

# p-value presheaves

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# p-value presheaves

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# 1. Introduction

In this preliminary work, we study the sheaf-ness properties of the p-value. In particular, p-values are presheaves and become sheaves if we add a bottom element to the initial topology. This trick is valid because p-values are fuzzy sets!

We also study the basic properties of the presheaf topos of p-values. Our purpose here is to exhibit the tools provided by topos theory to "approximate" p-values of Neyman-Pearson tests by p-values of other types of tests. In this report, we provide the basic tools but do not achieve the approximation yet!

# 2. Notation

$\text{Lbg}\left(\mathbb{R}^d\right)$ is the set of all Lebesgue-measurable subsets of $\mathbb{R}^d$.

# 3. P-values

**Definition 3.1** (Test)**.** A *family of tests*, or simply *test*, is a function $T : [0,1] \to \text{Lbg}\left(\mathbb{R}^d\right)$ such that $a \leqslant a' \Rightarrow T(a) \leqslant T(a')$.

*Remark* 3.2. If $T : [0,1] \to \text{Lbg}\left(\mathbb{R}^d\right)$ is a test, then for $a \in [0,1]$, $T(a)$ is the rejection region of the null hypothesis with size $a$.

**Definition 3.3** (p-value)**.** Let $T$ be a test.

For $x \in \mathbb{R}^n$, the p-value of $x$ is defined as:

$$\text{pval}_T(x) = \inf\left(\{a \in [0,1] \mid x \in T(a)\}\right)$$

The p-value of $x$ is the minimum size of the test that puts $x$ into the rejection region.

**Definition 3.4** (Presheaf)**.** Let $\mathscr{C}$ be a small category.

A presheaf is a functor $\mathscr{C}^{\text{op}} \to \textbf{Sets}$.

Now consider the total order $([0,1], \leqslant)$ (in fact, it is a complete lattice). It defines a small category that we will denote by $\mathscr{U}$.

**Definition 3.5** (P-value presheaf)**.** Let $T : [0,1] \to \text{Lbg}\left(\mathbb{R}^d\right)$ be a test.

The *p-value presheaf for* $T$ is the following functor:

$$P_T : \begin{cases} \mathscr{U}^{\text{op}} & \longrightarrow & \textbf{Sets} \\ a & \longmapsto & \left\{x \in \mathbb{R}^n \mid \text{pval}_T(x) \geqslant a\right\} \\ a \subseteq b & \longmapsto & \begin{cases} P_T(b) & \longrightarrow & P_T(a) \\ x & \longmapsto & x \end{cases} \end{cases}$$

## 3.1. This is not a sheaf

Let us introduce the definition of a sheaf. We generally use topological spaces for that purpose.

**Definition 3.6** (Sheaf)**.** Let $(X, \text{Op}(X))$ be a topological space, we consider the category $\text{Op}(X)^{\text{op}}$.

A sheaf $S : \text{Op}(X)^{\text{op}} \to \textbf{Sets}$ is a presheaf with the following condition, called *sheaf condition*:

(sheaf condition) For all open set $U \in \text{Op}(X)$, for all covering $U = \bigcup_{a \in A} U_a$, and for all family $(x_a)_{a \in A} \in \prod_{a \in A} S(U_a)$ such that for all $a, b \in A$, we have $x_{a\,|U_a \cap U_b} = x_{b\,|U_a \cap U_b}$, there exists a unique $x \in S(U)$ such that $\forall a \in A,\ x_{|a} = x_a$.

A family $(U_a)_{a \in A}$ such that $U = \bigcup_{a \in A} U_a$ is called a *covering of* $U$. A family $(x_a)_{a \in A} \in \prod_{a \in A} S(U_a)$ such that for all $a, b \in A$, we have $x_{a\,|U_a \cap U_b} = x_{b\,|U_a \cap U_b}$ is called a *matching family of* $S$*-sections*. When

$x_{a|U_a \cap U_b} = x_{b|U_a \cap U_b}$, we also say that $x_a$ and $x_b$ *agree* on their common domain. The unique $x \in S(U)$ such that $\forall a \in A,\ x_{|a} = x_a$ is called the *gluing* of $(x_a)_{a \in A}$.

However, our p-value presheaf is not defined on a topological space, but on a partial order. The notion of sheaves can be extended from a topological space to a complete Heyting algebra. Besides, our partial order is a complete Heyting algebra.

*Remark* 3.7 (Introducing morphisms for sheaves on Heyting algebras). The following is adapted from [1, page 376, section 15.5.3].

Let $(H, \leqslant, \wedge, \vee)$ be a complete Heyting algebra, and let $P : H^{\text{op}} \to \textbf{Sets}$ be a presheaf over $H$.

Let $(h_a)_{a \in A}$ a subset of elements of $H$ and let $h = \bigvee_{a \in A} h_a$ be their supremum (it always exists because $H$ is complete). We denote the restriction functions by $r^h_a : P(h) \to P(h_a)$. There is a unique function $r^h : P(h) \to \prod_{a \in A} P(h_a)$ such that $\pi_{P(h_a)} \circ r^h = r^h_a$ (definition of product). Similarly, for every $a, b \in A$, there are functions $p^h_{a,b} : P(h_a) \to P(h_a \wedge h_b)$ and $q^h_{a,b} : P(h_b) \to P(h_a \wedge h_b)$. These also combine into: $p^h, q^h : \prod_{a \in A} P(h_a) \to \prod_{a,b \in A} P(h_a \wedge h_b)$.

We can now define sheaves on Heyting algebras.

**Definition 3.8** (Sheaf on a complete Heyting algebra)**.** Let $(H, \leqslant, \wedge, \vee, 0_H, 1_H)$ be a complete Heyting algebra. Let $S : H^{\text{op}} \to \textbf{Sets}$ be a presheaf over $H$.

The presheaf $S$ is called a sheaf when the following condition is verified:

- for all $(h_a)_{a \in A} \in H^A$, if $h = \bigvee_{a \in A} h_a$ and $r^h : S(h) \to \prod_{a \in A} S(h_a)$, $p^h : \prod_{a \in A} S(h_a) \to \prod_{a,b \in A} S(h_a \wedge h_b)$ and $q^h : \prod_{a \in A} S(h_a) \to \prod_{a,b \in A} S(h_a \wedge h_b)$ are functions as defined in Remark 3.7, then the following diagram is an equaliser:

$$S(h) \xrightarrow{r^h} \prod_{a \in A} S(h_a) \underset{q^h}{\overset{p^h}{\rightrightarrows}} \prod_{a,b \in A} S(h_a \wedge h_b)$$

For the sake of understandability, we will keep the same terminology as introduced with topological spaces, namely:

- A family $(h_a)_{a \in A}$ such that $h = \bigvee_{a \in A} h_a$ is called a *covering of* $h$.
- A family $(x_a)_{a \in A} \in \prod_{a \in A} S(h_a)$ such that for all $a, b \in A$, we have $x_{a|h_a \wedge h_b} = x_{b|h_a \wedge h_b}$, is called a *matching family of S-sections*. When $x_{a|h_a \wedge h_b} = x_{b|h_a \wedge h_b}$, we also say that $x_a$ and $x_b$ *agree* on their common domain.
- The unique $x \in S(h)$ such that $\forall a \in A,\ x_{|h_a} = x_a$ is called the *gluing* of $(x_a)_{a \in A}$.

**Lemma 3.9.** *If* $S : H \to \textbf{Sets}$ *is a sheaf, then* $S(0_H) \cong 1$.

*Proof.* Consider the empty covering $0_H = \bigvee_{a \in \emptyset} h_a$. The products $\prod_{a \in A} S(h_a)$ and $\prod_{a,b \in A} S(h_a \wedge h_b)$ are empty: $\prod_{a \in A} S(h_a) \cong \prod_{a,b \in A} S(h_a \wedge h_b) \cong 1$. Thus, the arrows $p^0$ and $q^0$ are the same and are the identity $\text{id}_1$. The equaliser of:

$$S(0_H) \xrightarrow{r^0} 1 \underset{q^0}{\overset{p^0}{\rightrightarrows}} 1$$

is obviously a terminal object 1; in other words, if $S$ is a sheaf, then $S(0_H) \cong 1$. □

**Proposition 3.10.** *Let $T : [0,1] \to \mathrm{Lbg}\left(\mathbb{R}^d\right)$ be a test and let $P_T$ be its p-value presheaf.*
*Then, $P_T$ is not a sheaf.*

*Proof.* Suppose $P_T$ is a sheaf. By Lemma 3.9, $P_T(0) \cong 1$. However:

$$P_T(\emptyset) = \left\{x \in \mathbb{R}^n \mid \mathrm{pval}_T(x) \geqslant 0\right\} = \mathbb{R} \ncong 1$$

□

*Remark* 3.11. The fact that $P_T$ is not a sheaf also means that there are coverings $h = \bigvee_{a \in A} h_a$ that do not lead to a unique gluing (either there is none, or there is more than one). In our case, the empty covering is the unique covering that does not find its gluing. In fact, the p-value presheaf only fails to be a sheaf on one covering.

Let $h = \bigvee_{a \in A} h_a$ be a cover such that $A \neq \emptyset$. Let $(x_a)_{a \in A}$ be a matching family of $P_T$-sections over $(h_a)_{a \in A}$: $x_a \in P_T(h_a)$ and $x_{a\,|h_a \wedge h_b} = x_{b\,|h_a \wedge h_b}$.

By definition of $P_T$, we have:

$$x_{a\,|h_a \wedge h_b} = P_T\left(h_a \wedge h_b \leqslant h_a\right)(x_a) = x_a$$

which yields, for all $a, b \in A$, $x_a = x_b$. Let $x = x_a$; then $\forall a \in A$, we have $x \in P_T(h_a)$, so $\mathrm{pval}_T(x) \geqslant h_a$ and $\mathrm{pval}_T(x) \geqslant \sup_{a \in A}(h_a) = h$ and $x \in P_T(h) = P_T\left(\bigvee_{a \in A} h_a\right)$. Consequently, $x$ is the unique gluing of $(x_a)_{a \in A}$, and $P_T$ satisfies the sheaf condition whenever the index family $A$ is not empty.

When $A$ is empty, then the gluing $x \in A$ doesn't exist.

This problem was already known in [1, p401, part. 15.6.6], not for p-values, but in the context of fuzzy logic. We consider a Heyting algebra $H$. A fuzzy set (over $H$) is a pair $(S, s)$ such that $S \in \mathbf{Sets}$ and $s : S \to H$. The category of fuzzy sets over $H$ is simply the slice $\mathbf{Sets}/H$ (where $H$ is a lattice).

Let $(S, s)$ be a fuzzy set. We define the presheaf:

$$P : \left\{ \begin{array}{ccc} H & \longrightarrow & \mathbf{Sets} \\ h & \longmapsto & \{x \in S \mid s(x) \geqslant h\} \end{array} \right.$$

This presheaf is very similar to that of our p-value presheaf. However, both suffer from the same problem: they are almost sheaves, and the part that fails is the image of the empty set or least element, which contains the whole set instead of being the terminal object. The solution proposed by [1] is to change the Heyting algebra for another Heyting algebra, adding a new element $\bot$ that becomes the new least element, and by forcing $P(\bot) = 1$.

Let $\mathscr{U}^+ = \mathscr{U} \cup \{\bot\}$. Then, let $\leqslant_{\mathscr{U}^+}$ be the smallest partial order that contains:

$$\left\{(\bot, U) \in \mathscr{U}^+ \times \mathscr{U}^+\right\} \cup \subset_{\mathscr{U}}$$

where $\subset_{\mathscr{U}}$ is the inclusion of open subsets in $\mathscr{U}$.

It is easy to see that:

**Proposition 3.12.** *$\mathscr{U}^+$ is a complete Heyting algebra.*

In fact, the pair $(\mathscr{U}^+, \leqslant_{\mathscr{U}^+})$ is simply $\mathscr{U}$ with a new initial element. As it is still a Heyting algebra, we can define a sheaf on it.

**Definition 3.13** (P-value sheaf)**.** Let $T : [0,1] \to \mathrm{Lbg}\left(\mathbb{R}^d\right)$ be a test.

The *p-value sheaf for $T$* is the following functor:

$$P_T : \begin{cases} \mathscr{U}^{+^{\mathrm{op}}} & \longrightarrow & \textbf{Sets} \\ a & \longmapsto & \begin{cases} \{x \in \mathbb{R}^n \mid \mathrm{pval}_T(x) \geqslant a\} & \text{if } a \in [0,1] \\ \{\emptyset\} & \text{if } a = \bot \end{cases} \\ a \subseteq b & \longmapsto & \begin{cases} P_T(b) & \longrightarrow & P_T(a) \\ x & \longmapsto & x \end{cases} \end{cases}$$

**Proposition 3.14.** *The p-value sheaf is an actual sheaf.*

*Proof.* An application of [1, Proposition 15.6.8]. □

### 3.2. Study of the presheaf

The use of sheaves is beyond our purposes. We actually only need a topos, and that topos doesn't need to be a sheaf topos. A presheaf topos is enough. Consider the presheaf topos based on the complete Heyting algebra $[0,1]$: $\mathbf{PSh}([0,1])$. Its subobject classifier is:

$$\Omega : \begin{cases} \mathscr{U}^{+} & \longrightarrow & \textbf{Sets} \\ a & \longmapsto & \{a' \mid a' \leqslant a\} = [0,a] \\ a \leqslant b & \longmapsto & \begin{cases} \Omega(b) & \longrightarrow & \Omega(a) \\ c & \longmapsto & a \wedge c = \min(a,c) \end{cases} \end{cases}$$

Let $T$ be a test and $P_T$ its associated p-value presheaf. A presheaf morphism $p : P_T \to \Omega$ makes the following diagram commute for all $a \leqslant b$:

$$\begin{array}{ccc} a & & P_T(b) \xrightarrow{p_b} \Omega(b) \\ \downarrow{\scriptstyle u} & \rightsquigarrow & \downarrow{\scriptstyle P_T(u)} \quad \checkmark \quad \downarrow{\scriptstyle \Omega(u)} \\ b & & P_T(a) \xrightarrow[p_a]{} \Omega(a) \end{array}$$

For $x \in P_T(b)$, we have:

$$\begin{aligned} p_a \circ P_T(u)(x) &= \Omega(u) \circ p_b(x) \\ p_a(x) &= p_b(x) \wedge a \end{aligned}$$

The canonical example is the following natural transformation:

$$p_a : \begin{cases} P_T(a) & \longrightarrow & \Omega(a) \\ x & \longmapsto & a \end{cases}$$

## 4. First random results

### 4.1. About p-value sheaves

We have already concluded that $[0,1]$, with the usual $\wedge$ and $\vee$ operators, is a complete Heyting algebra. Considering it as a proset category $\mathscr{U}$, being a complete Heyting algebra says that:

**Proposition 4.1.** *$\mathscr{U}$ has all small limits and colimits.*

*Proof.* Let $D : \mathscr{I} \to \mathscr{U}^{\mathrm{op}}$ be a diagram in $\mathscr{U}^{\mathrm{op}}$. Then:

$$\mathrm{Colim}\,(D) = \sup_{i\in \mathrm{Ob}_{\mathscr{I}}} D(i) = \bigwedge_{i\in \mathrm{Ob}_{\mathscr{I}}} D(i)$$
$$\mathrm{Lim}\,(D) = \inf_{i\in \mathrm{Ob}_{\mathscr{I}}} D(i) = \bigvee_{i\in \mathrm{Ob}_{\mathscr{I}}} D(i)$$

Note that the sup becomes a $\bigwedge$, and the inf becomes a $\bigvee$, because we are considering $\mathscr{U}^{\mathrm{op}}$ and not $\mathscr{U}$. Also note that the infima and suprema always exist because we are in a complete Heyting algebra. □

**Proposition 4.2.** *For any test $T$, its p-value presheaf $P_T$ is continuous and cocontinuous.*

*Proof.* Let $D : \mathscr{I} \to \mathscr{U}^{\mathrm{op}}$ be any (small) diagram in $\mathscr{U}^{\mathrm{op}}$.

We only consider the case of a limit; the proof is very similar for colimits.

$$\begin{aligned}
P_T\left(\mathrm{Lim}\,(D)\right) &= P_T\left(\bigvee_{i\in\mathscr{I}} D(i)\right)\\
&= \left\{ x \in \mathbb{R}^d \;\middle|\; \mathrm{pval}_T(x) \geqslant \bigvee_{i\in\mathscr{I}} D(i) \right\}\\
&= \left\{ x \in \mathbb{R}^d \;\middle|\; \forall i \in \mathscr{I},\ \mathrm{pval}_T(x) \geqslant D(i) \right\}\\
&= \bigcap_{i\in\mathscr{I}} \left\{ x \in \mathbb{R}^d \;\middle|\; \mathrm{pval}_T(x) \geqslant D(i) \right\}\\
&= \bigcap_{i\in\mathscr{I}} P_T\left(D(i)\right)
\end{aligned}$$

We now have to check that $\bigcap_{i\in\mathscr{I}} P_T\left(D(i)\right) \cong \mathrm{Lim}\,(P_T \circ D)$.

For all $i \to j \in \mathscr{I}$, we have $P_T\left(D(i)\right) \subset P_T\left(D(j)\right)$. We denote by $\iota_{i,j} = P_T\left(D(j) \to D(i)\right) = P_T\left(D(i) \subset D(j)\right) : P_T\left(D(i)\right) \to P_T\left(D(j)\right)$ the inclusion mapping between $P_T\left(D(i)\right)$'s. We also denote by $\iota_i : \bigcap_{i\in\mathscr{I}} P_T\left(D(i)\right) \to P_T\left(D(i)\right)$ the inclusion mapping of the intersection. For all $i \to j \in \mathscr{I}$, we have $\iota_{i,j} \circ \iota_i = \iota_j$, so that $\iota = \left(\iota_i : \bigcap_{i\in\mathscr{I}} P_T\left(D(i)\right) \to P_T\left(D(i)\right)\right)_{i\in\mathscr{I}}$ is a cone to $P_T \circ D$.

Let $(A, \alpha)$ be any cone to $P_T \circ D$. We denote the $P_T\left(D(i)\right)$-components of $\alpha$ by $\alpha_i : A \to P_T\left(D(i)\right)$. For all $i \to j \in \mathscr{I}$, and for all $x \in X$, we have $\iota_{i,j} \circ \alpha_i(x) = \alpha_j(x) = \alpha_i(x)$.

$$\begin{array}{ccc}
P_T\left(D(i)\right) & \xleftarrow{\ \alpha_i\ } & A \\
{\scriptstyle \iota_{i,j}}\downarrow & \swarrow_{\alpha_j} & \\
P_T\left(D(j)\right) & &
\end{array}$$

So in fact, $\alpha_i(X) = \alpha_j(X) \subset P_T\left(D(i)\right)$, which yields that, for all $i \in \mathscr{I}$, $\alpha_i(X) \subset \bigcap_{i\in\mathscr{I}} P_T\left(D(i)\right)$.

Let $u$ be such that:

$$u : \left\{ \begin{array}{ccc} X & \longrightarrow & \bigcap_{i\in\mathscr{I}} P_T\left(D(i)\right) \\ x & \longmapsto & \alpha_i(x) \end{array} \right.$$

for any of the $i \in \mathscr{I}$, because $\alpha_j(x) = \alpha_i(x)$. Then for all $i \in \mathscr{I}$, we have $\iota_i \circ u = \alpha_i$.

It is also easy to check the unicity of that $u$: suppose $u, u' : X \to \bigcap_{i\in\mathscr{I}} P_T\left(D(i)\right)$, then for all $x \in X$, we have:

$$\iota_i \circ u(x) = \alpha_i(x) = \iota_i \circ u'(x)$$
$$u(x) = u'(x)$$

which leads to $u = u'$.
Consequently, $\bigcap_{i \in \mathscr{I}} P_T\left(D(i)\right)$ is the limit of $P_T \circ D$. □

We know that all p-value presheaves are continuous. The question is: is every continuous presheaf a p-value presheaf? In other words, is any continuous presheaf $P : \mathscr{U}^{\text{op}} \to \mathbf{Sets}$ a $P_T$ for some test $T$? We have to restrict our search for now.

**Definition 4.3** (Lebesgue presheaf)**.** Let $S : \mathscr{U}^{\text{op}} \to \mathbf{Sets}$ be a presheaf.
The presheaf $S$ is called a *Lebesgue presheaf* when for all $a \in \mathscr{U}^{\text{op}}$, $S(a) \in \text{Lbg}\left(\mathbb{R}^d\right)$.

**Definition 4.4** (Inclusive presheaf)**.** Let $S : \mathscr{U}^{\text{op}} \to \mathbf{Sets}$ be a presheaf.
The presheaf $S$ is called *inclusive* when it sends every mapping $a \leqslant b$ to the inclusion mapping $S(a) \subset S(b)$.

**Definition 4.5** (Presheaf from a test)**.** Let $S : \mathscr{U}^{\text{op}} \to \mathbf{Sets}$ be a presheaf and let $T : [0,1] \to \text{Lbg}\left(\mathbb{R}^d\right)$ be a test.
We say that $S$ *comes from* $T$ if there exists a natural isomorphism $\alpha : S \to P_T$. We say that $S$ *comes from a test* if there exists a test $T$ such that $S$ comes from $T$.

**Lemma 4.6.** *Every continuous inclusive Lebesgue presheaf comes from a continuous test.*

*Proof.* Let $S$ be a continuous inclusive Lebesgue presheaf. Define $T : [0,1] \to \text{Lbg}\left(\mathbb{R}^d\right)$ as:

$$T(a) = \mathbb{R}^d \backslash S(a)$$

The continuity of $S$ gives the continuity of $T$. Let $A \subset [0,1]$:

$$\begin{aligned}\bigcup_{a \in A} T(a) &= \bigcup_{a \in A} \mathbb{R}^d \backslash S(a) \\ &= \mathbb{R}^d \backslash \bigcap_{a \in A} S(a) \\ &= \mathbb{R}^d \backslash S\left(\sup(A)\right) \\ &= T\left(\sup(A)\right)\end{aligned}$$

(The proof is the same for intersections.)
Then, let us compare $S$ with the p-value presheaf associated with $T$:

$$\begin{aligned}P_T(a) &= \left\{ x \in \mathbb{R}^d \;\middle|\; \text{pval}_T\left(x\right) \geqslant a \right\} \\ &= \mathbb{R}^d \backslash \bigcup_{b < a} T(b) \\ &= \mathbb{R}^d \backslash T(a) \\ &= S(a)\end{aligned}$$

As $S$ is inclusive, we know that $S\left(a \leqslant b\right) = P_T\left(a \leqslant b\right)$. Consequently, $S$ comes from $T$. □

**Corollary 4.7.** *For every p-value presheaf $P_T$ associated to $T$, there exists a continuous test $T'$ such that $P_T = S_{T'}$.*

The following result is a small addition to the previous lemma.

**Definition 4.8** (Mono-preserving functor)**.** The functor $F : \mathscr{C} \to \mathbf{Sets}$ is called *mono-preserving* when for all monomorphism $m : A \to B$, $F(m)$ is also monic. (The image of a mono is a mono.)

In our case, the presheaf $S : \mathscr{U}^{\mathrm{op}} \to \mathbf{Sets}$ will be mono-preserving when for all $a, b \in [0,1]$ such that $a \leqslant b$, we have $S(a \leqslant b)$ monic. (The image of an inequality is a monomorphism.)

**Lemma 4.9.** *Continuous mono-inducing Lebesgue sheaves come from a test.*

*Proof.* Let $S$ be a continuous mono-inducing Lebesgue presheaf. For $a < b$ ($a, b \in [0,1]$), we denote by $\sigma_{a,b}$ the arrow $\sigma_{a,b} = S(a < b)$.

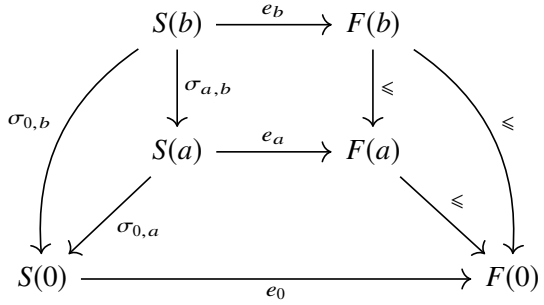
As $S$ is mono-inducing, $\sigma_{a,b}$ is monic. In **Sets**, every function $f : A \to B$ can be written as a $f = m \circ e$ where $e : A \to f(A)$ is epic, and $m : f(A) \to B$ is the canonical inclusion. But, if $f$ is monic, then $e$ is monic too, and in **Sets**, $e$ becomes an isomorphism.

Then, for all $a \in [0,1]$, the arrow $\sigma_{0,a} : S(a) \to S(0)$ decomposes to $\sigma_{0,a} = m \circ e_a$, where $e_a : S(a) \to \sigma_{0,a}(S(a))$ is an isomorphism and $m : \sigma_{0,a}(S(a)) \to S(0)$ is the canonical inclusion. Note that for $a = 0$, we have $\sigma_{0,0} = \mathrm{id}_{S(0)} = e_0$.

Let $F$ be the following functor:

$$F : \begin{cases} \mathscr{U}^{\mathrm{op}} & \longrightarrow & \mathbf{Sets} \\ a & \longmapsto & \sigma_{0,a}(S(a)) \\ a \leqslant b & \longmapsto & F(b) \subset F(a) \end{cases}$$

Let us study $e = (e_a)_{a \in [0,1]}$:




We have to prove that the square $(S(b), F(b), F(a), S(a))$ commutes, while we know that the other squares and triangles commute. For $x \in S(b)$, we have:

$$\begin{aligned} e_a \circ \sigma_{a,b}(x) &= F(0 \leqslant a) \circ e_a \circ \sigma_{a,b}(x) \\ &= e_0 \circ \sigma_{0,a} \circ \sigma_{a,b}(x) \\ &= e_0 \circ \sigma_{0,b}(x) \\ &= F(0 \leqslant b) \circ e_b(x) \\ &= e_b(x) \\ &= F(a \leqslant b) \circ e_b(x) \end{aligned}$$

So $e : S \to F$ is a natural transformation, and each component is an isomorphism, so $e$ is a natural isomorphism. Then, some properties of $S$ transfer to $F$: $F$ is a continuous presheaf. It is by definition inclusive and Lebesgue, so $F$ comes from a test and so does $S$. □

However, other presheaves unlikely come from a test. For example, any presheaf $\mathscr{U}^{\mathrm{op}} \to \mathbf{Sets}$ such that $\mathrm{card}(S(a)) > 2^{\aleph_0}$ for some $a \in \mathscr{U}$ will have trouble being isomorphic to $\mathbb{R}$.

## 4.2. About sheaf morphisms

Let us first study the sheaf morphisms between p-value sheaves $p : P_R \to P_T$.

**Proposition 4.10** (Nesting property)**.** *Let $R,T : [0,1] \to \mathrm{Lbg}\left(\mathbb{R}^d\right)$ be two tests (of the same size or not). Let $p : \mathbb{R}^d \to \mathbb{R}^d$ be a function.*

*The function $p$ defines a natural transformation $\bar{p} : P_R \to P_T = \left(p_{|P_R(a)}\right)_{a\in[0,1]}$ $\Leftrightarrow$ for all $a \in [0,1]$, $p\left(P_R(a)\right) \subset P_T(a)$.*

In other words, $p$ defines a function between nested open sets, as in Figure 1.

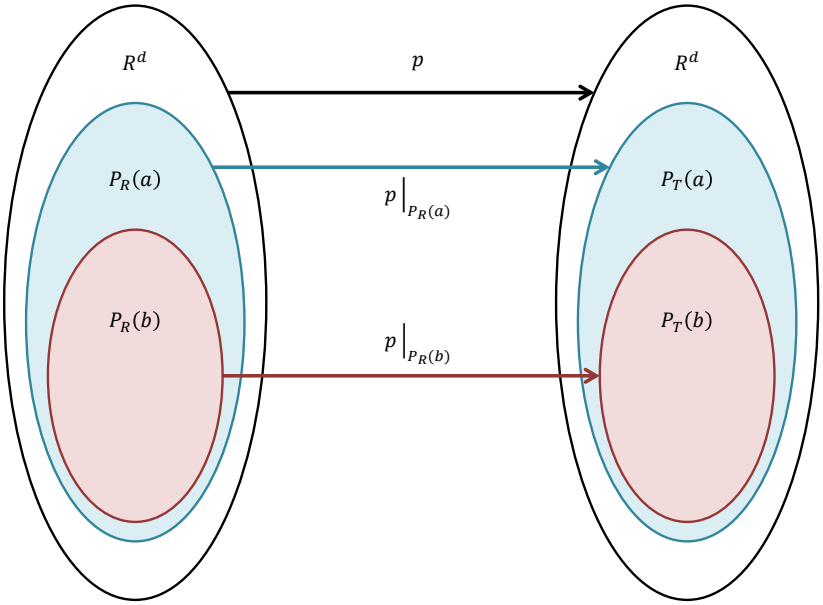



Figure 1: Illustration of the nesting property of $p$. *If $p : \mathbb{R}^d \to \mathbb{R}^d$ verifies this nesting property, then it defines a natural transformation between two p-value sheaves.*

*Proof.* Easy deduction from the following natural transformation diagram:

$$\begin{array}{ccc} a & & P_R(b) \xrightarrow{p_{|P_R(b)}} P_T(b) \\ \downarrow \subset & \rightsquigarrow & \downarrow \subset \quad \checkmark \quad \downarrow \subset \\ b & & P_R(a) \xrightarrow[p_{|P_R(a)}]{} P_T(a) \end{array}$$

□

**Corollary 4.11.** *If $\bar{p} = (p_a)_{a\in[0,1]}$ is a natural transformation $P_R \to P_T$, then $p = \bigcup_{a\in A} p_a$ is a function that verifies the nesting property.*

**Proposition 4.12.** *For all $a \in [0,1]$, $P_T(a) = \mathbb{R}^d \setminus \left(\bigcup_{b<a} T(b)\right)$.*

*Proof.* By computation:

$$\begin{aligned}
P_T(a) &= \left\{ x \in \mathbb{R}^d \;\middle|\; \mathrm{pval}_T(x) \geqslant a \right\} \\
&= \left\{ x \in \mathbb{R}^d \;\middle|\; \inf_{x \in T(b)} (b \in [0,1]) \geqslant a \right\} \\
&= \left\{ x \in \mathbb{R}^d \;\middle|\; \forall b < a, x \notin T(b) \right\} \\
&= \bigcap_{b<a} \mathbb{R}^d \backslash T(b) \\
&= \mathbb{R}^d \backslash \left( \bigcup_{b<a} T(b) \right)
\end{aligned}$$

□

**Corollary 4.13.** *For all $a \in [0,1]$, if $\bigcup_{b<a} T(b) = T(a)$ then $P_T(a) = \mathbb{R}^d \backslash T(a)$.*

*Example* 4.14. Note that we are considering very general tests $T : [0,1] \to \mathrm{Lbg}\left(\mathbb{R}^d\right)$, so $\left( \bigcup_{b<a} T(b) \right)$ has no reason to be equal to $T(a)$. Let us give a counterexample.

Consider the following functions:

$$\begin{aligned}
f : \left\{ \begin{array}{ccc} [0,1] & \longrightarrow & [0,1] \\ x & \longmapsto & \left\{ \begin{array}{ll} x & \text{if } x < \frac{1}{2} \\ \frac{1}{2}x + \frac{1}{2} & \text{if } x \geqslant \frac{1}{2} \end{array} \right. \end{array} \right. \\
g : \left\{ \begin{array}{ccc} ]0,1[ & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \tan\left(\pi x - \frac{\pi}{2}\right) \end{array} \right. \\
T : \left\{ \begin{array}{ccc} [0,1] & \longrightarrow & \mathrm{Lbg}\left(\mathbb{R}^d\right) \\ a & \longmapsto & \left\{ \begin{array}{ll} ]-g \circ f(a), g \circ f(a)[ & \text{if } a < 1 \\ \mathbb{R}^d & \text{if } a = 1 \end{array} \right. \end{array} \right.
\end{aligned}$$

The function $f$ is strictly increasing and establishes a bijection $[0,1] \to \left[0, \frac{1}{2}\right[ \cup \left[\frac{3}{4}, 1\right]$. Then $g$ is bijective and strictly increasing, so $g \circ f$ is striclty increasing and injective, and finally $T$ is stricly increasing in the Lebesgue set, so it is injective, and has a left inverse $\mathrm{Lbg}\left(\mathbb{R}^d\right) \to [0,1]$ (which is a size). In other words, $T$ is a test. However:

$$\begin{aligned}
\bigcup_{a<\frac{1}{2}} T(a) &= \bigcup_{a<\frac{1}{2}} ]-g \circ f(a), g \circ f(a)[ \\
&= \left] - \lim_{a<\frac{1}{2}} g \circ f(a), \lim_{a<\frac{1}{2}} g \circ f(a) \right[ \\
&= \left] -g\left( \lim_{a<\frac{1}{2}} f(a) \right), g\left( \lim_{a<\frac{1}{2}} f(a) \right) \right[ \\
&= \left] -g\left(\frac{1}{2}\right), g\left(\frac{1}{2}\right) \right[ \\
&\subsetneq T\left(\frac{1}{2}\right) = \left] -g\left(\frac{3}{4}\right), g\left(\frac{3}{4}\right) \right[
\end{aligned}$$

**Definition 4.15** (Continuous test)**.** Let $T$ be a test. We call $T$ a *continuous test* when, for all $A \subset [0,1]$, we have $\bigcup_{a \in A} T(a) = T(\sup(A))$ and $\bigcap_{a \in A} T(a) = T(\inf(A))$.

Note that this notion of continuity is closer to the categorical notion than the analytic notion.

In the following, we give an example of such a continuous test.

**Definition 4.16** (Likelihood ratio)**.** Let $f_0, f_1 : \mathbb{R}^d \to \mathbb{R}^+$ be two probability density functions, and let $x \in \mathbb{R}^d$.

The *likelihood ratio of $x$ being from $X \sim f_1$ instead of $X \sim f_0$*, written $\Lambda^{f_0,f_1}(x)$ or simply $\Lambda(x)$ when there is no ambiguity, is the following function:

$$\Lambda : \begin{cases} \mathbb{R}^d & \longrightarrow & \mathbb{R}^+ \cup \{\infty\} \\ x & \longmapsto & \begin{cases} \dfrac{f_1(x)}{f_0(x)} & \text{if } f_0(x) \neq 0 \\ \infty & \text{if } f_0(x) = 0 \text{ and } f_1(x) > 0 \\ 0 & \text{if } f_0(x) = 0 \text{ and } f_1(x) = 0 \end{cases} \end{cases}$$

In the following, we consider $X \sim f_0$, and we restrict ourselves to the cases where $\Lambda(X)$ is an absolutely continuous random variable. This happens, using Jacobi's transformation formula [2], when $f_0 > 0$ and when both $f_0$ and $f_1$ are differentiable, for example when $f_0$ and $f_1$ are both Gaussian distributions. In this case, the function $k \to \mathrm{P}[\Lambda(X) > k \mid X \sim f_0]$ (which is the false alarm probability) is continuous and stricly decreasing, and has a continuous and strictly decreasing inverse which is described in the following definition.

**Definition 4.17** (NP-threshold function)**.** Let $f_0, f_1 : \mathbb{R}^d \to \mathbb{R}^+$ be two probability density functions such that, if $X \sim f_0$ then $\Lambda(X)$ is absolutely continuous.

The *NP-threshold function*, written $k_{\mathrm{NP}}$ or simply $k$ in the following, is the function that assigns to each $a \in [0, 1]$, the threshold $k(a)$ such that:

$$a = \int_{\Lambda(x) > k(a)} f_0(x)\mathrm{d}x = \mathrm{P}[\Lambda(X) > k(a) \mid X \sim f_0]$$

*Remark* 4.18. The threshold $k(a)$ is the unique threshold whose false alarm probability is $a$.

As the function $k \to \mathrm{P}[\Lambda(X) > k \mid X \sim f_0]$ is continuous and stricly decreasing, it is easy to see that:

**Proposition 4.19.** *The NP-threshold function is continuous and strictly decreasing.*

**Definition 4.20** (Neyman-Pearson test)**.** Let $f_0, f_1 : \mathbb{R}^d \to \mathbb{R}^+$ be two probability density functions such that, if $X \sim f_0$ then $\Lambda(X)$ is absolutely continuous.

The *Neyman-Pearson test* is the following test:

$$\mathrm{NP} : \begin{cases} [0,1] & \longrightarrow & \mathrm{Lbg}\left(\mathbb{R}^d\right) \\ a & \longmapsto & \left\{x \in \mathbb{R}^d \;\middle|\; \Lambda(x) > k(a)\right\} \end{cases}$$

**Proposition 4.21.** *The test* NP *is continuous.*

*Proof.* This is due to the monotonicity of the threshold function $k$. Let $A \subset [0, 1]$:

$$\begin{aligned} \bigcup_{a \in A} \mathrm{NP}(a) &= \bigcup_{a \in A} \left\{x \in \mathbb{R}^d \;\middle|\; \Lambda(x) > k(a)\right\} \\ &= \left\{x \in \mathbb{R}^d \;\middle|\; \exists a \in A,\ \Lambda(x) > k(a)\right\} \\ &= \left\{x \in \mathbb{R}^d \;\middle|\; \Lambda(x) > \inf_{a \in A}(k(a))\right\} \\ &= \left\{x \in \mathbb{R}^d \;\middle|\; \Lambda(x) > k(\sup(A))\right\} \\ &= \mathrm{NP}(\sup(A)) \end{aligned}$$

The proof for the intersection is roughly the same. □

**Corollary 4.22.** $S_{\mathrm{NP}}(a) = \mathbb{R}\backslash\mathrm{NP}(a)$.

**Definition 4.23** (RDT context)**.** An RDT-context is a tuple $(d, \theta_0, C, \tau)$ where:

1. $d \in \mathbb{N}$ is the *dimension*;
2. $\theta_0 \in \mathbb{R}^d$ is called the *model*;
3. $C \in \mathrm{Mat}_{d,d}(\mathbb{R})$ is a positively definite square $d \times d$ real matrix;
4. $\tau \in \mathbb{R}_+^*$ is the *tolerance*.

**Definition 4.24** (RDT problem)**.** Let $(d, \theta_0, C, \tau)$ be an RDT-context.
The *RDT problem with context* $(\theta_0, C, \tau)$ consists in the following testing problem:

Observation $Y = \Theta + X$ where $X \sim \mathcal{N}\left(0, C^2\right)$, $\Theta$ is any random vector in $\mathbb{R}^d$ and $\Theta$ and $X$ are independent.

$\mathcal{H}_0$ $\|\Theta - Y\|_C \leqslant \tau$

$\mathcal{H}_1$ $\|\Theta - Y\|_C > \tau$

**Definition 4.25** (RDT threshold function)**.** Let $(d, \theta_0, C, \tau)$ be an RDT-context and let $a \in ]0, 1[$.
The *RDT threshold function with false alarm* $a$, written $\lambda_a$, is the following function:

$$\lambda_a : \left\{ \begin{array}{ccc} \mathbb{R}^+ & \longrightarrow & \mathbb{R}^+ \\ \tau & \longmapsto & \text{the unique solution in } \eta \text{ to the} \\ & & \text{equation: } a = 1 - \mathbb{F}_{\chi^2_d(\tau^2)}\left(\eta^2\right) \end{array} \right.$$

**Definition 4.26** (RDT test)**.** Let $(d, \theta_0, C, \tau)$ be an RDT-context.
The *RDT test with context* $(d, \theta_0, C, \tau)$ is the following test:

$$\mathrm{RDT} : \left\{ \begin{array}{ccc} [0,1] & \longrightarrow & \mathrm{Lbg}\left(\mathbb{R}^d\right) \\ a & \longmapsto & \left\{ \begin{array}{ll} \emptyset & \text{if } a = 0 \\ \mathbb{R}^d\backslash\{\theta_0\} & \text{if } a = 1 \\ \left\{x \in \mathbb{R}^d \;\middle|\; \lambda_a(\tau) < \|x - \theta_0\|_C\right\} & \text{otherwise} \end{array} \right. \end{array} \right.$$

**Proposition 4.27.** *The test* RDT *is continuous.*

*Proof.* Let $A \subset [0, 1]$. The proof is similar to the one for NP, however we have a few details to take care before.

$$\begin{aligned} \bigcup_{a \in A} \mathrm{RDT}(a) &= \bigcup_{a \in A} \left\{x \in \mathbb{R}^d \;\middle|\; \lambda_a(\tau) < \|x - \theta_0\|_C\right\} \\ &= \left\{x \in \mathbb{R}^d \;\middle|\; \exists a \in A,\ \lambda_a(\tau) < \|x - \theta_0\|_C\right\} \\ &= \left\{x \in \mathbb{R}^d \;\middle|\; \inf_{a \in A}\left(\lambda_a(\tau)\right) < \|x - \theta_0\|_C\right\} \end{aligned}$$

Then, $a \mapsto \lambda_a(\tau)$ is decreasing and continuous in $a$, so:

$$\inf_{a \in A}\left(\lambda_a(\tau)\right) = \lim_{\substack{x \in A \\ x \to \sup(A)}} \lambda_a(\tau) = \left\{ \begin{array}{ll} \lambda_{\sup(A)}(\tau) & \text{if } 1 \neq \sup(A) \\ 0 & \text{otherwise} \end{array} \right.$$

If $1 \neq \sup(A)$ then:

$$\begin{aligned}\bigcup_{a\in A}\mathrm{RDT}(a) &= \left\{x\in\mathbb{R}^d \;\middle|\; \inf_{a\in A}(\lambda_a(\tau)) < \|x-\theta_0\|_C\right\}\\ &= \left\{x\in\mathbb{R}^d \;\middle|\; \lambda_{\sup(A)}(\tau) < \|x-\theta_0\|_C\right\}\\ &= \mathrm{RDT}(\sup(A))\end{aligned}$$

Otherwise, if $1 = \sup(A)$ then:

$$\begin{aligned}\bigcup_{a\in A}\mathrm{RDT}(a) &= \left\{x\in\mathbb{R}^d \;\middle|\; \inf_{a\in A}(\lambda_a(\tau)) < \|x-\theta_0\|_C\right\}\\ &= \left\{x\in\mathbb{R}^d \;\middle|\; 0 < \|x-\theta_0\|_C\right\}\\ &= \mathbb{R}^d\backslash\{\theta_0\}\\ &= \mathrm{RDT}(\sup(A))\end{aligned}$$

□

**Proposition 4.28.**

## 4.3. About predicates

**Definition 4.29** (Predicate)**.** Let $S$ be a sheaf in $\mathbf{PSh}(\mathscr{U})$.

A *predicate* is a sheaf morphism $p : S \to \Omega$.

***Computing predicates***

Let $S : \mathscr{U}^{\mathrm{op}} \to \mathbf{Sets}$ be a sheaf, and let $p_1, p_2 : S \to \Omega$ be predicates. The predicates $p_1 \wedge p_2, p_1 \vee p_2, p_1 \to p_2, \neg p_1$ are all predicates $S \to \Omega$.

Let $a \in [0,1]$. For $x \in S(a)$, define $a_1 = p_{1,a}(x)$ and $a_2 = p_{2,a}(x)$. We have the trivial results:

$$\begin{aligned}(p_1 \wedge p_2)_a(x) &= \min(a_1, a_2)\\ (p_1 \vee p_2)_a(x) &= \max(a_1, a_2)\end{aligned}$$

Also:

$$\begin{aligned}(p_1 \to p_2)_a(x) &= p_{1,a}(x) \to p_{2,a}(x)\\ &= \bigvee_{R\wedge a_1 \leqslant a_2} R\\ &= \bigvee_{\substack{b\in[0,a]\\ b\wedge a_1\leqslant a_2}} b\\ &= \bigvee_{\substack{b\in[0,a]\\ \min(a_1,b)\leqslant a_2}} b\\ &= \bigvee_{\substack{b\in[0,a]\\ \min(a_1,b)\leqslant a_2}} b\\ &= U_c\end{aligned}$$

for some $c \in [0, a]$. Let us take a look at that $c$. We have:

$$c = \sup_{\substack{b \in [0,a] \\ \min(a_1, b) \leqslant a_2}} (b) = \begin{cases} a & \text{if } a_1 \leqslant a_2 \\ a_2 & \text{otherwise} \end{cases}$$

The negation $\neg p_1$ corresponds to the special case where $a_2 = U_0 = \emptyset$:

$$\begin{aligned} (\neg p_1)_a (x) &= p_{1,a}(x) \to \emptyset \\ &= \bigvee_{\substack{b \in [0,a] \\ \min(a_1, b) \leqslant 0}} b \\ &= U_c \end{aligned}$$

With $c$ being:

$$c = \sup_{\substack{b \in [0,a] \\ \min(a_1, b) \leqslant 0}} (b) = \begin{cases} a & \text{if } a_1 = 0 \\ 0 & \text{if } a_1 > 0 \end{cases}$$

In summary:

| Formula | Condition | Result |
|---|---|---|
| $p_{1,a}(x)$ | | $a_1$ |
| $p_{2,a}(x)$ | | $a_2$ |
| $(p_1 \wedge p_2)_a (x)$ | | $\min(a_1, a_2)$ |
| $(p_1 \vee p_2)_a (x)$ | | $\max(a_1, a_2)$ |
| $(p_1 \to p_2)_a (x)$ | if $a_1 \leqslant a_2$ | $a$ |
| | if $a_1 > a_2$ | $a_2$ |
| $(\neg p_1)_a (x)$ | if $a_1 = 0$ | $a$ |
| | if $a_1 > 0$ | $U_0 = \emptyset$ |

***Construction of a predicate***

How to write "$x \in \mathrm{B}(\theta_0, \rho)$" in a predicate? (We are considering an open ball)

We first need to find the right test. Then it will give us a sheaf, and we will design the right predicate.

The predicate should be a sheaf morphism: $p : P_T \to \Omega$, such that, for all $a \in [0, 1]$, it takes $x \in P_T(a)$ and returns the subset of $a$ for which $x \in \mathrm{B}(\theta_0, \rho)$.

As a first answer, we choose $\rho$ to be a parameter for the test. We define the following test:

$$T(a) = \mathrm{B}\left(\theta_0, \frac{a\rho}{1-a}\right)$$

Then $T$ is a continuous test, and $P_T(a) = \mathbb{R} \backslash \mathrm{B}\left(\theta_0, \frac{a\rho}{1-a}\right)$. Then the predicate should be $p : S \to \Omega$ such that $p_a(x) = \emptyset$ if $x \notin \mathrm{B}(\theta_0, \rho)$, and $p_a(x) = a \wedge U_{\frac{1}{2}}$ where $\frac{1}{2}$ is the solution to the equation in $a$: $\frac{a\rho}{1-a} = \rho$.

For $a < b$, we have $a \leqslant b$ and we need the following diagram to commute:

$$\begin{array}{ccc} S(b) & \xrightarrow{p_b} & \Omega(b) \\ \Big\downarrow{\scriptstyle \leqslant} & & \Big\downarrow{\scriptstyle \text{"}\wedge a\text{"}} \\ S(a) & \xrightarrow[p_a]{} & \Omega(a) \end{array}$$

If $x \in S(b)$, then:

$$\begin{aligned} \text{"} \wedge a\text{"} \circ p_b(x) &= \begin{cases} \emptyset \wedge a & \text{if } x \notin \mathrm{B}(\theta_0, \rho) \\ b \wedge a \wedge U_{\frac{1}{2}} & \text{if } x \in \mathrm{B}(\theta_0, \rho) \end{cases} \\ &= p_a(x) \end{aligned}$$

# 5. Study of the category of p-values

**Definition 5.1** (Category $\mathbf{PVal}(\mathscr{C})$)**.** Let $\mathscr{C}$ be a category representing a complete Heyting algebra.

The category of p-values on $\mathscr{C}$, denoted by $\mathbf{PVal}(\mathscr{C})$, is the full subcategory of $\mathbf{PSh}(\mathscr{C})$ such that every $P \in \mathbf{PVal}(\mathscr{C})$ comes from a test.

In the following we will study the properties of that category.

**Proposition 5.2.** *(Conjecture)*

1. $\mathbf{PVal}(\mathscr{U})$ *doesn't have all exponentials.*
2. $\mathbf{PVal}(\mathscr{U})$ *has all finite products.*
3. $\mathbf{PVal}(\mathscr{U})$ *has no terminal object.*

The problem with these conjectures is that the properties of $\mathbf{PSh}([0,1])$ don't transfer to their subcategories. A first guess was that, if $\mathbf{PVal}(\mathscr{U})$ had all exponentials, then the exponentials in $\mathbf{PVal}(\mathscr{U})$ should be isomorphic to the exponentials in $\mathbf{PSh}(\mathscr{U})$ given that $\mathbf{PVal}(\mathscr{U})$ is a full subcategory of $\mathbf{PSh}(\mathscr{U})$. In summary, it is thinkable that, if $[B \to C]$ is the exponential in $\mathbf{PVal}(\mathscr{U})$ and $C^B$ is the exponential in $\mathbf{PSh}(\mathscr{U})$, then:

$$\begin{aligned} \forall A, B, C \in \mathbf{PVal}(\mathscr{U}), \mathrm{Hom}_{\mathbf{PSh}(\mathscr{U})}(A, [B \to C]) \cong \mathrm{Hom}_{\mathbf{PSh}(\mathscr{U})}\left(A, C^B\right) \\ \Rightarrow [B \to C] \cong C^B \end{aligned}$$

However, this is not the case. Consider a topological space $(X, \mathscr{T})$ as a Heyting algebra. Then take any non-trivial topological subspace $(Y, \mathscr{T}_Y)$ of $X$, also viewed as a Heyting algebra. The exponential in a topological space is defined from the exponential in a Heyting algebra and is defined as:

$$U \cap V \leqslant W \Leftrightarrow V \leqslant \underbrace{U \to W}_{\text{exponential}}$$

A more explicit version of this exponential is a follows [3, Section 1.1.4., p21]:

$$\underbrace{U \to W}_{\text{in } \mathscr{T}} = \max\left(\{Z \in \mathscr{T} \mid Z \cap U \leqslant W\}\right)$$

Which is the interior of $W \cup (X \backslash U)$ in $X$.

If $U$ and $W$ are open subsets of $Y$ then, then the exponential $U \to W$ in $Y$ is the interior of $W \cup (Y \backslash U)$ which may be different to the interior of $W \cup (X \backslash U)$ (cf. Figure 2).

Finally, we can suspect there is no terminal object, because we don't see which object, other than $\Delta(1)$, where $\Delta : \mathscr{U} \to \mathbf{Sets}^{\mathscr{U}}$, could be the terminal object, and $\Delta(1)$ is not in $\mathbf{PVal}(\mathscr{U})$.

Otherwise, the product seems like an easy result.

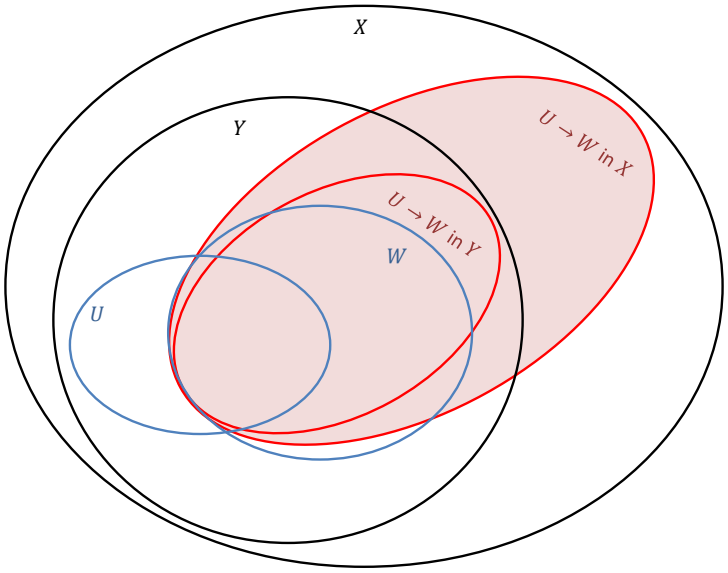



Figure 2: Illustration of what may happen when comparing the exponentials of a topological space with one of its subspaces. *Here, $Y$ is a topological subspace of $X$, $U$ and $W$ are open subsets of $Y$ (and thus of $X$), and the exponential $U \to W$ in $Y$ could be the exponential $(U \to W)_X \cap Y$ where $(U \to W)_X$ is the exponential in $X$.*

*Proof.* (**PVal** $(\mathscr{U})$ has all products) Let $d_1, d_2 \in \mathbb{N}$, and let $T_1 : [0,1] \to \mathrm{Lbg}\left(\mathbb{R}^{d_1}\right)$ and $T_2 : [0,1] \to \mathrm{Lbg}\left(\mathbb{R}^{d_2}\right)$ be two tests.

Then, for every $a \in [0,1]$, $T_1(a) \times T_2(a)$ is also in $\mathrm{Lbg}\left(\mathbb{R}^{d_1+d_2}\right)$. Then, $T_1 \times T_2 : [0,1] \to \mathrm{Lbg}\left(\mathbb{R}^{d_1+d_2}\right)$ is also a test.

Thus:

$$\begin{aligned} S_{T_1}(a) \times S_{T_2}(a) &= \left(\mathbb{R}^{d_1} \setminus \bigcup_{b<a} T_1(b)\right) \times \left(\mathbb{R}^{d_2} \setminus \bigcup_{b<a} T_2(b)\right) \\ &= \mathbb{R}^{d_1+d_2} \setminus \bigcup_{b<a} \left(T_1(b) \times T_2(b)\right) \\ &= \left(S_{T_1} \times S_{T_2}\right)(a) \end{aligned}$$

Thus, the product in **PVal** $(\mathscr{U})$ is the same as in **PSh** $(\mathscr{U})$. □

## 6. Natural transformation between NP and RDT

We will have to reformulate the problems they both answer to, in order to have two comparable tests.

Let $C$ be a positive-definite $d \times d$ matrix. Let $\mu \in \mathbb{R}^d$.

The NP problem amounts to testing:

$$\begin{aligned} \text{Observation}: &\quad Y \sim \mathcal{N}(\varepsilon, C) = \varepsilon + \mathcal{N}(0, C) \\ \mathcal{H}_0: &\quad \varepsilon = 0 \\ \mathcal{H}_1: &\quad \varepsilon = \mu \end{aligned}$$

While the RDT problem amounts to:

$$\begin{array}{rl} \text{Observation}: & Y = \varepsilon + X \text{ where } X \sim \mathcal{N}(0, C) \\ \mathcal{H}_0: & \left\|\varepsilon - \frac{1}{2}\mu\right\|_C \leqslant \left\|\frac{1}{2}\mu\right\|_C = \tau \\ \mathcal{H}_1: & \left\|\varepsilon - \frac{1}{2}\mu\right\|_C > \left\|\frac{1}{2}\mu\right\|_C = \tau \end{array}$$

We assume here that $\varepsilon$ can only take 0 or $\mu$ as values. The RDT problem is more general (as seen above in Definition 4.24) in that $\varepsilon$ could be any real random variable. The NP problem is also more general, because it could compare any two distributions. We also note that, in this formulation, the two versions of the null hypothesis are equivalent, and the two versions of the alternative hypothesis too. With this formulation, both problems are exactly the same. However, this doesn't mean that RDT and NP will give the same answer to a caught signal.

The goal of this section is to present the computations that lead to a natural transformation between $S_{\text{NP}}$ and $S_{\text{RDT}}$, so that NP and RDT do give the same answer to the same caught signal.

Let $a \in [0, 1]$.

$$\begin{aligned} x \in S_{\text{NP}}(a) &\Leftrightarrow \Lambda(x) \leqslant k(a) \\ &\Leftrightarrow \frac{\exp\left(-\frac{1}{2}\|x-\mu\|_C^2\right)}{\exp\left(-\frac{1}{2}\|x-0\|_C^2\right)} \leqslant k(a) \\ &\Leftrightarrow \frac{1}{2}\|x\|_C^2 - \frac{1}{2}\|x-\mu\|_C^2 \leqslant \ln(k(a)) \\ &\Leftrightarrow x^t C^{-1} x - x^t C^{-1} x + 2x^t C^{-1}\mu - \mu^t C^{-1}\mu \leqslant 2\ln(k(a)) \\ &\Leftrightarrow x^t C^{-1}\mu \leqslant \underbrace{\frac{1}{2}\mu^t C^{-1}\mu + \ln(k(a))}_{K(a)} \end{aligned}$$

As for RDT:

$$x \in S_{\text{RDT}}(a) \Leftrightarrow \left\|x - \frac{1}{2}\mu\right\|_C \leqslant \lambda_a(\tau)$$

The NP problem amounts to comparing a scalar product to a threshold: $x^t C^{-1}\mu \leqslant K(a)$, while the RDT problem amounts to comparing a distance to a threshold: $\left\|x - \frac{1}{2}\mu\right\|_C \leqslant \lambda_a(\tau)$. The expression of $K(a)$ and $\lambda_a(\tau)$ are not important here.

We are looking for a set of functions $f_a : S_{\text{RDT}}(a) \to S_{\text{NP}}(a)$ or $g_a : S_{\text{NP}}(a) \to S_{\text{RDT}}(a)$ that associate a ball to a hyperplane. Due to Proposition 4.10, $f_a$ and $g_a$ should verify the nesting property. We thus look functions $f : \mathbb{R}^d \to \mathbb{R}^d$ or $g : \mathbb{R}^d \to \mathbb{R}^d$ that respect this nesting property, and we will deduce the natural transformations from restrictions of these functions.

Namely, we are looking for a $g$ such that:

$$x^t C^{-1}\mu = K(a) \Leftrightarrow \left\|g(x) - \frac{1}{2}\mu\right\|_C = \lambda_a(\tau)$$

Let us start with dimension $d = 2$.

For a given $\mu \neq 0$, the set of $x^t C^{-1}\mu = K(a)$ defines a straight line that is perpendicular to $\mu$. Let $\mu_1$ be a vector orthogonal to $\mu$ with same length. We have $x = x^t C^{-1}\mu \cdot \mu + x^t C^{-1}\mu_1 \cdot \mu_1$.

Consider now the circle $\mathrm{B}\left(\frac{1}{2}\mu, \lambda_a(\tau)\right)$. We consider this circle as a trigonometric-like circle with radius $\lambda_a(\tau)$. As such, define the vector $\left(\frac{1}{2} + \lambda_a(\tau)\right)\mu$ as the trigonometric origin on this circle. Consider the vector with trigonometric coordinate $\alpha(x) = \arctan\left(x^t C^{-1}\mu_1\right)$; it is the following vector $g(x)$:

$$g(x) = \frac{1}{2}\mu + \lambda_a(\tau)\left(\frac{\cos(\alpha(x))}{\|\mu\|_C}\mu + \frac{\sin(\alpha(x))}{\|\mu_1\|_C}\mu_1\right)$$

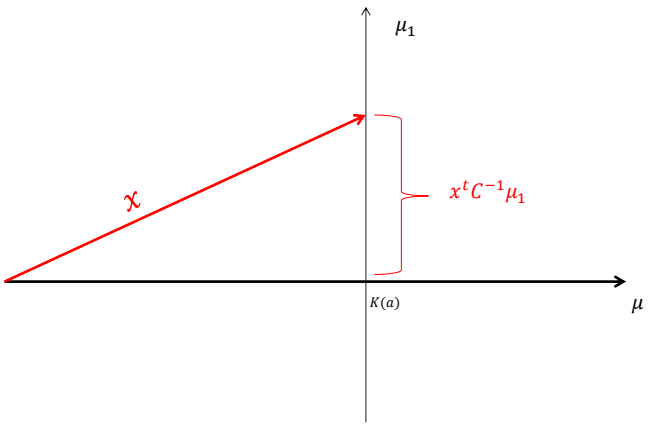



Figure 3: Illustration of the action of a 2-dimensional natural transformation between NP and RDT. *NP compares the projection of $x$ over $\mu$ to a threshold $K(a)$. As we are in finite dimension, there exists $\mu_1$ orthogonal to $\mu$ with the same length. The coordinates of $x$ in the basis $(\mu, \mu_1)$ are $\left(x^tC^{-1}\mu, x^tC^{-1}\mu_1\right)$. The goal of the natural transformation we are building is to send $x^tC^{-1}\mu_1$ to a trigonometric coordinate (see Figure 4)*

In dimension $d$, it amounts to sending a $d$-vector to another vector in the $d-1$-sphere with spherical coordinates. In dimension $d$, the spherical coordinates are $(r, \varphi_1, \ldots, \varphi_{d-1})$, such that, if $x$ has Cartesian coordinates $x_1, \ldots, x_d$:

$$\begin{aligned}
x_1 &= r\cos(\varphi_1)\\
x_2 &= r\sin(\varphi_1)\cos(\varphi_2)\\
x_3 &= r\sin(\varphi_1)\sin(\varphi_2)\cos(\varphi_3)\\
&\ldots\\
x_{d-1} &= r\sin(\varphi_1)\ldots\sin(\varphi_{d-2})\cos(\varphi_{d-1})\\
x_d &= r\sin(\varphi_1)\ldots\sin(\varphi_{d-2})\sin(\varphi_{d-1})
\end{aligned}$$

Then, in dimension $d$, given a positive-definite matrix $C$ and a vector $\mu$, we decide of a unique orthogonal basis $(\mu, \mu_1, \ldots, \mu_{d-1})$. For practical reasons, we do not assume that they have the same length 1 but $\|\mu\|_C$ (it doesn't change the result, only the easiness of the computations).

For a vector $x = (x_1, \ldots, x_d)$ such that $x^tC^{-1}\mu = K(a)$, we define a temporary vector $v(x)$ (which will play the role of the green one in Figure 4); $v(x)$ has spherical coordinates $(r, \varphi_1, \ldots, \varphi_{d-1})$ where:

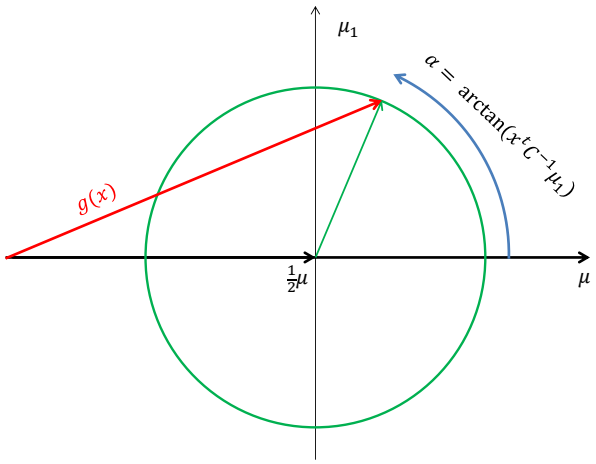



Figure 4: Illustration of the action of a 2-dimensional natural transformation between NP and RDT. *RDT compares the distance between $x$ and $\frac{1}{2}\mu$ to a threshold $\lambda_a$. So it defines a ball. In dimension 2, consider it as a trigonometric circle, in the sense that we want to place vectors on it. The vector with coordinates $\alpha = \arctan\left(x^t C^{-1}\mu_1\right)$ on the circle is the green vector. It gives the expression of $g(x)$ (in red) as the sum $\frac{1}{2}\mu + \lambda_a(\tau)\left(\frac{\cos(\alpha(x))}{\|\mu\|_C}\mu + \frac{\sin(\alpha(x))}{\|\mu_1\|_C}\mu_1\right)$.*

$$
\begin{aligned}
r &= \lambda_a(\tau) \\
\varphi_1 &= \arccos\left(\frac{x^t C^{-1}\mu_1}{\|x\|_C \, \|\mu_1\|_C}\right) \\
\varphi_2 &= \arccos\left(\frac{x^t C^{-1}\mu_2}{\|x\|_C \, \|\mu_2\|_C}\right) \\
&\dots \\
\varphi_{d-1} &= \arcsin\left(\frac{x^t C^{-1}\mu_{d-1}}{\|x\|_C \, \|\mu_{d-1}\|_C}\right)
\end{aligned}
$$

and then:

$$
g(x) = \frac{1}{2}\mu + v(x)
$$

Finally, the natural transformation $\bar{g} : S_{\text{NP}} \longrightarrow S_{\text{RDT}}$ is the following:

$$
\bar{g} : S_{\text{NP}} \longrightarrow S_{\text{RDT}} = \left(g_{|S_{\text{NP}}(a)} : \left\{ \begin{array}{ccc} S_{\text{NP}}(a) & \longrightarrow & S_{\text{RDT}}(a) \\ x & \longmapsto & g(x) \end{array} \right.\right)_{a\in[0,1]}
$$

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