightarrow & S_T(U_a) \\ x & \longmapsto & x \end{cases} \end{cases}$$

Proposition 3.10. *Let $T : [0,1] \to \mathcal{B}(\mathbb{R}^n)$ be a test and let S_T be its p-value sheaf.*
Then, S_T is an actual sheaf.

Proof. It's easy to see that S_T is a presheaf: it's a functor $\mathscr{T}^{\mathrm{op}} \to \textbf{Sets}$. We now have to check the sheaf condition.

Let $A \subset [0,1[$ and $(U_a)_{a \in A} \in \mathscr{T}^A$ be a cover of $U \in \mathscr{T}$. We have $U = \bigcup_{a \in A} U_a$. Let $(s_a)_{a \in A}$ be a matching family of S_T-sections over $(U_a)_{a \in A}$: $s_a \in S_T(U_a)$ and $s_a|_{U_a \cap U_b} = s_b|_{U_a \cap U_b}$.

By definition of S_T, we have:

$$s_a|_{U_a \cap U_b} = S_T(U_a \cap U_b \subset U_a)(s_a) = s_a$$

which yields, for all $a, b \in A, s_a = s_b$. Let $s = s_a$; then s satisfies $\forall a \in A$, $\mathrm{pval}_T(s) \geqslant a$, so $\mathrm{pval}_T(s) \geqslant \sup(A)$ and $s \in S_T\left(U_{\sup(A)}\right) = S_T\left(\bigcup_{a \in A} U_a\right)$. Consequently, s is the unique gluing of $(s_a)_{a \in A}$, and S_T satisfies the sheaf condition. □

Consider the sheaf topos based on $\mathscr{T}$: $\mathbf{Shv}([0,1[\,,\mathscr{T})$. Its subobject classifier is:

$$\Omega : \begin{cases} \mathscr{T}^{\mathrm{op}} & \longrightarrow & \mathbf{Sets} \\ U_a & \longmapsto & \{U_{a'} \mid a' \leqslant a\} \\ U_a \subset U_b & \longmapsto & \begin{cases} \Omega(U_b) & \longrightarrow & \Omega(U_a) \\ U_c & \longmapsto & U_a \cap U_c = U_{\min(a,c)} \end{cases} \end{cases}$$

Let T be a test and S_T its associated p-value sheaf. A sheaf morphism $p : S_T \to \Omega$ makes the following diagram commute for all $a \leqslant b$:

$$\begin{array}{ccc} U_a & & S_T(U_b) \xrightarrow{p_{U_b}} \Omega(U_b) \\ \downarrow{\scriptstyle u} & \rightsquigarrow & \downarrow{\scriptstyle S_T(u)} \quad \checkmark \quad \downarrow{\scriptstyle \Omega(u)} \\ U_b & & S_T(U_a) \xrightarrow[p_{U_a}]{} \Omega(U_a) \end{array}$$

For $x \in S_T(U_b)$, we have:

$$\begin{aligned} p_{U_a} \circ S_T(u)(x) &= \Omega(u) \circ p_{U_b}(x) \\ p_{U_a}(x) &= p_{U_b}(x) \cap U_a \end{aligned}$$

The canonical example is the following natural transformation:

$$p_{U_a} : \begin{cases} S_T(U_a) & \longrightarrow & \Omega(U_a) \\ x & \longmapsto & U_a \end{cases}$$

4. RDT and sheaves

Consider the set-theoretic integer $n \in \mathbb{N}$, $n = \{0, \ldots, n-1\}$. Endow n with the discrete topology $(n, \mathcal{P}(n))$. For $I \subset n$, $\mathbb{R}^I$ is the set of functions $I \to \mathbb{R}$; equivalently, it is the set of I-indexed sets of real numbers.

Let Π_n be the following functor:

$$\Pi_n : \begin{cases} \mathcal{P}(n) & \longrightarrow & \mathbf{Sets} \\ I & \longmapsto & \mathbb{R}^I \\ I \subset J & \longmapsto & \begin{cases} \Pi_n(J) & \longrightarrow & \Pi_n(I) \\ f & \longmapsto & f|_I \end{cases} \end{cases}$$

Proposition 4.1. *Π_n is a sheaf.*

For $I \subset n$, $\|-\|_{\mathbb{R}^I}$ is the Euclidean norm on $\mathbb{R}^I$.

Definition 4.2 (τ functor)**.** We define the functor τ by setting:

$$\tau : \begin{cases} \mathcal{P}(n) & \longrightarrow & \mathscr{T} \\ I & \longmapsto & \tau(I) \\ I \subset J & \longmapsto & \tau(I) \subset \tau(J) \end{cases}$$

where $\mathscr{T}$ is the topology defined in Definition 3.6. For $I \in \mathcal{P}(n)$, the unique a such that $\tau(I) = [0, a[$ is denoted τ_I, so that: $\tau(I) = [0, \tau_I[$.

The interest of this definition will tentatively be given later.

Let $f, \theta \in \mathbb{R}^n$. For all $I \in \mathcal{P}(n)$, the proposition: $\|\theta|_I - f|_I\|_{\mathbb{R}^I} \leqslant \tau_I$ is a proposition the truth of which can be known by simple inspection, provided that θ is known. In practice, we don't have access to θ but to a modified, noisy version $Y = Y(\theta)$ of it, and the question is then, for any given $I \subset n$, whether the proposition $\|\theta|_I - f|_I\|_{\mathbb{R}^I} \leqslant \tau_I$ is plausible in a certain sense or not, when we observe $Y|_I$.

Definition 4.3 (Optimal pair)**.** Let $f \in \mathbb{R}^n$, we call it model. Let Θ be a random vector whose distribution we don't know, and $X \sim \mathcal{N}\left(0, C^2\right)$ is a n-dimensional Gaussian vector. We observe $Y = \Theta + X$.

For all $I \subset n$, there exists a pair $\left(\alpha_i^{f,\tau}, T_I^{f,\tau}\right)$ [1], called *optimal pair restricted to* I, where:

1. $\alpha_i^{f,\tau} : \mathcal{B}\left(\mathbb{R}^I\right) \to [0,1]$ is a size
2. $T_I^{f,\tau} : [0,1] \to \mathcal{B}\left(\mathbb{R}^I\right)$ is a monotone retract of $\alpha_i^{f,\tau}$
3. $T_I^{f,\tau}$ is optimal for testing $\|\Theta|_I - f|_I\|_{\mathbb{R}^I} \leqslant \tau_I$ against its alternative $\|\Theta|_I - f|_I\|_{\mathbb{R}^I} \geqslant \tau_I$ when we observe $Y(\Theta)$.

Definition 4.4 (Restricted p-value)**.** We consider the τ functor described above. Let $I \subset n$ and let $\left(\alpha_i^{f,\tau}, T_I^{f,\tau}\right)$ be an optimal pair restricted to I.

The *I-restricted p-value of x* is the following value:

$$\mathrm{pval}_I^{f,\tau}(y) = \inf\left(\left\{a \in [0,1] \mid y \in T_I^{f,\tau}(a)\right\}\right)$$

The *I-restricted p-value sheaf* is the following sheaf:

$$S_I^{f,\tau} : \begin{cases} [0,1] & \longrightarrow & \mathbb{R}^I \\ U_a & \longmapsto & \left\{y \in \mathbb{R}^n \mid \mathrm{pval}_I^{f,\tau}(y) \geqslant a\right\} \\ U_a \subseteq U_b & \longmapsto & \begin{cases} S_T(U_b) & \longrightarrow & S_T(U_a) \\ x & \longmapsto & x \end{cases} \end{cases}$$

The p-value $\mathrm{pval}_I^{f,\tau}(y)$ measures the plausability of proposition $\|\Theta|_I - f|_I\|_{\mathbb{R}^I} \leqslant \tau_I$ when we observe $y \sim Y = \Theta + X$. The higher $\mathrm{pval}_I^{f,\tau}(y)$, the more plausible the proposition $\|\Theta|_I - f|_I\|_{\mathbb{R}^I} \leqslant \tau_I$.

5. First results

5.1. About p-value sheaves

It is easy to see that:

Proposition 5.1. *$(\mathscr{T}, \subset)$ is a total order with a minimal and maximal element (resp. $\emptyset$ and $\mathbb{R}^d$).*

It is also easy, but less easy, to see that:

Proposition 5.2. *$\mathscr{T}^{op}$ has all small limits and colimits.*

Proof. Let $D : \mathscr{I} \to \mathscr{T}^{\mathrm{op}}$ be a diagram in $\mathscr{T}^{\mathrm{op}}$. For all $i \in \mathscr{I}$, define $U_{d_i} = D(i)$. Then:

$$\mathrm{Colim}(D) = \sup_{i \in \mathrm{Ob}_{\mathscr{I}}} D_i = \bigcap_{i \in \mathrm{Ob}_{\mathscr{I}}} U_{d_i} = U_{\inf(\mathscr{I})}$$
$$\mathrm{Lim}(D) = \inf_{i \in \mathrm{Ob}_{\mathscr{I}}} D_i = \bigcup_{i \in \mathrm{Ob}_{\mathscr{I}}} U_{d_i} = U_{\sup(\mathscr{I})}$$

Note that the sup becomes an intersection, and the inf becomes a union, because we are considering $\mathscr{T}^{\mathrm{op}}$ and not $\mathscr{T}$. Also note that the infima and suprema always exist because these are infima and suprema of $\{d_i \mid i \in \mathrm{Ob}_{\mathscr{I}}\}$, which is a subset of $\mathbb{R}$. □

Proposition 5.3. *For any test T, its p-value sheaf S_T is continuous and cocontinuous.*

Proof. Let $D : \mathscr{I} \to \mathscr{T}^{\mathrm{op}}$ be any (small) diagram in $\mathscr{T}^{\mathrm{op}}$. For all $i \in \mathscr{I}$, define $U_{d_i} = D(i)$.

We only consider the case of a limit; the proof is very similar for colimits.

$$\begin{aligned} S_T\left(\mathrm{Lim}\,(D)\right) &= S_T\left(\bigcup_{i\in\mathscr{I}} U_{d_i}\right) \\ &= S_T\left(U_{\sup(d_i)}\right) \\ &= \left\{ x \in \mathbb{R}^d \;\middle|\; \mathrm{pval}_T(x) \geqslant \sup_{i\in\mathscr{I}}(d_i) \right\} \\ &= \left\{ x \in \mathbb{R}^d \;\middle|\; \forall i \in \mathscr{I},\ \mathrm{pval}_T(x) \geqslant d_i \right\} \\ &= \bigcap_{i\in\mathscr{I}} \left\{ x \in \mathbb{R}^d \;\middle|\; \mathrm{pval}_T(x) \geqslant d_i \right\} \\ &= \bigcap_{i\in\mathscr{I}} S_T\left(U_{d_i}\right) \end{aligned}$$

We now have to check that $\bigcap_{i\in\mathscr{I}} S_T\left(U_{d_i}\right) \cong \mathrm{Lim}\,(S_T \circ D)$.

For all $i \to j \in \mathscr{I}$, we have $S_T\left(U_{d_i}\right) \subset S_T\left(U_{d_j}\right)$. We denote by $\iota_{i,j} = S_T\left(U_{d_i} \subset U_{d_j}\right) : S_T\left(U_{d_i}\right) \to S_T\left(U_{d_j}\right)$ the inclusion mapping between $S_T\left(U_{d_i}\right)$'s. We also denote by $\iota_i : \bigcap_{i\in\mathscr{I}} S_T\left(U_{d_i}\right) \to S_T\left(U_{d_i}\right)$ the inclusion mapping of the intersection. For all $i \to j \in \mathscr{I}$, we have $\iota_{i,j} \circ \iota_i = \iota_j$, so that $\iota = \left(\iota_i : \bigcap_{i\in\mathscr{I}} S_T\left(U_{d_i}\right) \to S_T\left(U_{d_i}\right)\right)_{i\in\mathscr{I}}$ is a cone to $S_T \circ D$.

Let (A, α) be any cone to $S_T \circ D$. We denote the $S_T\left(U_{d_i}\right)$-components of α by $\alpha_i : A \to S_T\left(U_{d_i}\right)$. For all $i \to j \in \mathscr{I}$, and for all $x \in X$, we have $\iota_{i,j} \circ \alpha_i(x) = \alpha_j(x) = \alpha_i(x)$.

$$\begin{array}{ccc} S_T\left(U_{d_i}\right) & \xleftarrow{\ \alpha_i\ } & A \\ {\scriptstyle \iota_{i,j}}\downarrow & \swarrow_{\alpha_j} & \\ S_T\left(U_{d_j}\right) & & \end{array}$$

So in fact, $\alpha_i(X) = \alpha_j(X) \subset S_T\left(U_{d_i}\right)$, which yields that, for all $i \in \mathscr{I}$, $\alpha_i(X) \subset \bigcap_{i\in\mathscr{I}} S_T\left(U_{d_i}\right)$.

Let u be such that:

$$u : \left\{ \begin{array}{ccc} X & \longrightarrow & \bigcap_{i\in\mathscr{I}} S_T\left(U_{d_i}\right) \\ x & \longmapsto & \alpha_i(x) \end{array} \right.$$

for any of the $i \in \mathscr{I}$, because $\alpha_j(x) = \alpha_i(x)$. Then for all $i \in \mathscr{I}$, we have $\iota_i \circ u = \alpha_i$.

It is also easy to check the unicity of that u: suppose $u, u' : X \to \bigcap_{i\in\mathscr{I}} S_T\left(U_{d_i}\right)$, then for all $x \in X$, we have:

$$\begin{aligned} \iota_i \circ u(x) = \alpha_i(x) = \iota_i \circ u'(x) \\ u(x) = u'(x) \end{aligned}$$

which leads to $u = u'$.

Consequently, $\bigcap_{i\in\mathscr{I}} S_T\left(U_{d_i}\right) \cong \mathrm{Lim}\,(S_T \circ D)$. □

5.2. About sheaf morphisms

Let us first study the sheaf morphisms between p-value sheaves $p : S_R \to S_T$.

Proposition 5.4 (Nesting property)**.** *Let $R,T : [0,1] \to \mathcal{B}(\mathbb{R}^n)$ be two tests (of the same size or not). Let $p : \mathbb{R}^d \to \mathbb{R}^d$ be a function.*

The function p defines a natural transformation $\bar{p} : S_R \to S_T = \left(p_{|S_R(U_a)}\right)_{a\in[0,1]}$ $\Leftrightarrow$ for all $a \in [0,1]$, $p(S_R(U_a)) \subset S_T(U_a)$.

In other words, p defines a function between nested open sets, as in Figure 1.

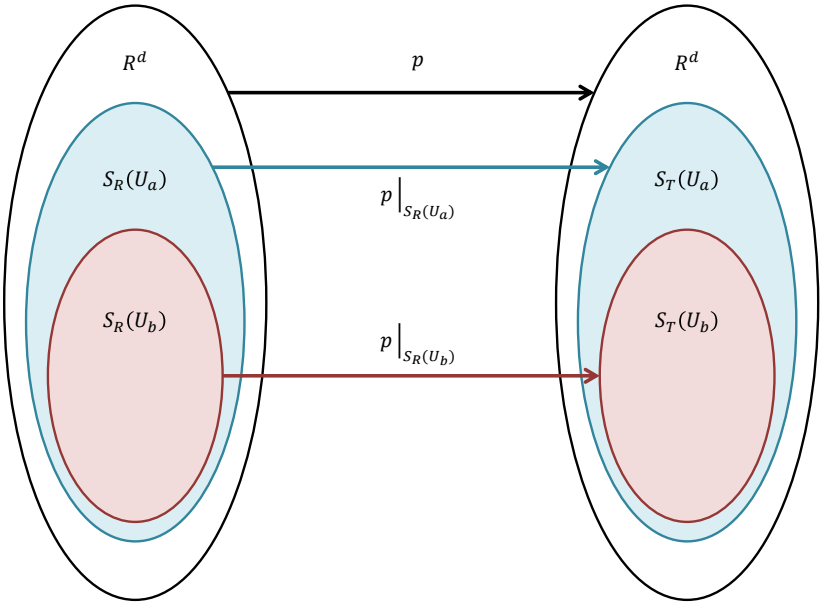

Figure 1: Illustration of the nesting property of p. *If $p : \mathbb{R}^d \to \mathbb{R}^d$ verifies this nesting property, then it defines a natural transformation between two p-value sheaves.*

Proof. Easy deduction from the following natural transformation diagram:

$$\begin{array}{ccc} U_a & & S_R(U_b) \xrightarrow{p|_{S_R(U_b)}} S_T(U_b) \\ \downarrow \subset & \rightsquigarrow & \downarrow \subset \quad \checkmark \quad \downarrow \subset \\ U_b & & S_R(U_a) \xrightarrow[p|_{S_R(U_a)}]{} S_T(U_a) \end{array}$$

□

Corollary 5.5. *If $\bar{p} = \left(p_{U_a}\right)_{a\in[0,1]}$ is a natural transformation $S_R \to S_T$, then $p = \bigcup_{a\in A} p_{U_a}$ is a function that verifies the nesting property.*

Proposition 5.6. *For all $a \in [0,1]$, $S_T(U_a) = \mathbb{R}^d \setminus \left(\bigcup_{b<a} T(b)\right)$.*

Proof. By computation:

$$\begin{aligned}
S_T(U_a) &= \left\{ x \in \mathbb{R}^d \;\middle|\; \mathrm{pval}_T(x) \geqslant a \right\} \\
&= \left\{ x \in \mathbb{R}^d \;\middle|\; \inf_{x \in T(b)} (b \in [0,1]) \geqslant a \right\} \\
&= \left\{ x \in \mathbb{R}^d \;\middle|\; \forall b < a, x \notin T(b) \right\} \\
&= \bigcap_{b<a} \mathbb{R}^d \backslash T(b) \\
&= \mathbb{R}^d \backslash \left(\bigcup_{b<a} T(b) \right)
\end{aligned}$$

□

Corollary 5.7. *For all $a \in [0,1]$, if $\bigcup_{b<a} T(b) = T(a)$ then $S_T(U_a) = \mathbb{R}^d \backslash T(a)$.*

Example 5.8. Note that we are considering very general tests $T : [0,1] \to \mathcal{B}(\mathbb{R}^n)$, so $\left(\bigcup_{b<a} T(b) \right)$ has no reason to be equal to $T(a)$. Let us give a counterexample.

Consider the following functions:

$$\begin{aligned}
f : \begin{cases} [0,1] & \longrightarrow & [0,1] \\ x & \longmapsto & \begin{cases} x & \text{if } x < \frac{1}{2} \\ \frac{1}{2}x + \frac{1}{2} & \text{if } x \geqslant \frac{1}{2} \end{cases} \end{cases} \\
g : \begin{cases}]0,1[& \longrightarrow & \mathbb{R} \\ x & \longmapsto & \tan\left(\pi x - \frac{\pi}{2}\right) \end{cases} \\
T : \begin{cases} [0,1] & \longrightarrow & \mathcal{B}(\mathbb{R}^n) \\ x & \longmapsto & \begin{cases}]-g \circ f(x), g \circ f(x)[& \text{if } a < 1 \\ \mathbb{R}^d & \text{if } a = 1 \end{cases} \end{cases}
\end{aligned}$$

The function f is strictly increasing and establishes a bijection $[0,1] \to \left[0, \frac{1}{2}\right[\cup \left[\frac{3}{4}, 1\right]$. Then g is bijective and strictly increasing, so $g \circ f$ is striclty increasing and injective, and finally T is stricly increasing in the Borel set, so it is injective, and has a left inverse $\mathcal{B}(\mathbb{R}^n) \to [0,1]$ (which is a size). In other words, T is a test. However:

$$\begin{aligned}
\bigcup_{a<\frac{1}{2}} T(a) &= \bigcup_{a<\frac{1}{2}}]-g \circ f(a), g \circ f(a)[\\
&= \left] - \lim_{a<\frac{1}{2}} g \circ f(a), \lim_{a<\frac{1}{2}} g \circ f(a) \right[\\
&= \left] -g\left(\lim_{a<\frac{1}{2}} f(a) \right), g\left(\lim_{a<\frac{1}{2}} f(a) \right) \right[\\
&= \left] -g\left(\frac{1}{2}\right), g\left(\frac{1}{2}\right) \right[\\
&\subsetneq T\left(\frac{1}{2}\right) = \left] -g\left(\frac{3}{4}\right), g\left(\frac{3}{4}\right) \right[
\end{aligned}$$

Definition 5.9 (Continuous test)**.** Let T be a test. We call T a *continuous test* when, for all $a \in [0,1]$, we have $\bigcup_{b<a} T(b) = T(a)$.

In the following, we give an example of such a continuous test.

Definition 5.10 (Likelihood ratio)**.** Let $f_0, f_1 : \mathbb{R}^d \to \mathbb{R}^+$ be two probability density functions, and let $x \in \mathbb{R}^d$.

The *likelihood ratio of x being from $X \sim f_0$ instead of $X \sim f_1$*, written $\Lambda^{f_0,f_1}(x)$ or simply $\Lambda(x)$ when there is no ambiguity, is the following function:

$$\Lambda : \left\{ \begin{array}{ccc} \mathbb{R}^d & \longrightarrow & \mathbb{R}^+ \cup \{\infty\} \\ x & \longmapsto & \left\{ \begin{array}{ll} \dfrac{f_0(x)}{f_1(x)} & \text{if } f_1(x) \neq 0 \\ \infty & \text{if } f_1(x) = 0 \text{ and } f_0(x) > 0 \\ 0 & \text{if } f_1(x) = 0 \text{ and } f_0(x) = 0 \end{array} \right. \end{array} \right. \tag{1}$$

In the practical case where f_1 is always stricly positive on its domain, Λ has values only in $\mathbb{R}^+$, and Λ becomes a measurable function.

Consider a random variable $X \sim f_0$. One can compute $\mathrm{P}\left[\Lambda(X) \leqslant k | X \sim f_0\right] = a$. When $k = 0$, we have $a = 1$. When $k \to \infty$, we have $a \to 0$.

Definition 5.11 (NP-threshold function)**.** Let $f_0, f_1 : \mathbb{R}^d \to \mathbb{R}^+$ be two probability density functions so that $\Lambda^{f_0,f_1} = \Lambda$ is continuous.

The *NP-threshold function*, written k_{NP} or simply k, is the function that assigns to each $a \in [0,1]$, threshold $k(a)$ such that:

$$a = \int\limits_{\Lambda(x) \leqslant k(a)} f_0(x)\mathrm{d}x = \mathrm{P}\left[\Lambda(X) \leqslant k | X \sim f_0\right]$$

Proposition 5.12. *The threshold function is continuous and decreasing.*

Proof. To be completed. □

Definition 5.13 (Neyman-Pearson test)**.** Let $f_0, f_1 : \mathbb{R}^d \to \mathbb{R}^+$ be two probability density functions so that $\Lambda^{f_0,f_1} = \Lambda$ is continuous.

The *Neyman-Pearson test* is the following test:

$$\mathrm{NP} : \left\{ \begin{array}{ccc} [0,1] & \longrightarrow & \mathcal{B}\left(\mathbb{R}^n\right) \\ a & \longmapsto & \left\{ x \in \mathbb{R}^d \;\middle|\; \Lambda(x) \leqslant k(a) \right\} \end{array} \right.$$

Proposition 5.14. *For all $a \in]0,1[$, we have $\bigcup\limits_{b<a} \mathrm{NP}(b) = \mathrm{NP}(a)$. In other words,* NP *is a continuous test.*

Proof. This is due to the continuity of the threshold function k, which remains to be proved:

$$\begin{aligned} \bigcup_{b<a} \mathrm{NP}(b) &= \bigcup_{b<a} \left\{ x \in \mathbb{R}^d \;\middle|\; \Lambda(x) \leqslant k(b) \right\} \\ &= \left\{ x \in \mathbb{R}^d \;\middle|\; \exists b < a,\ \Lambda(x) \leqslant k(b) \right\} \\ &= \left\{ x \in \mathbb{R}^d \;\middle|\; \Lambda(x) \leqslant \sup_{b<a} (k(b)) \right\} \\ &= \left\{ x \in \mathbb{R}^d \;\middle|\; \Lambda(x) \leqslant k(a) \right\} \\ &= \mathrm{NP}(a) \end{aligned}$$

□

Corollary 5.15. $S_{\mathrm{NP}}(U_a) = \mathbb{R} \backslash \mathrm{NP}(a)$.

5.3. About predicates

Definition 5.16 (Predicate)**.** Let S be a sheaf in **Shv**$([0,1[\,,\mathscr{T})$.

A *predicate* is a sheaf morphism $p : S \to \Omega$.

Computing predicates

Let $S : \mathscr{T}^{\text{op}} \to$ **Sets** be a sheaf, and let $p_1, p_2 : S \to \Omega$ be predicates. Let $a \in [0,1]$. For $x \in S(U_a)$, define $U_{a_1} = p_{1,U_a}(x)$ and $U_{a_2} = p_{2,U_a}(x)$. We have the trivial results:

$$(p_1 \wedge p_2)_{U_a}(x) = U_{\min(a_1,a_2)}$$
$$(p_1 \vee p_2)_{U_a}(x) = U_{\max(a_1,a_2)}$$

Also:

$$\begin{aligned}(p_1 \to p_2)_{U_a}(x) = p_{1,U_a}(x) &\to p_{2,U_a}(x)\\ &= \bigcup_{R \cap U_{a_1} \subset U_{a_2}} R\\ &= \bigcup_{\substack{b\in[0,a]\\ U_b \cap U_{a_1} \subset U_{a_2}}} U_b\\ &= \bigcup_{\substack{b\in[0,a]\\ U_{\min(a_1,b)} \subset U_{a_2}}} U_b\\ &= \bigcup_{\substack{b\in[0,a]\\ \min(a_1,b) \leqslant a_2}} U_b\\ &= U_c\end{aligned}$$

for some $c \in [0,a]$. Let us take a look at that c. We have:

$$c = \sup_{\substack{b\in[0,a]\\ \min(a_1,b)\leqslant a_2}}(b) = \begin{cases} a & \text{if } a_1 \leqslant a_2\\ a_2 & \text{otherwise}\end{cases}$$

The negation $\neg p_1$ corresponds to the special case where $U_{a_2} = U_0 = \emptyset$:

$$\begin{aligned}(\neg p_1)_{U_a}(x) = p_{1,U_a}(x) &\to \emptyset\\ &= \bigcup_{\substack{b\in[0,a]\\ \min(a_1,b)\leqslant 0}} U_b\\ &= U_c\end{aligned}$$

With c being:

$$c = \sup_{\substack{b\in[0,a]\\ \min(a_1,b)\leqslant 0}}(b) = \begin{cases} a & \text{if } a_1 = 0\\ 0 & \text{if } a_1 > 0\end{cases}$$

In summary:

Formula	Condition	Result
$p_{1,U_a}(x)$		U_{a_1}
$p_{2,U_a}(x)$		U_{a_2}
$(p_1 \wedge p_2)_{U_a}(x)$		$U_{\min(a_1,a_2)}$
$(p_1 \vee p_2)_{U_a}(x)$		$U_{\max(a_1,a_2)}$
$(p_1 \rightarrow p_2)_{U_a}(x)$	if $a_1 \leqslant a_2$ if $a_1 > a_2$	U_a U_{a_2}
$(\neg p_1)_{U_a}(x)$	if $a_1 = 0$ if $a_1 > 0$	U_a $U_0 = \emptyset$

References

[1] D. Pastor and Q.-T. Nguyen, "Random distortion testing and optimality of thresholding tests," *IEEE Transactions on Signal processing*, vol. 61, no. 16, pp. 4161–4171, 2013.

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