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## A crash course on Category Theory

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# **A crash course on Category Theory**

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## **Introduction**

This course will introduce category theory from an adjunction-driven point of view, which is somewhat unusual.

The goal is to introduce topoi.

## 1. Basic notions

This section will introduce some basic notions about category theory: categories, functors, opposite categories, natural transformations.

**Definition 1.1** (Category [1]). A category  $\mathcal{C}$  consists of the following data:

- A collection of *objects*, denoted  $\text{Ob}_{\mathcal{C}}$
- A collection of *morphisms*, or *arrows*, denoted  $\text{Mor}_{\mathcal{C}}$
- A map  $\text{dom} : \text{Mor}_{\mathcal{C}} \rightarrow \text{Ob}_{\mathcal{C}}$ ; for each morphism  $f$ ,  $\text{dom}(f)$  is called the *domain* of  $f$
- A map  $\text{cod} : \text{Mor}_{\mathcal{C}} \rightarrow \text{Ob}_{\mathcal{C}}$ ; for each morphism  $f$ ,  $\text{cod}(f)$  is called the *codomain* of  $f$
- For each morphism  $f \in \text{Mor}_{\mathcal{C}}$ , we write  $f : A \rightarrow B$  if  $A = \text{dom}(f)$  and  $B = \text{cod}(f)$
- A *composition law*  $\circ$  such that, for all  $f : A \rightarrow B$  and  $g : B \rightarrow C$ , there is a chosen morphism  $g \circ f : A \rightarrow C$
- For each object  $A \in \text{Ob}_{\mathcal{C}}$ , there is a chosen morphism  $1_A : A \rightarrow A$  called *identity morphism of A*

The composition law is required to be associative:  $\forall A, B, C, D \in \text{Ob}_{\mathcal{C}}, \forall f : A \rightarrow B$  and  $g : B \rightarrow C$  and  $h : C \rightarrow D$ ,  $(h \circ g) \circ f = h \circ (g \circ f)$ . Identity morphisms are required to act like identities:  $\forall A, B \in \text{Ob}_{\mathcal{C}}, \forall f : A \rightarrow B$ ,  $f \circ 1_A = 1_B \circ f = f$ .

In the rest of the course, a category  $\mathcal{C}$  will be described according to the following presentation:

**Objects:** An object in  $\mathcal{C}$  is...

**Morphisms:** A morphism in  $\mathcal{C}$  is...

**Identities:** An identity morphism is...

**Composition:** The composition law for morphisms is...

Usually, the description of morphisms suffices to implicitly define  $\text{dom}$  and  $\text{cod}$ , as in the following examples.

*Example 1.2.* One of the easiest categories is the category in of sets. We define the category **Sets** as the following:

**Objects:** An object in **Sets** is any set

**Morphisms:** A morphism in **Sets** is any function  $f : A \rightarrow B$

**Identities:** An identity morphism is an identity function  $\text{id}_A : A \rightarrow A$

**Composition:** The composition law for morphisms is the usual composition of functions

*Example 1.3.* Another different but useful example of category is the category based on a poset. If  $(P, \leq)$  is a partially-ordered set, then we can define the following category:

**Objects:** The objects are the elements of the set  $P$

**Morphisms:** There is an arrow  $p \rightarrow q$  if and only if  $p \leq q$

**Identities:** An identity morphism is an arrow  $p \rightarrow p$  representing the trivial equality  $p = p$

**Composition:** The composition law for morphisms is the transitivity of the partial order  $\leq$ : if  $p_0 \rightarrow p_1$  and  $p_1 \rightarrow p_2$  then the transitivity of  $\leq$  implies that  $p_0 \rightarrow p_2$

Note that here, the arrows have a very different meaning to the one in **Sets**. Arrows are not at all similar to functions, but rather the representation of the partial order. Note that there is at most one arrow between two objects in the poset.

This example will be useful not to base our intuition only on the category of **Sets**; **Sets** is a very nice category with lots of properties and examples, however, it does not represent the "generic" category. There are categories that behave differently and we need examples of them.

**Definition 1.4** (Hom-set). Let  $\mathcal{C}$  be a category, and let  $A$  and  $B$  be two objects of  $\mathcal{C}$ . We denote by  $\text{Hom}_{\mathcal{C}}(A, B)$  the collection of arrows  $A \rightarrow B$  in the category  $\mathcal{C}$ .

*Example 1.5.* In the category **Sets**,  $A$  and  $B$  are two sets, and  $\text{Hom}_{\mathbf{Sets}}(A, B)$  is the set of functions  $f : A \rightarrow B$ .

*Example 1.6.* In a poset  $(P, \leq)$ , we have  $\text{Hom}_P(p, q) = \{(p, q)\} \Leftrightarrow p \leq q$ ; otherwise,  $\text{Hom}_P(p, q) = \emptyset$ .

Note that nothing in definition 1.1 implies that  $\text{Ob}_{\mathcal{C}}$  or  $\text{Mor}_{\mathcal{C}}$  should be sets (nor should be  $\text{Hom}_{\mathcal{C}}(A, B)$ ). In fact,  $\text{Ob}_{\mathbf{Sets}}$  is not a set. In that sense, categories can be as *big* as possible. However, in the scope of this course, we will only use *somewhat small* categories, in the following sense.

**Definition 1.7** (Small, locally small and large categories [2]). A category  $\mathcal{C}$  is *small* if both  $\text{Ob}_{\mathcal{C}}$  and  $\text{Mor}_{\mathcal{C}}$  are sets; otherwise, it is *large*.

A category  $\mathcal{C}$  is *locally small* if, for all objects  $A, B \in \mathcal{C}$ , the Hom-set  $\text{Hom}_{\mathcal{C}}(A, B)$  is a set.

Note that small implies locally small. However, a large category may or may not be locally small.

*Example 1.8.* **Sets** is large but locally small.

*Example 1.9.* If  $(P, \leq)$  is a poset, then it is a small (thus locally small) category.

*Example 1.10.* The following example is inspired from set-theory. If  $V_{\alpha}$  is the  $\alpha$ -th set of the von Neumann hierarchy [3, Definition 2.1, p. 95], and if  $\lambda$  is a limit ordinal, then we define the category  $V_{\lambda}$  by:

**Objects:** An object in  $V_{\lambda}$  is any set  $A \in V_{\lambda}$

**Morphisms:** A morphism in  $V_{\lambda}$  is any function  $f : A \rightarrow B$  for  $A, B \in V_{\lambda}$

**Identities:** An identity morphism is an identity function  $\text{id}_A : A \rightarrow A$

**Composition:** The composition law for morphisms is the usual composition of functions

We can see  $V_{\lambda}$  as a truncated **Sets** category. The category  $V_{\lambda}$  is a small category.

*Example 1.11.* For an example of a large, non-locally small, see [4].

*Remark 1.12.* Small categories are locally small (because "sets contain sets").

We also define mappings somewhat similar to functions, or homomorphisms, between categories.

**Definition 1.13** (Functor [1]). Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories.

A *functor*  $F : \mathcal{C} \rightarrow \mathcal{D}$  is a mapping from  $\mathcal{C}$  to  $\mathcal{D}$  such that:

- $\forall C \in \text{Ob}_{\mathcal{C}}, F(C) \in \text{Ob}_{\mathcal{D}}$
- $\forall f : A \rightarrow B \in \text{Mor}_{\mathcal{C}}, F(f) : F(A) \rightarrow F(B) \in \text{Mor}_{\mathcal{D}}$
- $\forall A \in \text{Ob}_{\mathcal{C}}, F(1_A) = 1_{F(A)}$
- $\forall f : A \rightarrow B, g : B \rightarrow C \in \text{Mor}_{\mathcal{C}}, F(g \circ f) = F(g) \circ F(f)$

In other words, a functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  sends the objects (resp. morphisms) in  $\mathcal{C}$  to objects (resp. morphisms) in  $\mathcal{D}$ , preserving domains and codomains of morphisms, as well as identities and composition.

*Example 1.14.* If  $(P_1, \leq_1)$  and  $(P_2, \leq_2)$  are posets, then a functor between those two categories is a monotone function such that  $p \leq_1 q \Rightarrow F(p) \leq_2 F(q)$ .

One can interpret a functor  $\mathcal{C} \rightarrow \mathcal{D}$  as a way to have the *picture* of the category  $\mathcal{C}$  into the category  $\mathcal{D}$  ([1]). It is the idea behind diagrams as we will see in Section 6.

Sometimes, we come across some functors that behave strangely. Namely, sometimes a functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  may send  $c : C \rightarrow C'$  to  $F(c) : F(C') \rightarrow F(C)$  (note the inversion). We will give an example of such a functor. What is happening, is that  $F$  is actually not a functor  $\mathcal{C} \rightarrow \mathcal{D}$  but somehow defined on a similar, but "reversed" category of  $\mathcal{C}$ .

**Definition 1.15** (Opposite category [1]). Let  $\mathcal{C}$  be any category. We call *opposite, or dual category of  $\mathcal{C}$* , denoted by  $\mathcal{C}^{\text{op}}$ , the following category:

**Objects:** An object in  $\mathcal{C}^{\text{op}}$  is an object in  $\mathcal{C}$

**Morphisms:** An arrow  $f : B \rightarrow A$  in  $\mathcal{C}^{\text{op}}$  is an arrow  $f : A \rightarrow B$  in  $\mathcal{C}$

**Identities:** An identity in  $\mathcal{C}^{\text{op}}$  is an identity in  $\mathcal{C}$

**Composition:** The composition law in  $\mathcal{C}^{\text{op}}$  is the same as in  $\mathcal{C}$

Basically, the opposite category  $\mathcal{C}^{\text{op}}$  is the same category as  $\mathcal{C}$ , with inverted arrows.

If a functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  sends  $c : C \rightarrow C'$  to  $F(c) : F(C') \rightarrow F(C)$ , then  $F$  is not actually defined on  $\mathcal{C}$  but rather on  $\mathcal{C}^{\text{op}}$ :  $F : \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}$ . However, it is often simpler to consider only functors on  $\mathcal{C}$ , hence the following notions:

**Definition 1.16** (Covariant and contravariant functor). A functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is called *covariant* if it sends  $f : A \rightarrow B$  to  $F(f) : F(A) \rightarrow F(B)$ .

A functor  $G : \mathcal{C} \rightarrow \mathcal{D}$  is called *contravariant* if it sends  $f : A \rightarrow B$  to  $G(f) : G(B) \rightarrow G(A)$ , or equivalently, if  $G : \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}$  is a covariant functor.

Two examples of such functors are the following:

**Definition 1.17** (Covariant Hom-set functor [5]). Let  $\mathcal{C}$  be a (locally small) category, and let  $A \in \mathcal{C}$  be an object.

The mapping  $\text{Hom}_{\mathcal{C}}(A) - : \begin{cases} \mathcal{C} & \longrightarrow & \mathbf{Sets} \\ B & \longmapsto & \text{Hom}_{\mathcal{C}}(A, B) \end{cases}$  defines the *covariant Hom-set functor*. It sends an object  $B \in \mathcal{C}$  to the set  $\text{Hom}_{\mathcal{C}}(A, B)$  of arrows from  $A$  to  $B$ , and an arrow  $b : B \rightarrow B'$  to the arrow  $\text{Hom}_{\mathcal{C}}(A, b) : \text{Hom}_{\mathcal{C}}(A, B) \rightarrow \text{Hom}_{\mathcal{C}}(A, B')$  in **Sets**.

**Definition 1.18** (Contravariant Hom-set functor [5]). Let  $\mathcal{C}$  be a (locally small) category, and let  $B \in \mathcal{C}$  be an object.

The mapping  $\text{Hom}_{\mathcal{C}}(-, B) : \begin{cases} \mathcal{C}^{\text{op}} & \longrightarrow & \mathbf{Sets} \\ A & \longmapsto & \text{Hom}_{\mathcal{C}}(A, B) \end{cases}$  defines the *contravariant Hom-set functor*. It sends an object  $A \in \mathcal{C}^{\text{op}}$  to the set  $\text{Hom}_{\mathcal{C}}(A, B)$  of arrows from  $A$  to  $B$ , and an arrow  $a : A \rightarrow A'$  to the arrow  $\text{Hom}_{\mathcal{C}}(a, B) : \text{Hom}_{\mathcal{C}}(A', B) \rightarrow \text{Hom}_{\mathcal{C}}(A, B)$  in **Sets**.

*Remark 1.19.* Their names are not stolen:  $B \rightarrow \text{Hom}_{\mathcal{C}}(A) B$  is a covariant functor and  $A \rightarrow \text{Hom}_{\mathcal{C}}(A) B$  is a contravariant functor.

Note that both Hom-set functors imply  $\mathcal{C}$  to be locally small.

As stated a few paragraphs before, all the categories we will encounter in this course are locally small, unless stated otherwise.

**Lemma 1.20** (The simplest representation lemma). *Let  $\mathcal{X}, \mathcal{C}$  be categories, and let  $U : \mathcal{X} \rightarrow \mathcal{C}$  be a functor.*

1.  $\forall C \in \text{Ob}_{\mathcal{C}}, \forall x : X \rightarrow X' \in \text{Mor}_{\mathcal{X}}, \forall c \in \text{Hom}_{\mathcal{C}}(C, U(X)), \text{Hom}_{\mathcal{C}}(C, U(x))(c) = U(x) \circ c$
2.  $\forall X \in \text{Ob}_{\mathcal{X}}, \forall x : X' \rightarrow X'' \in \text{Mor}_{\mathcal{X}}, \forall y \in \text{Hom}_{\mathcal{X}}(X, X'), x \circ y = \text{Hom}_{\mathcal{X}}(X, x)(y)$
3.  $\forall X \in \text{Ob}_{\mathcal{X}}, \forall x : X \rightarrow X' \in \text{Mor}_{\mathcal{X}}, x = \text{Hom}_{\mathcal{X}}(X, x)(\text{id}_X)$

*Proof.* 1. By direct application of the definitions of a functor (Definition 1.13) and of the covariant Hom-set functor (Definition 1.17):

$$\text{Hom}_{\mathcal{C}}(C, U(x)) : \begin{cases} \text{Hom}_{\mathcal{C}}(C, U(X)) & \longrightarrow & \text{Hom}_{\mathcal{C}}(C, U(X')) \\ a & \longmapsto & \text{Hom}_{\mathcal{C}}(C, U(x))(a) \end{cases}$$

where  $\text{Hom}_{\mathcal{C}}(C, U(x))(a)$  is the morphism such that:

$$\begin{array}{ccc} & C & \\ a \swarrow & \checkmark & \searrow \text{Hom}_{\mathcal{C}, C}(U(x))(a) \\ U(X) & \xrightarrow{U(x)} & U(X') \end{array}$$

which gives  $\text{Hom}_{\mathcal{C}}(C, U(x))(a) = U(x) \circ a$ , hence the result.

2. Consequence of first part of the lemma with  $\mathcal{C} = \mathcal{X}$ ,  $U = \text{Id}_{\mathcal{C}}$ ,  $c = x$  and  $X = C$ .
3. Consequence of second part of the lemma with  $X = X'$  and  $y = \text{id}_X$ . □

We also define mappings between functors.

**Definition 1.21** (Natural transformation [1]). *Let  $\mathcal{C}$  and  $\mathcal{D}$  be two categories, and let  $F, G : \mathcal{C} \rightarrow \mathcal{D}$  be functors. A natural transformation  $\theta : F \rightarrow G$  consists of a collection of morphisms in  $\mathcal{D}$   $(\theta_C : F(C) \rightarrow G(C))_{C \in \text{Ob}_{\mathcal{C}}}$  such that, for all  $C, D \in \mathcal{C}$ , and for all  $h : C \rightarrow D$ , the following square commutes:*

$$\begin{array}{ccc} C & & F(C) \xrightarrow{\theta_C} G(C) \\ \downarrow h & \rightsquigarrow & \downarrow F(h) \quad \checkmark \quad \downarrow G(h) \\ D & & F(D) \xrightarrow{\theta_D} G(D) \end{array} \quad (1)$$

For each object  $C \in \mathcal{C}$ , the morphism  $\theta_C$  is called the  $C$ -component of  $\theta$ .

The natural transformation  $\theta : F \rightarrow G$  can be written in the following diagram:

$$\begin{array}{ccc} & F & \\ \mathcal{C} & \begin{array}{c} \curvearrowright \\ \Downarrow \theta \\ \curvearrowleft \end{array} & \mathcal{D} \\ & G & \end{array}$$

We denote by  $\text{Nat}(F, G)$  the collection of all natural transformations between  $F$  and  $G$ .

Depending on the context, and for the sake of readability, the  $C$ -component of a natural transformation  $\theta$  can be written  $\theta_C$  as above ( $C$  as an index) or  $\theta(C)$  ( $C$  as a parameter).

Natural transformations can be seen as a way to extract the parameters  $C, D$  and  $h$  from  $F(C), F(D)$  and  $F(h)$  and input them into  $G$ , while preserving arrows. It's a variable substitution.



**Definition 1.22** (Isomorphism [1]). Let  $\mathcal{C}$  be a category. A morphism  $f : A \rightarrow B \in \text{Mor}_{\mathcal{C}}$  is an *isomorphism* when there exists  $g : B \rightarrow A \in \text{Mor}_{\mathcal{C}}$  such that  $g \circ f = \text{id}_A$  and  $f \circ g = \text{id}_B$ . Such a  $g$  is denoted  $f^{-1}$ .

*Remark 1.23.* A few remarks can be made:

- If  $f : A \rightarrow B$  is an isomorphism in  $\mathcal{C}$  and  $F : \mathcal{C} \rightarrow \mathcal{D}$  is a functor, then  $F(f) : F(A) \rightarrow F(B)$  is still an isomorphism.
- An isomorphism in **Sets** is a function that is invertible. Thus, an isomorphism in **Sets** is a bijection.
- Similarly, in most "structured sets" categories (the category of vector spaces over a field  $\mathbb{K}$  **Lin** $_{\mathbb{K}}$ , the category of groups **Group**...), a bijective morphism is an isomorphism, just like in **Sets**. However, there exist bijective morphisms that are not isomorphisms (in **Top**, the category of topological spaces), and in more complicated categories, there exist isomorphisms that are not bijective (see the homotopy category of CW complexes). This is because bijectivity is *not* a property of morphisms that makes sense in terms of categories.

Before introducing the notion of natural isomorphism, we need to make something clear on the nature of natural transformations.

**Definition 1.24** (Composition of natural transformations). Let  $\mathcal{C}, \mathcal{D}$  be categories, and let  $F, G$  and  $H$  be functors  $\mathcal{C} \rightarrow \mathcal{D}$ .

If  $\theta : F \rightarrow G$  is the natural transformation  $\theta = \left( F(A) \xrightarrow{\theta_A} G(A) \right)_{A \in \mathcal{C}}$  and  $\eta : G \rightarrow H$  is the natural transformation  $\eta = \left( G(A) \xrightarrow{\eta_A} H(A) \right)_{A \in \mathcal{C}}$  then the composition of  $\theta$  by  $\eta$  is  $\eta \circ \theta : F \rightarrow H$ , defined by  $\eta \circ \theta = \left( F(A) \xrightarrow{\eta_A \circ \theta_A} H(A) \right)_{A \in \mathcal{C}}$ .

**Definition 1.25** (Functor category [1]). Let  $\mathcal{C}$  and  $\mathcal{D}$  be two categories. The *functor category*, denoted by **Func**  $(\mathcal{C}, \mathcal{D})$ , or by  $\mathcal{D}^{\mathcal{C}}$ , is the following category:

**Objects:** The objects are the functors  $F : \mathcal{C} \rightarrow \mathcal{D}$

**Morphisms:** A morphism between two functors  $F$  and  $G$  is a natural transformation  $\theta : F \rightarrow G = \left( F(A) \xrightarrow{\theta_A} G(A) \right)_{A \in \mathcal{C}}$

**Identities:** An identity on a functor  $F$  is the identity natural transformation  $\text{id}_F = \left( F(A) \xrightarrow{\text{id}_{F(A)}} F(A) \right)_{A \in \mathcal{C}}$

**Composition:** The composition law in **Func**  $(\mathcal{C}, \mathcal{D})$  is defined in Definition 1.24.

Natural transformations are morphisms between functors. Besides, if  $F, G : \mathcal{C} \rightarrow \mathcal{D}$  are two functors, then the notation  $\text{Nat}(F, G)$  actually stands for  $\text{Hom}_{\text{Func}(\mathcal{C}, \mathcal{D})}(F, G)$ ; however  $\text{Nat}(F, G)$  is usually more convenient.

Using Definition 1.25 (functor category), and Definition 1.22 (isomorphism), we deduce the definition of a natural isomorphism:

**Definition 1.26** (Natural isomorphism [1]). Let  $F, G : \mathcal{C} \rightarrow \mathcal{D}$  be functors. A *natural isomorphism*  $\theta : F \rightarrow G$  is a natural transformation that is an isomorphism in the functor category **Func**  $(\mathcal{C}, \mathcal{D})$ .

It is easy to see that:

**Lemma 1.27.** A natural transformation  $\theta : F \rightarrow G$  is a natural isomorphism whenever the  $C$ -components  $\theta_C : F(C) \rightarrow G(C)$  are isomorphisms.

This lemma gives a useful description of what a natural isomorphism is. It makes it easier to look for an inverse. We will use this lemma in the following section.

## 2. Yoneda lemma

Given a functor  $F : \mathcal{C} \rightarrow \mathbf{Sets}$ , can we transform it into a Hom-set functor? The answer is provided by the Yoneda lemma. The Yoneda lemma is based on a natural transformation, as illustrated by the following series of figures.

Diagramme incomplet !!!!

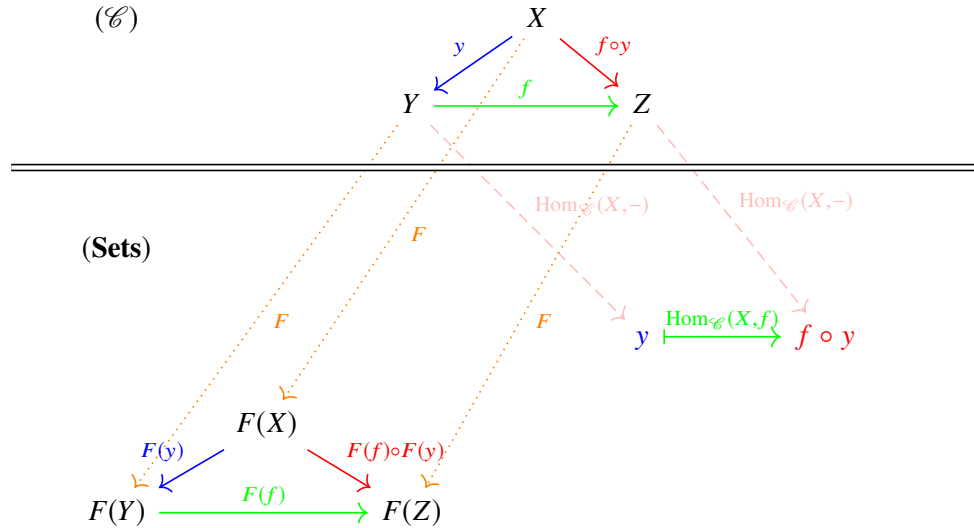
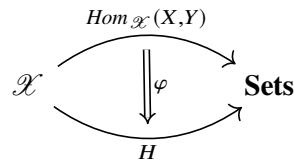


Diagramme incomplet !!!!

**Lemma 2.1.** Let  $\mathcal{X}, \mathcal{C}$  be two categories and let  $H : \mathcal{X} \rightarrow \mathbf{Sets}$  be a functor.

Given any  $X \in \text{Ob}_{\mathcal{X}}$  and any natural transformation  $\varphi = \left( \text{Hom}_{\mathcal{X}}(X, Y) \xrightarrow{\varphi_Y} H(Y) \right)_{Y \in \text{Ob}_{\mathcal{X}}}$  such that:



then  $\varphi_X(\text{id}_X)$  is the unique element  $e \in H(X)$  such that:

$$\forall Y \in \text{Ob}_{\mathcal{X}}, \forall y \in \text{Hom}_{\mathcal{X}}(X, Y), \varphi_Y(y) = H(y)(e)$$

*Proof.* Let  $Y \in \text{Ob}_{\mathcal{X}}$ , and let  $y \in \text{Hom}_{\mathcal{X}}(X, Y)$ .

By simplest representation lemma (Lemma 1.20, item 3), we have:

$$y = \text{Hom}_{\mathcal{X}}(X, y)(\text{id}_X)$$

Thus:

$$\begin{aligned} \varphi_Y(y) &= \varphi_Y(\text{Hom}_{\mathcal{X}}(X, y)(\text{id}_X)) \\ &= (\varphi_Y \circ \text{Hom}_{\mathcal{X}}(X, y))(\text{id}_X) \end{aligned}$$

Besides,  $\varphi : \text{Hom}_{\mathcal{X}}(X, -) \rightarrow H$  is a natural transformation; thus using Definition 1.21, diagram 1 with  $F = \text{Hom}_{\mathcal{X}}(X, -)$  and  $G = H$ , we have:

$$\varphi_Y \circ \text{Hom}_{\mathcal{X}}(X, y) = H(y) \circ \varphi_X$$

which yields:

$$\begin{aligned}\varphi_Y(y) &= (H(y) \circ \varphi_X)(\text{id}_X) \\ &= H(y)(\varphi_X(\text{id}_X))\end{aligned}$$

Hence the result. Now we have to prove that  $e = \varphi_X(\text{id}_X)$  is unique with that property. Let  $e' \in H(X)$  such that  $\forall Y \in \text{Ob}_{\mathcal{X}}, \forall y \in \text{Hom}_{\mathcal{X}}(X, Y), \varphi_Y(y) = H(y)(e') = H(y)(e)$ . Using  $X = Y$  and  $y = \text{id}_X$  yields:

$$\begin{aligned}H(\text{id}_X)(e') &= H(\text{id}_X)(e) \\ \text{id}_{H(X)}(e') &= \text{id}_{H(X)}(e) \\ e' &= e\end{aligned}$$

□

**Lemma 2.2.** Let  $\mathcal{X}, \mathcal{C}$  be two categories and let  $H : \mathcal{X} \rightarrow \mathbf{Sets}$  be a functor. Let  $X \in \text{Ob}_{\mathcal{X}}$  and  $e \in H(X)$ .

The mapping  $\varphi = \left( \text{Hom}_{\mathcal{X}}(X, Y) \xrightarrow{\varphi_Y} H(Y) \right)_{Y \in \text{Ob}_{\mathcal{X}}}$  defined by:

$$\varphi_Y : \begin{cases} \text{Hom}_{\mathcal{X}}(X, Y) & \longrightarrow & H(Y) \\ y & \longmapsto & H(y)(e) \end{cases}$$

is a natural transformation such that  $\varphi_X(\text{id}_X) = e$ .

*Proof.* We need to prove that, for any  $y : Y \rightarrow Y' \in \text{Mor}_{\mathcal{X}}$ , the following square commutes:

$$\begin{array}{ccc} Y & & \text{Hom}_{\mathcal{X}}(X, Y) \xrightarrow{\varphi_Y} H(Y) \\ \downarrow y & \rightsquigarrow & \downarrow \text{Hom}_{\mathcal{X}}(X, y) \quad \checkmark \quad \downarrow H(y) \\ Y' & & \text{Hom}_{\mathcal{X}}(X, Y') \xrightarrow{\varphi_{Y'}} H(Y') \end{array}$$

that is, we want:

$$\forall y : Y \rightarrow Y', \varphi_{Y'} \circ \text{Hom}_{\mathcal{X}}(X, y) = H(y) \circ \varphi_Y$$

Let  $y : Y \rightarrow Y'$  be an arrow in  $\mathcal{X}$ . For all  $x \in \text{Hom}_{\mathcal{X}}(X, Y) = \text{dom}(\text{Hom}_{\mathcal{X}}(X, y))$ :

$$\varphi_{Y'} \circ \text{Hom}_{\mathcal{X}}(X, y)(x) = \varphi_{Y'}(y \circ x) \tag{2}$$

$$= H(y \circ x)(e) \tag{3}$$

$$= H(y) \circ H(x)(e) \tag{4}$$

$$= H(y) \circ \varphi_Y(x) \tag{5}$$

Equation 2 is due to the simplest representation lemma (Lemma 1.20-2); Equations 3 and 5 are due to the definition of  $\varphi$  and Equation 4 comes from the definition of a functor (Definition 1.13).

Besides:

$$\begin{aligned}\varphi_X(\text{id}_X) &= H(\text{id}_X)(e) \\ &= \text{id}_{H(X)}(e) \\ &= e\end{aligned}$$

□

**Definition 2.3** (The  $\xi$  natural isomorphism). Let  $\mathcal{X}$  be a category, let  $H : \mathcal{X} \rightarrow \mathbf{Sets}$  be a functor and let  $X$  be an object in  $\text{Ob}_{\mathcal{X}}$ .

We define:

$$\xi_{H,X} : \begin{cases} \text{Nat}(\text{Hom}_{\mathcal{X}}(X, -), H) & \longrightarrow & H(X) \\ \varphi & \longmapsto & \varphi_X(\text{id}_X) \end{cases}$$

The  $\xi$  natural isomorphism is the mapping  $\xi : H, X \rightarrow \xi_{H,X}$ .

**Definition 2.4** (The  $\theta$  natural isomorphism). Let  $\mathcal{X}$  be a category, let  $H : \mathcal{X} \rightarrow \mathbf{Sets}$  be a functor and let  $X$  be an object in  $\text{Ob}_{\mathcal{X}}$ .

We define:

$$\theta_{H,X} : \begin{cases} H(X) & \longrightarrow & \text{Nat}(\text{Hom}_{\mathcal{X}}(X, -), H) \\ e & \longmapsto & \varphi_e^{H,X} = \left( \varphi_{e,Y}^{H,X} : \begin{cases} \text{Hom}_{\mathcal{X}}(X, Y) & \longrightarrow & H(Y) \\ y & \longmapsto & H(y)(e) \end{cases} \right)_{Y \in \text{Ob}_{\mathcal{X}}} \end{cases}$$

The  $\theta$  natural isomorphism is the mapping  $\theta : H, X \rightarrow \theta_{H,X}$ .

Please note that those two natural isomorphisms are standard in the demonstrations of the Yoneda lemma; however their notation isn't. We highlight those two isomorphisms because they will have several occurrences in the current course.

**Proposition 2.5.** *The mappings  $\xi$  and  $\theta$  are both actual natural isomorphisms, covariant in  $H$  and contravariant in  $X$ , and they are inverse of each other.*

**Corollary 2.6** (Yoneda lemma [2]). *Let  $\mathcal{X}$  be a category, let  $H : \mathcal{X} \rightarrow \mathbf{Sets}$  be a functor and let  $X$  be an object in  $\text{Ob}_{\mathcal{X}}$ .*

*Then,  $\text{Nat}(\text{Hom}_{\mathcal{X}}(X, -), H) \cong H(X)$ .*

*Proof.* Let  $H : \mathcal{X} \rightarrow \mathbf{Sets}$  and let  $X \in \text{Ob}_{\mathcal{X}}$ .

**[Inverse]**

We first prove that  $\theta$  is the inverse of  $\xi$ . Then, we will prove that both are natural transformations.

Let  $e \in H(X)$ .

$$\begin{aligned} \xi_{H,X} \circ \theta_{H,X}(e) &= \xi_{H,X} \left( \varphi_e^{H,X} \right) \\ &= \varphi_{e,X}^{H,X}(\text{id}_X) \\ &= e \end{aligned} \tag{6}$$

$$\tag{7}$$

The transition 6  $\Rightarrow$  7 comes from Lemma 2.2.

Similarly, let  $\varphi \in \text{Nat}(\text{Hom}_{\mathcal{X}}(X, -), H)$ . Note that, according to Lemma 2.1,  $\varphi$  is:

$$\varphi = \left( \varphi_Y : \begin{cases} \text{Hom}_{\mathcal{X}}(X, Y) & \longrightarrow & H(Y) \\ y & \longmapsto & H(y)(\varphi_X(\text{id}_X)) \end{cases} \right)_{Y \in \text{Ob}_{\mathcal{X}}}$$

Thus:

$$\begin{aligned} \theta_{H,X} \circ \xi_{H,X}(\varphi) &= \theta_{H,X}(\varphi_X(\text{id}_X)) \\ &= \left( \varphi_{\varphi_X(\text{id}_X), Y}^{H,X} : \begin{cases} \text{Hom}_{\mathcal{X}}(X, Y) & \longrightarrow & H(Y) \\ y & \longmapsto & H(y)(\varphi_X(\text{id}_X)) \end{cases} \right)_{Y \in \text{Ob}_{\mathcal{X}}} \\ &= \varphi \end{aligned}$$

Consequently,  $\theta$  and  $\xi$  are mutually inverses.

We only have to check their naturalities.

**[ $\xi$  is a natural transformation in  $H$ ]**

Let  $\alpha : H \rightarrow H'$  be a natural transformation. We want to check if the following diagram commutes:

$$\begin{array}{ccc}
 \begin{array}{c} H \\ \downarrow \alpha \\ H' \end{array} & \rightsquigarrow & \begin{array}{ccc} \text{Nat}(\text{Hom}_{\mathcal{X}}(X, -), H) & \xrightarrow{\xi_{H,X}} & H(X) \\ \downarrow \text{Nat}(\text{Hom}_{\mathcal{X}}(X, -), \alpha) & & \downarrow \alpha_X \\ \text{Nat}(\text{Hom}_{\mathcal{X}}(X, -), H') & \xrightarrow{\xi_{H',X}} & H'(X) \end{array}
 \end{array}$$

Let  $\varphi \in \text{Nat}(\text{Hom}_{\mathcal{X}}(X, -), H)$ .

$$\begin{aligned}
 \xi_{H,X} \circ \text{Nat}(\text{Hom}_{\mathcal{X}}(X, -), \alpha)(\varphi) &= \xi_{H,X}(\alpha \circ \varphi) \\
 &= (\alpha_X \circ \varphi_X)(\text{id}_X) \\
 &= \alpha_X(\varphi_X(\text{id}_X)) \\
 &= \alpha_X \circ \xi_{H,X}(\varphi)
 \end{aligned}$$

which gives the expected result.

**[ $\xi$  is a natural transformation in  $X$ ]**

Recall that  $\xi$  is contravariant in  $X$ . Let  $x : X' \rightarrow X \in \text{Mor}_{\mathcal{X}}$ . We want to check if the following diagram commutes:

$$\begin{array}{ccc}
 \begin{array}{c} X' \\ \downarrow x \\ X \end{array} & \rightsquigarrow & \begin{array}{ccc} \text{Nat}(\text{Hom}_{\mathcal{X}}(X, -), H) & \xrightarrow{\xi_{H,X}} & H(X) \\ \downarrow \text{Nat}(\text{Hom}_{\mathcal{X}}(x, -), H) & & \downarrow H(x) \\ \text{Nat}(\text{Hom}_{\mathcal{X}}(X', -), H) & \xrightarrow{\xi_{H,X'}} & H(X') \end{array}
 \end{array}$$

The arrows  $\text{Nat}(\text{Hom}_{\mathcal{X}}(x, -), H)$  and  $H(x)$  are inverted because  $\xi$  is supposed to be contravariant in  $X$ .

Let  $\varphi \in \text{Nat}(\text{Hom}_{\mathcal{X}}(x, -), H)$ .

On the one hand:

$$\begin{aligned}
 \xi_{H,X'} \circ \text{Nat}(\text{Hom}_{\mathcal{X}}(x, -), H)(\varphi) &= \xi_{H,X'}(\varphi \circ \text{Hom}_{\mathcal{X}}(x, -)) \\
 &= (\varphi \circ \text{Hom}_{\mathcal{X}}(x, -))_{X'}(\text{id}_{X'}) \\
 &= \varphi_{X'} \circ \text{Hom}_{\mathcal{X}}(x, X')(\text{id}_{X'}) \\
 &= \varphi_{X'}(\text{id}_{X'} \circ x) \\
 &= \varphi_{X'}(x)
 \end{aligned}$$

On the other hand, note that  $\varphi$  is also a natural transformation. Thus, for  $x : X' \rightarrow X$  in  $\mathcal{X}^{\text{op}}$  (note that we are in the opposite category), the following diagram does commute:

$$\begin{array}{ccc}
 X' & & \text{Hom}_{\mathcal{X}}(X, X) \xrightarrow{\varphi_X} H(X) \\
 \downarrow x & \rightsquigarrow & \downarrow \text{Hom}_{\mathcal{X}}(X, x) \quad \checkmark \quad \downarrow H(x) \\
 X & & \text{Hom}_{\mathcal{X}}(X, X') \xrightarrow{\varphi_{X'}} H(X')
 \end{array}$$

In particular:  $H(x) \circ \varphi_X = \varphi_{X'} \circ \text{Hom}_{\mathcal{X}}(X, x)$ , which gives:

$$\begin{aligned}
 H(x) \circ \xi_{H, X}(\varphi) &= H(x)(\varphi_X(\text{id}_X)) \\
 &= \varphi_{X'} \circ \text{Hom}_{\mathcal{X}}(X, x)(\text{id}_X) \\
 &= \varphi_{X'}(\text{id}_X \circ x) \\
 &= \varphi_{X'}(x) \\
 &= \xi_{H, X'} \circ \text{Nat}(\text{Hom}_{\mathcal{X}}(x, -), H)(\varphi)
 \end{aligned}$$

Consequently,  $\xi$  is natural in both its parameters  $X$  and  $H$ .

**[ $\theta$  is a natural transformation in  $H$ ]**

The idea is similar to  $\xi$ . Let  $\alpha : H \rightarrow H'$  be a natural transformation. We want to prove that  $\text{Nat}(\text{Hom}_{\mathcal{X}}(X, -), \alpha) \circ \theta_{H, X} = \theta_{H, X} \circ \alpha_X$ .

Let  $e \in H(X)$ .

$$\theta_{H, X} \circ \alpha_X(e) = \varphi_{\alpha_X(e)}^{H', X}$$

and:

$$\begin{aligned}
 \text{Nat}(\text{Hom}_{\mathcal{X}}(X, -), \alpha) \circ \theta_{H, X}(e) &= \text{Nat}(\text{Hom}_{\mathcal{X}}(X, -), \alpha) \left( \varphi_e^{H, X} \right) \\
 &= \alpha \circ \varphi_e^{H, X}
 \end{aligned}$$

where  $\alpha \circ \varphi_e^{H, X}$  is the natural transformation:

$$\alpha \circ \varphi_e^{H, X} = \left( \alpha_Y \circ \varphi_{e, Y}^{H, X} : \begin{array}{ccc} \text{Hom}_{\mathcal{X}}(X, Y) & \longrightarrow & H'(Y) \\ y & \longmapsto & \alpha_Y(H(y)(e)) \end{array} \right)_{Y \in \text{Ob}_{\mathcal{X}}} \quad (8)$$

$$= \left( \alpha_Y \circ \varphi_{e, Y}^{H, X} : \begin{array}{ccc} \text{Hom}_{\mathcal{X}}(X, Y) & \longrightarrow & H'(Y) \\ y & \longmapsto & H'(y) \circ \alpha_X(e) \end{array} \right)_{Y \in \text{Ob}_{\mathcal{X}}} \quad (9)$$

$$= \varphi_{\alpha_X(e)}^{H', X}$$

$$= \theta_{H, X} \circ \alpha_X(e)$$

The transition 8  $\Rightarrow$  9 is due to the naturality of  $\alpha$ .

**[ $\theta$  is a natural transformation in  $X$ ]**

Let  $x : X' \rightarrow X$  be a morphism in  $\mathcal{X}$ . We want  $\text{Nat}(\text{Hom}_{\mathcal{X}}(x, -), H) \circ \theta_{H, X} = \theta_{H, X'} \circ H(x)$ . Let  $e \in H(X)$ :

$$\begin{aligned}
 \text{Nat}(\text{Hom}_{\mathcal{X}}(x, -), H) \circ \theta_{H, X}(e) &= \text{Nat}(\text{Hom}_{\mathcal{X}}(x, -), H) \left( \varphi_e^{H, X} \right) \\
 &= \varphi_e^{H, X} \circ \text{Hom}_{\mathcal{X}}(x, -)
 \end{aligned}$$

where  $\varphi_e^{H,X} \circ \text{Hom}_{\mathcal{X}}(x, -)$  is the following natural transformation:

$$\varphi_e^{H,X} \circ \text{Hom}_{\mathcal{X}}(x, -) = \left( \varphi_{e,Y}^{H,X} \circ \text{Hom}_{\mathcal{X}}(x, Y) : \begin{cases} \text{Hom}_{\mathcal{X}}(X', Y) & \longrightarrow & H(Y) \\ y & \longmapsto & \varphi_{e,Y}^{H,X}(y \circ x) \end{cases} \right)_{Y \in \text{Ob}_{\mathcal{X}}}$$

However:

$$\begin{aligned} \varphi_{e,Y}^{H,X}(y \circ x) &= H(y \circ x)(e) \\ &= H(y)(H(x)(e)) \end{aligned}$$

which yields:

$$\begin{aligned} \varphi_e^{H,X} \circ \text{Hom}_{\mathcal{X}}(x, -) &= \varphi_{H(x)(e)}^{H,X'} \\ &= \theta_{H,X'} \circ H(x)(e) \end{aligned}$$

Consequently,  $\theta$  is natural in both its parameters  $X$  and  $H$ .

**[Conclusion]**

Both  $\xi$  and  $\theta$  are natural transformations in  $H$  and  $X$ , and they are mutually inverses. As a consequence,  $\xi$  and  $\theta$  are natural isomorphisms between  $\text{Nat}(\text{Hom}_{\mathcal{X}}(X, -), H)$  and  $H(X)$ . □

*Remark 2.7.* As stated in Definition 1.25,  $\text{Nat}(\text{Hom}_{\mathcal{X}}(X, -), H)$  corresponds to the Hom-set:  $\text{Hom}_{\mathbf{Func}(\mathcal{X}, \mathbf{Sets})}(\text{Hom}_{\mathcal{X}^{\text{op}}}(X, -), H)$ . Note that it's  $\mathcal{X}^{\text{op}}$  and not  $\mathcal{X}$ , because the natural isomorphism is contravariant in  $X$ .

*Remark 2.8.* The Yoneda lemma has a central role due to its various meanings and consequences.

1. First, depending on the "size" of  $\mathcal{X}$ , we have different interpretations. If  $\mathcal{X}$  is small, then  $\text{Nat}(\text{Hom}_{\mathcal{X}}(X, -), H)$  is a set because  $\mathbf{Sets}^{\mathcal{X}}$  becomes locally small. If  $\mathcal{X}$  is locally small, then it says nothing on  $\mathbf{Sets}^{\mathcal{X}}$ . However, the Yoneda lemma states that  $\text{Nat}(\text{Hom}_{\mathcal{X}}(X, -), H)$  is always a set. If  $\mathcal{X}$  is non-locally small, then the functor  $\text{Hom}_{\mathcal{X}}(X, -)$  doesn't exist and the Yoneda lemma doesn't hold there.
2. Secondly, from a set-theoretic point of view, the Yoneda lemma states that there are not *that many natural transformations*: there are exactly  $\text{card}(H(X))$  natural transformations  $\text{Hom}_{\mathcal{X}}(X, -) \rightarrow H$ , as each of these natural transformations is entirely determined by one element in  $H(X)$ .
3. Thirdly, according to the Yoneda lemma, if  $H = \text{Hom}_{\mathcal{X}}(Y, -)$ :

$$\text{Nat}(\text{Hom}_{\mathcal{X}}(X, -), \text{Hom}_{\mathcal{X}}(Y, -)) \cong \text{Hom}_{\mathcal{X}}(Y, X)$$

(Note the inversion) As stated in the previous paragraph, each element in  $\text{Hom}_{\mathcal{X}}(Y, X)$  characterises one natural transformation in  $\text{Nat}(\text{Hom}_{\mathcal{X}}(X, -), \text{Hom}_{\mathcal{X}}(Y, -))$ . Consequently, any natural transformation  $\text{Hom}_{\mathcal{X}}(X, -) \rightarrow \text{Hom}_{\mathcal{X}}(Y, -)$  is determined by an arrow  $Y \rightarrow X$  using the application  $\theta$  seen in the proof of the Yoneda lemma. Consequently, the only arrows  $\text{Hom}_{\mathcal{X}}(X, A) \rightarrow \text{Hom}_{\mathcal{X}}(Y, A)$  are of the form  $\text{Hom}_{\mathcal{X}}(f, A)$  for some  $f : Y \rightarrow X$ .

The dual version of the Yoneda lemma is as follows:

**Lemma 2.9** (Contravariant Yoneda lemma). *Let  $\mathcal{X}$  be a category, let  $G : \mathcal{X}^{\text{op}} \rightarrow \mathbf{Sets}$  be a contravariant functor and let  $X$  be an object in  $\text{Ob}_{\mathcal{X}}$ .*

*Then,  $\text{Nat}(\text{Hom}_{\mathcal{X}}(-, X), G) \cong G(X)$ .*

The functor  $X \rightarrow \text{Hom}_{\mathcal{D}}(X, -)$  has good properties. Let's spend some time studying them.

**Definition 2.10** (Yoneda embedding [1]). Let  $\mathcal{C}$  be a category.

The *Yoneda embedding* is the functor:

$$y : \begin{cases} \mathcal{C}^{\text{op}} & \longrightarrow & \mathbf{Func}(\mathcal{C}, \mathbf{Sets}) \\ C & \longmapsto & \text{Hom}_{\mathcal{C}}(C, -) \\ f : D \rightarrow C & \longmapsto & \text{Hom}_{\mathcal{C}}(f, -) : \text{Hom}_{\mathcal{C}}(D, -) \rightarrow \text{Hom}_{\mathcal{C}}(C, -) \end{cases}$$

*Remark 2.11.* Note that the Yoneda embedding is defined  $\mathcal{C}^{\text{op}}$ . Thus,  $f : C \rightarrow D$  in  $\mathcal{C}$  becomes  $f : D \rightarrow C$  in  $\mathcal{C}^{\text{op}}$ , and  $y(f)$  has the direction of  $f \in \mathcal{C}^{\text{op}}$ .

**Definition 2.12** (Injective, surjective, full, faithful, embedding [1]). Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor.

1. 1.1. The functor  $F$  is said *injective (resp. surjective) on objects* if  $\text{Ob}_F : \text{Ob}_{\mathcal{C}} \rightarrow \text{Ob}_{\mathcal{D}}$  is injective (resp. surjective).
- 1.2. The functor  $F$  is said *injective (resp. surjective) on arrows* if  $\text{Mor}_F : \text{Mor}_{\mathcal{C}} \rightarrow \text{Mor}_{\mathcal{D}}$  is injective (resp. surjective).
2. For all  $A, B \in \text{Ob}_{\mathcal{C}}$ , define the mapping:

$$F_{A,B} : \begin{cases} \text{Hom}_{\mathcal{C}}(A, B) & \longrightarrow & \text{Hom}_{\mathcal{D}}(F(A), F(B)) \\ f & \longmapsto & F(f) \end{cases}$$

- 2.1. The functor  $F$  is said *faithful* if  $\forall A, B \in \text{Ob}_{\mathcal{C}}$ ,  $F_{A,B}$  is injective.
- 2.2. The functor  $F$  is said *full* if  $\forall A, B \in \text{Ob}_{\mathcal{C}}$ ,  $F_{A,B}$  is surjective.
3. The functor  $F$  is called an *embedding* if it is injective on objects, full and faithful.

Difference between injective on arrows and faithful?

**Proposition 2.13.** *The Yoneda embedding is an actual embedding.*

*Proof.* The injectivity on objects is easy. Suppose  $y(C) = y(D)$ ; then:

$$\begin{aligned} y(C) &= y(D) \\ \text{Hom}_{\mathcal{C}}(C, -) &= \text{Hom}_{\mathcal{C}}(D, -) \\ \Rightarrow \text{Hom}_{\mathcal{C}}(C, C) &= \text{Hom}_{\mathcal{C}}(D, C) \end{aligned}$$

Those two sets are equal. Thus,  $\text{id}_C \in \text{Hom}_{\mathcal{C}}(C, C) \Rightarrow \text{id}_C \in \text{Hom}_{\mathcal{C}}(D, C) \Rightarrow C = D$ . Thus,  $y$  is injective on objects.

As noted in Remark 2.8-3, the Yoneda lemma implies that:

$$\begin{aligned} \text{Nat}(\text{Hom}_{\mathcal{C}}(C, -), \text{Hom}_{\mathcal{C}}(D, -)) &\cong \text{Hom}_{\mathcal{C}}(D, C) \\ \text{Hom}_{\mathbf{Sets}^{\mathcal{C}^{\text{op}}}}(y(C), y(D)) &\cong \text{Hom}_{\mathcal{C}}(D, C) \end{aligned}$$

Proposition 2.5 also states that the following natural transformation is an isomorphism:

$$\theta_{y(C), D} : \begin{cases} \text{Hom}_{\mathcal{C}}(C, D) & \longrightarrow & \text{Hom}_{\mathbf{Sets}^{\mathcal{C}^{\text{op}}}}(y(D), y(C)) \\ f & \longmapsto & \varphi_f^{y(C), D} \end{cases}$$

where  $\varphi_f$  is:



$$\begin{aligned}
 \varphi_f^{y(C),D} &= \left( \varphi_{f,X}^{y(C),D} : \left\{ \begin{array}{l} \text{Hom}_{\mathcal{C}}(D, X) \longrightarrow y(C)(X) \\ g \longmapsto y(C)(g)(f) \end{array} \right\}_{X \in \text{Ob}_{\mathcal{C}}} \right) \\
 &= \left( \varphi_{f,X}^{y(C),D} : \left\{ \begin{array}{l} \text{Hom}_{\mathcal{C}}(D, X) \longrightarrow \text{Hom}_{\mathcal{C}}(C, X) \\ g \longmapsto \text{Hom}_{\mathcal{C}}(C, g)(f) \end{array} \right\}_{X \in \text{Ob}_{\mathcal{C}}} \right) \\
 &= \left( \varphi_{f,X}^{y(C),D} : \left\{ \begin{array}{l} \text{Hom}_{\mathcal{C}}(D, X) \longrightarrow \text{Hom}_{\mathcal{C}}(C, X) \\ g \longmapsto g \circ f \end{array} \right\}_{X \in \text{Ob}_{\mathcal{C}}} \right)
 \end{aligned}$$

We compare with what  $y(f)$  is:

$$\begin{aligned}
 y(f) &= \left( y_X(f) : \left\{ \begin{array}{l} y(D)(X) \longrightarrow y(C)(X) \\ g \longmapsto y_X(f)(g) \end{array} \right\}_{X \in \text{Ob}_{\mathcal{C}}} \right) \\
 &= \left( y_X(f) : \left\{ \begin{array}{l} \text{Hom}_{\mathcal{C}}(D, X) \longrightarrow \text{Hom}_{\mathcal{C}}(C, X) \\ g \longmapsto \text{Hom}_{\mathcal{C}}(f, X)(g) \end{array} \right\}_{X \in \text{Ob}_{\mathcal{C}}} \right) \\
 &= \left( y_X(f) : \left\{ \begin{array}{l} \text{Hom}_{\mathcal{C}}(D, X) \longrightarrow \text{Hom}_{\mathcal{C}}(C, X) \\ g \longmapsto g \circ f \end{array} \right\}_{X \in \text{Ob}_{\mathcal{C}}} \right) \\
 &= \varphi_f^{y(C),D}
 \end{aligned}$$

Consequently:

$$\theta_{y(C),D} = \left\{ \begin{array}{l} \text{Hom}_{\mathcal{C}}(C, D) \longrightarrow \text{Hom}_{\text{Sets}_{\mathcal{C}^{\text{op}}}}(y(D), y(C)) \\ f \longmapsto y(f) \end{array} \right.$$

which yields that  $y$  is full and faithful. □

Using the Yoneda lemma (both covariant and contravariant) and the fact that the Yoneda embedding is an embedding, one can show the following corollaries:

**Corollary 2.14.** *Let  $\mathcal{C}$  be a locally small category. Then,  $\forall C, D \in \text{Ob}_{\mathcal{C}}, f : C \rightarrow D$  is an isomorphism  $\Leftrightarrow \text{Hom}_{\mathcal{C}}(-, f) : \text{Hom}_{\mathcal{C}}(-, C) \rightarrow \text{Hom}_{\mathcal{C}}(-, D)$  is an isomorphism.*

**Corollary 2.15.** *Let  $\mathcal{C}$  be a locally small category. Then,  $\forall C, D \in \text{Ob}_{\mathcal{C}}, f : C \rightarrow D$  is an isomorphism  $\Leftrightarrow \text{Hom}_{\mathcal{C}}(f, -) : \text{Hom}_{\mathcal{C}}(D, -) \rightarrow \text{Hom}_{\mathcal{C}}(C, -)$  is an isomorphism.*

**Corollary 2.16.** *Let  $\mathcal{C}$  be a locally small category. Then,  $\forall C, D \in \text{Ob}_{\mathcal{C}}, \text{Hom}_{\mathcal{C}}(C, -) \cong \text{Hom}_{\mathcal{C}}(D, -) \Rightarrow C \cong D$ .*

**Corollary 2.17.** *Let  $\mathcal{C}$  be a locally small category. Then,  $\forall C, D \in \text{Ob}_{\mathcal{C}}, \text{Hom}_{\mathcal{C}}(-, C) \cong \text{Hom}_{\mathcal{C}}(-, D) \Rightarrow C \cong D$ .*

### 3. Universal elements, universal arrows, representations

Introduction lol.

**Definition 3.1** (Universal element [2]). Let  $\mathcal{C}$  be a category, and  $F : \mathcal{C} \rightarrow \mathbf{Sets}$  a functor.

The pair  $(X^*, e^*) \in \text{Ob}_{\mathcal{C}} \times F(X^*)$  is a universal element for  $F$  if the natural transformation  $\theta_{F, X^*}(e^*) : \text{Hom}_{\mathcal{C}}(X^*, -) \rightarrow F$  is an isomorphism.

*Remark 3.2.* When one sees Definition 3.1, the two natural questions should be:

- Is this universal element unique?
- Does the Yoneda embedding have a universal element?

The answer to the second question is easy: no, the Yoneda embedding doesn't have a universal element, because it is a functor  $\mathcal{C} \rightarrow \mathbf{Func}(\mathcal{C}^{\text{op}}, \mathbf{Sets})$ . However, for  $C \in \mathcal{C}$ , its  $C$ -component  $y(C) : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Sets}$  could have one.

We are looking for a pair  $(X^*, e^*)$  such that:

$$\begin{aligned} \theta_{y(C), X^*}(e^*) &= \left( \varphi_X : \begin{array}{ccc} \text{Hom}_{\mathcal{C}}(X^*, X) & \longrightarrow & y(C)(X) \\ x & \longmapsto & y(C)(x)(e^*) \end{array} \right)_{X \in \text{Ob}_{\mathcal{C}}} \\ &= \left( \varphi_X : \begin{array}{ccc} \text{Hom}_{\mathcal{C}}(X^*, X) & \longrightarrow & \text{Hom}_{\mathcal{C}}(C, X) \\ x & \longmapsto & \text{Hom}_{\mathcal{C}}(C, x)(e^*) \end{array} \right)_{X \in \text{Ob}_{\mathcal{C}}} \\ &= \left( \varphi_X : \begin{array}{ccc} \text{Hom}_{\mathcal{C}}(X^*, X) & \longrightarrow & \text{Hom}_{\mathcal{C}}(C, X) \\ x & \longmapsto & x \circ e^* \end{array} \right)_{X \in \text{Ob}_{\mathcal{C}}} \end{aligned}$$

What could the pair  $(X^*, e^*)$  be for  $\theta_{y(C), X^*}(e^*)$  to be an isomorphism? There is one obvious answer: take  $(C, \text{id}_C)$ .

But is this answer unique? Probably not. But is it unique up to isomorphism? The answer to this question lies in Proposition 3.5. Before, we have to show an intermediate proposition.

*Remark 3.3.* If the functor  $F : \mathcal{C} \rightarrow \mathbf{Sets}$  is contravariant, then the universal element is a pair  $(X^*, e^*)$  such that  $\theta_{F, X^*}^{\text{op}}(e^*) : \text{Hom}_{\mathcal{C}}(-, X^*) \rightarrow F$ , where  $\theta_{F, X}^{\text{op}} : F(X) \rightarrow \text{Nat}(\text{Hom}_{\mathcal{C}}(-, X), F)$  is the dual of  $\theta_{F, X}$ .

**Proposition 3.4** (Universal mapping property [5]). *Let  $\mathcal{C}$  be a category and  $F : \mathcal{C} \rightarrow \mathbf{Sets}$  a functor.*

*The pair  $(C^*, e^*)$  is a universal element for  $F$  if and only if  $\forall X \in \text{Ob}_{\mathcal{C}}, \forall e \in F(X), \exists! x \in \text{Hom}_{\mathcal{C}}(C^*, X) e = F(x)(e^*)$ .*

*Proof.* Using Definition 3.1:

$$\begin{aligned} &(C^*, e^*) \text{ is a universal element for } F \\ \Leftrightarrow \theta_{F, X^*}(e^*) &= \left( \varphi_{e^*, X}^{F, X^*} : \begin{array}{ccc} \text{Hom}_{\mathcal{C}}(X^*, X) & \longrightarrow & F(X) \\ x & \longmapsto & F(x)(e^*) \end{array} \right)_{X \in \text{Ob}_{\mathcal{C}}} \text{ is an isomorphism} \\ \Leftrightarrow \forall X \in \text{Ob}_{\mathcal{C}}, \varphi_{e^*, X}^{F, X^*} &: \begin{array}{ccc} \text{Hom}_{\mathcal{C}}(X^*, X) & \longrightarrow & F(X) \\ x & \longmapsto & F(x)(e^*) \end{array} \text{ is an isomorphism} \\ \Leftrightarrow \forall X \in \text{Ob}_{\mathcal{C}}, \forall e \in F(X), \exists! x \in \text{Hom}_{\mathcal{C}}(C^*, X) & e = F(x)(e^*) \end{aligned}$$

□

**Proposition 3.5.** *Let  $\mathcal{C}$  be a category, and  $F : \mathcal{C} \rightarrow \mathbf{Sets}$  a functor.*

*If  $(X_0, e_0)$  and  $(X_1, e_1)$  are universal elements for  $F$ , then there exists a unique isomorphism  $\varphi : X_0 \rightarrow X_1$  such that  $F(\varphi)(e_0) = e_1$ .*

*Proof.* If  $(X_0, e_0)$  and  $(X_1, e_1)$  are universal elements for  $F$ , then by Universal Mapping Property (Proposition 3.4):

1. there is a unique  $\varphi_0 \in \text{Hom}_{\mathcal{C}}(X_0, X_1)$  such that  $F(\varphi_0)(e_0) = e_1$ .
2. there is a unique  $\varphi_1 \in \text{Hom}_{\mathcal{C}}(X_1, X_0)$  such that  $F(\varphi_1)(e_1) = e_0$ .
3. there exists a unique  $\psi_0 \in \text{Hom}_{\mathcal{C}}(X_0, X_0)$  such that  $F(\psi_0)(e_0) = e_0$ . However,  $\text{id}_{X_0}$  also has this property, so  $\psi_0 = \text{id}_{X_0}$ .
4. there exists a unique  $\psi_1 \in \text{Hom}_{\mathcal{C}}(X_1, X_1)$  such that  $F(\psi_1)(e_1) = e_1$ . However,  $\text{id}_{X_1}$  also has this property, so  $\psi_1 = \text{id}_{X_1}$ .

Now let us study  $\varphi_0 \circ \varphi_1$  and  $\varphi_1 \circ \varphi_0$ . Combining items 1 and 2, we have:

$$\begin{aligned} F(\varphi_0)(F(\varphi_1)(e_1)) &= F(\varphi_0)(e_1) \\ F(\varphi_0) \circ F(\varphi_1)(e_1) &= e_1 \\ F(\varphi_0 \circ \varphi_1)(e_1) &= e_1 \end{aligned} \tag{10}$$

$$\begin{aligned} F(\varphi_1)(F(\varphi_0)(e_0)) &= F(\varphi_1)(e_0) \\ F(\varphi_1) \circ F(\varphi_0)(e_0) &= e_0 \\ F(\varphi_1 \circ \varphi_0)(e_0) &= e_0 \end{aligned} \tag{11}$$

As  $\text{id}_{X_0}$  (resp.  $\text{id}_{X_1}$ ) is the unique arrow such that  $F(\text{id}_{X_0})(e_0) = e_0$  (resp.  $F(\text{id}_{X_1})(e_1) = e_1$ ), we deduce from Equation 10 (resp. from Equation 11) that  $\varphi_1 \circ \varphi_0 = \text{id}_{X_0}$  (resp.  $\varphi_0 \circ \varphi_1 = \text{id}_{X_1}$ ). Consequently,  $\varphi_0$  is the isomorphism described in the proposition.  $\square$

**Definition 3.6** (Representable functor). Let  $\mathcal{C}$  be a category, and  $F : \mathcal{C} \rightarrow \mathbf{Sets}$  a functor.

A *representation of  $F$*  is a pair  $(X^*, \psi)$  where:

- $X^* \in \text{Ob}_{\mathcal{C}}$  is called the *representing object of  $F$*
- $\psi : \text{Hom}_{\mathcal{C}}(X^*, -) \rightarrow F$  is a natural isomorphism.

The functor  $F$  is said *representable* if such a representation exists.

*Remark 3.7.* As in Remark 3.3, a representation of a contravariant functor  $F$  is a pair  $(X^*, \psi)$  such that  $\psi : \text{Hom}_{\mathcal{C}}(-, X^*) \rightarrow F$  is a natural isomorphism.

**Lemma 3.8.** Let  $\mathcal{C}$  be a category, and  $F : \mathcal{C} \rightarrow \mathbf{Sets}$  a functor.

If  $(X^*, e^*)$  is a universal element for  $F$  then  $(X^*, \theta_{F, X^*}(e^*))$  is a representation of  $F$ .

**Lemma 3.9.** Let  $\mathcal{C}$  be a category, and  $F : \mathcal{C} \rightarrow \mathbf{Sets}$  a functor.

If  $(X^*, \psi)$  is a representation of  $F$ , then  $(X^*, \psi_{X^*}(\text{id}_{X^*}))$  is a universal element for  $F$ .

**Theorem 3.10.** Let  $\mathcal{C}$  be a category, and  $F : \mathcal{C} \rightarrow \mathbf{Sets}$  a functor.

There exists a universal element for  $F \Leftrightarrow F$  is representable.

Theorem 3.10 is an immediate consequence of the two previous lemmas. Besides, the two lemmas give a way to convert a universal element into a representation.

*Proof of Lemma 3.8.* Let  $(X^*, e^*)$  be a universal element for  $F$ ; it follows from Definition 3.1 that  $\theta_{F, X^*}(e^*) : \text{Hom}_{\mathcal{C}}(X^*, -) \rightarrow F$  is a natural isomorphism. Thence,  $(X^*, \theta_{F, X^*}(e^*))$  is a representation of  $F$ .  $\square$

*Proof of Lemma 3.9.* Let  $(X^*, \psi)$  be a representation of  $F$ . By Proposition 2.5, we have  $\psi = \theta_{F, X^*}(\psi_{X^*}(\text{id}_{X^*}))$ . Besides, by definition of a representation,  $\psi = \theta_{F, X^*}(\psi_{X^*}(X^*))$  is an isomorphism, which gives that  $(X^*, \psi_{X^*}(\text{id}_{X^*}))$  is a universal element for  $F$ .  $\square$

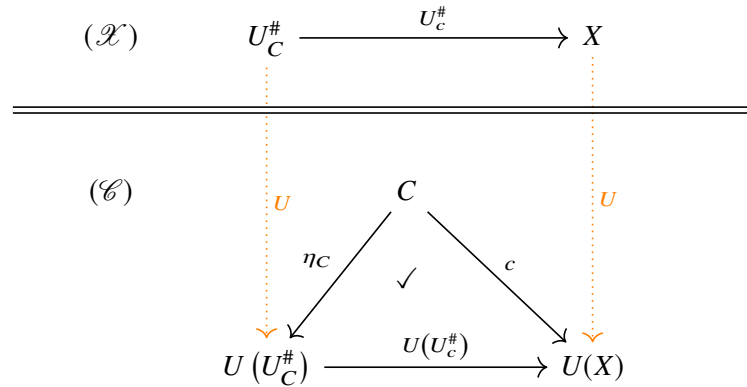
**Corollary 3.11.** *Representations of a functor  $F$  are unique up to isomorphism.*

**Definition 3.12** (Universal arrows [2]). Let  $\mathcal{X}, \mathcal{C}$  be two categories. Let  $U : \mathcal{X} \rightarrow \mathcal{C}$  be a functor and let  $C \in \text{Ob}_{\mathcal{C}}$ .

A *universal arrow from  $C$  to  $U$*  is a pair  $(U_C^\#, \eta_C)$ , where  $U_C^\# \in \text{Ob}_{\mathcal{X}}$  and  $\eta_C \in \text{Hom}_{\mathcal{C}}(C, U(U_C^\#))$ , such that, for all  $X \in \text{Ob}_{\mathcal{X}}$ , for all  $c \in \text{Hom}_{\mathcal{C}}(C, U(X))$ , there exists a unique  $x \in \text{Hom}_{\mathcal{X}}(U_C^\#, X)$  such that  $c = U(x) \circ \eta_C$ .

In the following, this unique  $x$  will be denoted  $U_c^\#$ , such that:  $c = U(U_c^\#) \circ \eta_C$ .

Here is a diagram that sums up the idea behind universal arrows:



**Lemma 3.13.** *Let  $\mathcal{X}, \mathcal{C}$  be two categories. Let  $U : \mathcal{X} \rightarrow \mathcal{C}$  be a functor and let  $C \in \text{Ob}_{\mathcal{C}}$ .*

*If  $(U_C^\#, \eta_C)$  is a universal arrow from  $C$  to  $U$ , then  $U_{\eta_C}^\# = \text{id}_{U_C^\#}$ .*

*Proof.* The arrow  $U_{\eta_C}^\#$  is the unique arrow that verifies:  $\eta_C = U(U_{\eta_C}^\#) \circ \eta_C$ . Of course,  $\eta_C = U(\text{id}_{U_C^\#}) \circ \eta_C$ , so  $\text{id}_{U_C^\#} = U_{\eta_C}^\#$ .  $\square$

**Proposition 3.14.** *Let  $\mathcal{X}, \mathcal{C}$  be two categories. Let  $U : \mathcal{X} \rightarrow \mathcal{C}$  be a functor and let  $C \in \text{Ob}_{\mathcal{C}}$ .*

1.  $(U_C^\#, \eta_C)$  is a universal arrow from  $C$  to  $U \Leftrightarrow \theta_{\text{Hom}_{\mathcal{C}}(C, U(-)), U_C^\#}(\eta_C)$  is a natural isomorphism.
2. If for all  $C \in \text{Ob}_{\mathcal{C}}$ , there exists  $X_C \in \text{Ob}_{\mathcal{X}}$  and a natural isomorphism  $\varphi : \text{Hom}_{\mathcal{X}}(X_C, -) \rightarrow \text{Hom}_{\mathcal{C}}(C, U(-))$ , then  $(X_C, \varphi_{X_C}(\text{id}_{X_C}))$  is a universal arrow from  $C$  to  $U$ .

*Proof.* **[Item 1]**

By definition of a universal arrow  $(U_C^\#, \eta_C)$ , for all  $X \in \text{Ob}_{\mathcal{X}}$ , for all  $c \in \text{Hom}_{\mathcal{C}}(C, U(X))$ , there exists a unique  $x \in \text{Hom}_{\mathcal{X}}(U_C^\#, X)$  such that  $c = U(x) \circ \eta_C$ ; equivalently, for all  $X \in \text{Ob}_{\mathcal{X}}$ , the function:

$$\varphi_X : \begin{cases} \text{Hom}_{\mathcal{X}}(U_C^\#, X) & \longrightarrow & \text{Hom}_{\mathcal{C}}(C, U(X)) \\ x & \longmapsto & U(x) \circ \eta_C \end{cases}$$

is an isomorphism; that is, those  $\varphi_X$  are the components of  $\theta_{\text{Hom}_{\mathcal{C}}(C, U(-)), U_C^\#}(\eta_C)$ , which is a natural isomorphism.

**[Item 2]**

Let  $C \in \mathcal{C}$  and  $X \in \mathcal{X}$ .

We have  $X_C \in \text{Ob}_{\mathcal{X}}$  and a natural isomorphism  $\varphi : \text{Hom}_{\mathcal{X}}(X_C, -) \rightarrow \text{Hom}_{\mathcal{C}}(C, U(-))$ . We have  $\varphi = \theta_{\text{Hom}_{\mathcal{C}}(C, U(-)), X_C}(\varphi_{X_C}(\text{id}_{X_C}))$ , which by Item 1 yields that  $(X_C, \varphi_{X_C}(\text{id}_{X_C}))$  is a universal arrow from  $C$  to  $U$ .  $\square$

When seeing the definitions of universal elements and arrows, we wonder what could be the link between both. In fact, universal arrows and universal elements are two very close notions.

**Proposition 3.15.** *Let  $\mathcal{X}, \mathcal{C}$  be two categories. Let  $U : \mathcal{X} \rightarrow \mathcal{C}$  be a functor and let  $C \in \text{Ob}_{\mathcal{C}}$ .*

*If  $(U_C^\#, \eta_C)$  is a universal arrow from  $C$  to  $U$ , then  $(U_C^\#, \eta_C)$  is also a universal element for  $\text{Hom}_{\mathcal{C}}(C, U(-))$ .*

*Proof.* This proposition directly follows from Definitions 3.1 (universal element) and 3.12 (universal arrow). In fact, by definition of a universal element for  $\text{Hom}_{\mathcal{C}}(C, U(-))$ , the following natural transformation should be an isomorphism:

$$\theta_{\text{Hom}_{\mathcal{C}}(C, U(-)), U_C^\#}(\eta_C) = \left( \varphi_Y : \begin{array}{ccc} \text{Hom}_{\mathcal{C}}(U_C^\#, Y) & \longrightarrow & \text{Hom}_{\mathcal{C}}(C, U(Y)) \\ x & \longmapsto & \text{Hom}_{\mathcal{C}}(C, U(x))(\eta_C) \end{array} \right)_{Y \in \text{Ob}_{\mathcal{C}}}$$

By simplest representation lemma (Lemma 1.20), we have:

$$\text{Hom}_{\mathcal{C}}(C, U(x))(\eta_C) = U(x) \circ \eta_C$$

By definition of a universal arrow,  $\forall Y \in \text{Ob}_{\mathcal{X}}, \forall c \in \text{Hom}_{\mathcal{C}}(C, U(Y)), \exists! c = U(x) \circ \eta_C = F_{\eta_C}(Y)(x)$ . Thus,  $\varphi_Y \in \text{Mor}_{\text{Sets}}$  is a bijection, thus an isomorphism; consequently, the natural transformation  $\theta_{\text{Hom}_{\mathcal{C}}(C, U(-)), X_C}(\eta_C)$  is also an isomorphism.  $\square$

Remember that universal elements are defined for a functor  $\mathcal{C} \rightarrow \mathbf{Sets}$ , and not just for a functor between any two categories. The converse proposition is a bit less general.

**Proposition 3.16.** *We denote by  $1$  the set  $\mathcal{P}(\emptyset) = \{\emptyset\} = 1$  where  $\emptyset$  is the empty set. For any set  $E$ , for any  $e \in E$ , we define:*

$$\delta_e^E : \begin{array}{ccc} 1 & \longrightarrow & E \\ x & \longmapsto & e \end{array}$$

*Let  $\mathcal{X}$  be a category, let  $U : \mathcal{X} \rightarrow \mathbf{Sets}$  be a functor.*

*If  $(X^*, e^*)$  is a universal element for  $U$ , then  $(X^*, \delta_{e^*}^{U(X^*)})$  is a universal arrow from  $1$  to  $U$ .*

*Proof.* By Proposition 3.4, if  $(X^*, e^*)$  is a universal element for  $U$ , then  $\forall X \in \mathcal{X}, \forall e \in U(X), \exists! x \in \text{Hom}_{\mathcal{X}}(X^*, X)$  such that:

$$\begin{aligned} e &= U(x)(e^*) \\ \Leftrightarrow \delta_e^{U(X)}(0) &= U(x)\left(\delta_{e^*}^{U(X^*)}(0)\right) \\ \Leftrightarrow \delta_e^{U(X)} &= U(x) \circ \delta_{e^*}^{U(X^*)} \end{aligned}$$

Consequently, we have:  $\forall X \in \mathcal{X}, \forall \delta_e^{U(X)} \in \text{Hom}_{\text{Sets}}(1, U(X)), \exists! x \in \text{Hom}_{\mathcal{X}}(X^*, X), \delta_e^{U(X)} = U(x) \circ \delta_{e^*}^{U(X^*)}$ , which yields that  $(X^*, \delta_{e^*}^{U(X^*)})$  is a universal arrow from  $1$  to  $U$ .  $\square$

We sum up the results into this theorem:

**Theorem 3.17.** *Let  $\mathcal{X}$  be a category, and let  $U : \mathcal{X} \rightarrow \mathbf{Sets}$  be a functor.*

1.  $(X^*, e^*)$  is a universal element for  $U \Leftrightarrow (X^*, \theta_{U, X^*}(e^*))$  is a representation of  $U$ .
2.  $(X^*, e^*)$  is a universal element for  $U \Leftrightarrow (X^*, \delta_{e^*}^{U(X^*)})$  is a universal arrow from  $1$  of  $U$ .
3.  $(X^*, \psi)$  is a representation of  $U \Leftrightarrow (X^*, \psi_X(\text{id}_X))$  is a universal element for  $U$ .
4.  $(U_C^\#, \eta_C)$  is a universal arrow from  $C$  to  $U \Leftrightarrow (U_C^\#, \eta_C)$  is a universal element for  $\text{Hom}_{\text{Sets}}(C, U(-))$ .

## 4. Towards adjunctions

**Definition 4.1** (Left adjoint - from universal arrows). Let  $\mathcal{X}, \mathcal{C}$  be two categories. Let  $U : \mathcal{X} \rightarrow \mathcal{C}$  be a functor. We suppose that for all  $C \in \text{Ob}_{\mathcal{C}}$ , there exists a universal arrow  $(U_C^\#, \eta_C)$  from  $C$  to  $U$ .

The *left adjoint* of  $U$ , denoted by  $U^*$ , is the mapping:

$$U^* : \begin{cases} \mathcal{C} & \longrightarrow & \mathcal{X} \\ C & \longmapsto & U_C^\# \\ c : C \rightarrow C' & \longmapsto & U_{\eta_C \circ c}^\# \end{cases}$$

Let's study some properties of the left adjoint.

**Lemma 4.2.** Let  $\mathcal{X}, \mathcal{C}$  be two categories. Let  $U : \mathcal{X} \rightarrow \mathcal{C}$  be a functor, and let  $U^*$  be the left adjoint of  $U$ .

For any  $c \in \text{Hom}_{\mathcal{C}}(C, C')$ ,  $U^*(c)$  is the unique solution in  $x \in \text{Hom}_{\mathcal{X}}(U_C^\#, U_{C'}^\#)$  to the equation:  $\eta_{C'} \circ c = U(x) \circ \eta_C$ .

*Proof.* We have  $\eta_{C'} \circ c \in \text{Hom}_{\mathcal{C}}(C, U(U_{C'}^\#))$ . By definition of a universal arrow  $(U_C^\#, \eta_C)$  from  $C$  to  $U$ , there exists a unique  $U_{\eta_C \circ c}^\# \in \text{Hom}_{\mathcal{C}}(C')$  such that:  $\eta_{C'} \circ c = U(U_{\eta_C \circ c}^\#) \circ \eta_C = U(U^*(c)) \circ \eta_C$ .  $\square$

**Proposition 4.3.** Let  $\mathcal{X}, \mathcal{C}$  be two categories. Let  $U : \mathcal{X} \rightarrow \mathcal{C}$  be a functor, and let  $U^*$  be the left adjoint of  $U$ .

The left adjoint  $U^* : \mathcal{C} \rightarrow \mathcal{X}$  is a functor.

*Proof.* The mapping  $U^*$  sends objects (resp. arrows) in  $\mathcal{C}$  to objects (resp. arrows) in  $\mathcal{X}$ .

Using Lemma 3.13, we check the behaviour of  $U^*$  on identity arrows:

$$U^*(\text{id}_C) = U_{\eta_C \circ \text{id}_C}^\# = U_{\eta_C}^\# = \text{id}_{U_C^\#}$$

As for the composition, let  $c : C \rightarrow C'$  and  $c' : C' \rightarrow C''$ . By definition of  $U^*$ , we have:

$$\begin{aligned} \eta_{C'} \circ c &= U(U^*(c)) \circ \eta_C \\ \eta_{C''} \circ c' &= U(U^*(c')) \circ \eta_{C'} \\ \eta_{C''} \circ c' \circ c &= U(U^*(c' \circ c)) \circ \eta_C \end{aligned} \tag{12}$$

But also:

$$\begin{aligned} \eta_{C''} \circ c' \circ c &= (U(U^*(c')) \circ \eta_{C'}) \circ c \\ &= U(U^*(c')) \circ (\eta_{C'} \circ c) \\ &= U(U^*(c')) \circ U(U^*(c)) \\ &= U(U^*(c') \circ U^*(c)) \end{aligned} \tag{13}$$

Equations 12 and 13, together with Lemma 4.2, yield:

$$U^*(c' \circ c) = U^*(c') \circ U^*(c)$$

$\square$

**Proposition 4.4.** Let  $\mathcal{X}, \mathcal{C}$  be two categories. Let  $U : \mathcal{X} \rightarrow \mathcal{C}$  be a functor, and let  $U^*$  be the left adjoint of  $U$ .

The mapping  $\eta = (\eta_C : C \rightarrow U \circ U^*(C))$  is a natural transformation  $\eta : \text{Id}_{\mathcal{C}} \rightarrow U \circ U^*$ .

*Proof.* We need to check if, for each  $c : C \rightarrow C'$ , the following diagram commutes:

$$\begin{array}{ccc}
 C & \xrightarrow{\eta_C} & U(U^*(C)) \\
 \downarrow c & \checkmark & \downarrow U(U^*(c)) \\
 C' & \xrightarrow{\eta_{C'}} & U(U^*(C'))
 \end{array}$$

That is, we need to check whether  $U(U^*(c)) \circ \eta_C = \eta_{C'} \circ c$ , which is the result of Lemma 4.2.  $\square$

**Proposition 4.5.** Let  $\mathcal{X}, \mathcal{C}$  be two categories. Let  $U : \mathcal{X} \rightarrow \mathcal{C}$  be a functor, and let  $U^*$  be the left adjoint of  $U$ .

For all  $X \in \mathcal{X}$ , for all  $C \in \mathcal{C}$ , we define:

$$\beta_{C,X} : \begin{cases} \text{Hom}_{\mathcal{X}}(U^*(C), X) & \longrightarrow & \text{Hom}_{\mathcal{C}}(C, U(X)) \\ x & \longmapsto & U(x) \circ \eta_C \end{cases}$$

The mapping  $\beta : C, X \mapsto \beta_{C,X}$  is a natural isomorphism  $\text{Hom}_{\mathcal{X}}(U^*(-), -) \rightarrow \text{Hom}_{\mathcal{C}}(-, U(-))$ , contravariant in  $C$  and covariant in  $X$ .

*Proof.* For  $C \in \mathcal{C}$ ,  $\beta_{C,-} : \text{Hom}_{\mathcal{X}}(U^*(C), -) \rightarrow \text{Hom}_{\mathcal{C}}(C, U(-))$  is the same function as  $\beta_{C,-} = \theta_{\text{Hom}_{\mathcal{C}}(C, U(-)), U^*(C)}(\eta_C)$  which we know is a natural isomorphism (cf. Proposition 3.14, item 1).

For  $X \in \mathcal{X}$ , we study  $\beta_{-,X} : \text{Hom}_{\mathcal{X}}(U^*(-), X) \rightarrow \text{Hom}_{\mathcal{C}}(-, U(X))$ . Let  $c : C \rightarrow C'$ , we want the following diagram to commute:

$$\begin{array}{ccccc}
 C & & \text{Hom}_{\mathcal{X}}(U^*(C'), X) & \xrightarrow{\beta_{C',X}} & \text{Hom}_{\mathcal{C}}(C', U(X)) \\
 \downarrow c & \rightsquigarrow & \downarrow \text{Hom}_{\mathcal{X}}(U^*(c), X) & & \downarrow \text{Hom}_{\mathcal{C}}(c, U(X)) \\
 C' & & \text{Hom}_{\mathcal{X}}(U^*(C), X) & \xrightarrow{\beta_{C,X}} & \text{Hom}_{\mathcal{C}}(C, U(X))
 \end{array}$$

(Note that  $\beta_{-,X}$  is supposed to be contravariant in  $C$ ).

Let  $f \in \text{Hom}_{\mathcal{X}}(U^*(C'), X)$ . On the one hand:

$$\begin{aligned}
 \beta_{C,X} \circ \text{Hom}_{\mathcal{X}}(U^*(c), X)(f) &= \beta_{C,X}(f \circ U^*(c)) \\
 &= U(f \circ U^*(c)) \circ \eta_C
 \end{aligned}$$

while on the other hand:

$$\begin{aligned}
 \text{Hom}_{\mathcal{C}}(c, U(X)) \circ \beta_{C',X}(f) &= \text{Hom}_{\mathcal{C}}(c, U(X))(U(f) \circ \eta_{C'}) \\
 &= U(f) \circ \eta_{C'} \circ c \\
 &= U(f) \circ U(U^*(c)) \circ \eta_C \\
 &= U(f \circ U^*(c)) \circ \eta_C \\
 &= \beta_{C,X} \circ \text{Hom}_{\mathcal{X}}(U^*(c), X)(f)
 \end{aligned}$$

Thus,  $\beta_{-,X}$  is a natural transformation. Note that each component in  $C$  of  $\beta_{-,X} = (\beta_{C,X})_{C \in \text{Ob}_{\mathcal{C}}}$  is an isomorphism (because  $\theta_{\text{Hom}_{\mathcal{C}}(C, U(-)), U^*(C)}(\eta_C)$ , for any  $C \in \text{Ob}_{\mathcal{C}}$ , is a natural isomorphism), thus so is  $\beta_{-,X}$ .  $\square$

**Proposition 4.6.** Let  $\mathcal{X}, \mathcal{C}$  be two categories. Let  $U : \mathcal{X} \rightarrow \mathcal{C}$  be a functor, and let  $U^*$  be the left adjoint of  $U$ .

If  $F : \mathcal{C} \rightarrow \mathcal{X}$  is a functor for which there exists a natural isomorphism  $\gamma : \text{Hom}_{\mathcal{X}}(F(-), -) \rightarrow \text{Hom}_{\mathcal{C}}(-, U(-))$ , then there exists a unique natural isomorphism  $\alpha : F \rightarrow U^*$ ; in other words, the left adjoint is unique up to a unique isomorphism.

*Proof.* Let  $F : \mathcal{C} \rightarrow \mathcal{X}$  such that  $\gamma : \text{Hom}_{\mathcal{X}}(F(-), -) \rightarrow \text{Hom}_{\mathcal{C}}(-, U(-))$  is a natural isomorphism.

For all  $C \in \mathcal{C}$ , we have  $\gamma_{C, -} = \theta_{\text{Hom}_{\mathcal{C}}(C, U(-)), F(C)}(e_C)$  for some  $e_C : C \rightarrow U(F(C)) = \gamma_{C, F(C)}(\text{id}_{F(C)})$ . We deduce from Proposition 3.14-2 that  $F$  is a left adjoint for  $U$ . As  $U^*$  and  $F$  are left adjoints for  $U$ , then  $(F(C), e_C)$  and  $(U^*(C), \eta_C)$  are universal arrows from  $C$  to  $U$  (Definition 4.1), so  $(F(C), e_C)$  and  $(U^*(C), \eta_C)$  are also universal elements for  $\text{Hom}_{\mathcal{C}}(C, U(-))$  (Proposition 3.15). According to Proposition 3.5, there exists a unique isomorphism  $\alpha_C : U^*(C) \rightarrow F(C)$  such that  $U(\alpha_C)(e_C) = \eta_C$ .

We now have to show that  $\alpha = (\alpha_C)_{C \in \text{Ob}_{\mathcal{C}}}$  is natural in  $C$ .

We have the following diagram:

$$\begin{array}{ccccc}
 & & C & & \\
 & \swarrow \eta_C & \downarrow & \searrow e_C & \\
 U(F(C)) & \xleftarrow{U(\alpha_C)} & & \xrightarrow{} & U(U^*(C)) \\
 \downarrow U(F(c)) & & \downarrow c & & \downarrow U(U^*(c)) \\
 & \swarrow \eta_{C'} & C' & \searrow e_{C'} & \\
 U(F(C')) & \xleftarrow{U(\alpha_{C'})} & & \xrightarrow{} & U(U^*(C'))
 \end{array}$$

where the following subdiagrams commute:

$$\begin{array}{ccc}
 & C & \\
 \eta_C \swarrow & \checkmark & \searrow e_C \\
 U(F(C)) & \xrightarrow{U(\alpha_C)} & U(U^*(C))
 \end{array}
 \qquad
 \begin{array}{ccc}
 & C' & \\
 \eta_{C'} \swarrow & \checkmark & \searrow e_{C'} \\
 U(F(C')) & \xrightarrow{U(\alpha_{C'})} & U(U^*(C'))
 \end{array}$$

due to the construction of  $\alpha_C$  and:

$$\begin{array}{ccccc}
 U(F(C)) & \xleftarrow{\eta_C} & C & \xrightarrow{e_C} & U(U^*(C)) \\
 \downarrow U(F(c)) & & \downarrow c & & \downarrow U(U^*(c)) \\
 U(F(C')) & \xleftarrow{\eta_{C'}} & C' & \xrightarrow{e_{C'}} & U(U^*(C'))
 \end{array}$$

due to the naturality of  $e : \text{Id}_{\mathcal{C}} \rightarrow U \circ F$  and  $\eta : \text{Id}_{\mathcal{C}} \rightarrow U \circ U^*$ .

By diagram chasing, we have:

$$\begin{aligned}
 U(\alpha_{C'}) \circ U(F(c)) \circ \eta_C &= U(U^*(c)) \circ U(\alpha_C) \circ \eta_C \\
 U(\alpha_{C'} \circ F(c)) \circ \eta_C &= U(U^*(c) \circ \alpha_C) \circ \eta_C \\
 \alpha_{C'} \circ F(c) &= U^*(c) \circ \alpha_C
 \end{aligned}$$

The last equation is due to  $\beta_{C, X}$  being an isomorphism. This equation makes the following diagram commute:



$$\begin{array}{ccc}
 U(F(C)) & \xrightarrow{\alpha_C} & U^*(C) \\
 \downarrow F(c) & \checkmark & \downarrow U^*(c) \\
 F(C') & \xrightarrow{\alpha_{C'}} & U^*(C')
 \end{array}$$

which makes  $\alpha = (\alpha_C)_{C \in \text{Ob}_{\mathcal{C}}}$  natural in  $C$ . □

**Definition 4.7** (Adjunction - official). Let  $\mathcal{X}, \mathcal{C}$  be two categories. Let  $U : \mathcal{X} \rightarrow \mathcal{C}$  and  $F : \mathcal{C} \rightarrow \mathcal{X}$  be two functors.

The 3-tuple  $(F, U, \beta)$  is called an *adjunction* whenever  $\beta$  is a natural isomorphism  $\beta : \text{Hom}_{\mathcal{X}}(F(-), -) \rightarrow \text{Hom}_{\mathcal{C}}(-, U(-))$ .

We also say that  $F$  is *left adjoint to  $U$*  and  $U$  is *right adjoint to  $F$* . We will refer to  $\beta$  as the *adjunction*<sup>1</sup> of  $F$  and  $U$ .

If  $(F, U, \beta)$  is an adjunction, we may write  $F \dashv U$  or  $\frac{F(C) \rightarrow X}{C \rightarrow U(X)} (\beta)$ .

The following lemma proves that if  $(F, U, \beta)$  is an adjunction, then  $F$  is actually left adjoint to  $U$  as defined in Definition 4.1. In fact, both definitions are equivalent.

**Lemma 4.8.** Let  $\mathcal{X}, \mathcal{C}$  be two categories. Let  $U : \mathcal{X} \rightarrow \mathcal{C}$  and  $F : \mathcal{C} \rightarrow \mathcal{X}$  be two functors.

$F \dashv U \Leftrightarrow$  there exists a natural transformation  $\eta : \text{Id}_{\mathcal{C}} \rightarrow U \circ F$  such that  $\forall C \in \text{Ob}_{\mathcal{C}}, (F(C), \eta_C)$  is a universal arrow from  $C$  to  $U$ .

*Proof.* **[Proof of  $\Leftarrow$ ]**

Suppose that we have a  $\eta : \text{Id}_{\mathcal{C}} \rightarrow U \circ F$  such that  $\forall C \in \text{Ob}_{\mathcal{C}}, (F(C), \eta_C)$  is a universal arrow from  $C$  to  $U$ . According to the definition of a left adjoint,  $F$  corresponds to a left adjoint on objects. We have to check if, for all  $X \in \mathcal{X}$ , for all  $c : C \rightarrow C'$ ,  $F(c)$  is the unique solution in  $x \in \text{Hom}_{\mathcal{X}}(F(C), X)$  to the equation:

$$\eta'_{C'} \circ c = U(x) \circ \eta_C \tag{14}$$

The natural transformation  $\eta : \text{Id}_{\mathcal{C}} \rightarrow U \circ F$  makes the following diagram commute:

$$\begin{array}{ccc}
 C & \xrightarrow{\eta_C} & U(F(C)) \\
 \downarrow c & \checkmark & \downarrow U(F(c)) \\
 C' & \xrightarrow{\eta_{C'}} & U(F(C'))
 \end{array}$$

which proves that  $F(c)$  is indeed a solution to Equation 14. The uniqueness of the solution comes from the definition of a universal arrow (Definition 3.12).

Consequently,  $F$  is a left adjoint to  $U$ . By Proposition 4.5, we can define a  $\beta$  from  $\eta$  that is a natural isomorphism  $\text{Hom}_{\mathcal{X}}(F(C), X) \rightarrow \text{Hom}_{\mathcal{C}}(C, U(X))$ . Finally,  $(F, U, \beta)$  is an adjunction.

**[Proof of  $\Rightarrow$ ]**

Suppose  $F \dashv U$ , and suppose  $(F, U, \beta)$  is an adjunction. Define  $\eta$  to be the natural transformation with components:

$$\eta_C = \beta_{C, F(C)}(\text{id}_{F(C)}) \in \text{Hom}_{\mathcal{C}}(C, U(F(C)))$$

---

<sup>1</sup>The  $\beta$  natural isomorphism appears to be unnamed in most references. However, in the rest of this course, it may be convenient to give it a name. Please note that nobody but the authors give this name to that isomorphism.

The naturality of  $\eta$  comes from the naturality of  $\beta = (\beta_{C,X})_{C \in \mathcal{C}, X \in \mathcal{X}}$  in both its variables. The naturality in  $X$  gives, when  $c : C \rightarrow C'$ , for  $F(c) : F(C) \rightarrow F(C') \in \mathcal{X}$ :

$$\begin{array}{ccc} \text{Hom}_{\mathcal{X}}(F(C), F(C)) & \xrightarrow{\beta_{C,F(C)}} & \text{Hom}_{\mathcal{X}}(C, U \circ F(C)) \\ \text{Hom}_{\mathcal{X}}(F(C), F(c)) \downarrow & \checkmark & \downarrow \text{Hom}_{\mathcal{X}}(C, U \circ F(c)) \\ \text{Hom}_{\mathcal{X}}(F(C), F(C')) & \xrightarrow{\beta_{C,F(C')}} & \text{Hom}_{\mathcal{X}}(C, U \circ F(C')) \end{array}$$

while the naturality in  $C$  gives, for  $c : C \rightarrow C'$ :

$$\begin{array}{ccc} \text{Hom}_{\mathcal{X}}(F(C'), F(C')) & \xrightarrow{\beta_{C',F(C')}} & \text{Hom}_{\mathcal{X}}(C', U \circ F(C')) \\ \text{Hom}_{\mathcal{X}}(F(c), F(C')) \downarrow & \checkmark & \downarrow \text{Hom}_{\mathcal{X}}(c, U \circ F(C')) \\ \text{Hom}_{\mathcal{X}}(F(C), F(C')) & \xrightarrow{\beta_{C,F(C')}} & \text{Hom}_{\mathcal{X}}(C, U \circ F(C')) \end{array}$$

From the first diagram, we obtain:

$$\begin{aligned} \beta_{C,F(C')} \circ \text{Hom}_{\mathcal{X}}(F(C), F(c)) (\text{id}_{F(C)}) &= \text{Hom}_{\mathcal{X}}(C, (U \circ F)(c)) \circ \beta_{C,F(C)} (\text{id}_{F(C)}) \\ \beta_{C,F(C')} \circ F(c) &= (U \circ F)(c) \circ \beta_{C,F(C)} (\text{id}_{F(C)}) \end{aligned} \quad (15)$$

and from the second diagram:

$$\begin{aligned} \beta_{C,F(C')} \circ \text{Hom}_{\mathcal{X}}(F(c), F(C')) (\text{id}_{F(C')}) &= \text{Hom}_{\mathcal{X}}(c, U \circ F(C')) \circ \beta_{C',F(C')} (\text{id}_{F(C')}) \\ \beta_{C,F(C')} \circ F(c) &= \beta_{C',F(C')} (\text{id}_{F(C')}) \circ c \end{aligned} \quad (16)$$

Combining Equations 15 and 16, we obtain:

$$\begin{aligned} (U \circ F)(c) \circ \beta_{C,F(C)} (\text{id}_{F(C)}) &= \beta_{C',F(C')} (\text{id}_{F(C')}) \circ c \\ (U \circ F)(c) \circ \eta_C &= \eta_{C'} \circ c \end{aligned}$$

which proves that  $\eta$  is a natural transformation.

We have to show that each  $(F(C), \eta_C)$  is a universal arrow from  $C$  to  $U$ . We have a natural isomorphism  $\beta_{C,-} : \text{Hom}_{\mathcal{X}}(F(C), -) \rightarrow \text{Hom}_{\mathcal{C}}(C, U(-))$ ; so according to Proposition 3.14,  $(F(C), \beta_{C,F(C)} (\text{id}_{F(C)})) = (F(C), \eta_C)$  is a universal arrow. Besides,  $F$  is a left adjoint to  $U$ .  $\square$

**Definition 4.9** (Unit of an adjunction). Let  $(F, U, \beta)$  be an adjunction.

The *unit of the adjunction*  $(F, U, \beta)$  is the natural transformation  $\eta : \text{Id}_{\mathcal{C}} \rightarrow U \circ F$  such that  $\forall C \in \text{Ob}_{\mathcal{C}}, \eta_C = \beta_{C,F(C)} (\text{id}_{F(C)})$ .

We will define the dual notion of a counit. However, we will not construct it explicitly as we did the unit (that is, using universal arrows, then left adjoints), because it is not that interesting. We will first compute the inverse of the adjunction  $\beta$ .

Note that Lemma 4.8 proves that Definitions 4.1 and 4.7 are not only equivalent, but also that we can construct the unit  $\eta$  from the adjunction  $\beta$  and conversely. The same goes from the counit  $\varepsilon$  that we will define right after the following lemma.

**Lemma 4.10.** *Let  $(F, U, \beta)$  be an adjunction.*

*For all  $C \in \text{Ob}_{\mathcal{C}}$ , define  $\eta_C = \beta_{C, F(C)}(\text{id}_{F(C)})$ . Then,  $\beta_{C, X}$  is:*

$$\beta_{C, X} : \begin{cases} \text{Hom}_{\mathcal{X}}(F(C), X) & \longrightarrow & \text{Hom}_{\mathcal{C}}(C, U(X)) \\ x & \longmapsto & U(x) \circ \eta_C \end{cases}$$

*For all  $X \in \text{Ob}_{\mathcal{X}}$ , define  $\varepsilon_X = \beta_{U(X), X}^{-1}(\text{id}_{U(X)})$ . Then,  $\beta_{C, X}^{-1}$  is:*

$$\beta_{C, X}^{-1} : \begin{cases} \text{Hom}_{\mathcal{C}}(C, U(X)) & \longrightarrow & \text{Hom}_{\mathcal{X}}(F(C), X) \\ c & \longmapsto & \varepsilon_X \circ F(c) \end{cases}$$

*Proof.* By naturality of  $\beta$  and  $\beta^{-1}$ , and for  $c : C \rightarrow U(X)$  and  $x : F(C) \rightarrow X$ , the following two diagrams commute:

$$\begin{array}{ccc} \text{Hom}_{\mathcal{X}}(F(C), F(C)) & \xrightarrow{\beta_{C, F(C)}} & \text{Hom}_{\mathcal{C}}(C, U \circ F(C)) \\ \text{Hom}_{\mathcal{X}}(F(C), x) \downarrow & \checkmark & \downarrow \text{Hom}_{\mathcal{X}}(C, U(x)) \\ \text{Hom}_{\mathcal{X}}(F(C), X) & \xrightarrow{\beta_{C, X}} & \text{Hom}_{\mathcal{C}}(C, U(X)) \\ \\ \text{Hom}_{\mathcal{C}}(U(X), U(X)) & \xrightarrow{\beta_{U(X), X}^{-1}} & \text{Hom}_{\mathcal{X}}(F \circ U(X), X) \\ \text{Hom}_{\mathcal{C}}(c, U(X)) \downarrow & \checkmark & \downarrow \text{Hom}_{\mathcal{X}}(F(c), X) \\ \text{Hom}_{\mathcal{C}}(C, U(X)) & \xrightarrow{\beta_{C, X}^{-1}} & \text{Hom}_{\mathcal{X}}(F(C), X) \end{array}$$

Suppose we have  $\beta_{C, X}(x) = c$  (or equivalently  $\beta_{C, X}^{-1}(c) = x$ ). Those two diagrams combine into this one:

$$\begin{array}{ccc} \text{Hom}_{\mathcal{X}}(F(C), F(C)) & & \text{Hom}_{\mathcal{C}}(U(X), U(X)) \\ \downarrow \beta_{C, F(C)} & & \downarrow \beta_{U(X), X}^{-1} \\ \text{Hom}_{\mathcal{C}}(C, U \circ F(C)) & & \text{Hom}_{\mathcal{X}}(F \circ U(X), X) \\ \downarrow \text{Hom}_{\mathcal{X}}(C, U(x)) & \text{Hom}_{\mathcal{C}}(c, U(X)) & \downarrow \text{Hom}_{\mathcal{X}}(F(c), X) \\ & \text{Hom}_{\mathcal{X}}(F(C), x) & \\ & \downarrow & \\ \text{Hom}_{\mathcal{C}}(C, U(X)) & \xrightarrow{\beta_{C, X}^{-1}} & \text{Hom}_{\mathcal{X}}(F(C), X) \\ & \checkmark & \\ & \xrightarrow{\beta_{C, X}} & \end{array}$$

Firstly, with  $\text{id}_{F(C)} \in \text{Hom}_{\mathcal{X}}(F(C), F(C))$ , we have:

$$\begin{aligned}\beta_{C,X}^{-1} \circ \text{Hom}_{\mathcal{C}}(C, U(x)) \circ \beta_{C,F(C)}(\text{id}_{F(C)}) &= \text{Hom}_{\mathcal{X}}(F(C), x)(\text{id}_{F(C)}) \\ \beta_{C,X}^{-1}(U(x) \circ \beta_{C,F(C)}(\text{id}_{F(C)})) &= x \circ \text{id}_{F(C)} \\ \beta_{C,X}^{-1}(U(x) \circ \eta_C) &= x\end{aligned}$$

Secondly, with  $\text{id}_{U(X)} \in \text{Hom}_{\mathcal{C}}(U(X), U(X))$ , we have:

$$\begin{aligned}\beta_{C,X} \circ \text{Hom}_{\mathcal{X}}(F(c), X) \circ \beta_{U(X),X}^{-1}(\text{id}_{U(X)}) &= \text{Hom}_{\mathcal{C}}(c, U(X))(\text{id}_{U(X)}) \\ \beta_{C,X} \circ \beta_{U(X),X}^{-1}(\text{id}_{U(X)}) \circ F(c) &= \text{id}_{U(X)} \circ c \\ \beta_{C,X}(\varepsilon_X \circ F(c)) &= c\end{aligned}$$

The first calculation shows that  $\beta_{C,X}(x) = U(x) \circ \eta_C$  and the second shows that  $\beta_{C,X}^{-1}(c) = \varepsilon_X \circ F(c)$ .  $\square$

**Lemma 4.11.** *Let  $\mathcal{X}, \mathcal{C}$  be two categories. Let  $U : \mathcal{X} \rightarrow \mathcal{C}$  and  $F : \mathcal{C} \rightarrow \mathcal{X}$  be two functors.*

*$F \dashv U \Leftrightarrow$  there exists a natural transformation  $\varepsilon : F \circ U \rightarrow \text{Id}_{\mathcal{X}}$  such that  $\forall X \in \text{Ob}_{\mathcal{X}}, \forall C \in \text{Ob}_{\mathcal{C}}$  and  $\forall x : F(C) \rightarrow X$ , there exists a unique arrow  $c : C \rightarrow U(X)$  such that:  $\varepsilon_X \circ F(c) = x$ .*

*Proof. [Proof of  $\Rightarrow$ ]*

If  $F \dashv U$ , then let  $(F, U, \beta)$  be the adjunction. We have  $\beta_{C,X}^{-1} : \text{Hom}_{\mathcal{X}}(-, U(-)) \rightarrow \text{Hom}_{\mathcal{X}}(F(-), -)$ .

Define  $\varepsilon = (\varepsilon_X)_{X \in \text{Ob}_{\mathcal{X}}}$  to be:

$$\varepsilon_X = \beta_{U(X),X}^{-1}(\text{id}_{U(X)})$$

A diagram chasing very similar to the one in the proof of Lemma 4.8 shows that  $\varepsilon$  is a natural transformation.

Let  $X \in \text{Ob}_{\mathcal{X}}$ , let  $C \in \text{Ob}_{\mathcal{C}}$  and let  $x : F(C) \rightarrow X$ . The existence and unicity of the  $c : C \rightarrow U(X)$  such that  $\varepsilon_X \circ F(c) = x$  comes from the bijectivity of  $\beta_{C,X}^{-1}$  as the equation is also:  $\beta_{C,X}^{-1}(c) = x$ . Of course, that  $c$  is  $c = \beta_{C,X}(x) = U(x) \circ \eta_C$ .

**[Proof of  $\Leftarrow$ ]**

Define:

$$\gamma_{C,X} : \begin{cases} \text{Hom}_{\mathcal{C}}(C, U(X)) & \longrightarrow & \text{Hom}_{\mathcal{X}}(F(C), X) \\ c & \longmapsto & \varepsilon_X \circ F(c) \end{cases}$$

The definition of  $\varepsilon$  states that each  $\gamma_{C,X}$  is an isomorphism. Now we have to prove that  $\gamma = (\gamma_{C,X})_{C \in \text{Ob}_{\mathcal{C}}, X \in \text{Ob}_{\mathcal{X}}}$  is natural (but contravariant) in  $C$  and (covariant) in  $X$ .

$$\begin{array}{ccccc} C & & \text{Hom}_{\mathcal{X}}(C', U(X)) & \xrightarrow{\gamma_{C',X}} & \text{Hom}_{\mathcal{X}}(F(C'), X) \\ \downarrow c & \rightsquigarrow & \downarrow & & \downarrow \text{Hom}_{\mathcal{X}}(F(c), X) \\ C' & & \text{Hom}_{\mathcal{X}}(C, U(X)) & \xrightarrow{\gamma_{C,X}} & \text{Hom}_{\mathcal{X}}(F(C), X) \end{array}$$

For  $f \in \text{Hom}_{\mathcal{X}}(C', U(X))$ , we have:

$$\begin{aligned}\text{Hom}_{\mathcal{X}}(F(c), X) \circ \gamma_{C',X}(f) &= \varepsilon_X \circ F(f) \circ F(c) \\ \gamma_{C,X} \circ \text{Hom}_{\mathcal{X}}(c, U(X))(f) &= \varepsilon_X \circ F(f \circ c)\end{aligned}$$

So  $\gamma$  is natural in  $C$ . As for the naturality in  $X$ :

$$\begin{array}{ccc}
 X & & \text{Hom}_{\mathcal{X}}(C, U(X)) \xrightarrow{\gamma_{C,X}} \text{Hom}_{\mathcal{X}}(F(C), X) \\
 \downarrow x & \rightsquigarrow & \downarrow \text{Hom}_{\mathcal{X}}(C, U(x)) \quad \quad \quad ? \quad \quad \quad \downarrow \text{Hom}_{\mathcal{X}}(F(C), x) \\
 X' & & \text{Hom}_{\mathcal{X}}(C, U(X')) \xrightarrow{\gamma_{C,X'}} \text{Hom}_{\mathcal{X}}(F(C), X')
 \end{array}$$

Let  $f \in \text{Hom}_{\mathcal{X}}(C, U(X))$ :

$$\begin{aligned}
 \gamma_{C,X} \circ \text{Hom}_{\mathcal{X}}(F(C), x)(f) &= x \circ \varepsilon_X \circ F(f) \\
 \text{Hom}_{\mathcal{X}}(C, U(x)) \circ \gamma_{C,X'}(f) &= \varepsilon_{X'} \circ F(U(x)) \circ F(f)
 \end{aligned}$$

Don't forget that  $\varepsilon$  is a natural transformation  $F \circ U \rightarrow \text{Id}_{\mathcal{X}}$ . Thence, we have the following commutative diagram:

$$\begin{array}{ccc}
 F \circ U(X) & \xrightarrow{\varepsilon_X} & X \\
 \downarrow F \circ U(x) & \checkmark & \downarrow x \\
 F \circ U(X') & \xrightarrow{\varepsilon_{X'}} & X
 \end{array}$$

which gives:

$$x \circ \varepsilon_X = \varepsilon_{X'} \circ F(U(x))$$

and finally:

$$\begin{aligned}
 x \circ \varepsilon_X \circ F(f) &= \varepsilon_{X'} \circ F(U(x)) \circ F(f) \\
 \gamma_{C,X} \circ \text{Hom}_{\mathcal{X}}(F(C), x)(f) &= \text{Hom}_{\mathcal{X}}(C, U(x)) \circ \gamma_{C,X'}(f)
 \end{aligned}$$

Thus,  $\gamma$  is a natural transformation in both  $X$  and  $C$ ; each component is an isomorphism, so  $\gamma$  is a natural isomorphism  $\text{Hom}_{\mathcal{C}}(-, U(-)) \rightarrow \text{Hom}_{\mathcal{X}}(F(-), -)$ ; so  $\gamma^{-1}$  is a natural isomorphism  $\text{Hom}_{\mathcal{X}}(F(-), -) \rightarrow \text{Hom}_{\mathcal{C}}(-, U(-))$ .

For all  $C \in \text{Ob}_{\mathcal{C}}$ , define  $\eta_C = \gamma_{C, F(C)}^{-1}(\text{id}_{F(C)})$ . By Proposition 3.14, item 2, we know that  $(F(C), \eta_C)$  is a universal arrow from  $C$  to  $U$ , which makes  $F$  the left adjoint of  $U$  by Lemma 4.8.  $\square$

**Definition 4.12** (Counit of an adjunction). Let  $(F, U, \beta)$  be an adjunction.

The *counit of the adjunction*  $(F, U, \beta)$  is the natural transformation  $\varepsilon : F \circ U \rightarrow \text{Id}_{\mathcal{X}}$  such that  $\forall X \in \text{Ob}_{\mathcal{X}}, \varepsilon_X = \beta_{U(X), X}^{-1}(\text{id}_{U(X)})$ .

The notion of adjunction appears everywhere in mathematics. As this notion is very important, we need to give many examples.

*Example 4.13* (Identity). Let  $\mathcal{C}$  be a category. Then the identity functor  $\text{Id}_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}$  is both left and right adjoint of itself; and  $\text{Id}_{-, -} : \text{Hom}_{\mathcal{C}}(\text{Id}_{\mathcal{C}}(-), -) \rightarrow \text{Hom}_{\mathcal{C}}(-, \text{Id}_{\mathcal{C}}(-))$  is the adjunctor. The unit and counit are:  $\eta, \varepsilon : \text{Id}_{\mathcal{C}} \rightarrow \text{Id}_{\mathcal{C}}$ .

*Example 4.14* (Isomorphisms). Let  $\mathcal{C}, \mathcal{X}$  be categories, and let  $F : \mathcal{C} \rightarrow \mathcal{X}$  be an isomorphism between those categories. Then  $F \dashv F^{-1} \dashv F$ . In fact,  $x : F(C) \rightarrow X \in \mathcal{X} \Leftrightarrow F^{-1}(x) : C \rightarrow F^{-1}(X)$  and  $c : F^{-1}(X) \rightarrow C \Leftrightarrow F(c) : X \rightarrow F(C)$ .

For the adjunction  $(F, F^{-1}, \beta)$ , the adjunctor  $\beta$  has components:

$$\beta_{C,X} : \begin{cases} \text{Hom}_{\mathcal{X}}(F(C), X) & \longrightarrow & \text{Hom}_{\mathcal{C}}(C, F^{-1}(X)) \\ x & \longmapsto & F^{-1}(x) \end{cases}$$

while for the second adjunction  $(F^{-1}, F, \gamma)$ , the adjunctor  $\gamma$  has components:

$$\gamma_{C,X} : \begin{cases} \text{Hom}_{\mathcal{X}}(F^{-1}(X), C) & \longrightarrow & \text{Hom}_{\mathcal{C}}(X, F(C)) \\ c & \longmapsto & F(c) \end{cases} = \beta_{C,X}^{-1}$$

The units and counits are the identity natural transformations.

*Example 4.15* (Increasing linear function). Let  $\mathcal{R} = (\mathbb{R}, \leq)$  be the category of the totally ordered set  $\mathbb{R}$ , equipped with the usual order on real numbers.

**Objects:** An object in  $\mathcal{R}$  is a real number  $x \in \mathbb{R}$

**Morphisms:** There is an arrow  $x_0 \rightarrow x_1$  if and only if  $x_0 \leq x_1$

**Identities:** An identity morphism is an arrow  $x \rightarrow x$

**Composition:** If  $x_0 \rightarrow x_1$  and  $x_1 \rightarrow x_2$  are two arrows, then there is one arrow  $x_0 \rightarrow x_2$

Note that there is only one arrow between two objects (real numbers)  $x_0, x_1$ ; if  $x_0 < x_1$  then  $\text{Hom}_{\mathbb{R}}(x_0, x_1)$  contains only one arrow, while  $\text{Hom}_{\mathbb{R}}(x_1, x_0)$  is empty. Similarly, there is only one arrow  $x \rightarrow x$ , and it is the identity. Finally, the composition law on arrows comes from the transitivity of the order relation  $\leq$ .

Let  $F : \mathcal{R} \rightarrow \mathcal{R}$  be the functor:  $F : x \rightarrow ax + b$  with  $a > 0$ . Let's check if  $F$  is actually a functor.

If  $x_0 \rightarrow x_1$ , then  $x_0 \leq x_1$ , which gives  $ax_0 + b \leq ax_1 + b$ , thus  $F(x_0) \rightarrow F(x_1)$ . If  $x \rightarrow x$  then  $F(x) \rightarrow F(x)$ . Finally, if  $x_0 \rightarrow x_1$  and  $x_1 \rightarrow x_2$  then  $F(x_0) \rightarrow F(x_1)$  and  $F(x_1) \rightarrow F(x_2)$  and  $F(x_0) \rightarrow F(x_2)$  (by transitivity of  $\leq$ ).

Now suppose you have  $x_0, x_1 \in \mathbb{R}$  such that:

$$\begin{aligned} F(x_0) \leq x_1 &\Leftrightarrow ax_0 + b \leq x_1 \\ &\Leftrightarrow x_0 \leq \frac{1}{a}x_1 - \frac{b}{a} \end{aligned}$$

Define  $U : \begin{cases} \mathcal{R} & \longrightarrow & \mathcal{R} \\ x & \longmapsto & \frac{1}{a}x_1 - \frac{b}{a} \end{cases}$ . Then  $F$  is left adjoint to  $U$ . The adjunctor  $\beta$  transforms arrows  $F(x_0) \rightarrow x_1$  to arrows  $x_0 \rightarrow U(x_1)$ . The unit  $\eta$  and counit  $\varepsilon$  are the identity natural transformations  $\text{Id}_{\mathcal{R}} \rightarrow \text{Id}_{\mathcal{R}}$ .

*Example 4.16* (Decreasing linear function). We can build a similar example of adjunctor using  $\mathcal{R} = (\mathbb{R}, \leq)$  and its opposite category  $\mathcal{R}^{\text{op}} = (\mathbb{R}, \geq)$ .

Let  $\mathcal{R}^{\text{op}} = (\mathbb{R}, \geq)$  be the category of the totally ordered set  $\mathbb{R}$ , equipped with the usual order on real numbers.

**Objects:** An object in  $\mathcal{R}^{\text{op}}$  is a real number  $x \in \mathbb{R}$

**Morphisms:** There is an arrow  $x_0 \rightarrow x_1$  if and only if  $x_0 \geq x_1$

**Identities:** An identity morphism is an arrow  $x \rightarrow x$

**Composition:** If  $x_0 \rightarrow x_1$  and  $x_1 \rightarrow x_2$  are two arrows, then there is one arrow  $x_0 \rightarrow x_2$

Note that  $(\mathbb{R}, \geq)$  is actually the opposite category of  $(\mathbb{R}, \leq)$ .

Let  $F : \mathcal{R}^{\text{op}} \rightarrow \mathcal{R}$  be the functor:  $F : x \rightarrow ax + b$  with  $a < 0$ . Let  $x_0, x_1 \in \mathbb{R}$  such that:

$$\begin{aligned} F(x_0) \geq x_1 &\Leftrightarrow ax_0 + b \geq x_1 \\ &\Leftrightarrow x_0 \leq \frac{1}{a}x_1 - \frac{b}{a} \\ &\Leftrightarrow x_0 \leq U(x_1) \end{aligned}$$

We can define the same  $U$  as in the previous example; and  $F$  is again left adjoint to  $U$ . The adjunction  $\beta$  transforms arrows  $F(x_0) \rightarrow x_1$  in  $\mathcal{R}^{\text{op}}$  to arrows  $x_0 \rightarrow U(x_1)$  in  $\mathcal{R}$ . The unit and counit are the identity natural transformations as  $U = F^{-1}$ .

*Example 4.17* (Image and inverse image of a function). Let  $f : X \rightarrow Y$  be a function between two sets  $X$  and  $Y$ . The two categories will be the partially ordered sets  $\mathcal{X} = (\mathcal{P}(X), \subseteq)$  and  $\mathcal{Y} = (\mathcal{P}(Y), \subseteq)$  equipped with the usual inclusion of sets.

Define the three functors:

$$\begin{aligned} F : \left\{ \begin{array}{l} \mathcal{X} \longrightarrow \mathcal{Y} \\ A \longmapsto f(A) = \{f(a) \mid a \in A\} \end{array} \right. \\ G : \left\{ \begin{array}{l} \mathcal{Y} \longrightarrow \mathcal{X} \\ B \longmapsto f^{-1}(B) = \{b \in X \mid f(b) \in B\} \end{array} \right. \\ F^* : \left\{ \begin{array}{l} \mathcal{X} \longrightarrow \mathcal{Y} \\ A \longmapsto \{y \in Y \mid f^{-1}(\{y\}) \subseteq A\} \end{array} \right. \end{aligned}$$

The functor  $F$  gives the image of a subset of  $X$ ,  $G$  gives the inverse image of a subset of  $Y$  and  $F^*$  gives the subset of inverse images of singletons of elements of  $Y$ . We let the reader check that those three functions are actually functors.

Suppose we have  $F(A) \subseteq B$ . For all  $a \in A$ ,  $f(a) \in F(A) \subseteq B$ , so for all  $a \in A$ ,  $a \in f^{-1}(B) = G(B)$ , which gives  $A \subseteq G(B)$ . Conversely, suppose we have  $A \subseteq G(B)$ . For all  $a \in A$ ,  $a \in G(B) = \{b \in X \mid f(b) \in B\}$ , so for all  $a \in A$ ,  $f(a) \in B$ , which gives  $F(A) \subseteq B$ .

We have  $F \dashv G$ :

$$F(A) \subseteq B \Leftrightarrow A \subseteq G(B)$$

The adjunction  $\beta$  transforms arrows  $F(A) \rightarrow B$  to arrows  $A \rightarrow G(B)$ . Note that  $A \subseteq G \circ F(A)$  but there is in general no reason why  $A$  should be equal to  $G \circ F(A)$  (except if  $f$  is injective). Consequently, the unit of the adjunction is:

$$\eta = \left( \eta_A : \left\{ \begin{array}{l} A \longrightarrow G \circ F(A) \\ a \longmapsto a \end{array} \right\} \right)_{A \subseteq X}$$

Similarly, note that  $F \circ G(B) \subseteq B$  but there is in general no reason why  $F \circ G(B)$  should be equal to  $B$  (except if  $f$  is surjective), so the counit is:

$$\varepsilon = \left( \varepsilon_B : \left\{ \begin{array}{l} F \circ G(B) \longrightarrow B \\ b \longmapsto b \end{array} \right\} \right)_{B \subseteq Y}$$

Besides, we also have  $G \dashv F^*$ . In fact, suppose we have  $G(A) \subseteq B$ . Then,  $\forall x \in A$ ,  $f^{-1}(\{x\}) \subseteq G(A) \subseteq B$ , so  $\forall x \in A$ ,  $x \in F^*(B)$ , hence  $A \subseteq F^*(B)$ . Conversely, if  $A \subseteq F^*(B)$  then  $\forall x \in G(A)$ , we have:

$$\begin{aligned}
 x \in G(A) &\Rightarrow f(x) \in A \subset F^*(B) \\
 &\Rightarrow f(x) \in \{y \in Y \mid f^{-1}(\{y\}) \subseteq B\} \\
 &\Rightarrow f^{-1}(\{f(x)\}) \subseteq B \\
 &\Rightarrow x \in B
 \end{aligned}$$

So we have  $G(A) \subseteq B$ .

The adjunctor  $\beta^*$  transforms arrows  $G(A) \rightarrow B$  to arrows  $A \rightarrow F^*(B)$ .

Before computing the unit and counit, note that, for  $B \subset Y$ :

$$\begin{aligned}
 F^*(G(B)) &= F^*(f^{-1}(B)) \\
 &= \{y \in Y \mid f^{-1}(\{y\}) \subseteq f^{-1}(B)\} \\
 &= \{y \in Y \mid \forall x \in f^{-1}(\{y\}), f(x) \in B\} \\
 &= \{y \in Y \mid y \in B \wedge \exists x \in X, y = f(x)\} \\
 &= B \cap f(X) \subseteq B
 \end{aligned}$$

The interpretation is the following:  $F^*(G(B))$  is the biggest subset of  $B$  that contains only images by  $f$ . Again,  $F^*(G(B))$  has no reason to be equal to  $B$ , except if  $f$  is surjective.

Also, for  $A \subset X$ :

$$\begin{aligned}
 G(F^*(A)) &= f^{-1}(F^*(A)) \\
 &= \{x \in X \mid f(x) \in F^*(A)\} \\
 &= \{x \in X \mid f(x) \in \{y \in Y \mid f^{-1}(\{y\}) \subseteq A\}\} \\
 &= \{x \in X \mid f^{-1}(\{f(x)\}) \subseteq A\} \\
 &= \bigcup \{C \subseteq X \mid f(C) \subseteq A\} \\
 &\supseteq A
 \end{aligned}$$

The interpretation of  $G(F^*(A))$  is as follows:  $G(F^*(A))$  is the biggest subset of  $X$  that gives  $f(A)$ . Again,  $G(F^*(A))$  has no reason to be equal to  $A$ , except if  $f$  is injective.

In this case, the unit and counit are not easy to write. In fact, we will need to create an equivalence relation over  $X$ , for example  $x = y \text{ mod } f \Leftrightarrow f(x) = f(y)$ . Then we will need a section function that sends an equivalence class to its representative. Such a choice of section function should be chosen to be compatible with what we want from the unit and counit.

*Example 4.18* (Galois connections). The previous three examples are special cases of monotone Galois connections. Every Galois connection between two posets is an adjunction.

Further examples of adjunctions will appear in the rest of the text. Our goal is to introduce topoi whose definition is easier with adjunctions.

**Definition 4.19** (Whiskering). Let  $F, F' : \mathcal{C} \rightarrow \mathcal{C}'$  and  $G, G' : \mathcal{C}' \rightarrow \mathcal{D}$  be functors, and let  $\alpha : F \rightarrow F'$  and  $\beta : G \rightarrow G'$  be natural transformations.

1. The whiskering of  $G$  and  $\alpha$ , denoted by  $G \circ \alpha$ , is the natural transformation:  $G \circ \alpha : G \circ F \rightarrow G \circ F'$  with components  $(G(\alpha_C) : G \circ F(C) \rightarrow G \circ F'(C))_{C \in \mathcal{C}}$ .
2. The whiskering of  $\beta$  and  $F$ , denoted by  $\beta \circ F$ , is the natural transformation:  $\beta \circ F : G \circ F \rightarrow G' \circ F$  with components  $(\alpha_{G(C)} : G \circ F(C) \rightarrow G' \circ F(C))_{C \in \mathcal{C}}$ .



It is easy to check that:

**Proposition 4.20.** *Let  $F, F' : \mathcal{C} \rightarrow \mathcal{C}'$  and  $G, G' : \mathcal{C}' \rightarrow \mathcal{D}$  be functors, and let  $\alpha : F \rightarrow F'$  and  $\beta : G \rightarrow G'$  be natural transformations.*

$$\begin{aligned} (H \circ G) \circ \beta &= H \circ (G \circ \beta) & \text{Id}_{\mathcal{C}'} \circ \beta &= \beta \\ \alpha \circ (F \circ H) &= (\alpha \circ F) \circ H & \alpha \circ \text{Id}_{\mathcal{C}} &= \alpha \end{aligned}$$

*Remark 4.21.* Proposition 4.20 simply states that whiskering (on the left or on the right) can be seen as a (left or right) monoid action of the monoid of functors (with composition) over the class of natural transformations.

**Proposition 4.22.** *Let  $F, F' : \mathcal{C} \rightarrow \mathcal{C}'$  and  $G, G' : \mathcal{C}' \rightarrow \mathcal{D}$  be functors, and let  $\alpha : F \rightarrow F'$  and  $\beta : G \rightarrow G'$  be natural transformations.*

*Then the following diagram commutes:*

$$\begin{array}{ccc} G \circ F & \xrightarrow{\beta \circ F} & G' \circ F \\ G \circ \alpha \downarrow & \checkmark & \downarrow G' \circ \alpha \\ G \circ F' & \xrightarrow{\beta \circ F'} & G' \circ F' \end{array}$$

*Proof.* For  $C \in \mathcal{C}$ , consider the following "implemented" diagram:

$$\begin{array}{ccc} G(F(C)) & \xrightarrow{\beta_{F(C)}} & G'(F(C)) \\ G(\alpha_C) \downarrow & & \downarrow G'(\alpha_C) \\ G(F'(C)) & \xrightarrow{\beta_{F'(C)}} & G'(F'(C)) \end{array}$$

This diagram commutes because  $\beta$  is a natural transformation  $G \rightarrow G'$  and  $\alpha_C$  is an arrow  $F(C) \rightarrow F'(C)$ . □

**Proposition 4.23** (Triangle identities). *Let  $\mathcal{X}, \mathcal{C}$  be two categories. Let  $U : \mathcal{X} \rightarrow \mathcal{C}$  and  $F : \mathcal{C} \rightarrow \mathcal{X}$  be two functors. Let  $\eta : \text{Id}_{\mathcal{C}} \rightarrow U \circ F$  and  $\varepsilon : F \circ U \rightarrow \text{Id}_{\mathcal{X}}$  be natural transformations.*

*The tuple  $(F, U, \eta, \varepsilon)$  is an adjunction iff the following triangles commute:*

$$\begin{array}{ccc} L & \xrightarrow{L \circ \eta} & L \circ R \circ L \\ & \searrow \text{id}_L & \downarrow \varepsilon \circ L \\ & & L \end{array} \qquad \begin{array}{ccc} R & \xrightarrow{\eta \circ R} & R \circ L \circ R \\ & \searrow \text{id}_R & \downarrow R \circ \varepsilon \\ & & R \end{array}$$

*Proof.* **[Proof of  $\Rightarrow$ ]** Suppose  $(F, U, \eta, \varepsilon)$  is an adjunction. According to Lemma 4.10, we can compute the adjunction from the unit and counit:

$$\begin{aligned} \beta_{C,X} : \left\{ \begin{array}{ll} \text{Hom}_{\mathcal{X}}(F(C), X) & \longrightarrow \text{Hom}_{\mathcal{C}}(C, U(X)) \\ x & \longmapsto U(x) \circ \eta_C \end{array} \right. \\ \beta_{C,X}^{-1} : \left\{ \begin{array}{ll} \text{Hom}_{\mathcal{C}}(C, U(X)) & \longrightarrow \text{Hom}_{\mathcal{X}}(F(C), X) \\ c & \longmapsto \varepsilon_X \circ F(c) \end{array} \right. \end{aligned}$$

Also, from Definition 4.9 and 4.12, we deduce the triangle identities:

$$\begin{aligned} \eta_C = \beta_{C,F(C)} (\text{id}_{F(C)}) &\Leftrightarrow \beta_{C,F(C)}^{-1} (\eta_C) = \text{id}_{F(C)} \\ &\Leftrightarrow \varepsilon_{F(C)} \circ F(\eta_C) = \text{id}_{F(C)} \\ &\Leftrightarrow (\varepsilon \circ F)_C \circ (F \circ \eta)_C = \text{id}_{F(C)} \end{aligned}$$

$$\begin{aligned} \varepsilon_X = \beta_{U(X),X}^{-1} (\text{id}_{U(X)}) &\Leftrightarrow \beta_{U(X),X} (\varepsilon_X) = \text{id}_{U(X)} \\ &\Leftrightarrow U(\varepsilon_X) \circ \eta_{U(X)} = \text{id}_{U(X)} \\ &\Leftrightarrow (U \circ \varepsilon)_X \circ (\eta \circ U)_X = \text{id}_{U(X)} \end{aligned}$$

**[Proof of  $\Leftarrow$ ]** If we have the triangle identities, we can define the adjunctor  $\beta$  from  $\eta$  and its inverse  $\gamma$  (in place of  $\beta^{-1}$ ) from  $\varepsilon$ , and we prove that  $\gamma$  is the inverse of  $\beta$ :

$$\begin{aligned} \beta_{C,X} : \begin{cases} \text{Hom}_{\mathcal{X}}(F(C), X) & \longrightarrow & \text{Hom}_{\mathcal{C}}(C, U(X)) \\ x & \longmapsto & U(x) \circ \eta_C \end{cases} \\ \gamma_{C,X} : \begin{cases} \text{Hom}_{\mathcal{C}}(C, U(X)) & \longrightarrow & \text{Hom}_{\mathcal{X}}(F(C), X) \\ c & \longmapsto & \varepsilon_X \circ F(c) \end{cases} \end{aligned}$$

And then, for  $x : F(C) \rightarrow X$ , we have:

$$\gamma_{C,X} \circ \beta_{C,X}(x) = \varepsilon_X \circ F(U(x) \circ \eta_C)$$

This is equal to  $x$  due to the following diagram:

$$\begin{array}{ccccc} F(C) & \xrightarrow{F(\eta_C)} & F \circ U \circ F(C) & \xrightarrow{F \circ U(x)} & F \circ U(X) \\ & \searrow \text{id}_{F(C)} & \downarrow \varepsilon_{F(C)} & & \downarrow \varepsilon_X \\ & & F(C) & \xrightarrow{x} & X \end{array}$$

The left-hand triangle commutes because of the triangle identities; the right-hand square commutes because it represents the naturality of  $\varepsilon : F \circ U \rightarrow \text{Id}_{\mathcal{X}}$ .

The converse equality is similarly proven:

$$\beta_{C,X} \circ \gamma_{C,X}(c) = U(\varepsilon_X \circ F(c)) \circ \eta_C$$

which is equal to  $c$  according to the following diagram:

$$\begin{array}{ccccc} C & \xrightarrow{c} & U(X) & & \\ \eta_C \downarrow & & \eta_{U(X)} \downarrow & \searrow \text{id}_{U(X)} & \\ U \circ F(C) & \xrightarrow{U \circ F(c)} & U \circ F \circ U(X) & \xrightarrow{U(\varepsilon_X)} & U(X) \end{array}$$

The left-hand square commutes because  $\eta$  is a natural transformation, and the right-square commutes because of the triangle identities.

Thus,  $\beta$  and  $\gamma$  are both natural isomorphisms (the proof of naturality is not interesting and is left to the reader) and are inverses of each other.  $\square$

We finally give a third definition of adjunction:

**Definition 4.24** (Adjoint - triangle identities). Let  $\mathcal{X}, \mathcal{C}$  be two categories. Let  $U : \mathcal{X} \rightarrow \mathcal{C}$  and  $F : \mathcal{C} \rightarrow \mathcal{X}$  be two functors. Let  $\eta : \text{Id}_{\mathcal{C}} \rightarrow U \circ F$  and  $\varepsilon : F \circ U \rightarrow \text{Id}_{\mathcal{X}}$  be natural transformations.

The tuple  $(F, U, \eta, \varepsilon)$  is called an adjunction if the following triangles commute:

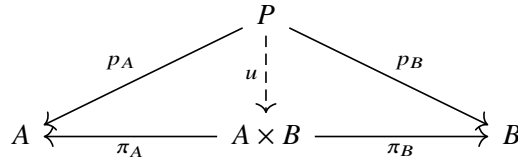
$$\begin{array}{ccc}
 L & \xrightarrow{L \circ \eta} & L \circ R \circ L \\
 & \searrow \text{id}_L & \downarrow \varepsilon \circ L \\
 & & L
 \end{array}
 \qquad
 \begin{array}{ccc}
 R & \xrightarrow{\eta \circ R} & R \circ L \circ R \\
 & \searrow \text{id}_R & \downarrow R \circ \varepsilon \\
 & & R
 \end{array}$$

## 5. Objects with some universality in them

A word about the UMP.

**Definition 5.1** (Product [1]). Let  $\mathcal{C}$  be a category and let  $A$  and  $B$  be objects in  $\mathcal{C}$ .

The *product of  $A$  and  $B$*  is 3-tuple  $(A \times B, \pi_A, \pi_B)$  where  $A \times B$  is an object in  $\mathcal{C}$ , and  $\pi_A : A \times B \rightarrow A$  and  $\pi_B : A \times B \rightarrow B$  are two arrows, such that, for all object  $P$  with two arrows  $p_A : P \rightarrow A$  and  $p_B : P \rightarrow B$ , there exists a unique arrow  $u : P \rightarrow A \times B$  such that  $\pi_A \circ u = p_A$  and  $\pi_B \circ u = p_B$ , that is, such that the following diagram commutes:



We call  $\pi_A, \pi_B$  projections, and we denote  $u$  by  $u = (p_A, p_B)$ .

The definition of the product can be interpreted as follows. Given three objects  $A, B, C$ , the "shorter" path from  $C$  to  $A$  and  $B$  at the same time, always passes through  $A \times B$ . In a sense,  $A \times B$  is an "optimised" link to  $A$  and  $B$ .

*Example 5.2.* In **Sets**, the product is the usual cartesian product of sets, and the projections are the usual projections  $(a, b) \mapsto a$  and  $(a, b) \mapsto b$ .

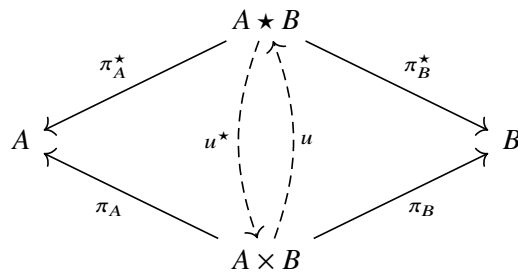
*Example 5.3.* In a poset category  $(P, \leq)$ , the product  $p \times q$  of two elements  $p$  and  $q$  verifies  $p \times q \leq p$  and  $p \times q \leq q$ , and for all  $r \leq p, r \leq q$ , we have  $r \leq p \times q$ . In fact,  $p \times q = \inf(p, q)$ .

**Proposition 5.4.** Let  $\mathcal{C}$  be a category and let  $A$  and  $B$  be objects in  $\mathcal{C}$ .

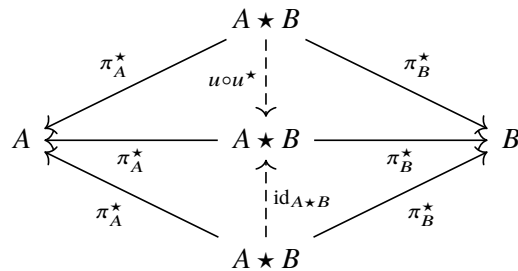
The product  $A \times B$  is unique up to isomorphism.

*Proof.* Let  $(A \times B, \pi_A, \pi_B)$  and  $(A \star B, \pi_A^*, \pi_B^*)$  be two products of  $A$  and  $B$ .

By definition of both products, there exists unique  $u, u^*$  such that the following diagram commutes:



We then have:  $\pi_A^* \circ u \circ u^* = \pi_A^* = \pi_A^* \circ \text{id}_{A \star B}$  and  $\pi_B^* \circ u \circ u^* = \pi_B^* = \pi_B^* \circ \text{id}_{A \star B}$ . The following diagram commutes:



By uniqueness condition, we have  $u \circ u^* = \text{id}_{A \star B}$ . A similar reasoning yields  $u^* \circ u = \text{id}_{A \times B}$ ; then  $u$  and  $u^*$  are isomorphisms.  $\square$

Consequently, it is natural to mention "the" product of two objects, instead of "a" product.

The definition of a product can be generalized from  $n = 2$  to any  $n \in \mathbb{N}$ . When  $n = 1$ , the product of  $C_1$  is just  $C_1$  and the projection  $\pi_1 : C_1 \rightarrow C_1$  is the identity. When  $n = 0$ , the empty product is an object  $*$  such that for all objects  $P$ , there exists a unique arrow  $u : P \rightarrow *$  (we will see later that this is the terminal object). Note that depending on the category  $\mathcal{C}$ , not all pairs  $(C_1, C_2)$  may have a product.

**Definition 5.5** (Category with finite products). The category  $\mathcal{C}$  is said to have *finite products* if  $\forall n \in \mathbb{N}, \forall (C_i)_{i \in n}$ , the product  $\prod_{i \in n} C_i$  exists.

The product of categories can also be defined. However, it is not always a product in the category of categories (if such a thing exists). It is still useful for further definitions.

**Definition 5.6** (Product of categories). Let  $\mathcal{C}$  and  $\mathcal{D}$  be two categories. We define the *category of pairs*, or the *product category*  $\mathcal{C} \times \mathcal{D}$  by:

**Objects:** An object in  $\mathcal{C} \times \mathcal{D}$  is a pair  $(C, D)$  where  $C \in \text{Ob}_{\mathcal{C}}$  and  $D \in \text{Ob}_{\mathcal{D}}$

**Morphisms:** A morphism in  $\mathcal{C} \times \mathcal{D}$  is a pair  $(c, d) : (C, D) \rightarrow (C', D')$  where  $c : C \rightarrow C' \in \text{Mor}_{\mathcal{C}}$  and  $d : D \rightarrow D' \in \text{Mor}_{\mathcal{D}}$

**Identities:** An identity morphism is a pair  $(\text{id}_C, \text{id}_D)$

**Composition:** The composition law for morphisms is pairwise:  $(c, d) \circ (c', d') = (c \circ c', d \circ d')$  (using the composition laws of  $\mathcal{C}$  and  $\mathcal{D}$ )

*Remark 5.7.* We can define the category **Cat** of small categories, where the morphisms are the functors between small categories. In that case, if  $\mathcal{C}$  and  $\mathcal{D}$  are small categories, then the product  $\mathcal{C} \times \mathcal{D}$  is an actual product in this category.

Besides, any two categories give birth to a product category, however this product is not necessarily an actual product in the categorical sense.

Before checking on the dual notion of the product, let us have a look at the behaviour of the covariant and contravariant Hom-set functors in relation to the product.

**Proposition 5.8.** *Let  $\mathcal{C}$  be a category with finite products. Then there is a natural isomorphism:*

$$\text{Hom}_{\mathcal{C}}(A, B \times C) \cong \text{Hom}_{\mathcal{C}}(A, B) \times \text{Hom}_{\mathcal{C}}(A, C)$$

in  $A, B$  and  $C$ .

*Proof.* As a product, there is a unique  $u : \text{Hom}_{\mathcal{C}}(A, B \times C) \rightarrow \text{Hom}_{\mathcal{C}}(A, B) \times \text{Hom}_{\mathcal{C}}(A, C)$  such that the following diagram commutes:

$$\begin{array}{ccccc}
 & & \text{Hom}_{\mathcal{C}}(A, B \times C) & & \\
 & \swarrow \text{Hom}_{\mathcal{C}}(A, \pi_B) & \downarrow u & \searrow \text{Hom}_{\mathcal{C}}(A, \pi_C) & \\
 \text{Hom}_{\mathcal{C}}(A, B) & \xleftarrow{\pi_{\text{Hom}_{\mathcal{C}}(A, B)}} & \text{Hom}_{\mathcal{C}}(A, B) \times \text{Hom}_{\mathcal{C}}(A, C) & \xrightarrow{\pi_{\text{Hom}_{\mathcal{C}}(A, C)}} & \text{Hom}_{\mathcal{C}}(A, C)
 \end{array}$$

Conversely, given  $(f_B, f_C) \in \text{Hom}_{\mathcal{C}}(A, B) \times \text{Hom}_{\mathcal{C}}(A, C)$ , by definition of the product  $B \times C$ , there is a unique  $v : A \rightarrow B \times C$  such that the following diagram commutes:

$$\begin{array}{ccccc}
 & & A & & \\
 & \swarrow \pi_B \circ f & \downarrow v & \searrow \pi_C \circ f & \\
 B & \xleftarrow{\pi_B} & B \times C & \xrightarrow{\pi_C} & C
 \end{array}$$

Besides,  $v = \langle f_B, f_C \rangle : A \rightarrow B \times C$ , so we define:

$$\alpha_{A,B,C} : \begin{cases} \text{Hom}_{\mathcal{C}}(A, B) \times \text{Hom}_{\mathcal{C}}(A, C) & \longrightarrow & \text{Hom}_{\mathcal{C}}(A, B \times C) \\ (f_B, f_C) & \longmapsto & \langle f_B, f_C \rangle \end{cases}$$

By a reasoning similar to the one in the proof of Proposition 5.4, we have  $\alpha_{A,B,C} = u^{-1}$ , so  $\alpha_{A,B,C}$  is a bijection.

The naturality is easy to check; let  $a : A \rightarrow A'$ :

$$\begin{array}{ccc} \begin{array}{c} A \\ \downarrow a \\ A' \end{array} & \rightsquigarrow & \begin{array}{ccc} \text{Hom}_{\mathcal{C}}(A', B) \times \text{Hom}_{\mathcal{C}}(A', C) & \xrightarrow{\alpha_{A',B,C}} & \text{Hom}_{\mathcal{C}}(A', B \times C) \\ \downarrow \text{Hom}_{\mathcal{C}}(a,B) \times \text{Hom}_{\mathcal{C}}(a,C) & & \downarrow \text{Hom}_{\mathcal{C}}(a, B \times C) \\ \text{Hom}_{\mathcal{C}}(A, B) \times \text{Hom}_{\mathcal{C}}(A, C) & \xrightarrow{\alpha_{A,B,C}} & \text{Hom}_{\mathcal{C}}(A, B \times C) \end{array} \end{array}$$

We check that the diagram commutes:

$$\begin{aligned} \alpha_{A,B,C} \circ \text{Hom}_{\mathcal{C}}(a,B) \times \text{Hom}_{\mathcal{C}}(a,C)(f_B, f_C) &= \alpha_{A,B,C}(f_B \circ a, f_C \circ a) \\ &= \langle f_B \circ a, f_C \circ a \rangle \\ \text{Hom}_{\mathcal{C}}(a, B \times C) \circ \alpha_{A',B,C}(f_B, f_C) &= \text{Hom}_{\mathcal{C}}(a, B \times C) \langle f_B, f_C \rangle \\ &= \langle f_B, f_C \rangle \circ a \\ &= \langle f_B \circ a, f_C \circ a \rangle \end{aligned}$$

The other naturalities are as easy to check. □

This property is not specific to the Hom-set functor.

**Definition 5.9** (Preserving products). The functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is said to *preserve products* when, for all  $A, B \in \text{Ob}_{\mathcal{C}}$ , if  $A \times B$  exists, then  $F(A \times B) \cong F(A) \times F(B)$ .

**Proposition 5.10.** *The covariant Hom-set functor  $\text{Hom}_{\mathcal{C}}(A, -)$  preserves products.*

A similar question could be asked about the contravariant Hom-set functor: is there a natural isomorphism  $\text{Hom}_{\mathcal{C}}(A \times B, C) \rightarrow \text{Hom}_{\mathcal{C}}(A, C) \times \text{Hom}_{\mathcal{C}}(B, C)$ ? In fact, the answer is no. The right isomorphism is this one:

**Proposition 5.11.** *Let  $\mathcal{C}$  be a category with finite products. Then there is a natural isomorphism:*

$$\text{Hom}_{\mathcal{C} \times \mathcal{C}}((A, A), (B, C)) \cong \text{Hom}_{\mathcal{C}}(A, B) \times \text{Hom}_{\mathcal{C}}(A, C)$$

in  $A, B$  and  $C$ .

The proof is very similar to that of the covariant Hom-set functor.

Combining Proposition 5.8 and Proposition 5.11, we have:

**Proposition 5.12.** *Let  $\mathcal{C}$  be a category with finite products. Then there is a natural isomorphism:*

$$\text{Hom}_{\mathcal{C}}((A, A), (B, C)) \cong \text{Hom}_{\mathcal{C}}(A, B \times C)$$

in  $A, B$  and  $C$ .

In other words, the diagonal functor:

$$\Delta_2 : \begin{cases} \mathcal{C} & \longrightarrow & \mathcal{C} \times \mathcal{C} \\ C & \longmapsto & (C, C) \\ c & \longmapsto & (c, c) \end{cases}$$

is right adjoint to the product functor  $- \times - : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ .

**Remark 5.13.** The unit of the adjunction  $\Delta_2 \dashv \times$  is:

$$\eta_C : C \rightarrow C \times C$$

and the counit is:

$$\varepsilon_{A,B} : (A \times B, A \times B) \rightarrow (A, B)$$

The dual notion of that of a product is the coproduct:

**Definition 5.14** (Coproduct). Let  $\mathcal{C}$  be a category and let  $A$  and  $B$  be objects in  $\mathcal{C}$ .

The *coproduct of  $A$  and  $B$*  is 3-tuple  $(A + B, i_A, i_B)$  where  $A + B$  is an object in  $\mathcal{C}$ , and  $i_A : A \rightarrow A + B$  and  $i_B : B \rightarrow A + B$  are two arrows, such that, for all object  $X$  with two arrows  $x_A : A \rightarrow X$  and  $x_B : B \rightarrow X$ , there exists a unique arrow  $u : A + B \rightarrow X$  such that the following diagram commutes:

$$\begin{array}{ccccc} & & X & & \\ & \nearrow^{x_A} & \uparrow u & \nwarrow_{x_B} & \\ A & \xrightarrow{i_A} & A + B & \xleftarrow{i_B} & B \end{array}$$

We call  $i_A, i_B$  *injections*, although they do not need to be injective.

**Example 5.15.** In **Sets**, the coproduct  $A + B$  corresponds to the disjoint union of  $A$  and  $B$ , for example defined as

$$A + B = \{(a, 0) \mid a \in A\} \cup \{(b, 1) \mid b \in B\}$$

with injections being:

$$\begin{aligned} i_A &: a \rightarrow (a, 0) \\ i_B &: b \rightarrow (b, 1) \end{aligned}$$

**Example 5.16.** In a poset category  $(P, \leq)$ , the coproduct  $p + q$  is  $p + q = \sup(p, q)$ .

It is easy to see that:

**Proposition 5.17.** Let  $\mathcal{C}$  be a category and let  $A$  and  $B$  be objects in  $\mathcal{C}$ .

$(A + B, i_A, i_B)$  is a coproduct in  $\mathcal{C}$  if and only if  $(A + B, i_A, i_B)$  is a product in  $\mathcal{C}^{op}$ .

**Corollary 5.18.** The coproduct is unique up to isomorphism.

The proof of the following is very similar to the proof of Proposition 5.8. We will just give the natural isomorphism to consider.

**Proposition 5.19.** Let  $\mathcal{C}$  be a category such that for all  $A, B$ , the coproduct  $A + B$  exists. Then, there is a natural isomorphism:

$$\text{Hom}_{\mathcal{C}}(A + B, C) \cong \text{Hom}_{\mathcal{C} \times \mathcal{C}}((A, B), (C, C))$$

In other words, the diagonal functor  $\Delta_2 : \mathcal{C} \mapsto (\mathcal{C}, \mathcal{C})$  is left adjoint to the coproduct functor  $- + - : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ .

*Proof.* If  $f : A + B \rightarrow C$  then by definition of the coproduct, there are two arrows  $i_A : A \rightarrow A + B$  and  $i_B : B \rightarrow A + B$  such that  $f = f \circ p_A + f \circ p_B$  ( $f \circ p_A$  can be seen as the restriction of  $f$  to  $A$ ). Consider the mapping:

$$\alpha_{A,B,C} : \begin{cases} \text{Hom}_{\mathcal{C}}(A + B, C) & \longrightarrow & \text{Hom}_{\mathcal{C} \times \mathcal{C}}((A, B), (C, C)) \\ f & \longmapsto & (f \circ p_A, f \circ p_B) \end{cases}$$

Then it is not hard (but quite long) to prove that  $\alpha_{A,B,C}$  defines a natural transformation  $\text{Hom}_{\mathcal{C}}(- + -, -) \rightarrow \text{Hom}_{\mathcal{C} \times \mathcal{C}}((-, -), \Delta(-))$ .

The unit is:

$$\eta_{A,B} : (A, B) \rightarrow (A + B, A + B)$$

and the counit is:

$$\varepsilon_C : C + C \rightarrow C$$

□

In summary:

**Theorem 5.20.**  $+ + \Delta_2 + \times$ .

**Definition 5.21** (Exponential [1]). Let  $\mathcal{C}$  be a category with finite products, and let  $B, C$  be objects of  $\mathcal{C}$ .

An exponential of  $B$  and  $C$  is a pair  $(C^B, \varepsilon)$  where  $C^B$  is an object in  $\mathcal{C}$  and  $\varepsilon : C^B \times B \rightarrow C$ , such that, for any arrow  $f : A \times B \rightarrow C$ , there exists a unique arrow  $f^c : A \rightarrow C^B$  such that the following diagram commutes:

$$\begin{array}{ccc} A & & A \times B \\ \downarrow f^c & & \downarrow f^c \times \text{id}_B \quad \searrow f \\ C^B & & C^B \times B \xrightarrow{\varepsilon} C \end{array}$$

The arrow  $\varepsilon$  is called *evaluation*; the arrow  $f^c$  is the (*exponential*) *transpose* of  $f$ .

*Remark 5.22.* Let's consider the category of sets  $\mathcal{C} = \mathbf{Sets}$ .

Let  $B, C$  be two sets; their exponential is  $C^B = \text{Hom}_{\mathbf{Sets}}(B, C)$  (note that this is specific to **Sets**). Let  $f : A \times B \rightarrow C$  be a function. As a function in two variables,  $f : a, b \mapsto f(a, b)$  can also be seen as a function  $f^c : a \mapsto f(a, -) : b \mapsto f(a, b)$ . The operation  $f^c : a \mapsto f(a, -)$  is a function  $A \rightarrow C^B$ , it is called *curryfication*; however, the operation  $g, b \mapsto g(b)$  is a function  $C^B, B \rightarrow C$  called *evaluation*. The exponential of two sets  $B$  and  $C$  is the pair  $(C^B, \varepsilon)$  where  $C^B = \text{Hom}_{\mathbf{Sets}}(B, C)$  and  $\varepsilon$  is the function:

$$\varepsilon : \begin{cases} C^B \times B & \longrightarrow & C \\ g, b & \longmapsto & g(b) \end{cases}$$

Thus, for all  $f : A \times B \rightarrow C$ , we have  $\varepsilon \circ (f^c \times \text{id}_B)(a, b) = \varepsilon(f(a, -), b) = f(a, b)$ . The goal of the exponential is to generalise these notions of curryfication and evaluation to other categories.

**Proposition 5.23.** Let  $\mathcal{C}$  be a category with finite products. We also suppose that  $X^Y$  exists for all objects  $X, Y \in \mathcal{C}$ .

Let  $A$  be an object of  $\mathcal{C}$ . Let  $P_A$  and  $E_A$  be the functors:

$$P_A : \begin{cases} \mathcal{C} & \longrightarrow & \mathcal{C} \\ X & \longmapsto & X \times A \\ x : X \rightarrow X' & \longmapsto & x \times \text{id}_A : X \times A \rightarrow X' \times A \end{cases}$$



$$E_A : \begin{cases} \mathcal{C} & \longrightarrow & \mathcal{C} \\ X & \longmapsto & X^A \\ x : X \rightarrow X' & \longmapsto & x^A : X^A \rightarrow X'^A \end{cases}$$

Then  $P_A \dashv E_A$ .

*Proof.* We let the reader check that  $P_A$  and  $E_A$  actually are functors.

We want to prove that there is a natural isomorphism  $\gamma$  with components:

$$\gamma_{X,Y} : \text{Hom}_{\mathcal{C}}(X, Y^A) \rightarrow \text{Hom}_{\mathcal{C}}(X \times A, Y)$$

For fixed  $Y$  and  $A$ , we consider the exponential  $(Y^A, \varepsilon_Y)$ .

Let  $f : X \times A \rightarrow Y$ ; then by definition of the exponential  $Y^A$ , there exists a unique  $f^c : X \rightarrow Y^A$  such that  $f = \varepsilon_Y \circ f^c \times \text{id}_A$ . Consequently, there is a bijection:

$$\gamma_{X,Y} : \begin{cases} \text{Hom}_{\mathcal{C}}(X, Y^A) & \longrightarrow & \text{Hom}_{\mathcal{C}}(X \times A, Y) \\ g & \longmapsto & \varepsilon_Y \circ g \times \text{id}_A \end{cases}$$

As for the naturality of  $\gamma$ , let  $x : X \rightarrow X'$ :

$$\begin{array}{ccc} \text{Hom}_{\mathcal{C}}(X', Y^A) & \xrightarrow{\gamma_{X',Y}} & \text{Hom}_{\mathcal{C}}(X' \times A, Y) \\ \text{Hom}_{\mathcal{C}}(x, Y^A) \downarrow & ? & \downarrow \text{Hom}_{\mathcal{C}}(x \times A, Y) \\ \text{Hom}_{\mathcal{C}}(X, Y^A) & \xrightarrow{\gamma_{X,Y}} & \text{Hom}_{\mathcal{C}}(X \times A, Y) \end{array}$$

For  $f \in \text{Hom}_{\mathcal{C}}(X', Y^A)$ , we have:

$$\begin{aligned} \text{Hom}_{\mathcal{C}}(x \times A, Y) \circ \gamma_{X',Y}(f) &= \text{Hom}_{\mathcal{C}}(x \times A, Y) \circ \varepsilon_Y \circ (f \times \text{id}_A) \\ &= \varepsilon_Y \circ (f \times \text{id}_A) \circ (x \times \text{id}_A) \\ &= \varepsilon_Y \circ (f \circ x \times \text{id}_A) \\ \gamma_{X,Y} \circ \text{Hom}_{\mathcal{C}}(x, Y^A)(f) &= \gamma_{X,Y}(f \circ x) \\ &= \varepsilon_Y \circ (f \circ x \times \text{id}_A) \\ &= \text{Hom}_{\mathcal{C}}(x \times A, Y) \circ \gamma_{X',Y}(f) \end{aligned}$$

Thus the diagram commutes.

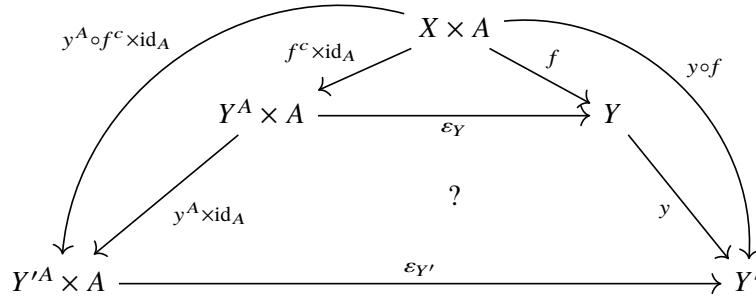
As for the naturality in  $Y$ , let  $y : Y \rightarrow Y'$ :

$$\begin{array}{ccc} \text{Hom}_{\mathcal{C}}(X, Y^A) & \xrightarrow{\gamma_{X,Y}} & \text{Hom}_{\mathcal{C}}(X \times A, Y) \\ \text{Hom}_{\mathcal{C}}(X, y^A) \downarrow & ? & \downarrow \text{Hom}_{\mathcal{C}}(X \times A, y) \\ \text{Hom}_{\mathcal{C}}(X, Y'^A) & \xrightarrow{\gamma_{X,Y'}} & \text{Hom}_{\mathcal{C}}(X \times A, Y') \end{array}$$

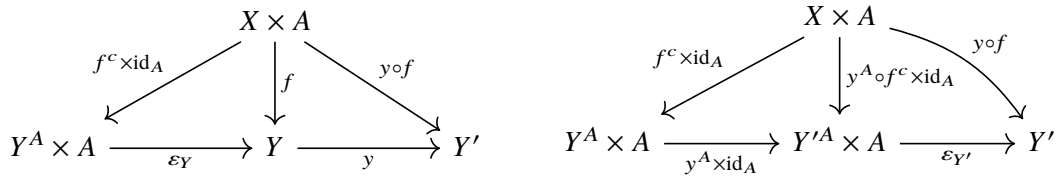
Let  $\varepsilon_{Y'}$  be the evaluation that comes with  $Y'^A$ . For  $f \in \text{Hom}_{\mathcal{C}}(X, Y^A)$ , we have:

$$\begin{aligned} \text{Hom}_{\mathcal{C}}(X \times A, y) \circ \gamma_{X,Y}(f) &= y \circ \varepsilon_Y \circ (f \times \text{id}_A) \\ \gamma_{X,Y'} \circ \text{Hom}_{\mathcal{C}}(X, y^A)(f) &= \gamma_{X,Y'}(y^A \circ f) \\ &= \varepsilon_{Y'} \circ (y^A \circ f \times \text{id}_A) \end{aligned}$$

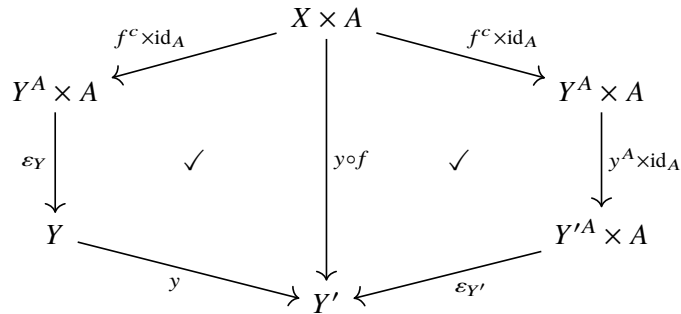
Consider the following diagram:



The following subdiagrams commute:



We deduce that the following diagram commutes too:



which proves the equality:

$$\begin{aligned} \varepsilon_{Y'} \circ (y^A \circ f \times \text{id}_A) &= y \circ \varepsilon_Y \circ (f \times \text{id}_A) \\ \Leftrightarrow \gamma_{X,Y'} \circ \text{Hom}_{\mathcal{C}}(X, y^A)(f) &= \text{Hom}_{\mathcal{C}}(X \times A, y) \circ \gamma_{X,Y}(f) \end{aligned}$$

and thus the naturality of  $\gamma$  in  $Y$ .

The inverse natural isomorphism  $\gamma^{-1}$  is the adjunctor between  $P_A$  and  $E_A$ . □

**Corollary 5.24.** *The exponential is unique up to isomorphism.*

*Proof.* Consequence of the unicity of the right adjoint up to isomorphism. □

**Remark 5.25.** In **Sets**, the two functors  $P_A$  and  $E_A$  would be:

$$P_A : \begin{cases} \mathcal{C} & \longrightarrow \\ X & \longmapsto \\ x : X \rightarrow X' & \longmapsto \end{cases} \quad \begin{cases} \mathcal{C} \\ X \times A \\ x \times \text{id}_A : \begin{cases} X \times A & \longrightarrow \\ e, a & \longmapsto \end{cases} \end{cases} \quad \begin{cases} X' \times A \\ (x(e), a) \end{cases}$$

$$E_A : \begin{cases} \mathcal{C} & \longrightarrow \\ X & \longmapsto \\ x : X \rightarrow X' & \longmapsto \end{cases} \quad \begin{cases} \mathcal{C} \\ X^A \\ x^A : \begin{cases} X^A & \longrightarrow \\ f & \longmapsto \end{cases} \end{cases} \quad \begin{cases} X'^A \\ x \circ f \end{cases}$$

**Remark 5.26.** If the category  $\mathcal{C}$  has all exponentials  $(Y^A, \varepsilon_Y)$  for all  $A, Y \in \text{Ob}_{\mathcal{C}}$ , then  $\varepsilon : (\varepsilon_Y)_{Y \in \text{Ob}_{\mathcal{C}}}$  is the counit of the adjunction  $P_A \dashv E_A$ . In fact, using the  $\gamma$  seen in the proof of Proposition 5.23, we have  $\gamma_{E_A(Y), Y}(\text{id}_{E_A(Y)}) = \varepsilon_Y \circ (\text{id}_{E_A(Y)}, \text{id}_A) = \varepsilon_Y$ .

**Definition 5.27** (Initial and terminal object [2]). Let  $\mathcal{C}$  be a category, and let  $I, T$  be objects of  $\mathcal{C}$ .

The object  $I$  is called *initial* when, for every  $C \in \text{Ob}_{\mathcal{C}}$ , there is only one arrow  $I \rightarrow C$ . The initial object is often denoted by  $0$ .

The object  $T$  is called *terminal* when, for every  $C \in \text{Ob}_{\mathcal{C}}$ , there is only one arrow  $C \rightarrow T$ . The terminal object is often denoted by  $1$ .

**Example 5.28.** In **Sets**, any singleton  $\{a\}$  is a terminal object, because there is only one function  $A \rightarrow \{a\}$  for every set  $A$  (the constant function  $x \mapsto a$ ). Besides, the empty set  $\emptyset$  is the unique initial object; for set-theoretic reasons, there is only one function  $\emptyset \rightarrow A$  (the empty function).

**Example 5.29.** If  $(P, \leq)$  is a poset, then the initial object is the minimal object  $\min(P)$  (if it exists) and the terminal object is the maximum  $\max(P)$  (if it exists).

**Proposition 5.30.** 1. Let  $\mathcal{C}$  be a category with initial object  $I$ . The initial object is unique up to unique isomorphism.

2. Let  $\mathcal{C}$  be a category with terminal object  $T$ . The terminal object is unique up to unique isomorphism.

*Proof.* **[Proof of 1]**

Let  $I$  and  $I'$  be two initial objects. Then there is only one arrow  $I \rightarrow I$ ,  $i : I \rightarrow I'$ ,  $i' : I' \rightarrow I$  and  $I' \rightarrow I'$ . We have  $i' \circ i : I \rightarrow I$ , but the only arrow  $I \rightarrow I$  is  $\text{id}_I$  so  $i' \circ i = \text{id}_I$ . Similarly, we have  $i \circ i' = \text{id}_{I'}$ , so  $i$  and  $i'$  are isomorphisms between  $I$  and  $I'$ .

**[Proof of 2]**

Same as with the initial objects. □

**Lemma 5.31.** If  $\mathcal{C}$  has finite products, then  $\mathcal{C}$  has a terminal object.

*Proof.* For any finite sequence of objects  $(A_i)_{i \in n}$  there is a product  $\prod_{i \in n} A_i$  together with projections  $\pi_{A_i} : \prod_{i \in n} A_i \rightarrow A_i$ .

If  $n = 0$ , we have an object  $1$  with no projections, such that for all  $C \in \mathcal{C}$ , there is a unique arrow  $!_C : C \rightarrow 1$  such that no diagram commutes<sup>2</sup>. □

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<sup>2</sup>In fact I don't really understand this proof. It comes from [1, p. 47], and is supported by <https://math.stackexchange.com/questions/1991522/terminal-objects-as-nullary-products>. I don't find the proof convincing because I feel like we can define the product of  $n$  objects, for  $n \geq 2$ , or even  $n = 1$ . However,  $n = 0$  seems like using the definition for a borderline case. As everyone seems to agree to this lemma (probably because it can be proven from elsewhere, using other tools), I mention it.

**Proposition 5.32.** *Let  $\mathcal{C}$  be a category.*

*If  $\mathcal{C}$  has a terminal object  $1$ , then  $C \cong C^1 \cong C \times 1$ .*

*Dually, if  $\mathcal{C}$  has an initial object  $0$ , then  $C + 0 \cong C$ .*

*Proof.* For the equivalence  $C \cong C \times 1$ , it suffices to show that  $C$  is also a product of  $C$  and  $1$ . For any  $p : P \rightarrow C$ , there is a unique arrow  $!_P : P \rightarrow 1$ . So, there is a unique arrow  $u$  such that the following diagram commutes:

$$\begin{array}{ccc} & P & \\ & \swarrow & \searrow \\ C & \xleftarrow{p} & C & \xrightarrow{!_C} & 1 \\ & \nwarrow & \downarrow u & \nearrow & \\ & C & & & \end{array}$$

and that  $u$  is  $u = p$ . So  $C$  and  $C \times 1$  are both products of  $C$  and  $1$ , so they are equivalent.

The same proof, with reverse arrows, yields that  $C + 0 \cong C$ .

As for the exponential, consider the adjunction  $\text{Hom}_{\mathcal{C}}(X, C^1) \cong \text{Hom}_{\mathcal{C}}(X \times 1, C) \cong \text{Hom}_{\mathcal{C}}(X, C)$ . By Corollary 2.16, we have  $C \cong C^1$ .

(Proof without Yoneda? Only by diagram chase? Exponentials are not unique up to iso, apparently.)

□

**Proposition 5.33.** *Let  $\mathcal{C}$  be a category and let  $F : \mathcal{C} \rightarrow \mathbf{Sets}$ . Then there is a natural isomorphism  $\text{Hom}_{\mathbf{Sets}}(1, F(C)) \cong F(C)$ , natural in both  $X$  and  $F$ .*

*Proof.* Recall that in  $\mathbf{Sets}$ ,  $1 = \{0\}$ .

Let  $C \in \mathcal{C}$ , we define the mapping:

$$\varphi_{F,C} : \begin{cases} \text{Hom}_{\mathbf{Sets}}(1, F(C)) & \longrightarrow & F(C) \\ f & \longmapsto & f(0) \end{cases}$$

Of course,  $\varphi_{F,C}$  is a bijection (isomorphism between sets):  $y \in F(C)$  then there is exactly one function  $f : 1 \rightarrow F(C)$  such that  $f(0) = y$ .

As for the naturality in  $C$ , if  $c : C \rightarrow C'$  then we check if the following diagram commutes:

$$\begin{array}{ccc} C & & \text{Hom}_{\mathbf{Sets}}(1, F(C)) \xrightarrow{\varphi_{F,C}} F(C) \\ \downarrow c & \rightsquigarrow & \downarrow \text{Hom}_{\mathbf{Sets}}(1, F(c)) \quad ? \quad \downarrow F(c) \\ C' & & \text{Hom}_{\mathbf{Sets}}(1, F(C')) \xrightarrow{\varphi_{F,C'}} F(C') \end{array}$$

For  $f \in \text{Hom}_{\mathbf{Sets}}(1, F(C))$ :

$$\begin{aligned} F(c) \circ \varphi_{F,C}(f) &= F(c)(f(0)) \\ \varphi_{F,C'} \circ \text{Hom}_{\mathcal{C}}(1, F(c))(f) &= \varphi_{F,C'}(F(c)(f)) \\ &= F(c)(f)(0) \\ &= F(c) \circ \varphi_{F,C}(f) \end{aligned}$$

hence the naturality in  $C$ .

Then, for a fixed  $C \in \mathcal{C}$ , if  $y : F \rightarrow F'$  is a natural transformation, we need to check if the following diagram commutes:

$$\begin{array}{ccc}
 \begin{array}{c} F \\ \downarrow \alpha \\ F' \end{array} & \rightsquigarrow & \begin{array}{ccc} \text{Hom}_{\mathcal{C}}(1, F(C)) & \xrightarrow{\varphi_{F,C}} & F(C) \\ \downarrow \text{Hom}_{\mathcal{C}}(1, \alpha_C) & & \downarrow \alpha_C \\ \text{Hom}_{\mathcal{C}}(1, F'(C)) & \xrightarrow{\varphi_{F',C}} & F'(C) \end{array}
 \end{array}$$

For  $f \in \text{Hom}_{\mathcal{C}}(1, F(C))$ , it does:

$$\begin{aligned}
 \alpha_C \circ \varphi_{F,C}(f) &= \alpha_C \circ f(0) \\
 \varphi_{F',C} \circ \text{Hom}_{\mathbf{Sets}}(1, \alpha_C)(f) &= \varphi_{F',C}(\alpha_C \circ f) \\
 &= \alpha_C \circ f(0) \\
 &= \alpha_C \circ \varphi_{F,C}(f)
 \end{aligned}$$

□

**Definition 5.34** (Cartesian closed category [2]). The category  $\mathcal{C}$  is called *Cartesian closed* whenever the following three conditions hold:

1. There is a terminal object 1
2.  $\mathcal{C}$  has finite products
3. For all objects  $C, D \in \text{Ob}_{\mathcal{C}}$ , the exponential  $C^D$  exists

*Example 5.35.* The category **Sets** is Cartesian closed.

Before introducing more "universal objects", let us study some properties of some morphisms. We already introduced the notion of an isomorphism, and we saw that in **Sets**, they were exactly the bijections (Remark 1.23). Thus, isomorphisms generalise the concept of bijection to other categories. Now, one could ask: how to generalise the concept of injections and surjections?

**Definition 5.36** (Epimorphisms and monomorphisms [1]). Let  $\mathcal{C}$  be a category and let  $c : C \rightarrow C'$  be an arrow in  $\mathcal{C}$ .

The arrow  $c$  is a *monomorphism*, or is *monic*, if, for all  $f, g : A \rightarrow C$ ,  $c \circ f = c \circ g \Rightarrow f = g$ :

$$A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} C \xrightarrow{c} C'$$

The arrow  $c$  is an *epimorphism*, or is *epic*, if, for all  $f, g : C' \rightarrow B$ ,  $f \circ c = g \circ c \Rightarrow f = g$ :

$$C \xrightarrow{c} C' \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} B$$

*Remark 5.37.* In **Sets**, suppose  $c : C \rightarrow C'$  is monic. Let  $x, y \in C$  such that  $c(x) = c(y)$ . Let  $f_x$  and  $f_y$  be the functions:

$$f_x : \begin{cases} 1 \rightarrow C \\ i \mapsto x \end{cases} \quad \text{and} \quad f_y : \begin{cases} 1 \rightarrow C \\ i \mapsto y \end{cases}$$

As  $c$  is monic, we have  $c \circ f_x = c \circ f_y \Rightarrow f_x = f_y \Rightarrow x = y$ .

Conversely, if  $c$  is injective, then for all  $f, g : X \rightarrow C$ , if  $c \circ f = c \circ g$ , then for all  $x \in X$ ,  $c \circ f(x) = c \circ g(x)$  which by injectivity means  $f(x) = g(x)$  and then  $f = g$ .

Now, if  $c : C \rightarrow C'$  is epic, let  $\chi_{c(C)} : C' \rightarrow 2$  be the characteristic function of  $c(C)$  (the image of  $c$ ), and let  $\text{cste}_1 : x \rightarrow 1$  be the constant function. We have  $\chi_{c(C)} \circ c = \text{cste}_1 \circ c$ , which by epicity gives  $\chi_{c(C)} = \text{cste}_1$ , and thus  $C' = c(C)$ , from which we deduce the surjectivity.

If  $c : C \rightarrow C'$  is surjective, let  $f, g : C' \rightarrow B$  such that  $f \circ c = g \circ c$ . For all  $y \in C'$ , there exists an  $x$  such that  $y = c(x)$  and  $f \circ c(x) = g \circ c(x) = f(y) = g(y)$ , which gives  $f = g$ , and  $c$  is epic.

In summary, in **Sets**, monomorphisms are exactly injective functions, and epimorphisms are exactly surjective functions.

*Remark 5.38.* In **Sets**, monomorphisms are exactly injective functions and epimorphisms are exactly surjective functions, but this is not necessary in most categories.

In most "structured sets" categories, for example, in **Monoids**, in **Groups**, in  $\mathbf{Vect}_{\mathbb{F}}$  (category of vector spaces based on a field  $\mathbb{F}$ ), the monomorphisms are exactly the injective morphisms. However, the epimorphisms are not exactly the surjective morphisms. For more information, see [1, Section 2.1, pp30-31].

In a poset category  $(P, \leq)$ , every arrow is monic and epic. This is due to the unicity of the arrow between two objects.

**Proposition 5.39.** *Let  $f : A \rightarrow B$  and  $g : B \rightarrow A$  such that  $g \circ f = \text{id}_A$ . Then  $f$  is monic while  $g$  is epic.*

*Proof.* Let  $a, a' : A' \rightarrow A$  such that  $f \circ a = f \circ a'$ , then  $g \circ f \circ a = g \circ f \circ a' \Rightarrow a = a'$ , so  $f$  is monic.

Let  $b, b' : A \rightarrow A'$  such that  $b \circ g = b' \circ g$ , then  $b \circ g \circ f = b' \circ g \circ f \Rightarrow b = b'$ , so  $g$  is epic.  $\square$

From Remark 5.37, we deduce that a function in **Sets** is an isomorphism if and only if it is both monic and epic. However, the "if and only if" does not hold for most categories (see [1, Section 2.1.1, pp32-33] for an example). What does hold is the following:

**Corollary 5.40.** *If  $c : C \rightarrow C'$  is an isomorphism, then  $c$  is both a monomorphism and an epimorphism.*

Let us apply it right now with in the following proposition.

**Proposition 5.41.** *Let  $\mathcal{X}$  and  $\mathcal{C}$  any categories. Let  $F, G : \mathcal{X} \rightarrow \mathcal{C}$  be two functors and let  $\alpha : F \rightarrow G$  be a natural transformation between those two functors.*

*If for all  $X \in \mathcal{X}$ ,  $\alpha_X : F(X) \rightarrow G(X)$  is monic (resp. epic), then so is  $\alpha : F \rightarrow G$ .*

*Proof.* Suppose that each  $X$ -component is monic. The proof is similar when we are considering epic components.

Consider  $\beta, \beta' : H \rightarrow F$  such that  $\alpha \circ \beta = \alpha \circ \beta'$ .

$$H \begin{array}{c} \xrightarrow{\beta} \\ \xrightarrow{\beta'} \end{array} F \xrightarrow{\alpha} G \quad \Leftrightarrow \quad H(X) \begin{array}{c} \xrightarrow{\beta_X} \\ \xrightarrow{\beta'_X} \end{array} F(X) \xrightarrow{\alpha_X} G(X)$$

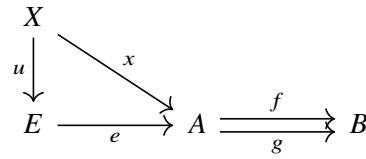
In terms of components, this means that for all  $X \in \mathcal{X}$ , we have  $\alpha_X \circ \beta_X = \alpha_X \circ \beta'_X$ . As every component is monic, this gives  $\beta_X = \beta'_X$ , and then  $\beta = \beta'$ . Thus,  $\alpha$  is monic.  $\square$

Surprisingly, the converse does not hold in general. In fact, it needs some more properties about the codomain category.

The digression about epimorphisms and monomorphisms is over now. We can get back to the examples of objects with universal properties.

**Definition 5.42** (Equalisers [1]). Let  $\mathcal{C}$  be a category, and let  $f, g : A \rightarrow B$  be two arrows.

An equaliser of  $f$  and  $g$  is a pair  $(E, e)$  with  $E \in \mathcal{C}$  and  $e : E \rightarrow A$ , such that  $f \circ e = g \circ e$  and, for all  $x : X \rightarrow A$  such that  $f \circ x = g \circ x$ , there exists a unique  $u : X \rightarrow E$  such that the following diagram commutes:



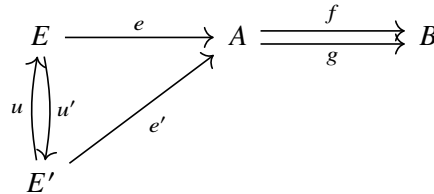
Example 5.43. (From [1]).

In **Sets**, given two functions  $f, g : A \rightarrow B$ , their equaliser is  $(E, e)$  where  $E = \{x \in A \mid f(x) = g(x)\}$  and  $e : E \rightarrow A$  is the canonic inclusion.

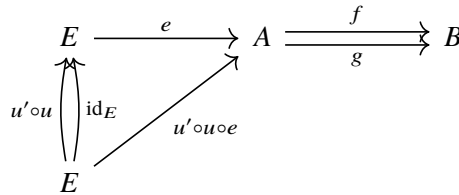
In a poset category  $(P, \leq)$ , there is at most one arrow  $p \rightarrow q$ . Thus, the equaliser of  $f, g : p \rightarrow q$ , with  $p = q$  is  $(p, \text{id}_p)$ .

**Proposition 5.44.** *The equaliser is unique up to isomorphism.*

*Proof.* Let  $(E, e)$  and  $(E', e')$  be equalisers of  $f$  and  $g$ . There exist unique  $u : E' \rightarrow E$  and  $u' : E \rightarrow E'$  such that  $e = u' \circ e'$  and  $e' = u \circ e$ , as in the following diagram:



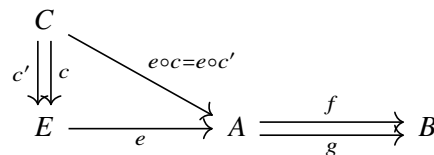
Thus, we have:  $e = u' \circ e' = u' \circ u \circ e$ , which gives the following diagram:



By unicity of the arrow  $E \rightarrow E$  which makes the diagram commute, we have  $u' \circ u = \text{id}_E$ . A similar reasoning yields  $u \circ u' = \text{id}_{E'}$ . □

**Proposition 5.45.** *Let  $(E, e)$  be an equaliser of  $f, g : A \rightarrow B$ . Then  $e$  is monic.*

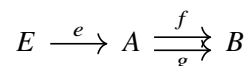
*Proof.* Let  $c, c' : C \rightarrow E$  such that  $e \circ c = e \circ c'$ .



By definition of an equaliser, we have  $f \circ e \circ c = g \circ e \circ c$ , so there exists a unique  $u : C \rightarrow E$  such that  $e \circ u = e \circ c = e \circ c'$ . By unicity of  $u$ , we have  $u = c = c'$ , hence  $e$  is monic. □

**Proposition 5.46.** *Let  $(E, e)$  be an equaliser of  $f, g : A \rightarrow B$ . If  $e$  is an epimorphism then  $e$  is an isomorphism.*

*Proof.* Suppose  $e$  is epic. As an equaliser, we have the following diagram:



and as an epimorphism, we deduce that  $f \circ e = g \circ e \Rightarrow f = g$ .

Thus, the identity  $\text{id}_A : A \rightarrow A$  verifies  $f \circ \text{id}_A = g \circ \text{id}_A$ . Consequently, there exists a unique  $u : A \rightarrow E$  such that the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{e} & A \xrightarrow[f]{g} B \\ \uparrow u & \nearrow \text{id}_A & \\ A & & \end{array}$$

from which we deduce  $e \circ u = \text{id}_A$ .

The same occurs with  $e : E \rightarrow A$ :

$$\begin{array}{ccc} E & \xrightarrow{e} & A \xrightarrow[f]{g} B \\ \uparrow u & \nearrow \text{id}_A & \\ A & & \\ \uparrow e & \nearrow e = e \circ u \circ e & \\ E & & \end{array}$$

We know that  $e = e \circ \text{id}_E = e \circ (u \circ e) = (e \circ u) \circ e = \text{id}_A \circ e$ . As an equaliser,  $e$  is monic, so  $u \circ e = \text{id}_E$ ;  $e$  is an isomorphism and  $e^{-1} = u$ .  $\square$

We deduce from this proposition what a *monic epimorphism* (or an *epic monomorphism*, or *monic/epic*) lacks to be an isomorphism:

**Corollary 5.47.** *Let  $c : C \rightarrow C'$  be any arrow.*

*The arrow  $c$  is an isomorphism  $\Leftrightarrow c$  is an epic equaliser.*

Take the arrows and reverse them; you get the definition of a coequaliser:

**Definition 5.48** (Coequalisers [1]). Let  $\mathcal{C}$  be a category, and let  $f, g : A \rightarrow B$  be two arrows.

A *coequaliser of  $f$  and  $g$*  is a pair  $(Q, q)$  with  $Q \in \mathcal{C}$  and  $q : B \rightarrow Q$ , such that  $q \circ f = q \circ g$  and, for all  $x : B \rightarrow X$  such that  $x \circ f = x \circ g$ , there exists a unique  $u : Q \rightarrow X$  such that the following diagram commutes:

$$\begin{array}{ccccc} A & \xrightarrow[f]{g} & B & \xrightarrow{q} & Q \\ & & \searrow x & & \downarrow u \\ & & & & X \end{array}$$

By duality, the following proposition holds:

**Proposition 5.49.** *The coequaliser is unique up to isomorphism.*

**Proposition 5.50.** *Let  $(Q, q)$  be a coequaliser of  $f, g : A \rightarrow B$ .*

*Then  $q$  is epic.*

**Proposition 5.51.** *Let  $(Q, q)$  be a coequaliser of  $f, g : A \rightarrow B$ .*

*If  $q$  is a monomorphism then  $q$  is an isomorphism.*

**Corollary 5.52.** *Let  $c : C \rightarrow C'$  be any arrow.*

*The arrow  $c$  is an isomorphism  $\Leftrightarrow c$  is a monic coequaliser.*





$$\begin{aligned}
 A \times_C B &= \{(a, b) \in A \times B \mid a = b\} \\
 &= \{(a, a) \in A \times B\} \\
 &\cong \{a \in A \mid a \in B\} \\
 &= A \cap B
 \end{aligned}$$

The intersection of sets consists in a pullback of inclusion mappings in **Sets**.

*Example 5.56* (Pullbacks in a poset). In a poset category  $(P, \leq)$ , as there is at most one arrow between two objects, we don't need to check that any diagram commutes. In fact, the pullback is exactly the same as a product; that is, a pullback between  $p \rightarrow q$  and  $p' \rightarrow q$  is  $p \times_q p' = p \times p' = \inf(p, p')$ .

**Proposition 5.57.** *Pullbacks are unique up to isomorphism.*

*Proof.* This proof is similar to the ones for products, coproducts, equalisers, coequalisers. □

Let us study some more properties related to pullbacks. For example, pullbacks allow for a different characterisation of monomorphisms in a category.

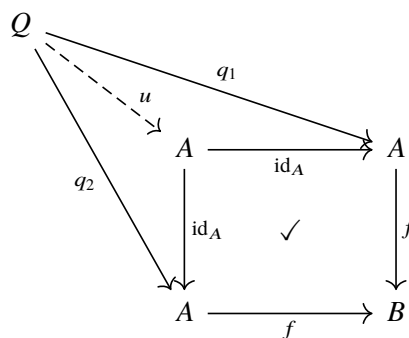
**Proposition 5.58.** *Let  $f : A \rightarrow B$  be an arrow. Then the following propositions are equivalent:*

1.  $f$  is a monomorphism
2. The pullback of  $f$  with itself exists and is  $(P, p, p')$  with  $p = p'$
3. The pullback of  $f$  with itself exists and is  $(A, \text{id}_A, \text{id}_A)$

*Proof.* [(1)  $\Rightarrow$  (3)] Suppose  $f$  is a monomorphism. Then for all  $c, c' : C \rightarrow A$  such that  $f \circ c = f \circ c'$ , we have  $c = c'$ .

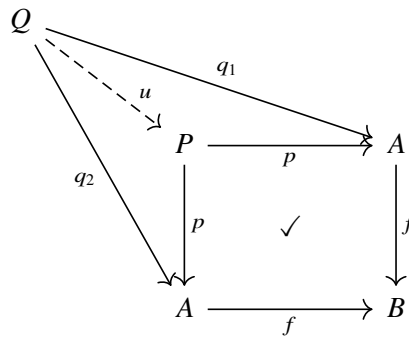
$$C \begin{array}{c} \xrightarrow{c} \\ \xrightarrow{c'} \end{array} A \xrightarrow{f} B$$

Then consider the triple  $(A, \text{id}_A, \text{id}_A)$ . Consider the following diagram:



such that  $f \circ q_1 = f \circ q_2$ . As  $f$  is monic, then  $q_1 = q_2$  and the unique  $u$  is  $u = q_1 = q_2$  and  $(A, \text{id}_A, \text{id}_A)$  is the pullback of  $A$  with itself.

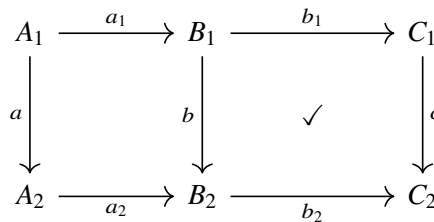
[(2)  $\Rightarrow$  (1)] Suppose  $(P, p, p)$  is the pullback of  $f$  with itself. Let  $Q$  be as in the following pullback diagram:



such that  $f \circ q_1 = f \circ q_2$ . By definition of a pullback, the unique  $u$  verifies:  $q_1 = p \circ u = q_2$ , hence the monicity of  $f$ .

[(3)  $\Rightarrow$  (2)] Obvious. □

**Proposition 5.59.** Consider the following diagram:



Suppose that  $(B_1, b_1, b)$  is the pullback of  $c$  with  $b_2$ . Then  $(A_1, a_1, a)$  is the pullback of  $b$  with  $a_2 \Leftrightarrow (A_1, b_1 \circ a_1, a)$  is the pullback of  $c$  with  $b_2 \circ a_2$ .

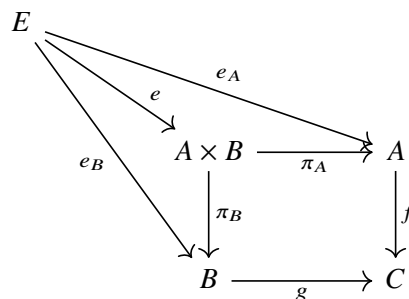
Suppose the right-hand square is a pullback; then the left-hand square is a pullback if and only if the whole rectangle is a pullback.

*Proof.* By diagram chase. □

This result proves sometimes to be useful, when some objects are defined in terms of pullbacks. More properties of pullbacks will come later. For now, let us just focus on the links between pullbacks, equalisers, products and terminal objects.

**Lemma 5.60.** Let  $\mathcal{C}$  be a category with products and equalisers. Let  $f : A \rightarrow C$  and  $g : B \rightarrow C$  be arrows.

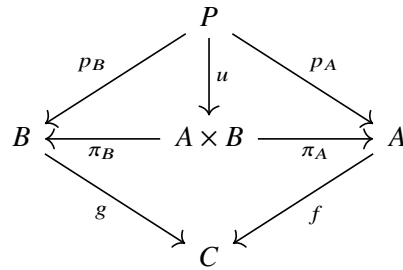
Let  $(A \times B, \pi_A, \pi_B)$  be the product of  $A$  and  $B$ ,  $E \in \mathcal{C}$ ,  $e : E \rightarrow A \times B$ ,  $e_A : E \rightarrow A$  and  $e_B : E \rightarrow B$ , as in the following diagram.



We suppose that  $e_A = \pi_A \circ e$  and  $e_B = \pi_B \circ e$ . The rest of the diagram is not supposed to commute otherwise.

$(E, e)$  is an equaliser of  $f \circ \pi_A$  and  $g \circ \pi_B \Leftrightarrow (E, e_A, e_B)$  is a pullback of  $f$  and  $g$ .

*Proof.* Suppose  $(E, e)$  is an equaliser of  $f \circ \pi_A$  and  $g \circ \pi_B$ . Let  $(P, p_A, p_B)$  such that  $p_A : P \rightarrow A$ ,  $p_B : P \rightarrow B$  and  $f \circ p_A = g \circ p_B$ . By definition of the product  $A \times B$ , there is a unique  $u : P \rightarrow A \times B$  such that the two upper triangles commute:



Now, we have:

$$\begin{aligned} f \circ p_A &= g \circ p_B \\ (f \circ \pi_A) \circ u &= (g \circ \pi_B) \circ u \end{aligned}$$

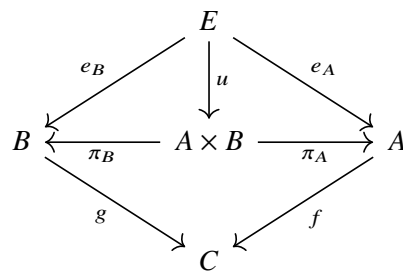
As  $(E, e)$  is an equaliser of  $f \circ \pi_A$  and  $g \circ \pi_B$ , there exists a unique  $v : P \rightarrow E$  such that  $e \circ v = u$ . Consequently, we have:

$$\begin{aligned} e \circ v &= u \\ \pi_A \circ e \circ v &= \pi_A \circ u \\ e_A \circ v &= p_A \end{aligned}$$

Similarly, for  $B$ , we have  $e_B \circ v = p_B$ . In summary, for any  $(P, p_A, p_B)$  such that  $f \circ p_A = g \circ p_B$ , there is a unique  $v : P \rightarrow E$  such that  $e_A \circ v = p_A$  and  $e_B \circ v = p_B$ ; thence,  $(E, e_A, e_B)$  is a pullback of  $f$  and  $g$ .

Conversely, suppose  $(E, e_A, e_B)$  is a pullback of  $f$  and  $g$ ; we have:  $f \circ e_A = g \circ e_B$ .

By definition of the product  $A \times B$ , there is a unique  $u : E \rightarrow A \times B$  such that the two upper triangles commute:



We already have  $e_A = \pi_A \circ e$  and  $e_B = \pi_B \circ e$ , so that unique  $u$  is  $e$ .

Let  $(P, p)$  be such that  $p : P \rightarrow A \times B$  and  $f \circ \pi_A \circ p = g \circ \pi_B \circ p$ . Such a  $p$  induces arrows  $p_A = \pi_A \circ p$  and  $p_B = \pi_B \circ p$ ; we have:

$$f \circ p_A = g \circ p_B$$

As  $(E, e_A, e_B)$  is a pullback of  $f$  and  $g$ , there is a unique  $v : P \rightarrow E$  such that  $p_A = e_A \circ v$  and  $p_B = e_B \circ v$ .

We deduce:

$$\begin{aligned} p_A &= e_A \circ v \Rightarrow \pi_A \circ p = \pi_A \circ e \circ v \\ p_B &= e_B \circ v \Rightarrow \pi_B \circ p = \pi_B \circ e \circ v \end{aligned}$$

which yields:

$$\begin{aligned} (\pi_A \circ p, \pi_B \circ p) &= (\pi_A \circ e \circ v, \pi_B \circ e \circ v) \\ (\pi_A, \pi_B) \circ p &= (\pi_A, \pi_B) \circ e \circ v \\ \text{id}_{A \times B} \circ p &= \text{id}_{A \times B} \circ e \circ v \\ p &= e \circ v \end{aligned}$$

To be an equaliser, there is one thing missing:  $f \circ \pi_A \circ e = g \circ \pi_B \circ e$ . This can be deduced from:

$$\begin{aligned} f \circ e_A &= g \circ e_B \\ f \circ \pi_A \circ e &= g \circ \pi_B \circ e \end{aligned}$$

□

**Corollary 5.61.** *If a category has finite products and equalisers, then it has pullbacks.*

**Lemma 5.62.** *Let  $\mathcal{C}$  be a category.*

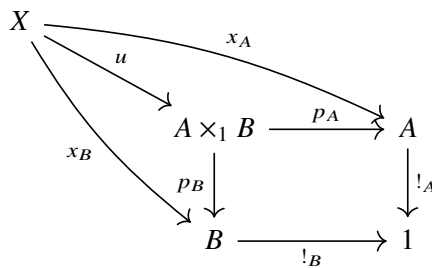
*If  $\mathcal{C}$  has pullbacks and a terminal object, then  $\mathcal{C}$  has finite products.*

*Proof.* Let  $A, B \in \text{Ob}_{\mathcal{C}}$ . There are unique arrows  $!_A : A \rightarrow 1$  and  $!_B : B \rightarrow 1$ . Let  $(A \times_1 B, p_A, p_B)$  be the pullback of  $!_A$  and  $!_B$ .

Let  $X$  be any object and let  $x_A : X \rightarrow A$  and  $x_B : X \rightarrow B$  be any arrows from  $X$  to  $A$  and  $B$ . By definition of a terminal object, there is a unique arrow  $!_X : X \rightarrow 1$ , so  $!_X$  is:

$$!_X = !_A \circ x_A = !_B \circ x_B$$

Then  $X$  qualifies for the existence of a unique  $u : X \rightarrow A \times_1 B$  such that the two triangles commute, as in the following diagram:



Thus,  $A \times_1 B$  is a product of  $A$  and  $B$ .

□

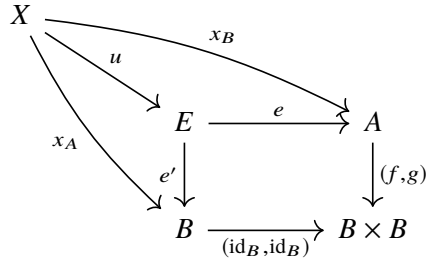
**Lemma 5.63.** *Let  $\mathcal{C}$  be a category.*

*If  $\mathcal{C}$  has pullbacks and finite products, then  $\mathcal{C}$  has equalisers.*

*Proof.* The proof again consists in finding the right pullback that will be the equaliser. As  $\mathcal{C}$  has products, we define  $B \times B$ . The pullback of  $(\text{id}_B, \text{id}_B) : B \rightarrow B \times B$  and  $(f, g) : A \rightarrow B \times B$  exists and is such that:

$$\begin{aligned}
 (\text{id}_B, \text{id}_B) \circ e' &= (f, g) \circ e \\
 (e', e') &= (f \circ e, g \circ e) \\
 \Rightarrow f \circ e &= g \circ e
 \end{aligned}$$

as in the diagram:

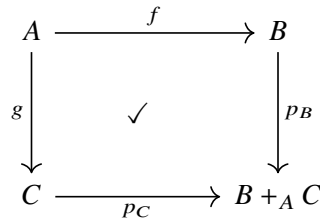


The universality of the equaliser comes from that of the pullback. □

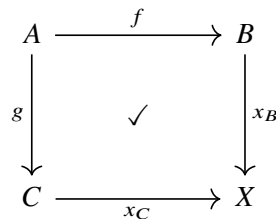
Finally, the dual notion of a pullback is a pushout:

**Definition 5.64** (Pushout [5]). Let  $\mathcal{C}$  be a category. Let  $f : A \rightarrow B$  and  $g : A \rightarrow C$  be arrows with same domain.

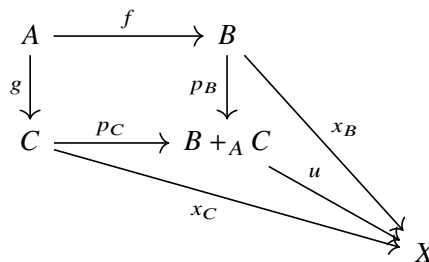
The *pushout of  $f$  and  $g$*  is a 3-tuple  $(B +_A C, p_B, p_C)$  such that the following diagram commutes:



and such that, for all  $(X, x_A, x_B)$  such that the following diagram commutes:



there is a unique arrow  $u : B +_A C \rightarrow X$  such that  $x_B = p_B \circ u$  and  $x_C = p_C \circ u$ , that is, such that the triangles and squares commute:



The arrows  $p_B : B \rightarrow B +_A C$  and  $p_C : C \rightarrow B +_A C$  are often called the inclusion mappings, just like in the coproduct.

*Example 5.65 (Pushout in Sets).* In **Sets**, consider the functions  $f : A \rightarrow B$  and  $g : A \rightarrow C$ . Then their pushout  $B +_A C$  is identified with a subset of  $B + C$ ; in fact, it is:

$$B +_A C = (B + C) / \equiv$$

where  $\equiv$  is the smallest equivalence relation on  $B + C$  such that for all  $a \in A$ ,  $f(a) \equiv g(a)$ .

Another interesting special case is the following. In Example 5.55, we defined the intersection  $A \cap B$  of two sets  $A$  and  $B$ . This intersection comes with trivial inclusion mappings  $i_A : \begin{cases} A \cap B \rightarrow A \\ x \mapsto x \end{cases}$  and  $i_B : \begin{cases} A \cap B \rightarrow B \\ x \mapsto x \end{cases}$ , so we can compute its pushout.

$$\begin{array}{ccc} A \cap B & \xrightarrow{i_A} & A \\ \downarrow i_B & \checkmark & \downarrow p_A \\ B & \xrightarrow{p_B} & A +_{A \cap B} B \end{array}$$

We have  $A +_{A \cap B} B = (A + B) / \equiv$  where  $\equiv$  is the smallest equivalence relation such that for all  $a \in A$ ,  $i_A(a) \equiv i_B(a)$ . In our case,  $i_A(a) = i_B(a) = a$ , so  $\equiv$  is simply the equality  $=$ . This means that, in the coproduct, which is a disjoint union in **Sets**, the pushout doesn't contain duplicates of the same element  $a$  if  $a$  is in both  $A$  and  $B$ . Thus, the pushout  $A +_{A \cap B} B$  is simply the union  $A \cup B$ .

The notions of equalisers and pullbacks will appear again in the rest of this course. The other two (coequalisers and pushouts) are introduced for the sake of completeness (and duality).

## 6. Your only colimit is yourself

Products/coproducts, initial/terminal objects, equalisers/coequalisers, pullbacks/pushouts are examples of the broader notion of limit. There are three ways to introduce limits, as illustrated in [2, Par. 9.2.6, p270]. We choose to introduce the limits using the characterisation with diagrams.

**Definition 6.1** (Diagram [6]). Let  $\mathcal{C}, \mathcal{I}$  be categories. A *diagram in  $\mathcal{C}$  of shape  $\mathcal{I}$*  is a functor  $\mathcal{I} \rightarrow \mathcal{C}$ .

The category  $\mathcal{I}$  is called the *index category* and it is usually (but not always!) small. If  $\mathcal{I}$  is finite, then the diagram is said *finite*.

In the following, the objects of  $\mathcal{I}$  will be denoted by  $i, j, k, \dots$  while the values of the functor  $\mathcal{I} \rightarrow \mathcal{C}$  will be denoted by  $D_i, D_j, D_k, \dots$

As explained in Section 1, a functor gives the picture of a category into another. A diagram  $\mathcal{I} \rightarrow \mathcal{C}$  is no more than that: just a picture of the category  $\mathcal{I}$  into the category  $\mathcal{C}$ , hence the name.

**Definition 6.2** (Category of diagrams). Let  $\mathcal{C}, \mathcal{I}$  be categories. The *category of diagrams in  $\mathcal{C}$  of shape  $\mathcal{I}$*  is the functor category  $\mathbf{Func}(\mathcal{I}, \mathcal{C}) = \mathcal{C}^{\mathcal{I}}$ .

**Definition 6.3** (Diagonal functor). Let  $\mathcal{C}, \mathcal{I}$  be categories.

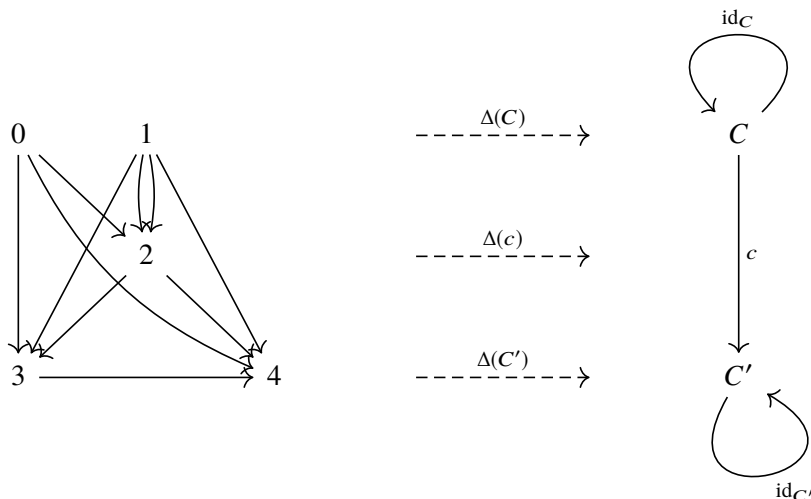
The *diagonal functor*  $\Delta$  is the functor  $\mathcal{C} \rightarrow \mathcal{C}^{\mathcal{I}}$  such that:

1. For all object  $C \in \mathcal{C}$ ,  $\Delta(C)$  is the diagram:

$$\Delta : \begin{cases} \mathcal{I} & \longrightarrow & \mathcal{C} \\ i & \longmapsto & C \\ i \rightarrow j & \longmapsto & \text{id}_C \end{cases}$$

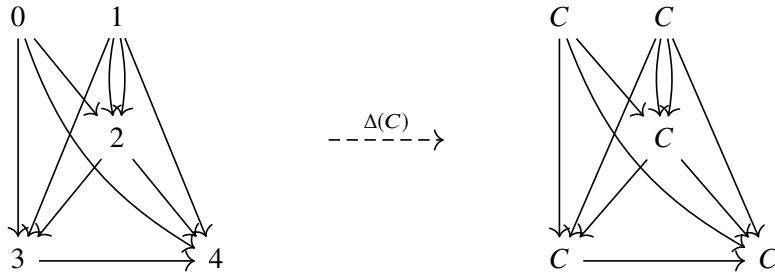
2. For all arrow  $c : C \rightarrow C' \in \mathcal{C}$ ,  $\Delta(c) : \Delta(C) \rightarrow \Delta(C')$  is the natural transformation  $\Delta(c) = (C \xrightarrow{c} C')_{i \in \mathcal{I}}$  (each component  $\Delta(c)_i$  is a copy of  $c$ ).

In summary, the functor  $\Delta(C)$  "collapses" the category  $\mathcal{I}$  into one element  $C$ . For example, if  $\mathcal{I}$  is the following five-element category:



One can also see  $\Delta(C)$  as a sequence of copies of  $C$ , indexed by the objects of  $\mathcal{I}$ . Here, the arrows of  $\mathcal{I}$  don't matter, as they always become  $\text{id}_C$ . If  $\mathcal{I}$  is a category with two objects, then  $\Delta(C) = (C, C)$ . A better view of the action of  $\Delta(C)$  is the following diagram:





where all the arrows in the right diagram are identity arrows.

**Definition 6.4** (Colimit). Let  $\mathcal{C}, \mathcal{I}$  be categories. Let  $\Delta : \mathcal{C} \rightarrow \mathcal{C}^{\mathcal{I}}$  be the diagonal functor and let  $D : \mathcal{I} \rightarrow \mathcal{C}$  be a diagram.

The pair  $(\text{Colim}(D), \eta_D)$  is the *colimit diagram for D* when  $(\text{Colim}(D), \eta_D)$  is a universal arrow from  $D$  to  $\Delta$ .

*Remark 6.5.* We have  $\text{Colim}(D) \in \text{Ob}_{\mathcal{C}}, \eta_D : D \rightarrow \Delta(\text{Colim}(D))$ ; that is,  $\eta_D$  is a natural transformation between the two diagrams  $D : \mathcal{I} \rightarrow \mathcal{C}$  and  $\Delta(\text{Colim}(D)) : \mathcal{I} \rightarrow \mathcal{C}$ .

*Example 6.6.* In **Sets**, a diagram  $D : \mathcal{I} \rightarrow \mathbf{Sets}$  is a functor that defines a small subcategory inside **Sets**. One might say it's a graph whose nodes are sets and whose arrows are functions such that the composite of two function is still an arrow in the graph. Note that, for all arrow  $u : i \rightarrow j$  in  $\mathcal{I}$ , the arrow  $D(u) : D_i \rightarrow D_j$  is a function between sets.

Define  $X = \bigsqcup_{i \in \text{Ob}_{\mathcal{I}}} D_i$  to be the coproduct in **Sets** of all  $D_i$ 's. For the sake of clarity, let us explicitly define this coproduct as:

$$X = \bigsqcup_{i \in \text{Ob}_{\mathcal{I}}} D_i = \{(i, x) \mid i \in \text{Ob}_{\mathcal{I}}, x \in D_i\}$$

(You can check that this actually is a coproduct.)

Define the preorder  $\simeq$  over  $X$  such that:  $(i, x) \simeq (i', x')$  iff there exists some  $u : i \rightarrow i'$  such that  $D(u)(x) = x'$ . Let  $\sim$  be the equivalence relation generated by this preorder.

Then, the colimit of the diagram  $D$  is the quotient set:

$$\text{Colim}(D) = X / \sim = \bigsqcup_{i \in \text{Ob}_{\mathcal{I}}} D_i / \sim$$

The natural transformation  $\eta_D$  is composed of the inclusion maps  $D_i \rightarrow \bigsqcup_{i \in \text{Ob}_{\mathcal{I}}} D_i / \sim$ .

Note that the coproduct of two sets corresponds to the special case where  $\text{card}(\text{Ob}_{\mathcal{I}}) = 2$  and there is no arrow between the two objects, so that the equivalence relation  $\sim$  is only the equality.

*Example 6.7.* In a poset category  $(P, \leq)$ , the diagram  $D : \mathcal{I} \rightarrow P$  defines a sub-order, and the colimit of that diagram, if it exists, is the sup of all  $D_i$ 's:  $\text{Colim}(D) = \sup_{i \in \text{Ob}_{\mathcal{I}}} D_i$ .

*Remark 6.8.* Let  $(\text{Colim}(D), \eta_D)$  be a colimit. By definition, it is a universal arrow from  $D$  to  $\Delta$ , so for all  $C \in \text{Ob}_{\mathcal{C}}$ , for all  $\alpha : D \rightarrow \Delta(C)$ , there is a unique  $x : \text{Colim}(D) \rightarrow C$  such that:

$$\alpha = \Delta(x) \circ \eta_D \tag{17}$$

Note that  $\alpha : D \rightarrow \Delta(C)$  is:

$$\alpha = (\alpha_i : D(i) \rightarrow \Delta(C)(i))_{i \in \mathcal{I}} = (\alpha_i : D_i \rightarrow C)_{i \in \mathcal{I}}$$

and  $\eta_D$  is:

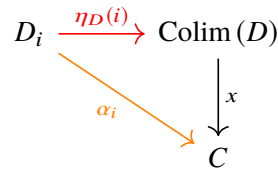
$$\eta_D = (\eta_D(i) : D_i \rightarrow \Delta(\text{Colim}(D))(i))_{i \in \mathcal{I}} = (\eta_D(i) : D_i \rightarrow \text{Colim}(D))_{i \in \mathcal{I}}$$

Finally, for  $x : \text{Colim}(D) \rightarrow C$ , we have  $\Delta(x) = (x : \text{Colim}(D) \rightarrow C)_{i \in \mathcal{I}}$ .

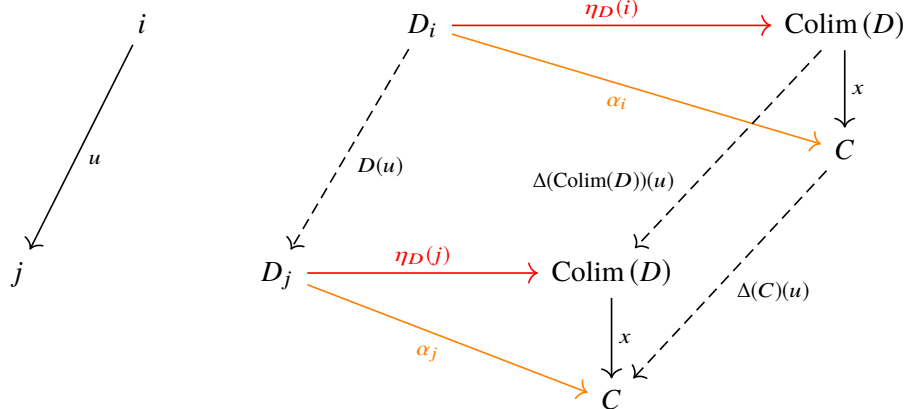
Thus, Equation 17 rewrites:

$$\begin{aligned} (\alpha_i : D_i \rightarrow C)_{i \in \mathcal{I}} &= (x : \text{Colim}(D) \rightarrow C)_{i \in \mathcal{I}} \circ (\eta_D(i) : D_i \rightarrow \text{Colim}(D))_{i \in \mathcal{I}} \\ &= \left( x \circ \left( D_i \xrightarrow{\eta_D(i)} \text{Colim}(D) \right) \right)_{i \in \mathcal{I}} \end{aligned}$$

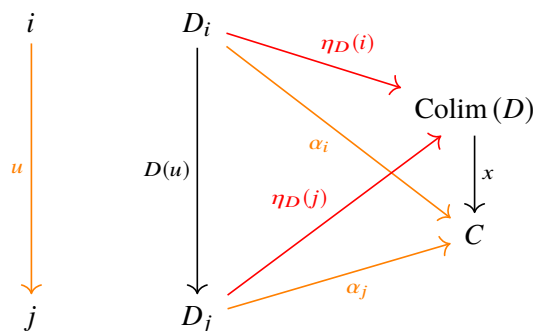
Therefore, for all  $i \in \mathcal{I}$ :



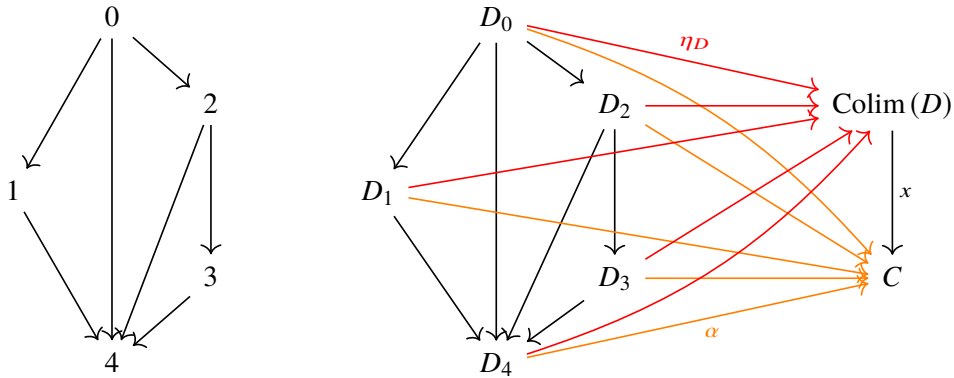
Besides, as  $\alpha : D \rightarrow \Delta(C)$  and  $\eta_D : D \rightarrow \Delta(\text{Colim}(D))$  are natural transformations, the following diagrams commute:



which simplifies to:



Now, using a more complex starting graph:



where all triangles commute.

We see some cone-like figures in red and orange, the base of which is the diagram with the  $D_i$ 's. We call these figures *cocones from  $D$  to  $C$* . Definition 6.10 makes it more formal.

It follows from the previous remark that:

**Proposition 6.9.** Let  $\mathcal{C}, \mathcal{I}$  be categories. Let  $\Delta : \mathcal{C} \rightarrow \mathcal{C}^{\mathcal{I}}$  be the diagonal functor and let  $D : \mathcal{I} \rightarrow \mathcal{C}$  be a diagram. Let  $C_D \in \text{Ob}_{\mathcal{C}}$  and  $\eta_D : D \rightarrow \Delta(C_D)$ .

The pair  $(C_D, \eta_D)$  is a colimit diagram for  $D \Leftrightarrow \forall C \in \text{Ob}_{\mathcal{C}}, \forall \alpha : D \rightarrow \Delta(C), \exists ! c \in \text{Hom}_{\mathcal{C}}(C_D, C)$  such that  $\forall i \in \text{Ob}_{\mathcal{I}}, \alpha_i = c \circ \eta_{D_i}$ .

*Proof.* See Remark 6.8. Otherwise, it follows from the definition of a colimit. □

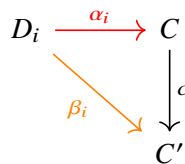
The notion of cocone was introduced in Remark 6.8. Here is the formal definition:

**Definition 6.10** (Category of cocones). Let  $\mathcal{C}, \mathcal{I}$  be categories. Let  $D : \mathcal{I} \rightarrow \mathcal{C}$  be a diagram.

The category **Cocones**( $D$ ) of cocones from  $D$  contains:

**Objects:** The objects are the natural transformations  $\alpha : D \rightarrow \Delta(C) = \left( D_i \xrightarrow{\alpha_i} C \right)_{i \in \text{Ob}_{\mathcal{I}}}$  for each  $C \in \text{Ob}_{\mathcal{C}}$ , called *cocones from  $D$  to  $C$*

**Morphisms:** Let  $\alpha : \left( D_i \xrightarrow{\alpha_i} C \right)_{i \in \text{Ob}_{\mathcal{I}}}$  and  $\beta : \left( D_i \xrightarrow{\beta_i} C' \right)_{i \in \text{Ob}_{\mathcal{I}}}$  be two cocones. An arrow  $c : \alpha \rightarrow \beta$  is an arrow  $c : C \rightarrow C'$  such that the following diagram commutes:



**Identities:** An identity morphism is an arrow  $\text{id}_C : C \rightarrow C$

**Composition:** The composition law for morphisms is the composition law for morphisms in  $\mathcal{C}$ .

*Example 6.11.* In a poset category, a cocone from  $D$  to  $C$  exists if and only if  $C$  is an upper bound of the  $D_i$ 's. Note that the colimit is the least upper bound of the  $D_i$ 's. This fact is made formal in the following proposition.

**Proposition 6.12.** Let  $\mathcal{C}, \mathcal{I}$  be categories. Let  $D : \mathcal{I} \rightarrow \mathcal{C}$  be a diagram. Let **Cocones**( $D$ ) be the category of cocones from  $D$ . Let  $C_D \in \mathcal{C}$  and  $\eta_D : D \rightarrow \Delta(C_D)$ .

$(C_D, \eta_D)$  is a colimit diagram for  $D \Leftrightarrow \eta_D$  is an initial object in **Cocones**( $D$ ).

*Proof.* Using Proposition 6.9, the proof is easy:

$$\begin{aligned}
 & \eta_D \text{ is an initial object in } \mathbf{Cocones}(D) \\
 \Leftrightarrow & \forall C \in \mathbf{Ob}_{\mathcal{C}} \forall \alpha : D \rightarrow \Delta(C), \exists ! c : \eta_D \rightarrow \alpha \\
 \Leftrightarrow & \forall C \in \mathbf{Ob}_{\mathcal{C}} \forall \alpha : D \rightarrow \Delta(C), \exists ! c : C_D \rightarrow C \text{ such that } \forall i \in \mathbf{Ob}_{\mathcal{J}} \alpha_i = c \circ \eta_D(i) \\
 \Leftrightarrow & \forall C \in \mathbf{Ob}_{\mathcal{C}} \forall \alpha : D \rightarrow \Delta(C), \exists ! c : C_D \rightarrow C \text{ such that } \alpha = \Delta(c) \circ \eta_D \\
 \Leftrightarrow & (C_D, \eta_D) \text{ is a universal arrow from } D \text{ to } \Delta \\
 \Leftrightarrow & (C_D, \eta_D) \text{ is a colimit diagram for } D
 \end{aligned}$$

□

**Corollary 6.13.**  $(C_D, \eta_D)$  is a colimit diagram for  $D \Leftrightarrow \eta_D$  is a cocone which is universal: for any cone  $\alpha : D \rightarrow \Delta(C)$ ,  $\exists ! c : C_D \rightarrow C$  such that  $\alpha = \Delta(c) \circ \eta_D$ .

We gave a characterisation of a colimit  $(C_D, \eta_D)$  based on some property of  $\eta_D$ . There is also a characterisation of a colimit based on the object  $C_D$ .

**Lemma 6.14.** If  $0$  is the initial object of  $\mathcal{C}$ , then the unique arrows  $i_X : 0 \rightarrow X$  define the unique natural transformation  $i : \Delta(0) \rightarrow \mathbf{Id}_{\mathcal{C}}$ .

If  $1$  is the terminal object of  $\mathcal{C}$ , then the unique arrows  $!_X : X \rightarrow 1$  define the unique natural transformation  $t : \mathbf{Id}_{\mathcal{C}} \rightarrow \Delta(1)$ .

*Proof.* Let  $f : X \rightarrow Y$ . We need to check if the following diagram commutes:

$$\begin{array}{ccc}
 \Delta(0)(X) & \xrightarrow{i_X} & \mathbf{Id}_{\mathcal{C}}(X) \\
 \Delta(0)(f) \downarrow & & \downarrow \mathbf{Id}_{\mathcal{C}}(f) \\
 \Delta(0)(Y) & \xrightarrow{i_Y} & \mathbf{Id}_{\mathcal{C}}(Y)
 \end{array}
 =
 \begin{array}{ccc}
 0 & \xrightarrow{i_X} & X \\
 \text{id}_0 \downarrow & & \downarrow f \\
 0 & \xrightarrow{i_Y} & Y
 \end{array}$$

We have  $f \circ i_X : 0 \rightarrow Y$ . By definition of an initial element, there is a unique arrow  $i_Y : 0 \rightarrow Y$ ; thus,  $i_Y = f \circ i_X = f \circ \text{id}_0 = f \circ i_X$ . Besides, this natural transformation  $i$  is unique due to the uniqueness of the arrows  $i_X$ .

The statement with the terminal objects has a similar proof. □

**Proposition 6.15.** Let  $\mathcal{C}$  be a category and let  $T \in \mathbf{Ob}_{\mathcal{C}}$ .

$T$  is terminal in  $\mathcal{C} \Leftrightarrow T$  is the colimit of  $\mathbf{Id}_{\mathcal{C}}$ .

*Proof.* [Proof of  $\Rightarrow$ ]

Suppose  $T$  is terminal in  $\mathcal{C}$ . By Lemma 6.14, there is a unique natural transformation  $i : \mathbf{Id}_{\mathcal{C}} \rightarrow \Delta(T)$ . This natural transformation is of course a cocone from  $T$  to  $\mathbf{Id}_{\mathcal{C}}$ .

Let  $\alpha : \mathbf{Id}_{\mathcal{C}} \rightarrow \Delta(C)$  be a cocone from  $C$  to  $D$  for some object  $C \in \mathcal{C}$ . We are looking for an  $x$  so that the following diagram commutes:

$$\begin{array}{ccc}
 X & \xrightarrow{!_X} & T \\
 \alpha_X \downarrow & \swarrow x & \\
 A & & 
 \end{array}$$

If  $X = T$ , we have:

$$\begin{array}{ccc}
 T & \xrightarrow{!_T} & T \\
 \alpha_T \downarrow & \swarrow x & \\
 A & & 
 \end{array}$$

There is a unique arrow  $!_T : T \rightarrow T$ , and  $!_T = \text{id}_T$ . Then, we have:  $\alpha_T = x \circ !_T = x$ . Consequently,  $\alpha_T$  is a morphism of cocones; but if  $m : A \rightarrow T$  is another morphism of cocones, the following diagram commutes:

$$\begin{array}{ccc}
 & & T \\
 & \nearrow \alpha_T & \downarrow \text{id}_T \\
 A & & T \\
 & \searrow m & 
 \end{array}$$

which gives  $m = \text{id}_T \circ \alpha_T = \alpha_T = x$ , hence the unicity of the  $x$ . Finally,  $(T, t)$  is the colimit of  $\text{Id}_{\mathcal{C}}$ .

**[Proof of  $\Leftarrow$ ]**

Let  $(T, \eta)$  be the colimit of  $\text{Id}_{\mathcal{C}}$ .

The first step consists in proving that  $\eta_T$  is  $\text{id}_T$ .

For all  $f : X \rightarrow Y$ , the following diagram commutes:

$$\begin{array}{ccc}
 X & & T \\
 f \downarrow & \searrow \eta_T & \nearrow \eta_Y \\
 Y & & 
 \end{array}$$

In particular, if  $f = \eta_X$ :

$$\begin{array}{ccc}
 X & & T \\
 \eta_T \downarrow & \searrow \eta_T & \nearrow \eta_X \\
 T & & 
 \end{array}$$

As this is true for any  $X$ , we conclude that  $\eta_T$  is a morphism of cocones  $\eta_T : \eta \rightarrow \eta$ . By Proposition 6.12,  $\eta$  is initial in  $\mathbf{Cocones}(\text{Id}_{\mathcal{C}})$ , so the arrow  $\eta_T$  is  $\eta_T = \text{id}_T$ .

The second step consists in showing the unicity of some arrow  $X \rightarrow T$ .

Let  $f : X \rightarrow T$ ; the following diagram commutes:

$$\begin{array}{ccc}
 \text{Id}_{\mathcal{C}}(X) & \xrightarrow{\eta_X} & \Delta(T)(X) \\
 \text{Id}_{\mathcal{C}}(f) \downarrow & \checkmark & \downarrow \Delta(T)(f) \\
 \text{Id}_{\mathcal{C}}(T) & \xrightarrow{\eta_T} & \Delta(T)(T)
 \end{array}
 =
 \begin{array}{ccc}
 X & \xrightarrow{\eta_X} & T \\
 f \downarrow & \checkmark & \downarrow \text{id}_T \\
 T & \xrightarrow{\text{id}_T} & T
 \end{array}$$

which gives  $\eta_X = f$ , hence the unicity of  $\eta_X$ . Note that  $\eta$  defines one arrow  $X \rightarrow T$  for each  $X \in \text{Ob}_{\mathcal{C}}$ ; so, for each  $X \in \text{Ob}_{\mathcal{C}}$ , there is only one arrow  $X \rightarrow T$ , so  $T$  is terminal.  $\square$

**Proposition 6.16.** Let  $\mathcal{C}$  be a category and let  $A, B \in \text{Ob}_{\mathcal{C}}$  such that the coproduct  $(A + B, c_A, c_B)$  exists.

Then  $A + B$  is the colimit of the diagram  $D : \begin{cases} 2 & \rightarrow & \mathcal{C} \\ 0 & \mapsto & A \\ 1 & \mapsto & B \end{cases}$ , where  $2$  is the category with two objects  $0, 1$  and no morphism between those two.

*Proof.* Note that for all  $X \in \mathcal{C}$ , the cocone  $\alpha : D \rightarrow \Delta(X)$  has only two components  $\alpha_A : A \rightarrow X$  and  $\alpha_B : B \rightarrow X$ . Besides, if  $A + B$  exists, then there is a unique  $u : A + B \rightarrow X$  such that  $\alpha_A = u \circ c_A$  and  $\alpha_B = u \circ c_B$ ; in other words,  $(A + B, c)$ , where  $c : D \rightarrow \Delta(A + B)$  is the natural transformation with components  $c_A$  and  $c_B$ , is the colimit of  $D$ .  $\square$

We now introduce the dual notion of a colimit, namely that of a limit. We will need to introduce cones (the dual notion of cocones) as well.

**Definition 6.17 (Limit).** Let  $\mathcal{C}, \mathcal{I}$  be categories. Let  $\Delta : \mathcal{C} \rightarrow \mathcal{C}^{\mathcal{I}}$  be the diagonal functor and let  $D : \mathcal{I} \rightarrow \mathcal{C}$  be a diagram.

The pair  $(\text{Lim}(D), \varepsilon_D)$  is the *limit diagram* for  $D$  when for all  $C \in \text{Ob}_{\mathcal{C}}$ , for all  $\alpha : \Delta(C) \rightarrow D$ , there is a unique  $x : C \rightarrow \text{Lim}(D)$  such that  $\alpha = \varepsilon_D \circ \Delta(x)$ .

We say that  $\mathcal{C}$  has *finite limits* if every diagram  $D : \mathcal{I} \rightarrow \mathcal{C}$  with finite index category  $\mathcal{I}$  has a limit.

*Remark 6.18.* We have  $\text{Lim}(D) \in \text{Ob}_{\mathcal{C}}$ ,  $\varepsilon_D : \Delta(\text{Lim}(D)) \rightarrow D$ ; that is,  $\varepsilon_D$  is a natural transformation between the two diagrams  $\Delta(\text{Lim}(D)) : \mathcal{I} \rightarrow \mathcal{C}$  and  $D : \mathcal{I} \rightarrow \mathcal{C}$ .

*Example 6.19.* As stated in Example 6.6, in **Sets**, a diagram  $D : \mathcal{I} \rightarrow \mathbf{Sets}$  is a functor that defines a small subcategory inside **Sets**.

The limit of  $D$  can be defined explicitly as:

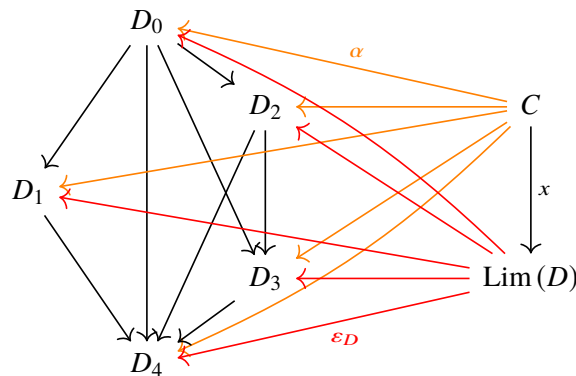
$$\text{Lim}(D) = \left\{ (s_i)_{i \in \text{Ob}_{\mathcal{I}}} \in \prod_{i \in \text{Ob}_{\mathcal{I}}} D_i \mid \forall u : i \rightarrow j, D(u)(s_i) = s_j \right\}$$

And the natural transformation  $\varepsilon_D$  is composed of each projection  $\prod_{i \in \text{Ob}_{\mathcal{I}}} D_i \rightarrow D_i$ .

Again, the product is a special case of limit, when  $\mathcal{I}$  is the category with only two objects and no arrow between them, so that the condition  $\forall u : i \rightarrow j, D(u)(s_i) = s_j$  is vacuously true.

*Example 6.20.* In a poset category  $(P, \leq)$ , the diagram  $D : \mathcal{I} \rightarrow P$  defines a sub-order, and the limit of that diagram, if it exists, is the inf of all  $D_i$ 's:  $\text{Lim}(D) = \inf_{i \in \text{Ob}_{\mathcal{I}}} D_i$ .

*Remark 6.21.* Using the same diagram as in the last example in Remark 6.8, and using the duality, a limit illustrates this way:



**Definition 6.22 (Cone).** Let  $D : \mathcal{I} \rightarrow \mathcal{C}$  be a diagram.

We define the category **Cones**( $D$ ) of cones to  $D$  as the following category:

**Objects:** The objects are the natural transformations  $\alpha : \Delta(C) \rightarrow D = \left( C \xrightarrow{\alpha_i} D_i \right)_{i \in \text{Ob}_{\mathcal{C}}}$  for  $C \in \text{Ob}_{\mathcal{C}}$ , called *cones from C to D*

**Morphisms:** Let  $\alpha : \left( C \xrightarrow{\alpha_i} D_i \right)_{i \in \text{Ob}_{\mathcal{C}}}$  and  $\beta : \left( C' \xrightarrow{\beta_i} D_i \right)_{i \in \text{Ob}_{\mathcal{C}}}$  be two cones. An arrow  $c : \alpha \rightarrow \beta$  is an arrow  $c : C \rightarrow C'$  such that the following diagram commutes:

$$\begin{array}{ccc} D_i & \xleftarrow{\alpha_i} & C \\ & \swarrow \beta_i & \downarrow c \\ & & C' \end{array}$$

**Identities:** An identity morphism is an arrow  $\text{id}_C : C \rightarrow C$

**Composition:** The composition law for morphisms is the composition law for morphisms in  $\mathcal{C}$ .

*Remark 6.23.* Note that a cocone is *from* the diagram  $D$  to the object  $C$ , while a cone is *from* the object  $C$  to the diagram  $D$ .

*Example 6.24.* In a poset category, a cone from  $C$  to  $D$  exists if and only if  $C$  is a lower bound of the  $D_i$ 's. Just like in Example 6.11, note that the limit is the greatest lower bound of the  $D_i$ 's. This fact is made formal in Proposition 6.25.

As the dual notion of colimit, we have the dual characterisations of limits:

**Proposition 6.25.** Let  $\mathcal{C}, \mathcal{I}$  be categories. Let  $D : \mathcal{I} \rightarrow \mathcal{C}$  be a diagram. Let  $\mathbf{Cones}(D)$  be the category of cones to  $D$ . Let  $C_D \in \mathcal{C}$  and  $\varepsilon_D : \Delta(C_D) \rightarrow D$ .

$(C_D, \varepsilon_D)$  is a limit diagram for  $D \Leftrightarrow \varepsilon_D$  is a terminal object in  $\mathbf{Cones}(D)$ .

*Proof.* Similar to the proof of Proposition 6.12. □

**Proposition 6.26.** Let  $\mathcal{C}$  be a category and let  $I \in \text{Ob}_{\mathcal{C}}$ .

$I$  is initial in  $\mathcal{C} \Leftrightarrow I$  is the limit of  $\text{Id}_{\mathcal{C}}$ .

*Proof.* Similar to the proof of Proposition 6.15. □

*Remark 6.27.* Colimits are the initial objects of the category of cocones (Proposition 6.12), while the terminal object of a category is the colimit of the identity functor (Proposition 6.15). Dually, limits are the terminal objects of the category of cones (Proposition 6.25), while the initial object of a category is the limit of the identity functor (Proposition 6.26). Try not to confuse!

In the following, we mention the other limit diagrams; the proof is essentially the same as Proposition 6.16.

**Proposition 6.28.** The following constructions are limits:

1. A terminal object in  $\mathcal{C}$  is the limit of the empty diagram  $D : \emptyset \rightarrow \mathcal{C}$ , with  $\emptyset$  as the empty category:

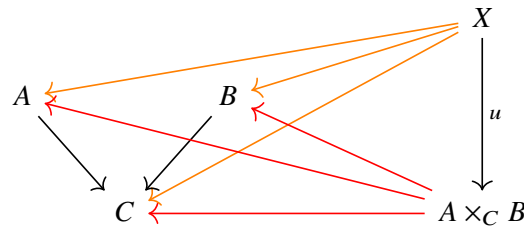
$$\begin{array}{c} X \\ \downarrow u \\ 1 \end{array}$$

2. A product  $A \times B$  in  $\mathcal{C}$  is the limit of the diagram  $D : \mathcal{C}_2 \rightarrow \mathcal{C}$ , with  $\mathcal{C}_2$  being the index category with two objects and no arrow between those two:

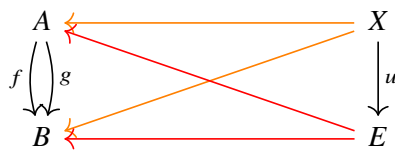
$$\begin{array}{ccc} A & \xrightarrow{\quad} & X \\ & \searrow & \downarrow u \\ B & \xrightarrow{\quad} & A \times B \end{array}$$

6. Your only colimit is yourself

3. A pullback of  $f : A \rightarrow C$  and  $g : B \rightarrow C$  is the limit of the diagram  $D : \mathcal{C}_3 \rightarrow \mathcal{C}$ , where  $\mathcal{C}_3$  is the index category described below:



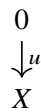
4. An equaliser of  $f, g : A \rightarrow B$  is the limit of the diagram  $D : \mathcal{C}_4 \rightarrow \mathcal{C}$  where  $\mathcal{C}_4$  is the index category described below:



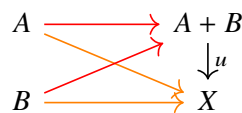
The dual statement is also true:

**Proposition 6.29.** The following constructions are colimits:

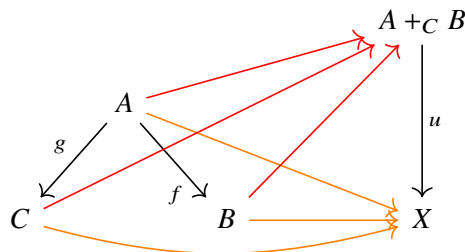
1. An initial object in  $\mathcal{C}$  is the colimit of the empty diagram  $D : \emptyset \rightarrow \mathcal{C}$ , with  $\emptyset$  as the empty category:



2. A coproduct  $A + B$  in  $\mathcal{C}$  is the colimit of the diagram  $D : \mathcal{C}_2 \rightarrow \mathcal{C}$ , with  $\mathcal{C}_2$  being the category with two objects and no arrow between those two:

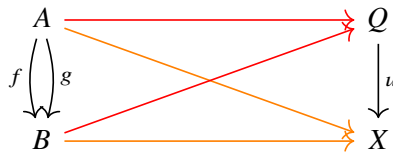


3. A pushout of  $f : A \rightarrow B$  and  $g : A \rightarrow C$  is the colimit of the diagram  $D : \mathcal{C}_3 \rightarrow \mathcal{C}$ , where  $\mathcal{C}_3$  is the category described below:



4. A coequaliser of  $f, g : A \rightarrow B$  is the colimit of the diagram  $D : \mathcal{C}_4 \rightarrow \mathcal{C}$  where  $\mathcal{C}_4$  is the category described below:





So in fact products, equalisers, terminal objects, pullbacks are special cases of limits (and their duals are special cases of colimits). Before exploring another link between those constructions, let us give two other instances of limits and colimits that may be useful in example-building.

*Remark 6.30.* Consider the category  $\mathcal{D}$  with only one object (and the identity morphism).

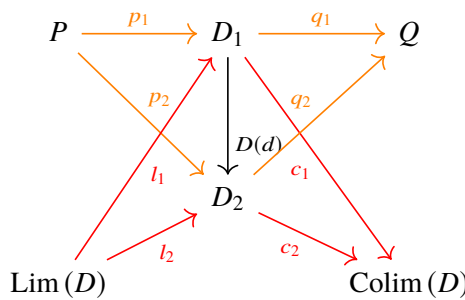
A diagram  $D : \mathcal{D} \rightarrow \mathcal{C}$  may be identified to the single object  $D(0) = D_0$ . It is easy to see that  $D_0$  is its own limit and colimit.

*Remark 6.31.* Consider the category  $\mathcal{D}$  consisting in two objects and one arrow:

$$\cdot \rightarrow \cdot$$

Consider a diagram  $D : \mathcal{D} \rightarrow \mathcal{C}$ ; its image will be an arrow  $D(d) : D_1 \rightarrow D_2$ . What could be the limit and colimit of this diagram?

Let  $(P, (p_1, p_2))$  be a cone to  $D$ , and suppose  $D$  has a limit  $(\text{Lim}(D), (l_1, l_2))$ .



The limit  $\text{Lim}(D)$  is such that there is a unique arrow  $x : P \rightarrow \text{Lim}(D)$  such that:

$$\begin{aligned} p_i &= l_i \circ x \quad (i = 1, 2) \\ p_2 &= D(d) \circ p_1 \\ l_2 &= D(d) \circ l_1 \end{aligned}$$

A cone to  $D$  defines two arrows  $p_1 : P \rightarrow D_1$  and  $p_2 : P \rightarrow D_2$  such that  $p_2 = D(d) \circ p_1$ . So, given a cone  $(P, (p_1, p_2))$  to  $D$ , there is a unique arrow  $p_1 : P \rightarrow D_1$  such that the diagram commutes. In fact,  $(D_1, (\text{id}_{D_1}, D(d)))$  is the limit of  $D$ .

With the same reasoning, it is easy to see that  $(D_2, (D(d), \text{id}_{D_2}))$  is the colimit of  $D$ .

This is a better way to state this remark (better for memory): considering an arrow  $A \rightarrow B$ ,  $A$  is its limit and  $B$  is its colimit. The limit is the domain, and the colimit is the codomain of the arrow.

*Remark 6.32.* Categories may or may not have all limits or all colimits, maybe for some diagrams and not others. However, diagrams from the one-object and the two-object categories always have a limit and a colimit in any category.

There is another, stronger link between products, equalisers, terminal objects and pullbacks. A similar link exists between their dual counterparts, see Theorem 6.35.

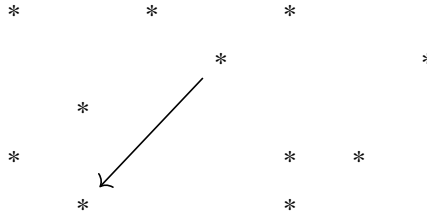
**Lemma 6.33.** *Let  $\mathcal{C}$  be any category.*

*If  $\mathcal{C}$  has finite products and equalisers, then  $\mathcal{C}$  has finite limits.*

*Proof.* (The proof written here is a resolution of [2, Exercise 3, Section 2.13, Chapter 9])

We will start the proof with one special case of index category. We then give a hint for a second special case. Those two proofs generalise to any index category.

Suppose  $\mathcal{I}$  is any finite category with only one non-identity arrow  $a : j \rightarrow k$ . It will then look like this category:



Now let  $D : \mathcal{I} \rightarrow \mathcal{C}$  be any diagram.

As  $\mathcal{C}$  has finite products, the product  $\prod_{i \in \mathcal{I}} D_i$  with arrows  $\pi_i : \prod_{i \in \mathcal{I}} D_i \rightarrow D_i$  exists. As  $\mathcal{C}$  has equalisers, consider the equaliser  $(E, e)$  of  $D(a) \circ \pi_j$  and  $\pi_k$ .

$$E \xrightarrow{e} \prod_{i \in \mathcal{I}} D_i \xrightarrow[\pi_k]{D(a) \circ \pi_j} D_k$$

Define  $\varepsilon = (e_i = \pi_i \circ e)_{i \in \mathcal{I}}$ . By definition of  $(E, e)$ , we have:

$$D(a) \circ \pi_j \circ e = \pi_k \circ e$$

which proves that  $\varepsilon$  is a natural transformation  $\Delta(\text{Lim}(D)) \rightarrow D$  (there is only one arrow to check).

We now prove that  $(E, \varepsilon)$  is the limit of the diagram  $D : \mathcal{I} \rightarrow \mathcal{C}$ . Let  $\alpha : \Delta(C) \rightarrow D$  be a cone to  $D$ ; we have  $D(a) \circ \alpha_j = \alpha_k$ .

Consider the function  $\Pi\alpha : C \rightarrow \prod_{i \in \mathcal{I}} D_i$  such that  $\forall i \in \mathcal{I}, \pi_i \circ \Pi\alpha = \alpha_i$ . We have:

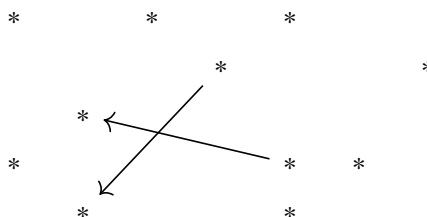
$$\begin{aligned} D(a) \circ \alpha_j &= \alpha_k \\ D(a) \circ \pi_j \circ \Pi\alpha &= \pi_k \circ \Pi\alpha \end{aligned}$$

As  $(E, e)$  is an equaliser of  $D(a) \circ \pi_j$  and  $\pi_k$ , there exists a unique  $u : C \rightarrow E$  such that  $e \circ u = \Pi\alpha$ , from which we infer, for all  $i \in \mathcal{I}$ :

$$\begin{aligned} e \circ u &= \Pi\alpha \\ \pi_i \circ e \circ u &= \pi_i \circ \Pi\alpha \\ e_i \circ u &= \alpha_i \\ \Rightarrow \varepsilon \circ \Delta(u) &= \alpha \end{aligned}$$

So  $(E, \varepsilon)$  is the limit of  $D$ .

Now suppose  $\mathcal{I}$  is any finite category with only two non-identity arrow  $a_0 : j_0 \rightarrow k_0$  and  $a_1 : j_1 \rightarrow k_1$ . It will then look like this category:



Note that no assumption is made about  $a_0$  and  $a_1$  being distinct; we only suppose that  $k_0 \neq j_1$  and  $j_0 \neq k_1$ ; otherwise they would compose and give birth to a third arrow.

For a diagram  $D : \mathcal{J} \rightarrow \mathcal{C}$ , we also build the product  $\prod_{i \in \mathcal{J}} D_i$  with its projections  $\pi_i : \prod_{i \in \mathcal{J}} D_i \rightarrow D_i$ .

We also define the following arrows:

$$\begin{aligned} r_0 &= D(a_0) \circ \pi_{j_0} \\ r_1 &= D(a_1) \circ \pi_{j_1} \\ s_0 &= \pi_{k_0} \\ s_1 &= \pi_{k_1} \\ r &= (r_0, r_1) \\ s &= (s_0, s_1) \end{aligned}$$

As  $\mathcal{C}$  has equalisers, consider the equaliser  $(E, e)$  of  $r : \prod_{i \in \mathcal{J}} D_i \rightarrow D_{k_0} \times D_{k_1}$  and  $s : \prod_{i \in \mathcal{J}} D_i \rightarrow D_{k_0} \times D_{k_1}$ . The proof is very similar to the previous one. If  $\alpha : \Delta(C) \rightarrow D$  is a cone to  $D$ , then we define  $\Pi\alpha$  to be the concatenation of the components of  $\alpha$ :  $\forall i \in \mathcal{J}, \alpha_i = \pi_i \circ \Pi\alpha$ . We check that  $s \circ \Pi\alpha = r \circ \Pi\alpha$  using the fact that  $\alpha$  is a natural transformation. As  $(E, e)$  is an equaliser, there exists a unique  $u : C \rightarrow E$  such that  $e \circ u = \Pi\alpha$ , and we conclude that  $\varepsilon \circ \Delta(u) = \Pi\alpha$ , with  $\varepsilon = (e_i = \pi_i \circ e)_{i \in \mathcal{J}}$  (which is a natural transformation  $\Delta(E) \rightarrow D$ ). Finally,  $(E, \varepsilon)$  is the limit of  $D$ .

As the final case, let  $\mathcal{J}$  be any finite category. Again,  $\mathcal{C}$  has finite products, so we define  $\prod_{i \in \mathcal{J}} D_i$  and its projections  $\pi_i$ . As the set of arrows in  $\mathcal{J}$  is also finite, we can consider all arrows  $a : j \rightarrow k \in \text{Mor}_{\mathcal{J}}$  and define the product  $\prod_{a: j \rightarrow k \in \text{Mor}_{\mathcal{J}}} D_k$ , that is, the product of all codomains of all arrows in  $\mathcal{J}$ . For  $b : m \rightarrow n$ , the projection of index  $a$  will be denoted  $\pi_a : \prod_{a: j \rightarrow k \in \text{Mor}_{\mathcal{J}}} D_k \rightarrow D_n$ .

We now define:

$$r, s : \prod_{i \in \mathcal{J}} D_i \rightarrow \prod_{a: j \rightarrow k \in \text{Mor}_{\mathcal{J}}} D_k$$

such that, for all  $a : j \rightarrow k \in \text{Mor}_{\mathcal{J}}$ , we have:

$$\begin{aligned} \pi_a \circ r &= D(a) \circ \pi_j = D(a) \circ \pi_{\text{dom } a} \\ \pi_a \circ s &= \pi_k = \pi_{\text{cod } a} \end{aligned}$$

Let  $(E, e)$  be an equaliser of  $r$  and  $s$ ; the rest of the proof is very similar to the previous two ones.  $\square$

We can now bring together all the lemmas that we disseminated throughout the last two sections, and prove:

**Theorem 6.34.** *Let  $\mathcal{C}$  be any category. The following propositions are equivalent:*

1.  $\mathcal{C}$  has finite products and equalisers
2.  $\mathcal{C}$  has pullbacks and a terminal object
3.  $\mathcal{C}$  has finite limits

*Proof.* (1  $\Rightarrow$  2)

By Corollary 5.61, products and equalisers give pullbacks, while by Lemma 5.31, having all finite products gives a terminal object.

(2  $\Rightarrow$  1)

By Lemma 5.62, pullbacks and terminal object give products, while by Lemma 5.63, pullbacks and products give equalisers.

(1  $\Rightarrow$  3)

By Lemma 6.33, products and equalisers give limits.

(3  $\Rightarrow$  1)

By Proposition 6.28, products and equalisers are special cases of limits. □

Of course, the dual theorem is also true:

**Theorem 6.35.** *Let  $\mathcal{C}$  be any category. The following propositions are equivalent:*

1.  $\mathcal{C}$  has finite coproducts and coequalisers
2.  $\mathcal{C}$  has pushouts and an initial object
3.  $\mathcal{C}$  has finite colimits

Note that the theorems we mentioned with limits used any index category, be it small or large. Some results we proved only for finite limits, but, for example, Proposition 6.34 generalises to any cardinality (and thus, for any small category):

**Theorem 6.36.** *Let  $\kappa$  be a cardinal and let  $\mathcal{C}$  be any category.*

*$\mathcal{C}$  has all products of cardinality  $\leq \kappa$  and equalisers  $\Leftrightarrow \mathcal{C}$  has all limits of cardinality  $\leq \kappa$ .*

*In other words,  $\mathcal{C}$  has all small products and equalisers  $\Leftrightarrow \mathcal{C}$  has all small limits.*

## 7. Limits and adjunctions

We will now study some properties of adjunctions and their behaviour with regards to limits.

We start with a remark. We saw in Theorem 5.20 that  $+ \dashv \Delta_2 \dashv \times$ . There is a more general statement, that we mention but will only give a sketch of proof.

Let  $\mathcal{C}$  be a category with finite limits, and let  $\mathcal{I}$  be any finite category. The mappings:

$$\begin{aligned} \text{Lim}(-) : \begin{cases} \mathcal{C}^{\mathcal{I}} & \longrightarrow & \mathcal{C} \\ D & \longmapsto & \text{Lim}(D) \end{cases} \\ \text{Colim}(-) : \begin{cases} \mathcal{C}^{\mathcal{I}} & \longrightarrow & \mathcal{C} \\ D & \longmapsto & \text{Colim}(D) \end{cases} \end{aligned}$$

are in fact functors. What is best, is that if  $\Delta_{\mathcal{I}} : \mathcal{C} \rightarrow \mathcal{C}^{\mathcal{I}}$  is the diagonal functor of shape  $\mathcal{I}$ , then we have:

**Theorem 7.1.**  $\text{Colim}(-) \dashv \Delta_{\mathcal{I}} \dashv \text{Lim}(-)$

*Sketch of proof.* This can be deduced from the definitions of a limit (resp. of a colimit), due to the existence and unicity of the arrow  $C \rightarrow \text{Lim}(D)$  (resp.  $\text{Colim}(D) \rightarrow C$ ) whenever we have a cone  $\Delta_{\mathcal{I}}(C) \rightarrow D$  (resp. a cocone  $D \rightarrow \Delta_{\mathcal{I}}(C)$ ). This gives the bijectivity between  $\text{Hom}_{\mathcal{C}^{\mathcal{I}}}(\Delta_{\mathcal{I}}(C), D)$  and  $\text{Hom}_{\mathcal{C}}(C, \text{Lim}(D))$  (resp.  $\text{Hom}_{\mathcal{C}}(\text{Colim}(D), C)$  and  $\text{Hom}_{\mathcal{C}^{\mathcal{I}}}(D, \Delta_{\mathcal{I}}(C))$ ). We then have to check that this defines a natural transformation. The (contravariant) naturality in  $C$  is easy, due to the definition of  $\Delta_{\mathcal{I}}$ , while the naturality in  $D$  requires a bit more attention not to confuse between cones and a natural transformation  $\delta : D \rightarrow D'$ .  $\square$

The main question we will tackle in this section is the following. Suppose we have a functor  $F : \mathcal{C} \rightarrow \mathcal{D}$ . Does it have an adjoint? How to know if it does or not? And if it does, how to find it?

A first step may be to look at some properties of adjoints.

We deduce from Definition 4.1 and Theorem 3.17 that:

**Proposition 7.2.** *Let  $U : \mathcal{C} \rightarrow \mathbf{Sets}$  be functors.*

*$U$  has a left adjoint  $\Leftrightarrow$  for all  $C \in \mathcal{C}$ ,  $\text{Hom}_{\mathbf{Sets}}(C, U(-))$  has a universal element.*

Another interesting property of adjoints is described right after the following definition. Just as some functors preserve products (see Definition 5.9), some functors preserve limits:

**Definition 7.3** (Preserving limits and colimits). Let  $\mathcal{I}$  be an index category, and let  $\mathcal{C}, \mathcal{D}$  be categories.

We say that the functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  *preserves all limits* (resp. *small limits*; resp. *finite limits*) when, for all index category (resp. small index category; resp. finite index category)  $\mathcal{I}$ , for all diagram  $D : \mathcal{I} \rightarrow \mathcal{C}$ , if the limit  $(\text{Lim}(D), \eta_D)$  exists, then  $(F(\text{Lim}(D)), F(\eta_D))$  is the limit of the diagram  $F \circ D$ .

Dually, we say that the functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  *preserves all colimits* (resp. *small colimits*; resp. *finite colimits*) when, for all index category (resp. small index category; resp. finite index category)  $\mathcal{I}$ , for all diagram  $D : \mathcal{I} \rightarrow \mathcal{C}$ , if the colimit  $(\text{Colim}(D), \varepsilon_D)$  exists, then  $(F(\text{Colim}(D)), F(\varepsilon_D))$  is the colimit of the diagram  $F \circ D$ .

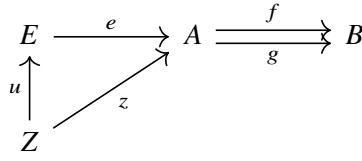
**Proposition 7.4.** *Let  $\mathcal{C}$  be a category with finite limits and let  $C \in \text{Ob}_{\mathcal{C}}$ .*

*The covariant Hom-set functor  $\text{Hom}_{\mathcal{C}}(C, -)$  preserves all finite limits.*

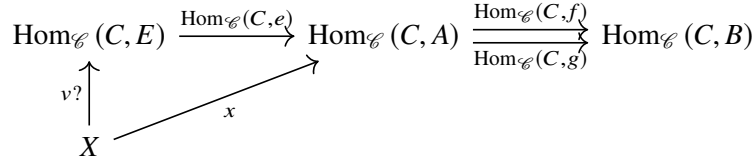
*Proof.* By Theorem 6.34, it suffices to show that  $\text{Hom}_{\mathcal{C}}(C, -)$  preserves finite products and equalisers.

We already know from Proposition 5.10 that  $\text{Hom}_{\mathcal{C}}(C, -)$  preserves binary products. For it to preserve finite products, we need to show that it preserves the terminal object. If  $T$  is the terminal object in  $\mathcal{C}$  then  $\text{Hom}_{\mathcal{C}}(C, T)$  contains only one arrow (the unique arrow  $C \rightarrow T$  in  $\mathcal{C}$ ). Consequently,  $\text{Hom}_{\mathcal{C}}(C, T) \cong 1$  (where  $1$  is the terminal object of  $\mathbf{Sets}$ ) and  $\text{Hom}_{\mathcal{C}}(C, -)$  preserves the terminal object.

Now, let  $(E, e)$  be the equaliser of  $f, g : A \rightarrow B$ . For all  $z : Z \rightarrow A$  such that  $f \circ z = g \circ z$ , there is a unique  $u : Z \rightarrow E$  such that the following diagram commutes:



The hom-set functor preserves the diagram (this is a property of functors). We need to check whether  $(\text{Hom}_{\mathcal{C}}(C, E), \text{Hom}_{\mathcal{C}}(C, e))$  is an equaliser of  $\text{Hom}_{\mathcal{C}}(C, f)$  and  $\text{Hom}_{\mathcal{C}}(C, g)$ . Let  $h : X \rightarrow \text{Hom}_{\mathcal{C}}(C, A)$  such that  $\text{Hom}_{\mathcal{C}}(C, f) \circ h = \text{Hom}_{\mathcal{C}}(C, g) \circ h$  as in the diagram:



We need to find a  $v : X \rightarrow \text{Hom}_{\mathcal{C}}(C, E)$  such that  $\text{Hom}_{\mathcal{C}}(C, e) \circ v = h$ .  
 Let  $x \in X$ . We have  $h(x) : C \rightarrow A$  and:

$$\begin{aligned}
 \text{Hom}_{\mathcal{C}}(C, f) \circ h(x) &= \text{Hom}_{\mathcal{C}}(C, g) \circ h(x) \\
 f \circ (h(x)) &= g \circ (h(x))
 \end{aligned}$$

So, the equaliser in  $\mathcal{C}$  applies here: there is a unique  $u(x) : C \rightarrow E$  such that  $e \circ (u(x)) = h(x)$ . Define  $u$  to be:

$$u : \begin{cases} X & \longrightarrow & \text{Hom}_{\mathcal{C}}(C, E) \\ x & \longmapsto & u(x) \end{cases}$$

Then, by construction, for all  $x \in X$ ,  $h(x) = e \circ (u(x)) = (\text{Hom}_{\mathcal{C}}(C, e) \circ u)(x)$  and  $u$  is unique. Consequently,  $(\text{Hom}_{\mathcal{C}}(C, E), \text{Hom}_{\mathcal{C}}(C, e))$  is still an equaliser.  $\square$

**Corollary 7.5.** *Representable functors preserve all finite limits.*

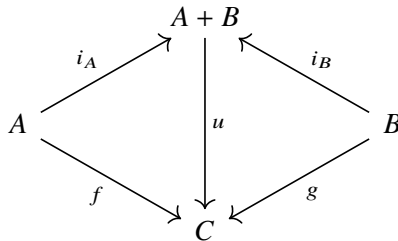
The dual version of this theorem is the following:

**Proposition 7.6.** *The contravariant Hom-set functor  $\text{Hom}_{\mathcal{C}}(-, C)$  sends finite colimits to finite limits.*

*Proof.* We have to show that the contravariant sends the initial object to the terminal object, the coproduct to the product, and the coequalisers to equalisers.

We have  $\text{Hom}_{\mathcal{C}}(0, C) \cong 1$  because there is only one arrow  $0 \rightarrow C$  (definition of an initial object).

Let  $f : A \rightarrow C$  and  $g : B \rightarrow C$  be two arrows; by definition of the coproduct  $A + B$ , there is a unique  $u : A + B \rightarrow C$  such that the following diagram commutes:



This exactly says that there is a bijection (an isomorphism in **Sets**):

$$\alpha_{A,B} : \begin{cases} \text{Hom}_{\mathcal{C}}(A + B, C) & \longrightarrow & \text{Hom}_{\mathcal{C}}(A, C) \times \text{Hom}_{\mathcal{C}}(B, C) \\ u & \longmapsto & (u \circ i_A, u \circ i_B) \end{cases}$$

This isomorphism is natural in  $A + B$ :

$$\begin{array}{ccccc}
 A & & \text{Hom}_{\mathcal{C}}(A' + B, C) & \xrightarrow{\alpha_{A',B}} & \text{Hom}_{\mathcal{C}}(A', C) \times \text{Hom}_{\mathcal{C}}(B, C) \\
 \downarrow a & \rightsquigarrow & \downarrow & & \downarrow \text{Hom}_{\mathcal{C}}(a, C) \times \text{Hom}_{\mathcal{C}}(B, C) \\
 A' & & \text{Hom}_{\mathcal{C}}(A + B, C) & \xrightarrow{\alpha_{A,B}} & \text{Hom}_{\mathcal{C}}(A, C) \times \text{Hom}_{\mathcal{C}}(B, C)
 \end{array}$$

Let  $f \in \text{Hom}_{\mathcal{C}}(A' + B, C)$ :

$$\begin{aligned}
 \text{Hom}_{\mathcal{C}}(a, C) \times \text{Hom}_{\mathcal{C}}(B, C) \circ \alpha_{A',B}(f) &= \text{Hom}_{\mathcal{C}}(a, C) \times \text{Hom}_{\mathcal{C}}(B, C)(f \circ i_{A'}, f \circ i_B) \\
 &= (f \circ i_{A'} \circ a, f \circ i_B) \\
 \alpha_{A,B} \circ \text{Hom}_{\mathcal{C}}(A + B, C)(f) &= \alpha_{A,B}(f \circ (a + \text{id}_B)) \\
 &= (f \circ (a + \text{id}_B) \circ i_A, f \circ (a + \text{id}_B) \circ i_B)
 \end{aligned}$$

Seeing the coproduct as a colimit, we deduce that the following diagram commutes:

$$\begin{array}{ccccc}
 & & A + B & & \\
 & \nearrow i_A & \downarrow a + \text{id}_B & \nwarrow i_B & \\
 A & & & & B \\
 \downarrow a & & & & \downarrow \text{id}_B \\
 & \searrow i_{A'} & A' + B & \swarrow i_B & \\
 A' & & & & B
 \end{array}$$

(Seeing  $i = (i_A : A \rightarrow A + B)_{A \in \mathcal{C}}$  as a natural transformation  $\text{Id}_{\mathcal{C}} \rightarrow \text{Id}_{\mathcal{C}} + \Delta(B)$ )

We have:

$$\begin{aligned}
 (a + \text{id}_B) \circ i_A &= i_{A'} \circ a \\
 (a + \text{id}_B) \circ i_B &= i_B \circ \text{id}_B \\
 \Rightarrow (f \circ (a + \text{id}_B) \circ i_A, f \circ (a + \text{id}_B) \circ i_B) &= (f \circ i_{A'} \circ a, f \circ i_B)
 \end{aligned}$$

The naturality in  $B$  is similar.

Finally, as for seeing that the contravariant Hom-set functor sends coequalisers to equalisers, the proof is very similar to showing that the covariant Hom-set functor preserves equalisers.  $\square$

In fact, these theorems are not only true for finite limits, but also for small limits. **As there is something I don't understand here, because a product of any set of sets could be empty (without the Axiom of Choice) (but always exists?)**, we will trust Awodey [1, Chapter 5, Proposition 5.25, p107] and admit the following proposition and corollaries:

**Proposition 7.7.** *Let  $\mathcal{C}$  be a category with small limits and let  $C \in \text{Ob}_{\mathcal{C}}$ .*

*The covariant Hom-set functor  $\text{Hom}_{\mathcal{C}}(C, -)$  preserves all small limits.*

**Corollary 7.8.** *Representable functors preserve all small limits.*

**Proposition 7.9.** *The contravariant Hom-set functor  $\text{Hom}_{\mathcal{C}}(-, C)$  sends small colimits to small limits.*

**Proposition 7.10** (Right Adjoints Preserve Limits [2], [1]). *Let  $(F, U, \beta)$  be an adjunction. Then  $F$  preserves colimits and  $U$  preserves limits.*

This proposition is commonly referred to as the "RAPL" ("Right Adjoints Preserve Limits").

*Proof.* Suppose  $F : \mathcal{C} \rightarrow \mathcal{X}$ ,  $U : \mathcal{X} \rightarrow \mathcal{C}$  and let  $D : \mathcal{I} \rightarrow \mathcal{X}$  be a diagram with a limit  $\text{Lim}(D)$ . For  $C \in \mathcal{C}$ , we have:

$$\begin{aligned} \text{Hom}_{\mathcal{C}}(C, U(\text{Lim}(D))) &\cong \text{Hom}_{\mathcal{C}}(F(C), \text{Lim}(D)) \\ &\cong \text{Lim}(\text{Hom}_{\mathcal{C}}(F(C), D(-))) \\ &\cong \text{Lim}(\text{Hom}_{\mathcal{C}}(C, U \circ D(-))) \\ &\cong \text{Hom}_{\mathcal{C}}(C, \text{Lim}(U \circ D)) \end{aligned}$$

The first and third equations are due to the adjunction, while the second and fourth are due to the preservation of limits by the Hom-set functor (Proposition 7.4). As a consequence of Yoneda Lemma (Corollary 2.16), we deduce:

$$U(\text{Lim}(D)) \cong \text{Lim}(U \circ D)$$

Similarly, if  $D : \mathcal{I} \rightarrow \mathcal{X}$  has a colimit  $\text{Colim}(D)$ , and for  $X \in \mathcal{X}$ :

$$\begin{aligned} \text{Hom}_{\mathcal{C}}(F(\text{Colim}(D)), X) &\cong \text{Hom}_{\mathcal{C}}(\text{Colim}(D), U(X)) \\ &\cong \text{Lim}(\text{Hom}_{\mathcal{C}}(D(-), U(X))) \\ &\cong \text{Lim}(\text{Hom}_{\mathcal{C}}(F \circ D(-), X)) \\ &\cong \text{Hom}_{\mathcal{C}}(\text{Colim}(F \circ D), X) \end{aligned}$$

Which also gives (by Corollary 2.17):

$$F(\text{Colim}(D)) \cong \text{Colim}(F \circ D) \tag{18}$$

□

**Definition 7.11** (Complete category). A category  $\mathcal{C}$  is said *complete* (resp. *cocomplete*) when it has all small limits (resp. all small colimits).

*Example 7.12.* The category **Sets** is complete and cocomplete. Of course **Sets** has all finite limits (because it has products, equalisers, a terminal object and pullbacks) and colimits (because it has coproducts, coequalisers, an initial object and pushouts), but in the rest of this course, we will just admit that **Sets** has all small limits and colimits.

**Definition 7.13** (Continuous functor). A functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is called *continuous* (resp. *cocontinuous*) if it preserves all small limits (resp. all small colimits).

*Example 7.14.* The covariant Hom-set functor is continuous but not cocontinuous.

We now move towards the next important theorem: the adjoint functor theorem. There are a few details to expand before.

**Definition 7.15** (Weakly initial [6]). Let  $\mathcal{C}$  be any category.

A set  $C^* = \{C_i \in \text{Ob}_{\mathcal{C}} \mid i \in I\}$  of objects in  $\mathcal{C}$  is a *weakly initial set* when for all  $C \in \mathcal{C}$ , there is an  $i \in I$  such that there is an arrow  $C_i \rightarrow C$ .



"There exists a set of objects that connects to any other object of the category" or "there is a (non-necessarily connected) subgraph that is connected to the rest of the category", or "there is some *weakly initial* subset of objects".

**Lemma 7.16** ([6], [1]). *Let  $\mathcal{C}$  be a locally small, complete category.*

*$\mathcal{C}$  has an initial object  $\Leftrightarrow$  There is a weakly initial set of objects in  $\mathcal{C}$*

*Proof.* If  $\mathcal{C}$  has an initial object, then any set containing that initial object is weakly initial.

Suppose that  $(C_i)_{i \in I}$  is a weakly initial set in  $\mathcal{C}$ .

Consider the category  $\mathcal{C}_I$  defined by:

**Objects:** The objects of  $\mathcal{C}_I$  are the  $C_i$  for  $i \in I$

**Morphisms:** If  $c \in \text{Mor}_{\mathcal{C}}$  is an arrow  $C_i \rightarrow C_j$  for  $i, j \in I$ , then  $c \in \mathcal{C}_I$

**Identities:** An identity morphism of an object  $C_i$  is an identity morphism  $\text{id}_{C_i} \in \mathcal{C}$

**Composition:** The composition law for morphisms is the usual composition in  $\mathcal{C}$

It is easy to see that  $\mathcal{C}_I$  is a small subcategory of  $\mathcal{C}$ . Then, the inclusion mapping:

$$F : \begin{cases} \mathcal{C}_I & \longrightarrow & \mathcal{C} \\ C & \longmapsto & C \\ c & \longmapsto & c \end{cases}$$

is a functor; or rather, as  $\mathcal{C}_I$  is small,  $F$  is a small diagram. As  $\mathcal{C}$  is complete, it has a limit  $\text{Lim}(F)$ .

We now show that  $\text{Lim}(F)$  is initial. Clearly, for all  $C \in \mathcal{C}$ , there is an arrow  $c_i : C_i \rightarrow C$ . As  $\text{Lim}(F)$  is the limit of  $F$ , there is also an arrow  $\eta_i : \text{Lim}(F) \rightarrow C_i$ , so for all  $C \in \mathcal{C}$ , there is an arrow:  $\text{Lim}(F) \rightarrow C$ , but this arrow is not necessarily unique.

Let  $f, g : \text{Lim}(F) \rightarrow C$  be two arrows, and let  $(E, e)$  be an equaliser of  $f$  and  $g$ . There is an  $i \in I$  such that:

$$\begin{array}{ccccc} & & E & \xrightarrow{e} & \text{Lim}(F) & \xrightarrow[f]{g} & C \\ & \nearrow^{c_i} & & & \nearrow^{\text{id}_{\text{Lim}(F)}} & & \\ C_i & & & & & & \\ & \searrow_{\eta_i} & & & & & \\ & & \text{Lim}(F) & & & & \end{array}$$

Besides, by unicity of the arrow  $\text{Lim}(F) \rightarrow \text{Lim}(F)$ , we deduce that:

$$e \circ c_i \circ \eta_i = \text{id}_{\text{Lim}(F)}$$

which gives:

$$\begin{aligned} f &= f \circ e \circ c_i \circ \eta_i \\ &= g \circ e \circ c_i \circ \eta_i \\ &= g \end{aligned}$$

using the fact that  $f \circ e = g \circ e$ .

Consequently, there is a unique arrow  $\text{Lim}(F) \rightarrow C$  for any  $C \in \mathcal{C}$ ; so  $\text{Lim}(F)$  is an initial object.  $\square$

**Definition 7.17** (Comma-category). Let  $F : \mathcal{C} \rightarrow \mathcal{X}$  and  $G : \mathcal{D} \rightarrow \mathcal{X}$  be two functors.

The *comma-category*  $(F | G)$  is the category described below:

**Objects:** The objects of  $(F | G)$  are triples  $(C, f, D)$  such that  $f : F(C) \rightarrow G(D)$ ,  $C \in \mathcal{C}$  and  $D \in \mathcal{D}$

**Morphisms:** A morphism  $(C, f, D) \rightarrow (C', f', D')$  is a pair  $(c, d)$  such that  $c : C \rightarrow C'$ ,  $d : D \rightarrow D'$  and the following square commutes:

$$\begin{array}{ccc} F(C) & \xrightarrow{F(c)} & F(C') \\ f \downarrow & \checkmark & \downarrow f' \\ G(D) & \xrightarrow{G(d)} & G(D') \end{array}$$

**Identities:** The identity morphism of an object  $(C, f, D)$  is the pair  $(\text{id}_C, \text{id}_D)$

**Composition:** The composition law for morphisms is the usual composition  $(c, d) \circ (c', d') = (c \circ c', d \circ d')$

If  $F$  is the diagonal functor  $\Delta(C)$  for some  $C \in \mathcal{C}$ , then the comma-category is written  $(C | G)$  and simplifies to:

**Objects:** The objects of  $(C | G)$  are pairs  $(D, f)$  such that  $f : C \rightarrow G(D)$  and  $D \in \mathcal{D}$

**Morphisms:** A morphism  $(D, f) \rightarrow (D', f')$  is an arrow  $d : D \rightarrow D'$  such that the following square commutes:

$$\begin{array}{ccc} & & G(D) \\ & \nearrow h & \downarrow G(d) \\ F(C) & & G(D') \\ & \searrow h' & \end{array}$$

**Identities:** The identity morphism of an object  $(D, f)$  is an identity arrow  $\text{id}_D$

**Composition:** The composition law for morphisms is the usual composition in  $\mathcal{D}$

Now, we consider the comma-category  $(C | U)$  where  $U : \mathcal{X} \rightarrow \mathcal{C}$  and  $C \in \mathcal{C}$ .

**Lemma 7.18.** *Let  $(C | U)$  where  $U : \mathcal{X} \rightarrow \mathcal{C}$  and  $C \in \mathcal{C}$ .*

*If  $\mathcal{X}$  is locally small, then the comma-category  $(C | U)$  is also locally small.*

*Proof.* Just note that, for any  $X, X' \in \mathcal{X}$ ,  $\text{Hom}_{\mathcal{C}}(X, X')$  is a set. Also, note that  $\text{Hom}_{\mathcal{C}}((X, f), (X', f')) \subset \text{Hom}_{\mathcal{C}}(X, X')$ . □

**Lemma 7.19.** *Let  $(C | U)$  where  $U : \mathcal{X} \rightarrow \mathcal{C}$  and  $C \in \mathcal{C}$ .*

*If  $\mathcal{X}$  is complete and  $U$  preserves small limits, then  $(C | U)$  is also complete.*

*Proof.* It is easy to check that, due to the preservation of limits by  $U$ , the comma-category  $(C | U)$  has products and equalisers. □

J'ai la flemme

**Lemma 7.20.** *Let  $(C | U)$  where  $U : \mathcal{X} \rightarrow \mathcal{C}$  and  $C \in \mathcal{C}$ .*

*The universal arrows from  $C$  to  $U$  are the initial objects of  $(C | U)$ .*

*Proof.* Let us recall the definition of a universal arrow from  $C$  to  $U$ : it is a pair  $(U_C^\#, \eta_C)$  such that:

- $U_C^\# \in \mathcal{X}$  and  $\eta_C : C \rightarrow U(U_C^\#)$
- for all  $X \in \mathcal{X}$ , for all  $c : C \rightarrow U(X)$ , there is a unique  $x : U_C^\# \rightarrow X$  such that  $c = U(x) \circ \eta_C$

An initial object in  $(C | U)$  is a pair  $(I, i)$  where  $I \in \mathcal{X}$  and  $i : C \rightarrow U(I)$  such that for all object  $(X, c) \in (C | U)$ , there is a unique arrow  $x$  such that the following triangle commutes:

$$\begin{array}{ccc}
 & & U(I) \\
 & \nearrow i & \downarrow U(x) \\
 C & & U(X) \\
 & \searrow c & \\
 & & 
 \end{array}$$

Both definitions are equivalent, hence the result. □

**Lemma 7.21.** *Let  $U : \mathcal{X} \rightarrow \mathcal{C}$ .*

*$U$  has a left adjoint  $\Leftrightarrow$  for each  $C \in \mathcal{C}$ , the comma-category  $(C | U)$  has an initial object.*

*Proof.* Combine the previous lemma (Lemma 7.20) and Definition 4.1. □

We can finally prove the following version of the Adjoint Functor Theorem, as it appears in [1]:

**Theorem 7.22** (Adjoint Functor Theorem - Awodey version [1]). *Let  $\mathcal{X}$  be locally small and complete. Let  $\mathcal{C}$  be any category and let  $U : \mathcal{X} \rightarrow \mathcal{C}$  be a continuous functor.*

*$U$  has a left adjoint  $\Leftrightarrow$  for each object  $C \in \mathcal{C}$ , the comma-category  $(C | U)$  has a weakly initial set.*

*Proof.* By Lemma 7.21,  $U$  has a left adjoint iff for each  $C \in \mathcal{C}$ , the comma-category  $(C | U)$  has an initial object. As  $\mathcal{X}$  is locally small and complete, by Lemma 7.16, for each  $C \in \mathcal{C}$ , the comma-category  $(C | U)$  has an initial object iff for each object  $C \in \mathcal{C}$ , the comma-category  $(C | U)$  has a weakly initial set. □

The following variant is also called Adjoint Functor Theorem:

**Theorem 7.23** (Adjoint Functor Theorem - Leinster version [6]). *Let  $\mathcal{X}$  be locally small and complete. Let  $\mathcal{C}$  be any category and let  $U : \mathcal{X} \rightarrow \mathcal{C}$  be a functor such that for each object  $X \in \mathcal{X}$ , the comma-category  $(X | U)$  has a weakly initial set.*

*$U$  has a left adjoint  $\Leftrightarrow U$  preserves limits.*

*Proof.* **[Proof of  $\Rightarrow$ ]**

Direct consequence of Proposition 7.10.

**[Proof of  $\Leftarrow$ ]**

By Lemma 7.18,  $\mathcal{X}$  is locally small, so is  $(X | U)$ . By Lemma 7.19, as  $U$  preserves limits and as  $\mathcal{X}$  is complete,  $(X | U)$  is complete. We can then use Lemma 7.16: for each object  $X \in \mathcal{X}$ ,  $(X | U)$  has a weakly initial set, so for all  $X \in \text{Ob } \mathcal{X}$ ,  $(X | U)$  has an initial object. By Lemma 7.21,  $U$  has a left adjoint. □

Note that if the category  $\mathcal{X}$  is small (instead of only locally small) then the condition on the weakly initial set is useless. We then have the following corollary:

**Corollary 7.24.** *Let  $\mathcal{X}$  be small (not only locally small) and complete. Let  $U : \mathcal{X} \rightarrow \mathbf{Sets}$  be a functor. The following propositions are equivalent:*

1.  $U$  is continuous
2.  $U$  has a left adjoint
3.  $U$  is representable

*Proof.* The equivalence  $1 \Leftrightarrow 2$  is obvious.

The proof of  $3 \Leftrightarrow 1$  is easy: if  $U$  is representable, then  $U \cong \text{Hom}_{\mathcal{C}}(C_U, -)$  for some  $C_U \in \mathcal{C}$ ; and  $\text{Hom}_{\mathcal{C}}(C_U, -)$  is continuous (Proposition 7.4).

Je ne vois pas pour 1 ou 2 implique 3...

□

*Remark 7.25. A word on forgetful functors:* See Awodey, p243-245 for an explanation, and Mac Lane, chapter V, for a proof.

The following remark is beyond the scope of this course, so we will not go into the technical details.

Chercher comment Cori et Lascar définissent les langages en théorie des modèles

An important application of the Adjoint functor theorem is the following:

If  $T$  is a finite theory, with  $T - \mathbf{Models}$  being the category of the models of  $T$  and homomorphisms between them (in the model-theoretic sense), then the forgetful functor  $T - \mathbf{Models} \rightarrow \mathbf{Sets}$  has a left adjoint.

This is powerful because it means that we can, in a sense, add some structure to a Set in order to make it a group or a ring (not exactly because axiom of choice)

If  $U : T - \mathbf{Models} \rightarrow \mathbf{Sets}$  is the forgetful functor, then there is an adjoint  $F : \mathbf{Sets} \rightarrow T - \mathbf{Models}$  such that, for all  $S \in \mathbf{Sets}$ , for all  $M \in T - \mathbf{Models}$ , we have:

$$\text{Hom}_{T-\mathbf{Models}}(F(S), M) \cong \text{Hom}_{\mathbf{Sets}}(S, U(M))$$

The adjoint of  $U$  is called the free functor. In short, the free functor  $F : \mathbf{Sets} \rightarrow T - \mathbf{Models}$  is the functor that maps a set  $S$  to a structure "generated" by that set  $S$  (for example, if  $T$  is the theory of vector spaces, then the free functor  $F$  will consider that a given set  $S$  is a basis, and will build a vector space using this basis).

See MacLane Chapter IV, pp87-88 for a list of adjoints, some of them being between forgetful and free functors.

According to nLab: A general way to construct free functors is with a transfinite construction of free algebras (in set-theoretic foundations), or with an inductive type or higher inductive type (in type-theoretic foundations).

*Remark 7.26.* Consider two categories  $\mathbf{Sets}^{\mathcal{C}}$  and  $\mathcal{D}$ , and a functor  $L : \mathbf{Sets}^{\mathcal{C}} \rightarrow \mathcal{D}$ . Suppose we want to find  $R : \mathcal{D} \rightarrow \mathbf{Sets}^{\mathcal{C}}$  the right adjoint of  $L$  (we suppose that such an adjoint exists). We will study the behaviour of  $R$  on objects and arrows.

Let  $D \in \mathcal{D}$ ; we have  $R(D) \in \mathbf{Sets}^{\mathcal{C}}$ : it is a contravariant functor  $R(D) : \mathcal{C} \rightarrow \mathbf{Sets}$ . By Yoneda Lemma, for some  $C \in \mathcal{C}$ , we have:

$$\begin{aligned} R(D)(C) &\cong \text{Nat}(\text{Hom}_{\mathcal{C}}(C, -), R(D)) \\ &\cong \text{Nat}(L(\text{Hom}_{\mathcal{C}}(C, -)), D) \end{aligned}$$

where the second equation is the definition of an adjunction. The simplest choice of  $R(D)$  should be:

$$R(D) = \text{Nat}(L(\text{Hom}_{\mathcal{C}}(-, -)), D) \tag{19}$$

So we have the behaviour of  $R(D)$  on objects. On arrows  $c : C \rightarrow C'$ , we suppose that  $R(D)$  is a functor  $\mathcal{C} \rightarrow \mathbf{Sets}$ ; so, by Yoneda lemma, the following diagram should commute:

$$\begin{array}{ccc} C & & R(D)(C) \xrightarrow{\theta_{R(D),C}} \text{Nat}(\text{Hom}_{\mathcal{C}}(C, -), R(D)) \\ \downarrow c & \rightsquigarrow & \downarrow R(D)(c) \quad \quad \quad \downarrow \text{Nat}(\text{Hom}_{\mathcal{C}}(c, -), R(D)) \\ C' & & R(D)(C') \xrightarrow{\theta_{R(D),C'}} \text{Nat}(\text{Hom}_{\mathcal{C}}(C', -), R(D)) \\ & & \quad \quad \quad \downarrow \xi_{R(D),C'} \end{array}$$

where  $\xi_{R(D),C}$  is the Yoneda isomorphism  $\text{Nat}(\text{Hom}_{\mathcal{C}}(C,-),R(D)) \rightarrow R(D)(C)$  and  $\theta_{R(D),C}$  is its inverse (cf. Definitions 2.3 and 2.4).

We deduce a (brutal) formula for  $R(D)(c)$ :

$$R(D)(c) = \xi_{R(D),C'} \circ \text{Nat}(\text{Hom}_{\mathcal{C}}(c,-),R(D)) \circ \theta_{R(D),C}$$

Finally, we want the behaviour of  $R(d)$ . For  $C \in \mathcal{C}$ , we have:

$$R(d)(D) = \text{Nat}(L(\text{Hom}_{\mathcal{C}}(C,-)),d)$$

So we have now described the functor  $R : \mathcal{D} \rightarrow \mathbf{Sets}^{\mathcal{C}}$  in terms of functors and natural transformations whose expression we know.

In the same vein of the Adjoint Functor Theorem, the following proposition is sometimes useful when we have to prove that some functor has a right adjoint.

In the special case where we only have finite limits, we have the converse to Proposition 5.23 (exponential is right adjoint to product).

**Proposition 7.27.** *Let  $\mathcal{C}$  be a category with all finite limits, and  $C \in \mathcal{C}$ .*

*Let  $P_C$  be the functor:*

$$P_C : \begin{cases} \mathcal{C} & \longrightarrow & \mathcal{C} \\ X & \longmapsto & X \times C \\ x : X \rightarrow X' & \longmapsto & x \times \text{id}_C : X \times C \rightarrow X' \times C \end{cases}$$

*Then, there exists a right adjoint  $E_C$  to  $P_C \Leftrightarrow$  for all  $A$ , the exponential  $A^C$  exists.*

*Proof.* The proof falls beyond the scope of this course. See [2, Chapter 13, Section 13.3, Exercise 5, p359] for an exercise that will guide you into the proof of  $\Rightarrow$ .

Note that Proposition 5.23 is exactly  $\Leftarrow$ . □

This proposition will be useful later in order to prove that some functor in a category with finite limits, has a right adjoint.

## 8. Sets-like categories

Besides adjoints, elementary topoi (plural of "topos" in Greek) are the second big part of this course. Before exploring this notion, we have to introduce some amount of notions around the following theme: introduction of set-like elements in categories.

We start with the categorical equivalent of a subset.

In **Sets**, when  $X \subset Y$ , if  $x : X \rightarrow Y$  is the inclusion, then  $(X, x)$  is an equaliser, and  $x$  is a monomorphism.

In several categories based on sets (for example, the category of groups, the category of graphs, the category of rings; of "sets with structure"), when we have  $X \subset Y$ , if  $x : X \rightarrow Y$  is a monic inclusion (that is, "an inclusion that respects the structure"), then  $x(X)$  is a sub-"set with structure" (for example, a subgroup, a subgraph, a subring...) of  $Y$ .

**Definition 8.1** (Category of subobjects). Let  $\mathcal{C}$  be a category and let  $C$  be an object of  $\mathcal{C}$ .

The *category of subobjects of  $C$* , denoted by  $\mathbf{SubObj}_{\mathcal{C}}(C)$  is the following category:

**Objects:** A subobject of  $C$  is a monomorphism  $m : M \rightarrow C$

**Morphisms:** A morphism between subobjects  $m : M \rightarrow C$  and  $m' : M' \rightarrow C$  is an arrow  $f : M \rightarrow M'$  such that the following diagram commutes:

$$\begin{array}{ccc} M & & C \\ & \searrow m & \\ & & \nearrow m' \\ M' & & \end{array}$$

**Identities:** The identity morphism  $m : M \rightarrow C$  is the identity morphism  $\text{id}_M : M \rightarrow M$

**Composition:** The composition law for morphisms is the usual composition of morphisms in the category  $\mathcal{C}$

*Example 8.2.* Consider  $\mathbb{R}$  the set of real numbers; let's study  $\mathbf{SubObj}_{\mathbf{Sets}}(\mathbb{R})$ .

The subobjects of  $\mathbb{R}$  are any injections  $x : X \rightarrow \mathbb{R}$ . Consequently, the subobjects of  $\mathbb{R}$  are not only the (inclusions of) subsets of  $\mathbb{R}$  but also any injection from  $X$  to  $\mathbb{R}$  where  $\text{card}(X) \leq \text{card}(\mathbb{R})$ .

Note that if  $\text{card}(X) \leq \text{card}(\mathbb{R})$ , then there are  $\text{card}(\mathbb{R}^X)$  injections from  $X \rightarrow \mathbb{R}$ , and each injection is a different subobject. As the collection of all sets with a certain cardinality is large (not a set) we deduce that  $\mathbf{SubObj}_{\mathbf{Sets}}(\mathbb{R})$  is a large category (but locally small, according to the next proposition).

**Proposition 8.3.** Let  $\mathcal{C}$  be a category and let  $C$  be an object of  $\mathcal{C}$ . The category  $\mathbf{SubObj}_{\mathcal{C}}(C)$  is a preorder.

*Proof.* As any two subobjects  $m : M \rightarrow C$  and  $m' : M' \rightarrow C$  are monic, there is at most one arrow  $f$  such that  $m' \circ f = m$ . A category where there is at most one arrow is a preorder.  $\square$

**Definition 8.4** (Inclusion and equivalence of subobjects). Let  $\mathbf{SubObj}_{\mathcal{C}}(C)$  be a category of subobjects.

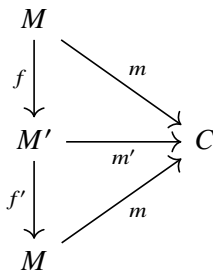
For  $m, m' \in \mathbf{SubObj}_{\mathcal{C}}(C)$ , we define the inclusion relation as:  $m \subset m' \Leftrightarrow$  there exists some arrow  $f : m \rightarrow m'$ .

If  $m \subset m'$  and  $m' \subset m$ , then we say that  $m$  and  $m'$  are equivalent and we write  $m \equiv m'$ .

**Proposition 8.5.** For  $m, m' \in \mathbf{SubObj}_{\mathcal{C}}(C)$ ,  $m \equiv m' \Leftrightarrow m \cong m'$ .

*In other words: two subobjects of  $C$  are equivalent iff they are isomorphic.*

*Proof.* Suppose  $m \equiv m'$ ; then there are two arrows  $f : M \rightarrow M'$  and  $f' : M' \rightarrow M$  such that the following diagram commutes:



There are two arrows  $f' \circ f, \text{id}_M : M \rightarrow M$ ; as  $\mathbf{SubObj}_{\mathcal{C}}(C)$  is a preorder, there is at most one arrow between two subobjects, so  $f' \circ f = \text{id}_M$ . Similarly,  $f' \circ f' = \text{id}_{M'}$ , so  $f' = f'^{-1}$  and  $m \equiv m'$ .

If  $m \equiv m'$ , then let  $f : m \rightarrow m'$  be an isomorphism; we deduce that  $m \subset m'$ . Also,  $f^{-1}$  is an isomorphism, so  $m' \subset m$ , and  $m \equiv m'$ . □

**Corollary 8.6.** *Equivalent subobjects have isomorphic domains.*

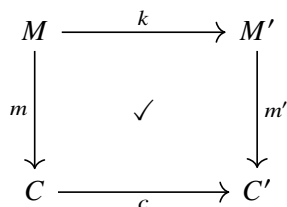
*Remark 8.7.* If  $(E, e)$  is the equaliser of  $f, g : A \rightarrow B$ , then  $E$  is a subobject of  $A$  (cf. Proposition 5.45).

*Remark 8.8.* In  $\mathbf{SubObj}_{\mathbf{Sets}}(\mathbb{R})$ , take  $x : X \rightarrow \mathbb{R}$  and  $x' : X' \rightarrow \mathbb{R}$ . If  $x \equiv x'$ , then there is a bijection between both; equivalently,  $\text{card}(X) = \text{card}(X')$ . Consequently, the equivalence classes of the subobjects of  $\mathbb{R}$  are the cardinals  $\kappa \leq \text{card}(\mathbb{R})$ . If you consider a category **Sets** with the Continuum Hypothesis, then  $\kappa \in \mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\}$ .

If **Sets** respects the axiom of choice, then this is true for any set: the equivalence classes of the subobjects of a set  $X$  are the cardinals  $\kappa \leq \text{card}(X)$ .

**Proposition 8.9.** *Let  $\mathcal{C}$  be a category. Let  $c : C \rightarrow C'$  be an arrow, and let  $m' : M' \rightarrow C'$  be a subobject of  $C'$ .*

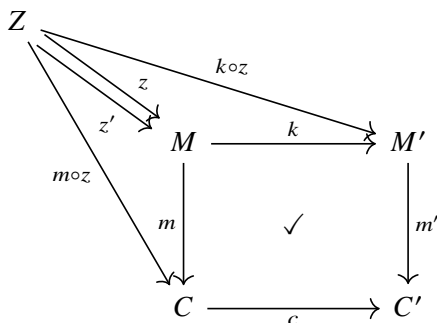
*Suppose that the following diagram is a pullback:*



*Then  $m : M \rightarrow C$  is also a subobject of  $C$ .*

*In other words: the pullback of a subobject is a subobject (or more generally: the pullback of a monomorphism is a monomorphism).*

*Proof.* Suppose there are two arrows  $z, z' : Z \rightarrow M$  such that  $m \circ z = m \circ z'$ .



As  $(M, m, k)$  is a pullback, we have  $c \circ m = m' \circ k$ , which yields:

$$\begin{aligned} c \circ m \circ z &= m' \circ k \circ z \\ &= m' \circ k \circ z' \\ &= c \circ m \circ z' \end{aligned}$$

By the universality of the pullback, the arrow  $z : Z \rightarrow M$  is unique, so  $z = z'$ . Thus,  $m$  is monic.  $\square$

**Proposition 8.10.** *Let  $\mathcal{C}$  be a category. Let  $c : C \rightarrow C'$  be an arrow, let  $m' : M' \rightarrow C'$  and  $m'_0 : M'_0 \rightarrow C'$  be subobjects of  $C'$ .*

*Suppose that the two following squares are pullbacks:*

$$\begin{array}{ccc} M & \xrightarrow{k} & M' \\ \downarrow m & \checkmark & \downarrow m' \\ C & \xrightarrow{c} & C' \end{array} \quad \begin{array}{ccc} M_0 & \xrightarrow{k_0} & M'_0 \\ \downarrow m_0 & \checkmark & \downarrow m'_0 \\ C & \xrightarrow{c} & C' \end{array} \quad (20)$$

*If  $m' \equiv m'_0$ , then  $m \equiv m_0$ .*

*Proof.* Let  $f'_0$  be the isomorphism  $f'_0 : m'_0 \rightarrow m'$ . Consider the following diagram:

$$\begin{array}{ccc} M_0 & \xrightarrow{k_0} & M'_0 \\ & \searrow f'_0 & \downarrow \\ & & M' \\ & \swarrow f'_0 \circ k_0 & \downarrow m' \\ & & C' \\ & \swarrow m_0 & \downarrow m \\ & & C \end{array}$$

*(Note: The diagram above is a simplified representation of the complex commutative diagram in the image, showing the relationships between  $M_0, M'_0, M, M', C, C'$  and arrows  $k_0, k, m_0, m, m', m', c, c, f'_0, f'_0 \circ k_0, m_0$ .)*

As  $f'_0 : m'_0 \rightarrow m'$  is an isomorphism between subobjects, we have  $m' \circ f'_0 = m'_0$ . We then deduce from the diagrams 20 that:

$$c \circ m_0 = m'_0 \circ k_0 = m' \circ f'_0 \circ k_0$$

As  $M$  is the pullback of  $c$  and  $m'$ , there is a unique arrow  $u : M_0 \rightarrow M$  such that  $k \circ u = f'_0 \circ k_0$  and  $m \circ u = m_0$ . With the same reasoning, we have a unique  $u' : M \rightarrow M_0$  such that  $k_0 \circ u' = f'^{-1}_0 \circ k$  and  $m_0 \circ u' = m'$ . We then have to prove that  $u' = u^{-1}$ ; this is because there is a unique arrow  $\text{id}_M : M \rightarrow M$  such that the diagram commutes; from which we deduce  $u \circ u' = \text{id}_M$  and  $u' \circ u = \text{id}_{M_0}$ .  $\square$

As stated in Remark 8.8, in **Sets**, subobjects can be grouped into equivalence classes, the representative of a given equivalence class being the cardinal of the subobjects. The collection of cardinals lower than a certain other cardinal is a set, while the collection of all subobjects generally is not. This is then easier, and more practical, not to refer to the collection of all subobjects  $\text{SubObj}_X(\text{Sets})$ , but rather, to the set of the equivalence classes of the subobjects:



**Definition 8.11** (Set of subobjects). Let  $\mathcal{C}$  be a category.

For  $C \in \text{Ob}_{\mathcal{C}}$ , we define  $\text{SubObj}_{\mathcal{C}}(C)$  to be the set of all equivalence classes<sup>3</sup> of subobjects of  $C$ ; more explicitly:

$$\text{SubObj}_{\mathcal{C}}(C) = \text{Ob}_{\text{SubObj}_C(\mathcal{C})} / \equiv$$

where  $\equiv$  is the equivalence of subobjects (Definition 8.4).

In the rest of this course, we will refer to equivalence classes of subobjects, instead of bare subobjects. So, a set of subobjects is to be understood as the set of equivalence classes of subobjects.

*Remark 8.12.* As noticed in Proposition 8.10, the pullbacks of equivalent subobjects are equivalent. So, in a category with pullbacks, given an arrow  $c : C \rightarrow C'$ , for any subobject  $m' : M' \rightarrow C'$  of  $C'$ , there is a subobject  $m : M \rightarrow C$  of  $C$  such that the following square is a pullback:

$$\begin{array}{ccc} M & \xrightarrow{k} & M' \\ m \downarrow & \checkmark & \downarrow m' \\ C & \xrightarrow{c} & C' \end{array}$$

For an arrow  $c : C \rightarrow C'$ , there is some function that sends any subobject of  $C'$  to one subobject of  $C$  in a way that gives the above pullback. This function is denoted as  $\text{SubObj}_{\mathcal{C}}(c) : \text{SubObj}_{\mathcal{C}}(C') \rightarrow \text{SubObj}_{\mathcal{C}}(C)$  (beware of the inversion!).

**Definition 8.13** (Subobject functor). Let  $\mathcal{C}$  be a category with pullbacks.

The *subobject functor*  $\text{SubObj}_{\mathcal{C}}$  (simply written  $\text{SubObj}$  when there is no doubt about the category) is the contravariant functor:

$$\text{SubObj}_{\mathcal{C}} : \begin{cases} \mathcal{C} & \longrightarrow & \mathbf{Sets} \\ C & \longmapsto & \text{SubObj}_{\mathcal{C}}(C) \\ c : C \rightarrow C' & \longmapsto & \text{SubObj}_{\mathcal{C}}(c) \end{cases}$$

where  $\text{SubObj}_{\mathcal{C}}(C)$  is the set of subobjects (Definition 8.11) and  $\text{SubObj}_{\mathcal{C}}(c) : \text{SubObj}_{\mathcal{C}}(C') \rightarrow \text{SubObj}_{\mathcal{C}}(C)$  is the function introduced in Remark 8.12.

We now generalise the notion of characteristic function with the following definition.

**Definition 8.14** (Subobject classifier). Let  $\mathcal{C}$  be a category with all finite limits.

A *subobject classifier* in  $\mathcal{C}$  is a pair  $(\Omega, t)$  where  $\Omega \in \text{Ob}_{\mathcal{C}}$  and  $t : 1 \rightarrow \Omega$  such that, for all  $C \in \mathcal{C}$ , and for any  $m : M \rightarrow C$  subobject of  $C$ , there is a unique arrow  $u : C \rightarrow \Omega$  such that the following diagram is a pullback:

$$\begin{array}{ccc} M & \longrightarrow & 1 \\ m \downarrow & \checkmark & \downarrow t \\ C & \xrightarrow{u} & \Omega \end{array}$$

The arrow  $u : C \rightarrow \Omega$  is called the *classifying arrow* for  $m$  and is generally written  $\chi_M$ .

---

<sup>3</sup>In fact, an equivalence class in this case, might not be a set. For example, in **Sets**, the collection of all subobjects of  $\mathbb{R}$  of cardinality  $\aleph_0$  is not a set. As a consequence, the collection of equivalence classes is not a set, as a set can only contain sets, at least in the set-theoretic sense (hereditarily: sets contain sets that contain sets and so on; there should not be any proper class in between). However, each representative is a cardinal, and there is only a set of cardinals below some cardinal. It is then more correct to refer to  $\text{SubObj}_{\mathcal{C}}(C)$  as the set of representatives.

In a sense, the subobject classifier is a "universal subobject".

**Proposition 8.15.** *Let  $\mathcal{C}$  be a category with a subobject classifier  $(\Omega, t)$ . The subobject classifier is unique up to isomorphism.*

*Proof.* Let  $(\Omega, t)$  and  $(\Omega', t')$  be two subobjects classifiers.

Note that an arrow  $1 \rightarrow X$  is necessarily monic (since there is only one arrow to the terminal object). So,  $t : 1 \rightarrow \Omega$  is a subobject of  $\Omega$ . By definition of a subobject classifier, there exist unique  $\chi_1 : \Omega \rightarrow \Omega'$  and  $\chi'_1 : \Omega' \rightarrow \Omega$  such that following diagram commutes:

$$\begin{array}{ccccc}
 1 & \xrightarrow{\text{id}_1} & 1 & \xrightarrow{\text{id}_1} & 1 \\
 \downarrow t & & \downarrow t' & & \downarrow t \\
 \Omega & \xrightarrow{\chi_1} & \Omega' & \xrightarrow{\chi'_1} & \Omega
 \end{array}$$

As there is a unique arrow  $\Omega \rightarrow \Omega$ , which already is  $\text{id}_\Omega$  (same for  $\Omega'$ ), we deduce that  $\chi'_1 = \chi_1^{-1}$ .  $\square$

*Example 8.16.* What could the subobject classifier be in **Sets**? For now, let us justify the notation  $\chi_M$ .

Suppose the simplest case. Consider a set  $X$ , a subset of  $Y$  (an actual subset, not only a subobject), and its canonical inclusion morphism  $i : Y \rightarrow X$ .

The terminal object in **Sets** is any one-element set; take the set-theoretic  $1 = \{0\}$ . Besides, there is a unique arrow  $!_Y : Y \rightarrow 1$ . We are looking for  $\Omega$  and  $t : 1 \rightarrow \Omega$ . Let's be explicit:  $!_Y$  and  $t$  are the functions:

$$!_Y : \begin{cases} Y & \longrightarrow & 1 \\ y & \longmapsto & 0 \end{cases} \quad t : \begin{cases} 1 & \longrightarrow & \Omega \\ 0 & \longmapsto & \omega \end{cases}$$

where  $\omega$  is some element in  $\Omega$ .

As a subobject classifier, the following square is a pullback:

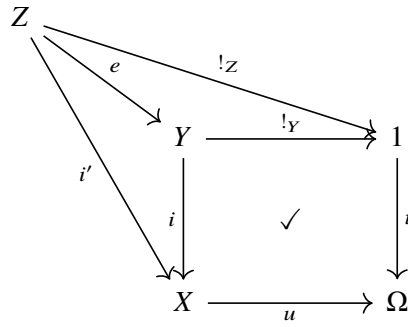
$$\begin{array}{ccc}
 Y & \xrightarrow{!_Y} & 1 \\
 \downarrow i & & \downarrow t \\
 X & \xrightarrow{u} & \Omega
 \end{array}$$

Let us observe the diagram in explicit terms. The function  $u$  is such that, for all  $y \in Y$ :

$$\begin{aligned}
 u \circ i(y) &= t \circ !_Y(y) \\
 u(y) &= t(0) \\
 u(y) &= \omega
 \end{aligned}$$

That is, for each  $y \in Y$ , the function  $u$  gives the same constant  $\omega$ .

Now take another subset  $Z \subset X$  with its inclusion mapping  $i' : Z \rightarrow X$  and its terminal arrow  $!_Z : Z \rightarrow 1$ . We also suppose that  $u \circ i' = t \circ !_Z$ . As  $Y$  is a pullback, there is a unique  $e : Z \rightarrow Y$  such that the following diagram commutes:



For all  $z \in Z$ , we have:

$$\begin{aligned} i'(z) &= i \circ e(z) \\ z &= e(z) \end{aligned}$$

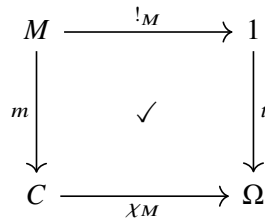
so  $e$  does not "alter"  $z$ ; it is an inclusion mapping:  $Z \subset Y$ . So, for all  $Z \subset X$  such that, for all  $z \in Z$ , we have  $u(z) = \omega$ , we deduce that  $Z \subset Y$ . Such a behaviour indicates that  $u$  should be the characteristic function  $\chi_Y$  of  $Y$ ,  $\Omega = 2 = \{0, 1\}$  and  $t : 0 \mapsto 1$  (the constant function that assigns 1 to its unique element 0).

In fact, we can check that  $(2, 0 \mapsto 1)$  is the subobject classifier of **Sets**. In fact, the subobject classifier is designed to generate classifying arrows, which are the categorical generalisation of characteristic functions.

**Proposition 8.17.** *Let  $\mathcal{C}$  be a category with all finite limits.*

*$\mathcal{C}$  has a subobject classifier  $(\Omega, t) \Leftrightarrow$  the subobject functor  $\text{SubObj}_{\mathcal{C}}$  is representable.*

*Proof.* By definition of a subobject classifier, for all  $C$ , for all  $m : M \rightarrow C \in \text{SubObj}_{\mathcal{C}}(C)$  subobject of  $C$ , there is a unique arrow  $\chi_M : C \rightarrow \Omega$  such that the following diagram is a pullback:



By definition of the subobject functor,  $m = \text{SubObj}(\chi_M)(t)$ . By Proposition 3.4,  $(\Omega, t)$  is a universal element of  $\text{SubObj}$ , and by Theorem 3.17,  $(\Omega, \theta_{\text{SubObj}, \Omega}(t))$  is a representation of  $\text{SubObj}$ .  $\square$

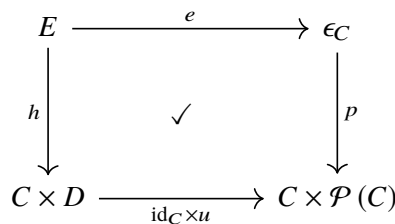
**Corollary 8.18.** *If the subobject functor  $\text{SubObj}_{\mathcal{C}}$  is representable, then its universal element is  $(\Omega, t)$ .*

**Corollary 8.19.**  $\text{SubObj}_{\mathcal{C}}(-) \cong \text{Hom}_{\mathcal{C}}(-, \Omega)$ .

We finish the section with the notion of a power object.

**Definition 8.20** (Power object). Let  $\mathcal{C}$  be a category with finite limits. Let  $C \in \mathcal{C}$ .

The *power object* of  $C$  is a pair  $(\mathcal{P}(C), p)$  where  $\mathcal{P}(C)$  is an object of  $\mathcal{C}$  and  $p : \epsilon_C \rightarrow C \times \mathcal{P}(C)$  is a subobject of  $C \times \mathcal{P}(C)$ , such that, for all  $D \in \mathcal{C}$ , for all subobject  $h : E \rightarrow C \times D$  of  $C \times D$ , there is a unique  $u : D \rightarrow \mathcal{P}(C)$  such that the following diagram is a pullback:



*Remark 8.21.* In terms of sets, the power object  $\mathcal{P}(C)$  is the powerset of  $C$  (the set of all subsets of  $C$ ). The object  $D$  is interpreted to be a family of subsets of  $C$  (a subset of  $\mathcal{P}(C)$ ). The interpretation of  $p : \epsilon_C \rightarrow C \times \mathcal{P}(C)$  will come later.

**Proposition 8.22.** *Let  $\mathcal{C}$  be a category with finite limits. Let  $C \in \mathcal{C}$ .  
The power object of  $C$  is unique up to isomorphism.*

*Proof.* Let  $(\mathcal{P}(C), p)$  and  $(\mathcal{P}', p')$  be two power objects of  $C$ .

$$\begin{array}{ccccccc}
 \epsilon_C & \longrightarrow & \epsilon'_C & \longrightarrow & \epsilon_C & \longrightarrow & \epsilon'_C \\
 \downarrow p & & \downarrow p' & & \downarrow p & & \downarrow p' \\
 C \times \mathcal{P}(C) & \xrightarrow{\text{id}_C \times u'} & C \times \mathcal{P}' & \xrightarrow{\text{id}_C \times u} & C \times \mathcal{P}(C) & \xrightarrow{\text{id}_C \times u'} & C \times \mathcal{P}' \\
 & \searrow \text{id}_C \times \text{id}_{\mathcal{P}(C)} & & \swarrow \text{id}_C \times \text{id}_{\mathcal{P}'} & & & \\
 & & & & & & 
 \end{array}$$

Considering the unicity of  $u, u', \text{id}_{\mathcal{P}(C)}$  and  $\text{id}_{\mathcal{P}'}$ , and noticing that each adjacent two pullbacks are pullbacks, we have  $u' = u^{-1}$ .  $\square$

**Proposition 8.23.** *Let  $\mathcal{C}$  be a category with finite limits. Let  $C \in \mathcal{C}$  with a power object  $\mathcal{P}(C)$ .  
For all  $X \in \mathcal{C}$ ,  $\text{Hom}_{\mathcal{C}}(X, \mathcal{P}(C)) \cong \text{SubObj}_{\mathcal{C}}(X \times C)$  naturally in  $X$ .*

*Proof.* By definition of a power object, there is a bijection  $\varphi_X : \text{Hom}_{\mathcal{C}}(X, \mathcal{P}(C)) \rightarrow \text{SubObj}_{\mathcal{C}}(X \times C)$ .  
Let  $x : X \rightarrow X'$  be an arrow; we check whether the following diagram commutes:

$$\begin{array}{ccccc}
 X & & \text{Hom}_{\mathcal{C}}(X', \mathcal{P}(C)) & \xrightarrow{\varphi_{X'}} & \text{SubObj}_{\mathcal{C}}(X' \times C) \\
 \downarrow & \rightsquigarrow & \downarrow & & \downarrow \text{SubObj}_{\mathcal{C}}(x \times C) \\
 X' & & \text{Hom}_{\mathcal{C}}(X, \mathcal{P}(C)) & \xrightarrow{\varphi_X} & \text{SubObj}_{\mathcal{C}}(X \times C) \\
 & & & & ?
 \end{array}$$

We want:

$$\text{SubObj}_{\mathcal{C}}(x \times C) \circ \varphi_{X'} = \varphi_X \circ \text{Hom}_{\mathcal{C}}(x, \mathcal{P}(C))$$

Let  $u : X' \rightarrow \mathcal{P}(C)$ ; there is a unique subobject  $h' : E' \rightarrow C \times X'$  such that the following is a pullback:

$$\begin{array}{ccc}
 E' & \longrightarrow & \epsilon_C \\
 \downarrow h' & \checkmark & \downarrow p \\
 C \times X' & \xrightarrow{\text{id}_C \times u} & C \times \mathcal{P}(C)
 \end{array}$$

Besides,  $\text{SubObj}_{\mathcal{C}}(x \times C)(h')$  is defined as the unique  $h$  that makes the following diagram a pullback:

$$\begin{array}{ccc}
 E & \longrightarrow & E' \\
 \downarrow h & \checkmark & \downarrow h' \\
 C \times X & \xrightarrow{\text{id}_C \times x} & C \times X'
 \end{array} \tag{21}$$

We deduce:

$$\begin{aligned} \text{SubObj}_{\mathcal{C}}(x \times C) \circ \varphi_{X'}(u) &= \text{SubObj}_{\mathcal{C}}(x \times C)(h') \\ &= h \end{aligned}$$

Then:

$$\varphi_X \circ \text{Hom}_{\mathcal{C}}(x, \mathcal{P}(C))(u) = \varphi_X(u \circ x) = h''$$

where  $h''$  is the unique arrow such that the following diagram is a pullback:

$$\begin{array}{ccc} E'' & \xrightarrow{\quad} & \epsilon_C \\ \downarrow h'' & \checkmark & \downarrow p \\ C \times X & \xrightarrow{\text{id}_C \times (u \circ x)} & C \times \mathcal{P}(C) \end{array} \quad (22)$$

Note that the above diagram decomposes into this:

$$\begin{array}{ccccc} E'' & \xrightarrow{\quad} & E' & \xrightarrow{\quad} & \epsilon_C \\ \downarrow h'' & \checkmark & \downarrow h' & \checkmark & \downarrow p \\ C \times X & \xrightarrow{\text{id}_C \times x} & C \times X' & \xrightarrow{\text{id}_C \times u} & C \times \mathcal{P}(C) \end{array}$$

(The arrow  $E'' \rightarrow E'$  can be obtained knowing that the right square is a pullback.)

By Proposition 5.59, if the right square  $E' \epsilon_C(C \times \mathcal{P}(C))(C \times X')$  and the outer rectangle  $E'' \epsilon_C(C \times \mathcal{P}(C))(C \times X)(C \times X)$  are pullbacks, then the left square  $E'' E'(C \times X')(C \times X)$  is also a pullback; by unicity of  $h''$  (from Diagram 22) and  $h$  (from Diagram 21), we deduce that  $h = h''$ , which let us conclude:

$$\begin{aligned} h &= h'' \\ \text{SubObj}_{\mathcal{C}}(x \times C) \circ \varphi_{X'}(u) &= \varphi_X \circ \text{Hom}_{\mathcal{C}}(x, \mathcal{P}(C))(u) \end{aligned}$$

□

In **Sets**, we know that  $\mathcal{P}(X) \cong 2^X$  (because each characteristic function  $X \rightarrow 2$  defines a subset of  $X$  and conversely). There is a similar result with the categorical equivalents of a subobject classifier and the power object.

**Proposition 8.24.** *Let  $\mathcal{C}$  be a category with finite limits and with a subobject classifier  $(\Omega, t)$ . Let  $C \in \mathcal{C}$ . Then,  $\mathcal{P}(C) \cong \Omega^C$ .*

*Proof.* By Proposition 8.23, we have:

$$\text{Hom}_{\mathcal{C}}(-, \mathcal{P}(C)) \cong \text{SubObj}_{\mathcal{C}}(- \times C)$$

By Corollary 8.18:

$$\text{SubObj}_{\mathcal{C}}(-) \cong \text{Hom}_{\mathcal{C}}(-, \Omega)$$

By Theorem 5.23:

$$\mathrm{Hom}_{\mathcal{C}}(- \times C, -) \cong \mathrm{Hom}_{\mathcal{C}}(-, (-)^C)$$

Combining those three results:

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}}(-, \mathcal{P}(C)) &\cong \mathrm{SubObj}_{\mathcal{C}}(- \times C) \\ &\cong \mathrm{Hom}_{\mathcal{C}}(- \times C, \Omega) \\ &\cong \mathrm{Hom}_{\mathcal{C}}(-, \Omega^C)\end{aligned}$$

By Corollary 2.17, we deduce that  $\mathcal{P}(C) \cong \Omega^C$ . □

**Corollary 8.25.** *If a category has all power objects, then  $\mathcal{P}(1)$  is the subobject classifier.*

*Proof.* Just write the definition of the power object of 1. □

## 9. Elementary topoi

We can now introduce topoi. We will first see the general notion of elementary topoi (which we will simply call topoi) and we will then study the notion of a Grothendieck topos.

**Definition 9.1** (Elementary topos (1) [1]). An *elementary topos* (or *topos* for short) is a category  $\mathcal{E}$  that has finite limits, all exponentials, and a subobject classifier.

**Definition 9.2** (Elementary topos (2) [2]). An *elementary topos* (or *topos* for short) is a category  $\mathcal{E}$  that is cartesian closed, has finite limits, and a representable subobject functor.

**Definition 9.3** (Elementary topos (3) [7]). An *elementary topos* (or *topos* for short) is a category  $\mathcal{E}$  that has finite limits and such that every object has a power object.

Of course:

**Proposition 9.4.** *The three definitions of a topos are equivalent.*

In fact, it is easy to see that (2)  $\Leftrightarrow$  (1)  $\Rightarrow$  (3); however (3)  $\Rightarrow$  (1) requires some more work. Let us just admit this proposition.

*Example 9.5.* As we have seen through this course, **Sets** has finite limits, all exponentials, a subobject classifier, every set has a powerset: **Sets** is an elementary topos. In fact, topoi are thought as categories that "roughly" behave like **Sets**.

As a first property of elementary topoi, we give the following theorem, but will not prove it as it is far from the scope of this course.

**Proposition 9.6.** *A topos has finite colimits.*

*Proof.* According to [2, Theorem 15.2.8, p389], this is very hard to prove, and the demonstration requires many notions that are beyond the scope of this course. We will just admit this theorem.  $\square$

**Proposition 9.7.** *Let  $\mathcal{E}$  be a topos. A monomorphism in  $\mathcal{E}$  is an equaliser.*

*Proof.* According to the definition of a subobject classifier  $(\Omega, t)$ , a monomorphism  $m : M \rightarrow C$  is indeed the equaliser of  $\chi_M : C \rightarrow \Omega$  and  $!_C \circ t$ .  $\square$

**Proposition 9.8.** *Let  $\mathcal{E}$  be a topos. The isomorphisms in  $\mathcal{E}$  are exactly the monic/epic.*

*Proof.* By Corollary 5.40, every isomorphism is both monic and epic; this is true in any category.

The converse comes from the topos-ness of  $\mathcal{E}$ . In a topos, all monomorphisms are equalisers, thus all monic/epic are epic equalisers, and by Proposition 5.47, all epic equalisers are isomorphisms.  $\square$

**Definition 9.9** (Image of an arrow [8]). Let  $f : A \rightarrow C$  be an arrow.

We say that  $f$  *factors through*  $m : B \rightarrow C$  when there exists  $e : A \rightarrow B$  such that  $f = m \circ e$ .

The *image of  $f$*  is a monomorphism  $m : B \rightarrow C$  such that  $f$  factors through  $m$ , and for all monomorphism  $m'$ , if  $f$  factors through  $m'$ , then so does  $m$ .

*Remark 9.10* (Explicit definition). The image of  $f$  is a monomorphism  $m : B \rightarrow C$  such that there exists  $e : A \rightarrow B$  such that  $f = m \circ e$ , and for all monomorphism  $m' : B' \rightarrow C$ , if there exists  $e' : A \rightarrow B'$  such that  $f = m' \circ e'$ , then there exists  $e'' : A \rightarrow B'$  such that  $m = m' \circ e''$ .

In a sense, the image of  $f$  is the "smallest" subobject of  $C$  through which  $f$  can factor.

In **Sets**, the image of a function  $f : X \rightarrow Y$  is the inclusion mapping  $f(X) \rightarrow Y$ .

**Definition 9.11** (Epi-mono factorisation). Let  $f : A \rightarrow C$  be an arrow in a category  $\mathcal{C}$ .

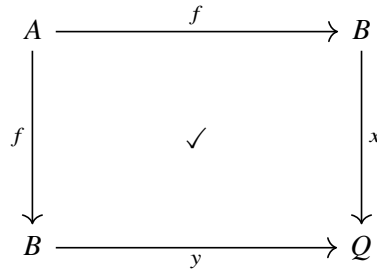
An *epi-mono factorisation of  $f$*  is a pair  $(m, e)$  where  $m : B \rightarrow C$  is a monomorphism and  $e : A \rightarrow B$  is an epimorphism such that  $f = m \circ e$ .

A category  $\mathcal{C}$  is said to *have epi-mono factorisations* when every arrow  $f$  has an epi-mono factorisation.

In **Sets**, this property is very easy to see. Let  $f : X \rightarrow Y$ ; then we can define  $e : X \rightarrow f(X)$  (the restriction of  $f$  to its image) and then  $m : f(X) \rightarrow Y$  as the inclusion function. Then  $e$  is epic (i.e. a surjection) and  $m$  is monic (i.e. an injection). As said before, topoi generalise in a sense the category of sets; and the following proposition holds:

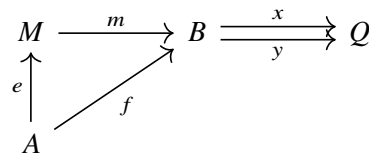
**Proposition 9.12.** *Let  $\mathcal{E}$  be a topos. Then  $\mathcal{E}$  has epi-mono factorisations.*

*Proof.* We need the fact that a topos has colimits. Let  $f : A \rightarrow B$  be a function; the pushout of  $f$  with  $f$  exists:

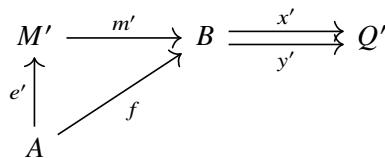


and  $x \circ f = y \circ f$ .

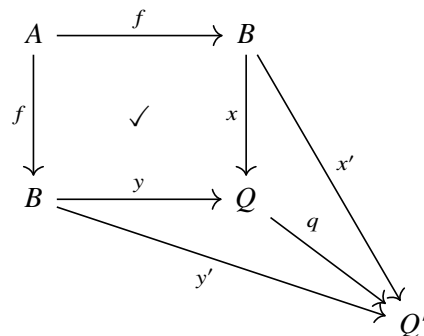
Let  $(M, m)$  be the equaliser of  $x$  and  $y$ ; there exists a unique  $e$  such that  $f = m \circ e$ :



We now show that  $m$  is the image of  $f$ . Suppose  $f = m' \circ e'$  for some monic  $m' : M' \rightarrow B$ . As we are in a topos, the monic  $m'$  is an equaliser of some  $x', y' : B \rightarrow Q'$ , as in the diagram:



As  $x' \circ f = y' \circ f$ , and using the fact that  $Q$  is a pushout of  $f$  with itself, there is a unique arrow  $q : Q \rightarrow Q'$  such that  $x' = q \circ x$  and  $y' = q \circ y$ , as in the following diagram:



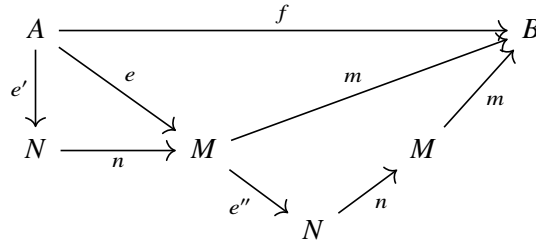
Now, we have:

$$x' \circ m = q \circ x \circ m = q \circ y \circ m = y' \circ m$$

so,  $m'$  being the equaliser of  $x'$  and  $y'$ , there is a unique  $e'' : M \rightarrow M'$  such that  $m = m' \circ e''$ ; so  $m$  is indeed the image of  $f$ .



The only fact to prove now is the epicness of  $e$ . To do so, let us introduce and explain the following diagram:



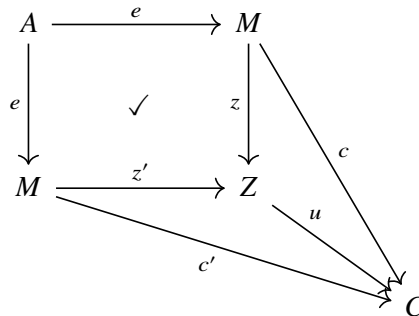
We have proven that every arrow factors through some monomorphism thanks to some pushout; then there are two arrows  $z, z' : M \rightarrow Q$  such that  $z$  and  $z'$  are the pushout of  $e$  with itself, and there exist  $e' : A \rightarrow N$  and a monomorphism  $n : N \rightarrow M$  such that  $e = n \circ e'$ , and  $n$  is the equaliser of  $z$  and  $z'$ .

First, let us focus on  $e'$  and  $n$ . We have  $f = m \circ e = m \circ n \circ e'$ ; so  $f$  factors through  $m \circ n$ , which is monic (composition of monomorphisms is monic). As  $m$  is the image of  $f$ ,  $m$  also does: there exists a  $e'' : M \rightarrow N$  such that  $m = m \circ n \circ e''$ ; as  $m$  is monic, we deduce:

$$\text{id}_M = n \circ e'' \Rightarrow n = n \circ e'' \circ n \Rightarrow e'' \circ n = \text{id}_N$$

as  $n$  is also monic; but as  $e'' \circ n = \text{id}_N$  and  $n \circ e'' = \text{id}_M$ , we deduce that  $n$  is an isomorphism.

Now, consider  $z$  and  $z'$ , the pushout of  $e$  with itself, with  $n$  being the equaliser of  $z$  and  $z'$ . As  $n$  is an isomorphism, we have  $z = z'$ . Now let  $c, c' : M \rightarrow C$  be such that  $c \circ e = c' \circ e$ ; as  $z$  and  $z'$  are a pushout of  $e$  with itself, there exists a unique  $u$  such that:



and  $c = u \circ z = u \circ z' = c'$ . Thus,  $e$  is epic. □

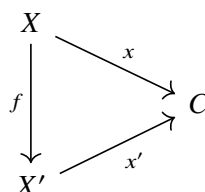
For the following property of a topos, let us get back to slice categories. We did not introduce slice categories as such, but rather, comma-categories (Definition 7.17).

**Definition 9.13** (Slice category). Let  $\mathcal{C}$  be a category, and let  $C \in \mathcal{C}$  be an object.

The *slice category of  $\mathcal{C}$  over  $C$* , simply written  $\mathcal{C}/C$ , is the following category:

**Objects:** The objects are the arrows  $x : X \rightarrow C \in \text{Mor}_{\mathcal{C}}$

**Morphisms:** For  $x : X \rightarrow C$  and  $x' : X' \rightarrow C$ , a morphism  $f : x \rightarrow x'$  is defined as an arrow  $f : X \rightarrow X' \in \text{Mor}_{\mathcal{C}}$  such that the following triangle commutes:



**Identities:** The identity morphism of  $x : X \rightarrow C$  is simply  $\text{id}_X$

**Composition:** The composition law for morphisms is the usual composition law in  $\mathcal{C}$

Confronting Definition 7.17 with Definition 9.13, it is easy to see that:

**Proposition 9.14.** *Let  $\mathcal{C}$  be a category and  $C \in \mathcal{C}$  be an object.*

*The slice category  $\mathcal{C}/C$  is the comma-category  $(\text{Id}_{\mathcal{C}} \mid \Delta(C))$ .*

**Proposition 9.15.** *Let  $\mathcal{C}$  be a category with a terminal object  $1$ . Then  $\mathcal{C}/1 \cong \mathcal{C}$ .*

*Proof.* The proof is quite easy. Define the functors:

$$F : \begin{cases} \mathcal{C} & \longrightarrow & \mathcal{C}/1 \\ C & \longmapsto & !_C : C \rightarrow 1 \\ c : C \rightarrow C' & \longmapsto & c \end{cases}$$

$$U : \begin{cases} \mathcal{C}/1 & \longrightarrow & \mathcal{C} \\ !_C : C \rightarrow 1 & \longmapsto & C \\ c : !_C \rightarrow !_C & \longmapsto & c \end{cases}$$

Then  $U$  clearly is the inverse of  $F$ ; thus  $F$  defines an isomorphism between  $\mathcal{C}$  and  $\mathcal{C}/1$ . □

Another obvious fact is that any slice category has a terminal object:

**Proposition 9.16.** *In the slice category  $\mathcal{C}/C$ ,  $\text{id}_C$  is the terminal object.*

**Proposition 9.17.** *Let  $\mathcal{C}$  be a category and  $C$  be an object in  $\mathcal{C}$ .*

*Consider the following diagram:*

$$\begin{array}{ccc} P & \xrightarrow{p_2} & X_2 \\ p_1 \downarrow & \searrow q & \downarrow c_2 \\ X_1 & \xrightarrow{c_1} & C \end{array}$$

*The previous diagram is a pullback in  $\mathcal{C} \Leftrightarrow q = c_1 \circ p_1 = c_2 \circ p_2$  is a product in  $\mathcal{C}/C$ .*

*Proof.* Just compare the universal properties of  $P$  (as a pullback) and  $q$  (as a product). □

Let us spend some time studying examples of slice categories, as they will turn crucial in our understanding of some properties of topoi.

*Example 9.18* (Slice category in a poset). Consider the poset  $(P, \leq)$ . Let  $p \in P$ ; what does slice category  $P/p$  look like? It is the following category:

**Objects:** The objects are the arrows  $q : q \rightarrow p$ ; that is, the objects are the pairs  $q = (q, p)$  such that  $q \leq p$ .

**Morphisms:** The arrows  $x : q \rightarrow q'$  are such that the following diagram commutes:

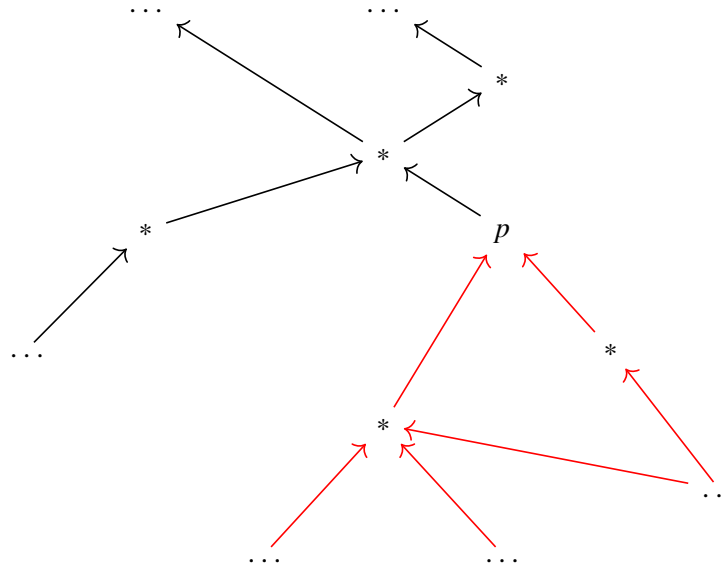
$$\begin{array}{ccc} q & & p \\ x \downarrow & \searrow q & \\ q' & \nearrow q' & \end{array}$$

That is, arrows  $x : q \rightarrow q'$  are simply pairs  $(q, q')$  such that  $q \leq q' \leq p$ .

**Identities:** The identity morphism of  $q : q \rightarrow p$  is simply the reflexivity rule of a partial order.

**Composition:** The composition law for morphisms is the transitivity of the preorder.

In summary, the slice category of a poset  $P/p$  is a truncated version of the poset  $P$ , made with only the objects that are below  $p$  (which have arrows that target  $p$ ) (in red in the following diagram).



*Example 9.19* (Slice category in **Sets**). In **Sets**, a slice category is less obvious. Let  $A$  be a set.

If  $A = 1$  then by Proposition 9.15, we have  $\mathbf{Sets}/1 \cong \mathbf{Sets}$ . If  $A = 0 = \emptyset$  then  $\mathbf{Sets}/0 \cong \mathcal{C}_1$ , where  $\mathcal{C}_1$  is the category with only one object and one identity morphism.

Now suppose  $A$  is not a trivial set:  $A \neq 0$  and  $A \neq 1$ .

Let us start from the easy questions. What is the size of  $\mathbf{Sets}/A$ ?

The objects of  $\mathbf{Sets}/A$  are all the functions  $a : X \rightarrow A$ . As there are always  $\text{card}(A)^{\text{card}(X)} > \text{card}(X)$  functions between  $X$  and  $A$ , for each set  $X$ , there are  $\text{card}(A)^{\text{card}(X)}$  arrows  $X \rightarrow A$ . Thus,  $\mathbf{Sets}/A$  is a large category.

As for the morphisms; an arrow  $x : a \rightarrow a'$  between  $a : X \rightarrow A$  and  $a' : X' \rightarrow A$  is an arrow  $x : X \rightarrow X'$  with some additional properties (the commutative triangle). We deduce that  $\text{Hom}(a, a') \subset \text{Hom}(X, X')$ . Therefore,  $\mathbf{Sets}/A$  is locally small, because **Sets** is.

Another question one may ask is: can we have two arrows  $a : X \rightarrow A$  and  $a' : X \rightarrow A$  and  $a \neq a'$ , that is, two different arrows from the same source? Consider the following functions:

$$\text{mod}_2 : \begin{cases} \mathbb{N} & \longrightarrow & 2 \\ n & \longmapsto & n \pmod 2 \end{cases}$$

$$\text{dom}_2 : \begin{cases} \mathbb{N} & \longrightarrow & 2 \\ n & \longmapsto & (n + 1) \pmod 2 \end{cases}$$

where  $2 = \{0, 1\}$  (the set-theoretic natural number 2). They are different functions and thus, different objects in  $\mathbf{Sets}/2$ .

There is also an arrow between them. The successor function:

$$\text{succ} : \begin{cases} \mathbb{N} & \longrightarrow & \mathbb{N} \\ n & \longmapsto & n + 1 \end{cases}$$

is an arrow  $\text{succ} : \text{mod}_2 \rightarrow \text{dom}_2$  in  $\mathbf{Sets}/2$ .

Intuitively, the slice category  $\mathbf{Sets}/A$  is a kind of "zoom" on how  $A$  "sees"  $\mathbf{Sets}$ . This zoom somehow individualises the arrows that target  $A$ ; in fact,  $\text{mod}_2$  and  $\text{dom}_2$  are indiscernible in  $\mathbf{Sets}$  (they are arrows from the same source  $\mathbb{N}$ ) while in  $\mathbf{Sets}/2$ , they are different objects.

Now consider the functions:

$$\text{mod}_3 : \begin{cases} \mathbb{N} & \longrightarrow & 3 \\ n & \longmapsto & n \bmod 3 \end{cases}$$

$$\text{mod}_2 : \begin{cases} \mathbb{N} & \longrightarrow & 2 \\ n & \longmapsto & n \bmod 2 \end{cases}$$

where  $\text{mod}_2$  is the obvious extension of  $\text{mod}_2 : \mathbb{N} \rightarrow 2$  to  $\mathbb{N} \rightarrow 3$  (we could have used different symbols but it was only making the notations inconvenient).

Is there an arrow  $f : \text{mod}_2 \rightarrow \text{mod}_3$ ? That is, a function such that  $n \bmod 2 = f(n) \bmod 3$ . The function  $\text{mod}_2 : \mathbb{N} \rightarrow \mathbb{N}$  (extension of  $\text{mod}_2$ ) does the job, but there are an infinity of functions that would do the job as well, for example:

$$\begin{cases} \mathbb{N} & \longrightarrow & \mathbb{N} \\ n & \longmapsto & \begin{cases} 3k & \text{if } n = 2k \\ 3k + 1 & \text{if } n = 2k + 1 \end{cases} \end{cases}$$

What about an arrow  $f : \text{mod}_3 \rightarrow \text{mod}_2$ ? That is, a function such that  $n \bmod 3 = f(n) \bmod 2$ . Such a function does not exist, because there is no  $m$  such that  $m \bmod 2 = 2 \bmod 3 = 2$  (note that we are not considering  $\mathbb{Z}/(2\mathbb{Z})$ , so the modulo operation only applies once). We conclude that there is no arrow  $\text{mod}_3 \rightarrow \text{mod}_2$ .

What condition makes it possible to have an arrow between two functions  $a : X \rightarrow A$  and  $a' : X' \rightarrow A$ ?

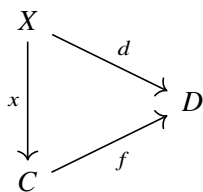
The previous example becomes obvious once we see that  $\text{mod}_2(\mathbb{N}) \subsetneq \text{mod}_3(\mathbb{N})$ . Consequently, there cannot be arrows  $\text{mod}_3 \rightarrow \text{mod}_2$ . This seems to be the condition we are looking for. In fact, we can show that, in  $\mathbf{Sets}/A$ , there is an arrow  $x : a \rightarrow a'$  between  $a : X \rightarrow A$  and  $a' : X' \rightarrow A$  if and only if  $a(X) \subset a'(X')$ . As a corollary, there are arrows  $x : a \rightarrow a'$  and  $x' : a' \rightarrow a$  if and only if  $a(X) = a'(X')$ .

In other words, the slice category induces a preorder  $\lesssim$  on the functions that target  $A$ : for  $a : X \rightarrow A$  and  $a' : X' \rightarrow A$ , we have  $a \lesssim a' \Leftrightarrow a(X) \subset a'(X')$ . This preorder gives the general structure of a slice category  $\mathbf{Sets}/A$ . We will see in another example (Example 9.32) another interpretation of a slice category in  $\mathbf{Sets}$ .

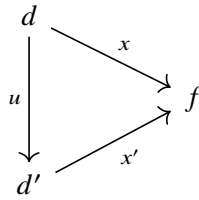
*Remark 9.20 (Slice of slice).* Let  $\mathcal{C}$  be a category and  $f : C \rightarrow D$  be an arrow in  $\mathcal{C}$ .

Consider the category  $\mathcal{C}/D$ . The arrow  $f : C \rightarrow D$  is an object in  $\mathcal{C}/D$ . We can keep "slicing" the category. Let's have a closer look at  $(\mathcal{C}/D)/f$ .

An object in  $(\mathcal{C}/D)/f$  is an arrow  $x : d \rightarrow f \in \text{Mor}_{\mathcal{C}/D}$ , that is, such that the following triangle commutes:



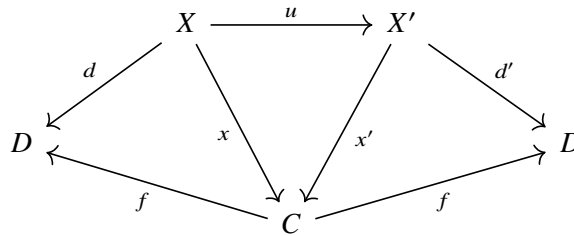
For  $x : d \rightarrow f$  and  $x' : d' \rightarrow f$ , an arrow  $u : x \rightarrow x'$  in  $(\mathcal{C}/D)/f$  is an arrow  $u : d \rightarrow d'$  such that the following diagram commutes:



But the arrows  $x$  and  $x'$  also make the following diagrams commute:

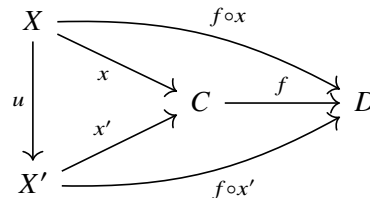


Finally,  $u : d \rightarrow d'$  is an arrow  $u : X \rightarrow X'$  such that the following diagram commutes:



In summary, an object in  $(\mathcal{C}/D)/f$  is an arrow  $x : X \rightarrow C$ , that is, an object in  $\mathcal{C}/C$ , and an arrow in  $(\mathcal{C}/D)/f$  is in fact an arrow in  $\mathcal{C}/C$ .

Conversely, an arrow  $x : X \rightarrow C$  (object in  $\mathcal{C}/C$ ) easily converts to an arrow  $x : x \circ f \rightarrow f$  (object in  $(\mathcal{C}/D)/f$ ). Besides, an arrow  $u : x \rightarrow x'$  in  $\mathcal{C}/C$  gives the following diagram:



Thus, an arrow  $u : x \rightarrow x'$  easily becomes an arrow  $u : f \circ x \rightarrow f \circ x'$ .

These two observations define two functors that are clearly inverses of each other. Consequently, we have proven the following proposition.

**Proposition 9.21** (Slice of a slice is a slice). *Let  $\mathcal{C}$  be a category, and let  $f : C \rightarrow D$  be an arrow in  $\mathcal{C}$ . Then,  $(\mathcal{C}/D)/f \cong \mathcal{C}/C$ .*

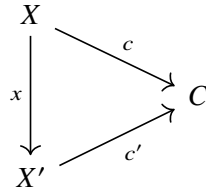
*Proof.* See Remark 9.20. □

**Definition 9.22** (Composition functor / dependent sum). Let  $\mathcal{C}$  be a category with all pullbacks, and let  $f : C \rightarrow D$  be an arrow in  $\mathcal{C}$ .

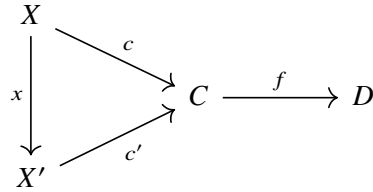
The *composition functor of  $f$* , or *dependent sum relative to  $f$* , written  $\Sigma_f$ , is the following functor:

$$\Sigma_f : \begin{cases} \mathcal{C}/C & \longrightarrow & \mathcal{C}/D \\ c & \longmapsto & f \circ c \\ x : c \rightarrow c' & \longmapsto & x : f \circ c \rightarrow f \circ c' \end{cases}$$

*Remark 9.23.* Note that, in the definition of the composition functor, for  $x : c \rightarrow c'$ , we have the following diagram:



which gives, by composition by  $f$ :



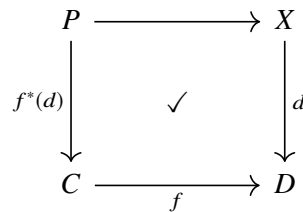
Consequently, an arrow  $x \in \mathcal{C}/C$  is also an arrow  $x \in \mathcal{C}/D$ .

**Definition 9.24** (Pullback functor). Let  $\mathcal{C}$  be a category with all pullbacks, and let  $f : C \rightarrow D$  be an arrow in  $\mathcal{C}$ .

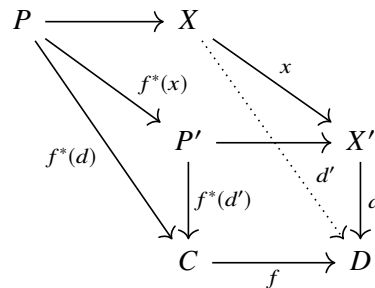
The *pullback functor*  $f^*$  is the following functor:

$$f^* : \begin{cases} \mathcal{C}/D & \longrightarrow & \mathcal{C}/C \\ d & \longmapsto & f^*(d) \\ x : d \rightarrow d' & \longmapsto & f^*(x) \end{cases}$$

where, for  $d : X \rightarrow D$ ,  $f^*(d)$  is such that the following square is a pullback:



and for an arrow  $x : d \rightarrow d'$ ,  $f^*(x)$  is the unique arrow  $P \rightarrow P'$  between pullbacks such that the following diagram commutes:



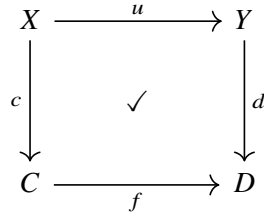
**Proposition 9.25.** Let  $\mathcal{C}$  be a category with pullbacks. Let  $f : A \rightarrow B$  be an arrow in  $\mathcal{C}$ .

Then,  $\Sigma_f \dashv f^*$ ; that is, the composition functor is left adjoint to the pullback functor.

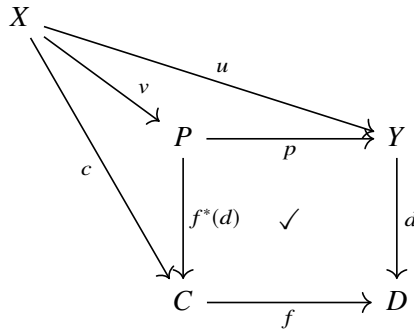
*Proof.* Let  $c : X \rightarrow C$  and  $d : Y \rightarrow D$ . We have to check that there is a natural isomorphism:

$$\text{Hom}_{\mathcal{C}/D}(\Sigma_f(c), d) \cong \text{Hom}_{\mathcal{C}/C}(c, f^*(d))$$

Let  $u \in \text{Hom}_{\mathcal{C}}(\Sigma_f(c), d)$ ; the following square commutes:



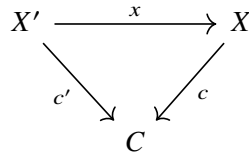
By definition of  $f^*(d)$ , the following diagram is a pullback; as a consequence, there is a unique  $v : X \rightarrow P$  such that the diagram commutes:



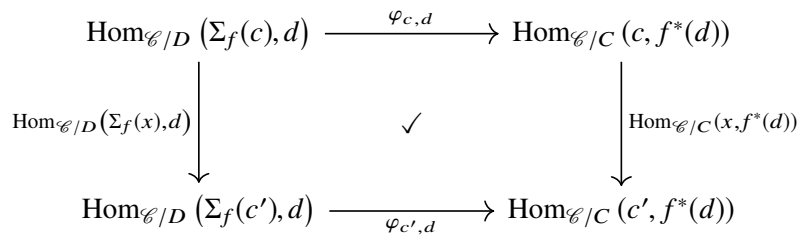
So, for all  $u \in \text{Hom}_{\mathcal{C}/D}(\Sigma_f(c), d)$ , there is a unique  $v \in \text{Hom}_{\mathcal{C}/C}(c, f^*(d))$  such that the above diagram commutes; in other words, the mapping  $\varphi_{c,d} : u \mapsto v$  is a bijection<sup>4</sup>.

We now have to check the naturality in  $c$  and  $d$ .

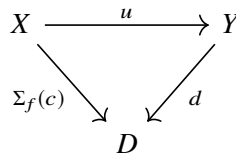
Let  $x : c' \rightarrow c$  be an arrow in  $\mathcal{C}/C$ ; thus  $x$  makes the following diagram commute:



We have to check whether the following diagram commutes:



Let  $u \in \text{Hom}_{\mathcal{C}/D}(\Sigma_f(c), d)$ ;  $u$  makes the following diagram commute:



So:

<sup>4</sup>In fact the formula establishes the reverse bijection, but this bijection will do.





The adjunction  $\Sigma_f \dashv f^*$  may sometimes be completed with a third functor  $\Pi_f$ , called the dependent product functor, that is right adjoint to the pullback functor  $f^*$ . However, this does not occur often; the existence of this right adjoint depends on some property of the category  $\mathcal{C}$ .

**Definition 9.26** (Locally Cartesian closed). A category  $\mathcal{C}$  is called *locally Cartesian closed* whenever, for all object  $C \in \mathcal{C}$ , the slice category  $\mathcal{C}/C$  is Cartesian closed.

From the definition and Proposition 9.15, it is easy to see that:

**Proposition 9.27.** *If  $\mathcal{C}$  is locally Cartesian closed and has a terminal object, then  $\mathcal{C}$  is Cartesian closed.*

Another result that is easy to see is the following:

**Proposition 9.28.** *Let  $\mathcal{C}$  be a category.*

*If  $\mathcal{C}$  is locally Cartesian closed, then so is every slice of  $\mathcal{C}$ .*

*Proof.* For all  $C \in \mathcal{C}$ ,  $\mathcal{C}/C$  is Cartesian closed. Then keep slicing the category by  $c : X \rightarrow C$ ; we have  $(\mathcal{C}/C)/c \cong (\mathcal{C}/X)$  (by Proposition 9.21), and  $\mathcal{C}/X$  is Cartesian closed, hence the result.  $\square$

The property that matters to us now is the following:

**Proposition 9.29.** *Let  $\mathcal{C}$  be a category with all pullbacks.*

*Then,  $\mathcal{C}$  is locally Cartesian closed  $\Leftrightarrow$  for all arrow  $f$ , the pullback functor  $f^*$  has a right adjoint  $\Pi_f$ .*

*Proof.* [**Proof of  $\Leftarrow$** ] Let  $f : C \rightarrow D$  be an arrow, and let  $\Pi_f$  be its right adjoint.

We have to find the terminal object, the products and the exponentials in  $\mathcal{C}/D$ . By Proposition 9.16, we know that the terminal object in  $\mathcal{C}/D$  is  $\text{id}_D$ . By Proposition 9.17, as  $\mathcal{C}$  has all pullbacks, we know that  $\mathcal{C}/D$  has all products.

Consider the following pullback:

$$\begin{array}{ccc} P & \longrightarrow & X \\ f^*(d) \downarrow & \checkmark & \downarrow d \\ C & \xrightarrow{f} & D \end{array}$$

By Proposition 9.25, the composition functor  $\Sigma_f$  is left adjoint to the pullback functor  $f^*$ . By Proposition 9.17, the pullback of  $f$  and  $d$  in  $\mathcal{C}$  corresponds to a product in  $\mathcal{C}/D$ . Then:

$$f \times d = f \circ f^*(d) = \Sigma_f(f^*(d))$$

We deduce the following equivalence of hom-sets:

$$\begin{aligned} \text{Hom}_{\mathcal{C}/D}(f \times d, u) &= \text{Hom}_{\mathcal{C}/D}(\Sigma_f(f^*(d)), u) \\ &\cong \text{Hom}_{\mathcal{C}/C}(f^*(d), f^*(u)) \\ &\cong \text{Hom}_{\mathcal{C}/D}(d, \Pi_f(f^*(u))) \end{aligned}$$

Then, in  $\mathcal{C}/D$ , by Proposition 5.23 (exponential is right adjoint to product), the exponential  $u^f$  can only be  $u^f = \Pi_f(f^*(u))$ . As such an exponential always exists (because  $f^*$  always does, and  $\Pi_f$  does by assumption),  $\mathcal{C}/D$  is Cartesian closed.

Thus,  $\mathcal{C}$  is locally Cartesian closed.

[**Proof of  $\Rightarrow$** ] Assume that  $\mathcal{C}$  is locally Cartesian closed. Then each slice category is Cartesian closed, so each slice category  $\mathcal{C}/D$  of  $\mathcal{C}$  has products. By Proposition 9.21, a slice of a slice is a slice. By the same reasoning, each slice category  $(\mathcal{C}/D)/f$  of  $\mathcal{C}/D$  has products.

By Proposition 9.17, each product in  $(\mathcal{C}/D)/f$  is a pullback in  $\mathcal{C}/D$ , so  $\mathcal{C}/D$  has all pullbacks. By Proposition 9.16, each slice category  $\mathcal{C}/D$  has a terminal object  $\text{id}_D$ .

Each slice category  $\mathcal{C}/D$  has pullbacks and a terminal object, so by Proposition 6.34, it has all finite limits. By Proposition 7.27, as every slice category has finite limits and exponentials, we deduce that the pullback functor  $f^*$  (in  $\mathcal{C}$ , that is the "product" functor in  $\mathcal{C}/D$ ) has a right adjoint.  $\square$

**Corollary 9.30.** *If  $\mathcal{C}$  has all pullbacks and is locally Cartesian closed, then each slice category  $\mathcal{C}/C$  has finite limits.*

**Definition 9.31** (Dependent product). Let  $\mathcal{C}$  be a category with all pullbacks and locally Cartesian closed. Let  $f : C \rightarrow D$  be an arrow in  $\mathcal{C}$ .

The *dependent product*  $\Pi_f$  is the right adjoint to the pullback functor  $f^*$ .

Before studying more properties, maybe we should take a break and look at how this functor behaves.

*Example 9.32* (Slice category in **Sets**, again). Consider **Sets** and some function  $f : C \rightarrow D$ .

Let  $c : X \rightarrow C$  be an object in **Sets**/ $C$ . It is a function, indeed, but the point of view that makes more sense in this context is the following. For all  $y \in C$ , we can define the set  $c^{-1}(y) = \{x \in X \mid c(x) = y\}$ . In this case, the function  $c : X \rightarrow C$  becomes an  $C$ -indexed set  $(c^{-1}(y))_{y \in C}$  where  $X = \sum_{y \in C} c^{-1}(y)$

(coproduct). Then, a morphism  $h : c \rightarrow c'$  in **Sets**/ $C$  is a function between  $C$ -indexed sets, such that  $h = \sum_{y \in C} h_y : c^{-1}(y) \rightarrow c'^{-1}(y)$ .

Note that this point of view explains why there cannot be any arrow from  $c : X \rightarrow C$  to  $c' : X' \rightarrow C$  whenever  $c'(X') \subsetneq c(X)$ . In fact, take  $y \in c(X) \setminus c'(X')$ ; we have  $c'^{-1}(y) = \emptyset$ , so the function  $h_y : c^{-1}(y) \rightarrow c'^{-1}(y)$  is a function to the empty set, what does not exist unless  $c^{-1}(y) = \emptyset$  too.

What about  $\Sigma_f$ ? If  $c : X \rightarrow C \in \mathcal{C}/C$ , then  $\Sigma_f(c) = f \circ c : X \rightarrow D$ ; that is,  $\Sigma_f(c)$  can be seen as the  $D$ -indexed set  $\left( (f \circ c)^{-1}(y) \right)_{y \in D}$ . The function between  $C$ -indexed sets  $h : c \rightarrow c' \simeq h : \sum_{y \in C} h_y : c^{-1}(y) \rightarrow c'^{-1}(y)$  (morphism in  $\mathcal{C}/C$ ) is sent to  $\Sigma_f(h) : f \circ c \rightarrow f \circ c' \simeq \Sigma_f(h) : \sum_{y \in D} (f \circ h)_y : (f \circ c)^{-1}(y) \rightarrow (f \circ c')^{-1}(y)$ . Beware of the notation:  $\sum_{y \in D}$  is a coproduct symbol.

As for  $\Pi_f$ . Let  $y \in D$  and  $c : X \rightarrow C$  be an object in  $\mathcal{C}/C$ . We call *partial section of  $f$  for  $y$  along  $c$*  any function  $s : f^{-1}(y) \rightarrow X$  such that the following diagram commutes:

$$\begin{array}{ccc} f^{-1}(y) & \xrightarrow{s} & X \\ & \searrow \subseteq & \downarrow c \\ & & C \end{array}$$

that is, for all  $x \in f^{-1}(y)$ ,  $c \circ s(x) = x$ . Note that  $s$  is essentially a right inverse of  $c$  on the reverse image of  $y \in D$  by  $f$ , and such an  $s$  may not be unique. Also, a partial section along  $c : X \rightarrow C$  requires that  $f^{-1}(y) \subset c(X)$ ; so  $f$  may not have partial sections for all  $y$  along  $c$ , for example if  $f^{-1}(y) \setminus c(X) \neq \emptyset$ .

Awodey [1, p233, below Proposition 9.18] states that  $\Pi_f(c) : S \rightarrow D$  where  $S$  is:

$$S = \{s : f^{-1}(y) \rightarrow X \mid y \in D \text{ and } s \text{ is a partial section of } f \text{ for } y \text{ along } c\}$$

$$S \subset \bigcup_{y \in D} \text{Hom}(f^{-1}(y), X)$$

and for all  $s : f^{-1}(y) \rightarrow X \in S$ ,  $\Pi_f(c)(s) = y$  (that is,  $\Pi_f(c)$  "projects" a partial section  $s : f^{-1}(y) \rightarrow X$  to the base of the inverse image on which the section takes place).

Finally, the pullback functor is easier to see. Remember that, in **Sets**, the pullback between  $f : C \rightarrow D$  and  $d : X \rightarrow D$  is the set  $C \times_D X = \{(c, x) \in C \times X \mid f(c) = d(x)\}$  (cf. Example 5.55). The pullback functor  $f^*$  sends  $d : X \rightarrow D$  to the projection of  $C \times_D X$  to  $C$ :

$$f^*(d) : \begin{cases} C \times_D X & \longrightarrow C \\ (c, x) & \longmapsto c \end{cases}$$

One may wonder where  $d$  appears in the construction of  $f^*(d)$ ; just remember the above definition of  $C \times_D X$  in **Sets**.

**Proposition 9.33.** *Let  $\mathcal{E}$  be a topos. Then every slice of  $\mathcal{E}$  is a topos as well.*

*Proof.* Let  $E \in \text{Ob}_{\mathcal{E}}$ . By Corollary 9.30, the slice category  $\mathcal{E}/E$  has finite limits. We now have to show that every object has a power object.

So, the proof can be found in [8, Chapter IV, Section 7, Theorem 1, p190] and in [7, Chapter 5, Section 2, Theorem 2.1, p149], but I don't want to spend too much time on it. The proofs are very long. We will just admit this proposition.  $\square$

**Corollary 9.34.** *A topos is locally Cartesian closed.*

*Proof.* In fact, as every slice of a topos is a topos, then in particular, each slice of a topos is Cartesian closed.  $\square$

The fact that a topos is locally Cartesian closed is crucial in theoretical computer science and logic, because it means that any topos has an internal type theory.

Let us sum up the properties of a topos that we have seen:

**Theorem 9.35** (Properties of a topos). *Let  $\mathcal{E}$  be a topos. Then  $\mathcal{E}$  has all the following properties:*

- *It has all finite limits*
- *It has all finite colimits*
- *It has all exponentials*
- *Every object has a power object*
- *It has a subobject classifier*
- *It is Cartesian closed*
- *It is locally Cartesian closed*
- *Its slices are Cartesian closed*
- *Its slices are locally Cartesian closed*
- *Its slices are topoi*
- *Its isomorphisms are exactly the monic/epic*
- *It has all epi-mono factorisations*

## 10. Presheaves, sheaves, sheaf topoi

Incomplete section. We introduce the category of presheaves and prove it's a topos, but the Crash Course stops here for now (2019-01-18).

For now we have only see one example of topos. Let us introduce another example.

**Definition 10.1** (Presheaf). Let  $\mathcal{C}$  be a small category. A *presheaf on  $\mathcal{C}$*  is a functor  $\mathcal{C}^{\text{op}} \rightarrow \mathbf{Sets}$ .

The *presheaf category*, written  $[\mathcal{C}^{\text{op}}, \mathbf{Sets}]$ ,  $\mathbf{Sets}^{\mathcal{C}^{\text{op}}}$  or simply  $\mathbf{PSh}(\mathcal{C})$ , is the functor category  $\mathbf{Func}(\mathcal{C}^{\text{op}}, \mathbf{Sets})$ .

The following propositions will aim at proving that the presheaf category  $\mathbf{PSh}(\mathcal{C})$  is a topos.

**Lemma 10.2.** *Let  $\mathcal{C}$  be a small category. Then for all diagram  $D : \mathcal{I} \rightarrow \mathbf{PSh}(\mathcal{C})$ , we have  $\text{Lim}(D)(C) \cong \text{Lim}(D(-)(C))$ ; in other words, limits in a presheaf category are computed objectwise.*

*Proof.* Let  $D : \mathcal{I} \rightarrow \mathbf{PSh}(\mathcal{C})$  be a diagram in  $\mathbf{PSh}(\mathcal{C})$ . The proof lies on the (admitted) fact that  $\mathbf{Sets}$  has all small limits (admitted in Example 7.12).

Note that there is a canonical equivalence of categories:  $\mathbf{Func}(\mathcal{I}, \mathbf{Func}(\mathcal{C}^{\text{op}}, \mathbf{Sets})) \cong \mathbf{Func}(\mathcal{I} \times \mathcal{C}^{\text{op}}, \mathbf{Sets})$ .

Call  $\widehat{D}$  the equivalent diagram  $\widehat{D} : \mathcal{I} \times \mathcal{C}^{\text{op}} \rightarrow \mathbf{Sets}$ .

For a given  $C \in \mathcal{C}^{\text{op}}$ ,  $\widehat{D}(-, C)$  is a diagram from  $\mathcal{I} \rightarrow \mathbf{Sets}$ , and as  $\mathbf{Sets}$  has all small limits, it has a limit  $(\text{Lim}(\widehat{D}(-, C)), \varepsilon_{D, C})$ . Then, if  $c : C \rightarrow C' \in \mathcal{C}$ , then, as  $\widehat{D}$  is a (contravariant) functor, there is an arrow:  $\widehat{D}(-, c) : \widehat{D}(-, C') \rightarrow \widehat{D}(-, C)$  between cones to  $D$  and by property of limits, there is a unique arrow  $\text{Lim}(\widehat{D}(-, c)) : \text{Lim}(\widehat{D}(-, C')) \rightarrow \text{Lim}(\widehat{D}(-, C))$ , such that the following diagram commutes:

$$\begin{array}{ccc}
 \begin{array}{c} C \\ \downarrow c \\ C' \end{array} & \rightsquigarrow & \begin{array}{ccc} C' & \xrightarrow{\varepsilon_{D, C'}} & \text{Lim}(\widehat{D}(-, C')) \\ \downarrow c & \checkmark & \downarrow \text{Lim}(\widehat{D}(-, c)) \\ C & \xrightarrow{\varepsilon_{D, C}} & \text{Lim}(\widehat{D}(-, C)) \end{array}
 \end{array}$$

Note that we are considering the right-hand square diagram in  $\mathcal{C}^{\text{op}}$ ; we have the naturality of  $\varepsilon_{D, C}$  in  $C$ .

There remains to show that  $\text{Lim}(\widehat{D}(-, C))$  is indeed a limit. Let  $\alpha : \Delta(P) \rightarrow D$  be a cone to  $D$ . Here, the diagonal functor is:  $\Delta : \mathbf{PSh}(\mathcal{C}) \rightarrow \mathbf{PSh}(\mathcal{C})^{\mathcal{I}}$ . We are looking for a unique  $\gamma : P \rightarrow \text{Lim}(\widehat{D}(-, -))$  such that, for all objects  $i \in \mathcal{I}$  and  $C \in \mathcal{C}$ , the following diagram commutes:  $\alpha_i(C) = \varepsilon_{D, C, i} \circ \gamma(C)$ .

$$\begin{array}{ccc}
 & & D(i)(C) \\
 & \nearrow \alpha_i(C) & \downarrow \gamma(C) \\
 \Delta(P)(i)(C) & & \text{Lim}(\widehat{D}(-, C)) \\
 & \searrow \varepsilon_{D, C, i} &
 \end{array}$$

Such a (unique)  $\gamma(C)$  always exists due to the universal property of limits in  $\mathbf{Sets}$ . We only have to check that this  $\gamma = (\gamma(C))_{C \in \mathcal{C}}$  is natural in  $C$ . It is due to the naturality of the other natural transformations  $\alpha$  and  $\varepsilon_{D, -}$ . Finally,  $\gamma$  is unique due to the uniqueness of each  $\gamma(C)$ .

Thus, for every diagram  $D : \mathcal{I} \rightarrow \mathbf{PSh}(\mathcal{C})$ , there is a limit, and for all  $C \in \mathcal{C}$ ,  $\text{Lim}(D)(C) \cong \text{Lim}(\widehat{D}(-, C)) \cong \text{Lim}(D(-)(C))$ .  $\square$

In fact, we proved something stronger:

**Corollary 10.3.**  $\mathbf{PSh}(\mathcal{C})$  has all small limits.

**Corollary 10.4.** For all  $P, Q \in \mathbf{PSh}(\mathcal{C})$ ,  $P \times Q$  is the functor:

$$P \times Q : \begin{cases} \mathcal{C}^{op} & \longrightarrow & \mathbf{Sets} \\ C & \longmapsto & P(C) \times Q(C) \\ c & \longmapsto & P(c) \times Q(c) \end{cases}$$

**Corollary 10.5.** The terminal object in  $\mathbf{PSh}(\mathcal{C})$  is the constant presheaf  $\Delta(1)$  where  $\Delta$  is the diagonal functor  $\Delta : \mathbf{Sets} \rightarrow \mathbf{PSh}(\mathcal{C})$ .

**Proposition 10.6.** Let  $\alpha : P \rightarrow Q$  be a morphism between two presheaves in  $\mathbf{PSh}(\mathcal{C})$ .

Then,  $\alpha$  is monic  $\Leftrightarrow$  for all  $X \in \mathcal{X}$ ,  $\alpha_X : P(X) \rightarrow Q(X)$  is monic.

*Proof.* The proof of  $\Leftarrow$  has already been given in Proposition 5.41.

Conversely, suppose that  $\alpha$  is monic. The characterisation of monics by pullbacks (Proposition 5.58 states that the pullback of  $\alpha$  with itself is  $(P, \text{id}_P, \text{id}_P)$ . As limits, and thus pullbacks, are computed objectwise, we deduce that the pullback of each  $C$ -component  $\alpha_C : P(C) \rightarrow Q(C)$  with itself is also a triple  $(P(C), \text{id}_{P(C)}, \text{id}_{P(C)})$ , making each component monic.  $\square$

*Remark 10.7.* We know that a presheaf category has all small limits. In particular, it has all binary products, so maybe it has exponentials.

Let  $P, Q \in \mathbf{PSh}(\mathcal{C})$  be presheaves. Suppose their exponential  $Q^P$  exists; let us study it.

By adjunction product/exponential (Proposition 5.23), we know that  $\text{Hom}_{\mathbf{PSh}(\mathcal{C})}(X \times P, Q) \cong \text{Hom}_{\mathbf{PSh}(\mathcal{C})}(X, Q^P)$ . By Yoneda lemma, we have:

$$\text{Hom}_{\mathbf{PSh}(\mathcal{C})}(\text{Hom}_{\mathcal{C}}(-, C), Q^P) \cong Q^P(C)$$

which defines the functor  $Q^P$  as:

$$Q^P(C) \cong \text{Hom}_{\mathbf{PSh}(\mathcal{C})}(\text{Hom}_{\mathcal{C}}(-, C) \times P, Q)$$

We have to check that this actually defines an exponential; that is, for every  $f : X \times P \rightarrow Q$ , there is a unique  $\hat{f} : X \rightarrow Q^P$  such that the following diagram commutes:

$$\begin{array}{ccc} X & & P \times X \\ \downarrow f & & \downarrow \text{id}_B \times \hat{f} \quad \searrow f \\ Q^P & & P \times Q^P \xrightarrow{\text{eval}} Q \end{array}$$

Note that  $f : X \times P \rightarrow Q$  is a natural transformation, and as limits (hence, products) are computed objectwise, the previous diagram becomes:

$$\begin{array}{ccc} X(C) & & P(C) \times X(C) \\ \downarrow \hat{f}_C & & \downarrow \text{id}_{B,C} \times \hat{f}_C \quad \searrow f_C \\ Q^P(C) & & P(C) \times Q^P(C) \xrightarrow{\text{eval}_C} Q(C) \end{array}$$

We define the natural transformation  $\text{eval}_C$  as:

$$\text{eval}_C : \begin{cases} P(C) \times \text{Nat}(\text{Hom}_{\mathcal{C}}(-, C) \times P, Q) & \longrightarrow & Q(C) \\ (p, \alpha) & \longmapsto & \alpha_C(\text{id}_C, p) \end{cases}$$

Note that  $\text{eval}$  is the counit of the adjunction  $\text{Hom}_{\mathbf{PSh}(\mathcal{C})}(X \times P, Q) \cong \text{Hom}_{\mathbf{PSh}(\mathcal{C})}(X, Q^P)$ .

Then, note that  $\text{eval}_C$  is defined such that  $\text{eval}_C \circ (\text{id}_P \times \hat{f})_C(x, p) = \text{eval}_C(p, \hat{f}_C(x)) = \hat{f}_C(x)(\text{id}_C, p) = f_C(x, p)$  by commutativity of the previous diagrams.

We now have to find the expression of  $\hat{f}$ . For now we focus on  $\hat{f}_C(x)$ . We already know that  $\hat{f}_C(x)(\text{id}_C, p) = f_C(x, p)$ . For  $c : C' \rightarrow C$ , the following diagram commutes:

$$\begin{array}{ccc}
 C' & & \text{Hom}_{\mathcal{C}}(C, C) \times P(C) \xrightarrow{\hat{f}_C(x)_C} Q(C) \\
 \downarrow c & \rightsquigarrow & \downarrow \text{Hom}_{\mathcal{C}}(c, C) \times P(c) \quad \checkmark \quad \downarrow Q(c) \\
 C & & \text{Hom}_{\mathcal{C}}(C', C) \times P(C') \xrightarrow{\hat{f}_C(x)_{C'}} Q(C')
 \end{array}$$

that is:

$$Q(c) \circ \hat{f}_C(x)_C(u, p) = \hat{f}_C(x)_{C'} \circ (\text{Hom}_{\mathcal{C}}(c, C) \times P(c))(u, p)$$

and in particular, when  $u = \text{id}_C$ :

$$\begin{aligned}
 Q(c) \circ \hat{f}_C(x)_C(\text{id}_C, p) &= \hat{f}_C(x)_{C'} \circ (\text{Hom}_{\mathcal{C}}(c, C) \times P(c))(\text{id}_C, p) \\
 Q(c)(f_C(c, p)) &= \hat{f}_C(x)_{C'}(c, P(c)(p))
 \end{aligned}$$

which defines  $\hat{f}_C(x)_{C'}(c, p')$  on pairs  $(c, p')$  such that  $p' = P(c)(p)$  (which is enough for our purposes). The naturality of  $\hat{f}_C(x)$  in  $C$  is immediate.

Finally, we have defined the exponential in  $\mathbf{PSh}(\mathcal{C})$ .

**Definition 10.8** (Exponential in a presheaf category). Let  $P, Q$  be two presheaves in  $\mathbf{PSh}(\mathcal{C})$ . Their exponential  $Q^P$  is defined as:

$$Q^P : \begin{cases} \mathcal{C}^{\text{op}} & \longrightarrow & \mathbf{Sets} \\ C & \longmapsto & \text{Hom}_{\mathbf{PSh}(\mathcal{C})}(\text{Hom}_{\mathcal{C}}(-, C) \times P, Q) \\ c & \longmapsto & \text{Hom}_{\mathbf{PSh}(\mathcal{C})}(\text{Hom}_{\mathcal{C}}(-, c) \times P, Q) \end{cases}$$

**Lemma 10.9.** A presheaf category  $\mathbf{PSh}(\mathcal{C})$  has all exponentials.

*Proof.* The construction in Remark 10.7 holds for any presheaves  $P$  and  $Q$ .  $\square$

**Corollary 10.10.** A presheaf category is Cartesian closed.

We could have defined the exponential in a presheaf category, and then prove that the presheaf it defines actually is an exponential, but we preferred showing how the definition naturally made sense.

So, we have all small (thus finite) limits, and all exponentials. The only thing missing is the subobject classifier. To this extend, we define:

**Definition 10.11** (Sieve [1]). Let  $\mathcal{C}$  be a small category, and let  $C$  be an object in  $\mathcal{C}$ .

A sieve on  $C$  is a set  $S \in \mathbf{Sets}$  such that:

$$\begin{aligned}
 S &= \{f : X \rightarrow C \mid \text{for some arrows } f : X \rightarrow C\} \\
 &\cup \{f \circ g : Y \rightarrow C \mid Y \in \mathcal{C}, g : Y \rightarrow X \text{ and } f \in S\}
 \end{aligned}$$

In other words,  $S$  is a set of (some) arrows of  $\mathcal{C}$  with codomain  $C$  (left-hand part of the union), stable by precomposition (right-hand part of the union), that is, for all  $g : Y \rightarrow X$  and  $f \in S$ , we have  $f \circ g \in S$ . Note that  $S$  doesn't necessarily contain all arrows  $X \rightarrow C$ .

**Definition 10.12** (Sieve presheaf). Let  $\mathcal{C}$  be a small category.

For  $C \in \mathcal{C}$ , we define  $\text{Sieve}(C) = \{S \in \mathbf{Sets} \mid S \text{ is a sieve on } C\}$ .

For  $c : C \rightarrow C' \in \mathcal{C}$ , we define:

$$\text{Sieve}(c) : \begin{cases} \text{Sieve}(C') & \longrightarrow & \text{Sieve}(C) \\ S & \longmapsto & \{g : X \rightarrow C \mid c \circ g \in S\} \end{cases}$$

The *sieve presheaf*<sup>5</sup>, written  $\text{Sieve}$ , is the following contravariant functor:

$$\text{Sieve} : \begin{cases} \mathcal{C} & \longrightarrow & \mathbf{Sets} \\ C & \longmapsto & \text{Sieve}(C) \\ c : C \rightarrow C' & \longmapsto & \text{Sieve}(c) \end{cases}$$

**Lemma 10.13.** Let  $\mathcal{C}$  be a small category and let  $\text{Sieve} : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Sets}$  be its sieve presheaf. And let  $\Delta(1)$  be the terminal object in  $\mathbf{PSh}(\mathcal{C})$ .

Then there exists  $t : \Delta(1) \rightarrow \text{Sieve}$  such that  $(\text{Sieve}, t)$  is a subobject classifier of  $\mathbf{PSh}(\mathcal{C})$ .

*Proof.* The morphism (natural transformation)  $t$  is:

$$t = \left( t_C : \begin{cases} 1 & \longrightarrow & \text{Sieve}(C) \\ x & \longmapsto & \{f : X \rightarrow C \mid X \in \mathcal{C}, f : X \rightarrow C\} \end{cases} \right)_{C \in \mathcal{C}}$$

That is,  $t_C$  is the function that selects the (unique) sieve that contains all arrows whose codomain is  $C$  (remember that sieves need not contain all arrows). The naturality of  $t$  is quite obvious once we remember that  $\Delta(1)(C) = 1$ .

We now have to check that  $(\text{Sieve}, t)$  is a subobject classifier. **Awodey describes the classifying arrow of  $\pi : Q \rightarrow P$  as  $\alpha$  such that:**

$$\alpha_C : \begin{cases} P(C) & \longrightarrow & \text{Sieve}(C) \\ x & \longmapsto & \{f : X \rightarrow C \mid P(f)(x) \in Q(X)\} \end{cases}$$

**but I am having trouble finding out that this defines a pullback. I leave the proof for now.** □

**Proposition 10.14.** Let  $\mathcal{C}$  be a small category. Then  $\mathbf{PSh}(\mathcal{C})$  is a topos.

*Proof.* By Corollary 10.3, a presheaf category has all small limits; in particular, it has all finite limits. By Lemma 10.9, a presheaf category has all exponentials. Finally, by Lemma 10.13, it has a subobject classifier. Consequently,  $\mathbf{PSh}(\mathcal{C})$  deserves its title of topos. □

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<sup>5</sup>Note that this name is not standard.

## 11. David's riddles

**Proposition 11.1.** Consider  $\mathcal{C}$  and its slice  $\mathcal{C}/C$ .

The coproduct of  $a_1 : A_1 \rightarrow C$  and  $a_2 : A_2 \rightarrow C$  (objects of the slice) is  $a_1 + a_2 : A_1 + A_2 \rightarrow C$ , where  $a_1 + a_2$  is the unique arrow from the UMP of the coproduct.

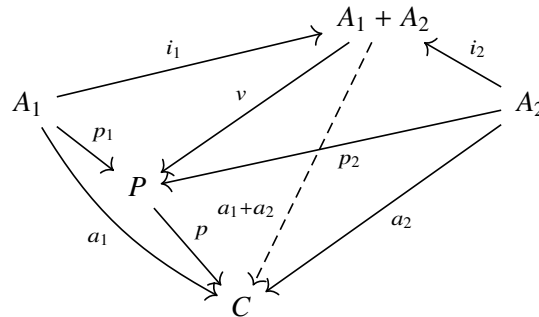
*Proof.* The proof is mainly diagram chase.

Let  $A_1 + A_2$  be the coproduct of  $A_1$  and  $A_2$ , the following diagram commutes for a unique arrow  $u$ :

$$\begin{array}{ccccc}
 & & C & & \\
 & a_1 \nearrow & \uparrow u & \nwarrow a_2 & \\
 A_1 & \xrightarrow{i_1} & A_1 + A_2 & \xleftarrow{i_2} & A_2
 \end{array} \tag{25}$$

This ensures that the inclusion maps  $i_1$  and  $i_2$  actually are arrows in  $\mathcal{C}/C$ .

We denote this  $u = a_1 + a_2$ . We now have to check that this actually defines a coproduct. Let  $P$  as in the diagram:



The fact that  $A_1 + A_2$  is a coproduct in  $\mathcal{C}$  gives that unique arrow  $v : A_1 + A_2 \rightarrow P$  such that  $v \circ i_1 = p_1$  and  $v \circ i_2 = p_2$ . We then have to check that  $v$  actually is an arrow in  $\mathcal{C}/C$ , that is, that  $p \circ v = a_1 + a_2$ . In fact,  $a_1 + a_2$  is the unique arrow  $A_1 + A_2 \rightarrow C$  that makes Diagram 25 commute;  $p \circ v$  also does, so the equality must hold.  $\square$

**Proposition 11.2** (Coproduct of pullbacks is a pullback). Let  $\mathcal{C}$  be locally Cartesian closed.

Consider the following diagram:

$$\begin{array}{ccc}
 \begin{array}{ccc} A_1 & \xrightarrow{b_1} & B_1 \\ a_1 \downarrow & & \downarrow d_1 \\ C & \xrightarrow{c} & D \end{array} & + & \begin{array}{ccc} A_2 & \xrightarrow{b_2} & B_2 \\ a_2 \downarrow & & \downarrow d_2 \\ C & \xrightarrow{c} & D \end{array} \\
 \Rightarrow & & \begin{array}{ccc} A_1 + A_2 & \xrightarrow{b_1+b_2} & B_1 + B_2 \\ a_1+a_2 \downarrow & & \downarrow d_1+d_2 \\ C & \xrightarrow{c} & D \end{array}
 \end{array}$$

If the left-hand squares are pullbacks, then the right-hand square is a pullback.

*Proof.* The category  $\mathcal{C}$  is locally Cartesian closed. Then the pullback functor  $c^* : \mathcal{C}/D \rightarrow \mathcal{C}/C$  based on  $c : C \rightarrow D$  has a right adjoint. Thus, it is a left adjoint. By Proposition 7.10, left adjoints preserve colimits, so in particular, coproducts. In consequence:

$$c^*(d_1 + d_2) = c^*(d_1) + c^*(d_2) = a_1 + a_2 \tag{26}$$

which gives the result.  $\square$



## 12. To do

Sur le fond :

1. A word on how special the exponential is: every limit is unique up to isomorphism due to the UMP, while we needed adjunctions in order to make the exponential unique up to iso. It's because there is something more profound behind products and such, that the exponential doesn't share and we will see that in the next section. The nature of the exponential is different.
2. L'adjonction  $\exists$  -?? -  $\forall$
3. Introduire les monades (juste après les adjoints). C'est assez facile, il y a plein d'exemples.
4. ETCS ? Elementary Theory of the Category of Sets
5. Natural numbers objects + construction of  $\mathbb{Z}$ ,  $\mathbb{Q}$  and  $\mathbb{R}$ ... Cf David's draft on discrete time behaviour type (given while at MIT): the corresponding sheaves are constant, except for reals.
6. Well-pointed topos ?
7. Présenter les topoi de sheaves, sur une topologie, sur une Heyting algebra.
8. Préciser topos  $\supset$  presheaf topos  $\supset$  sheaf topos = Gorthendieck topos (+ presheaf topos is the sheaf topos for the trivial topology)
9. Bon. J'adore les topoi, du coup peut-être que je ne me rends pas compte si je veux en rajouter trop ou pas. Donc voilà. Une fois qu'on a un topos (de presheaves par exemple), on peut définir une logique. Puis on peut définir des modalités (local operators?) qui sont des arrows  $\Omega \rightarrow \Omega$  qui vérifient quelques propriétés. Chaque modalité définit un subtopos dont la logique vérifie cette modalité (cf. Seven sketches, chap 7) Cf aussi [ce post](#)
10. Kan extensions (ça a l'air vraiment accessible)
11. I like [this exercise](#)
12. [Infinite chain of adjoint functors : \(un peu compliqué\)](#)
13. Une chaîne rigolote d'adjoints : cf cahier, le 15/05/2019.
14. [Encore plus d'exemples d'adjonctions](#)
15. [Beautiful examples of adjunctions](#)
16. [Adjunctions in fundamental theorems](#)
17. Question: Si  $F$  est adjoint à  $U$  est adjoint à  $F$ , que dire de  $F$  et  $U$ ? Hypothèse: c'est une équivalence de catégories.; je cherche donc à montrer que les unit/counits sont inverses l'une de l'autre; mais je n'y arrive pas.

Sur la forme :

1. Donner des noms aux théorèmes (dire en gros ce qu'ils disent)
2. Présentation d'une catégorie pour les enfants ?
3. Harmoniser les notations: est-ce que la catégorie par défaut est  $\mathcal{C}$  ou  $\mathcal{X}$  ? Est-ce que l'objet par défaut est  $C$  ou  $X$  ? Est-ce que l'adjonction par défaut est  $U^*$  ou  $F$  ?

4. Comment lire un diagramme commutatif
5. Mettre des exemples partout!
  - 5.1. A locally Cartesian closed category that is not Cartesian closed (LH, category of local homeomorphisms, doesn't have a terminal object), cf [https://en.wikipedia.org/wiki/Cartesian\\_closed\\_category#Examples](https://en.wikipedia.org/wiki/Cartesian_closed_category#Examples)
  - 5.2. Examples of topoi:  $\text{FinSets}$ ,  $V_\omega$ ,  $V_\alpha$  for  $\alpha$  limit ordinal (small topos), cf: <https://math.stackexchange.com/questions/116023/example-of-a-small-topos>
  - 5.3. The category of topological spaces lacks some properties: <https://ncatlab.org/nlab/show/nice+category+of+spaces>

**Symbols**

$\beta$  Adjunct of two functors  $F \dashv U$

$\text{Colim}(D)$  Colimit of the diagram  $D$

**Cocones**  $(D)$  Category of cocones from  $D$

**Cones**  $(D)$  Category of cones from  $D$

$(F | G)$  Comma-category

$\mathcal{C}/C$  Slice category over  $C$

$\Delta$  Diagonal functor

$\varepsilon$  Counit of an adjunction  $F \dashv U$

$C^B$  Exponential

$E_A$  Exponential functor

$f^*$  Pullback functor

**Func**  $(\mathcal{C}, \mathcal{D})$  The functor category

$\eta$  Unit of an adjunction  $F \dashv U$

$\text{Lim}(D)$  Limit of the diagram  $D$

**Nat**  $(F, G)$  Set of natural transformations  $F \rightarrow G$

$\mathcal{C}^{\text{op}}$  Opposite category

$1$  Terminal object

$\Omega$  Subobject classifier

$A \times B$  Product of objects

$A + B$  Coproduct of objects

$P_A$  Product functor

$A \times_C B$  Pullback of  $f : A \rightarrow C$  and  $g : B \rightarrow C$

$B +_A C$  Pushout of  $f : A \rightarrow B$  and  $g : A \rightarrow C$

$\mathcal{P}(C)$  Power object of  $C$

$\Pi_f$  Dependent product

**Sets** The category of sets

**SubObj** $_{\mathcal{C}}(C)$  Subobject category

$\text{SubObj}_{\mathcal{C}}$  Subobject functor

$\Sigma_f$  Dependent sum (or composition functor)

$\theta_{H,X}$  The Yoneda natural isomorphism  $H(X) \rightarrow \text{Nat}(\text{Hom}_{\mathcal{C}}(X, -), H)$

$U^*$  The left adjoint of a functor  $U$

$V_\lambda$  The  $\lambda$ -th set from the von Neumann hierarchy

$\xi_{H,X}$  The Yoneda natural isomorphism  $\text{Nat}(\text{Hom}_{\mathcal{C}}(X, -), H) \rightarrow H(X)$

$y$  The Yoneda embedding

$0$  Initial object

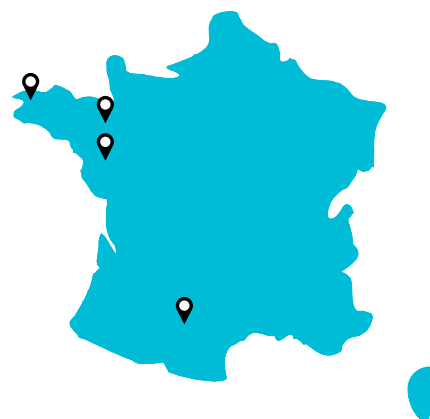
## References

- [1] S. Awodey, *Category Theory*, 2nd ed., ser. Oxford Logic Guides. Oxford University Press, Oxford, 2010, vol. 52.
- [2] M. Barr and C. Wells, *Category Theory for Computing Science*. Upper Saddle River, NJ, USA: Prentice-Hall, Inc., 1998.
- [3] K. Kunen, *Set theory - An introduction to independence proofs*, 7th ed., ser. Studies in logic and the foundations of mathematics. North-Holland Publishing Company, 1999, vol. 102.
- [4] D. E. Speyer, “What’s a reasonable category that is not locally small?” Question asked by aorq, replied by David E. Speyer; last accessed: 08-november-2018: <https://mathoverflow.net/questions/3278/whats-a-reasonable-category-that-is-not-locally-small>, 2009.
- [5] S. MacLane, *Categories for the Working Mathematician*, 2nd ed., ser. Graduate Texts in Mathematics. Springer-Verlag, New York, 1998, vol. 5.
- [6] T. Leinster, *Basic Category Theory*, ser. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2016, vol. 143. [Online]. Available: [www.arXiv.org/abs/arXiv:1612.09375](http://www.arXiv.org/abs/arXiv:1612.09375)
- [7] M. Barr and C. Wells, *Topos, triples and theories*, 2nd ed., ser. Reprints in Theory and Applications of Categories. Springer-Verlag, New York, 2005, vol. 12.
- [8] S. MacLane and I. Moerdijk, *Sheaves in Geometry and Logic*, 1st ed., ser. Universitext. Springer-Verlag New York, 1994.

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