ON THE EXTENDED GREEN-NAGHDI SYSTEM FOR AN UNEVEN BOTTOM WITH SURFACE TENSION

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BASHAR KHORBATLY*‡ AND SAMER ISRWAI*

ABSTRACT. In this paper, a derivation of the two-dimensional asymptotic nonlinear highly dispersive shallow-water extended Green-Naghdi system for an uneven bottom is represented. Then we consider the one-dimensional case of this model taking into consideration the effect of a small surface tension. We show that the construction of solution with a standard Picard iterative scheme can be accomplished in which the well-posedness in $X^s = H^{s+2}(\mathbb{R}) \times H^{s+2}(\mathbb{R})$ for some $s > \frac{3}{2}$, of the modified extended one-dimensional system for a finite large time existence $t = O(\frac{1}{\varepsilon \vee \beta})$ is proved.

1. Introduction

The water-wave problem in its simplest form concerns two-dimensional motion of an irrotational and incompressible inviscid liquid with a free surface, acted on only by gravity and surface tension. Assume that the fluid is of constant density ρ and denote by $\Omega_t = \{(X, z) \in \mathbb{R}^d \times \mathbb{R}, -h_0 + b(X) < z < \zeta(t, X)\}$ the domain of the fluid for each time variable t where the surface of the fluid is a graph parametrized by ζ and its bottom is parametrized by $-h_0 + b(X)$ independent of time with h_0 the depth. Knowing that $d = 1, 2$ the spatial dimension of the surface of the fluid where $X \in \mathbb{R}^d$ the spatial variable is written as $X = (x, y)$ when $d = 2$ and $X = x$ when $d = 1$, while the vertical variable is denoted by z.
The motion of an ideal moving fluid is described by the free surface Euler equations for steady flow along a streamline (their well-posedness were recognized after the work of Nalimov [15], Yosihara [3], Craig [16], Wu [10, 11] and Lannes [2]) which is a connection between the velocity V, the pressure P, and the density ρ of the fluid that is based on the Newton's second law of motion and can be written under the form

$$\partial_t V + (V \cdot \nabla_{X,z})V = -g\overrightarrow{e}_z - \frac{1}{\rho}\nabla_{X,z}P \qquad \text{in } (X, z) \in \Omega_t \ , \ t \geq 0. \tag{1}$$

The incompressibility of the fluid is expressed by

$$\nabla_{X,z} \cdot V = 0 \qquad \text{in } (X, z) \in \Omega_t \ , \ t \geq 0, \tag{2}$$

knowing that the irrotationality of the fluid means that

$$\nabla_{X,z} \times V = 0 \qquad \text{in } (X, z) \in \Omega_t \ , \ t \geq 0, \tag{3}$$

where $V : \mathbb{R}_+ \times \Omega_t \longrightarrow \mathbb{R}^d \times \mathbb{R}$ is the fluid velocity, $P : \mathbb{R}_+ \times \Omega_t \longrightarrow \mathbb{R}$ is the fluid pressure term at point $(X, z) \in \Omega_t$ and instant $t \geq 0$, $-g\overrightarrow{e}_z$ is the gravitational field which is acting vertically downward with g greater than zero and $\overrightarrow{e}_z$ is a unit vector in vertical direction. These equations are complemented with the dynamic condition and is given by

$$P - P_{atm} = \sigma\kappa(\zeta) \qquad \text{at } z = \zeta(t, X) \ , \ t \geq 0, \tag{4}$$

that expresses a balance of forces across the free surface and denoting by P_{atm} the (constant) atmospheric pressure. In addition, to the kinematic condition that is a boundary condition at the surface and states that

Date: February 16, 2019.
Key words and phrases. Water waves, shallow-water approximation, bottom topography, Extended Green-Naghdi equations, highly-dispersive model, large amplitude model, surface tension.

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the free surface moves with the fluid is given by

$$\partial_t\zeta - \sqrt{1+|\nabla_X\zeta|^2}V\cdot n_+ = 0 \qquad \text{at } z=\zeta(t,X)\ ,\ t\geq 0, \tag{5}$$

with the boundary condition on the velocity at the bottom and announces that the fluid particles do not cross the bottom is given by

$$V\cdot n_- = 0 \qquad \text{at } z=-h_0+b(X)\ ,\ t\geq 0, \tag{6}$$

where the outward unit normal vector to the free surface and the outward unit normal to the lower boundary of Ω_t are respectively denoted by

$$n_+ = \frac{1}{\sqrt{1+|\nabla_X\zeta|^2}}\big(\nabla_X\zeta^T,1\big)^T, \qquad \text{and} \qquad n_- = \frac{1}{\sqrt{1+|\nabla_X b|^2}}\big(\nabla_X b^T,-1\big)^T,$$

while $\sigma>0$ is the surface tension coefficient, and $\kappa(\zeta)=-\nabla\cdot\big(\frac{\nabla\zeta}{\sqrt{1+|\nabla\zeta|^2}}\big)$ is the mean curvature of the surface denoting by $\nabla=\nabla_X$. The last assumption (7) states that the fluid is at rest at infinity and is given by

$$\lim_{|(X,z)|\to\infty}|\zeta(X,z)|+|V(t,X,z)|=0 \quad \text{ in } (X,z)\in\Omega_t\ ,\ t\geq 0. \tag{7}$$

In this paper, we consider the Eulerian specification of the fluid motion that focuses on specific locations in the space through which the fluid flows as time passes rather than working in the Lagrangian approach, since it is most simple to deal with. In particular, when approximate features are investigated. More precisely, concerning asymptotic properties generates the occurrence of Green-Naghdi equations (see [19, 18, 22]) that takes into consideration neglected rotational effects (*i.e.* $0=\nabla_{X,z}\times V$), which are significant for wind driven waves, waves riding upon a sheared current, waves near a ship, or tsunami waves approaching a shore and ensures the existence of $\varphi\colon \mathbb{R}_+\times\Omega_t\longrightarrow\mathbb{R}$ the velocity potential flow of the fluid such that $\nabla_{X,z}\varphi=V$ in Ω_t. This plays a great role in writing the Euler system under Bernoulli's formulation

$$\begin{cases} \Delta_{X,z}\varphi=0 & \text{at} \quad -h_0+b(X)<z<\zeta(t,X),\\ \partial_z\varphi-\nabla_X b\cdot\nabla_X\varphi=0 & \text{at} \quad z=-h_0+b(X),\\ \partial_t\zeta+\nabla_X\varphi\cdot\nabla_X\zeta-\partial_z\varphi=0 & \text{at} \quad z=\zeta(t,X),\\ \partial_t\varphi+\dfrac{1}{2}|\nabla_{xyz}\varphi|^2+g\zeta=-\dfrac{\sigma}{\rho}\kappa(\zeta) & \text{at} \quad z=\zeta(t,X), \end{cases} \tag{8}$$

where the Laplacian equation is obtained by taking the divergence of the potential velocity after using the incompressibility condition.The second and third equations can be written using the boundary condition at the bottom (6) and the kinematic condition (5) respectively. While The last equation is established by commuting $V=\nabla_{X,z}\varphi$ in (1).

Presently, in order to solve the Laplacian equation we need information from the boundary that moves with time and its location is determined by two coupled nonlinear partial differential equations which is a basic difficulty. This difficulty leads us to derive some (much simpler) asymptotic models to this system which requires the identification of small parameters that it is often possible to deduce from their values some insight on the behavior of the flow. More precisely, let us introduce the following dimensionless parameters

$$\varepsilon=\frac{a}{h_0}, \qquad \mu=\frac{h_0^2}{\lambda^2}, \qquad \beta=\frac{b_0}{h_0}, \qquad \mathrm{Bo}=\frac{\rho g\lambda^2}{\sigma},$$

where the parameter $0\leq\varepsilon\leq 1$ is often called nonlinearity parameter, while $0\leq\mu\ll 1$ is the shallowness parameter, $0\leq\beta\leq 1$ is the typical amplitude of the bottom deformations (topography parameter) and Bo is the *classical Bond number* which measures the ratio of gravity forces over capillary forces, knowing that a is the of amplitude of the wave, λ the wave-length of the wave, b_0 the order of amplitude of the variations of the bottom topography, h_0 the reference depth, ρ the density of the fluid and σ is the surface tension coefficient. We now execute the classical shallow water ($\mu\ll 1$) non-dimensionalization using the following relations:

$$X=\lambda X', \qquad z=h_0 z', \qquad \zeta=a\zeta',$$
$$\varphi=\frac{a}{h_0}\lambda\sqrt{gh_0}\varphi', \qquad b=b_0 b', \qquad t=\frac{\lambda}{\sqrt{gh_0}}t'.$$

Remarking that the wave dispersion in water waves refers to the property that longer waves (large wavelength) have lower frequencies and travel faster than short waves and their maximum speed of propagation is $\sqrt{gh_0}$. Thus, those long waves have common speed $\sqrt{gh_0}$ called the linear phase velocity .
Therefore, the equations of motion $(1)-(7)$ then become (after eliminating the primes for sake of clarity) under the dimensionless Bernoulli's formulation

$$
\left\{\begin{array}{lll}
\mu\partial_x^2\varphi+\mu\partial_y^2\varphi+\partial_z^2\varphi=0 & \text{at} & -1+\beta b(X)<z<\varepsilon\zeta(t,X),\\
\partial_z\varphi-\mu\beta\nabla_X b\cdot\nabla_X\varphi=0 & \text{at} & z=-1+\beta b(X),\\
\partial_t\zeta-\dfrac{1}{\mu}(-\mu\varepsilon\nabla_X\zeta\cdot\nabla_X\varphi+\partial_z\varphi)=0 & \text{at} & z=\varepsilon\zeta(t,X),\\
\partial_t\varphi+\dfrac{1}{2}(\varepsilon|\nabla_X\varphi|^2+\dfrac{\varepsilon}{\mu}(\partial_z\varphi)^2)+\zeta=-\dfrac{1}{\mathrm{Bo}}\dfrac{\kappa(\varepsilon\sqrt{\mu}\zeta)}{\varepsilon\sqrt{\mu}} & \text{at} & z=\varepsilon\zeta(t,X),
\end{array}\right. \tag{9}
$$

Now, in order to reduce the dimensionless free surface Bernoulli's equations (9) into a system where all functions are evaluated at the free surface (in $\mathbb{R}_+\times\mathbb{R}^d$) and it is known as the dimensionless version of Zakharov/Criag-Sulem [14] formulation of the water-waves equations with surface tension. The demonstration is commenced by introducing $\psi\colon \mathbb{R}_+\times\mathbb{R}^d\longrightarrow\mathbb{R}$ the trace of the velocity potential at the free surface

$$
\psi(t,X)=\varphi\big(t,X,\varepsilon\zeta(t,X)\big)=\varphi_{|_{z=\varepsilon\zeta}}, \tag{10}
$$

and the Dirichlet-Neumann operator $\mathcal{G}_\mu[\varepsilon\zeta,\beta b]\cdot$ is defined by

$$
\mathcal{G}_\mu[\varepsilon\zeta,\beta b]\psi=-\mu\big(\varepsilon\nabla\zeta\big)\cdot\big(\nabla\varphi\big)_{|_{z=\varepsilon\zeta}}+\big(\partial_z\varphi\big)_{|_{z=\varepsilon\zeta}}=\sqrt{1+\mu\varepsilon^2\big|\nabla\zeta\big|^2}\big(\partial_n\varphi\big)_{|_{z=\varepsilon\zeta}}, \tag{11}
$$

with φ solving (see [1] for accurate analysis) the boundary value problem

$$
\left\{\begin{array}{l}
\mu\partial_x^2\varphi+\mu\partial_y^2\varphi+\partial_z^2\varphi=0 \quad \text{in} \quad -1+\beta b(X)<z<\varepsilon\zeta(t,X),\\
\partial_n\varphi_{|_{z=-1+\beta b}}=0,\\
\varphi_{|_{z=\varepsilon\zeta}}=\psi(t,X),
\end{array}\right. \tag{12}
$$

where $\partial_n\varphi=n_-\cdot\nabla_{X,z}\varphi$ refers to the upward normal derivative at the bottom. A set of two equations on the free surface parametrization ζ and the trace of the velocity potential at the surface $\psi=\varphi_{|_{z=\varepsilon\zeta}}$ involving the Dirichlet-Neumann operator is introduced as

$$
\left\{\begin{array}{l}
\partial_t\zeta-\dfrac{1}{\mu}\mathcal{G}_\mu[\varepsilon\zeta,\beta b]\psi=0,\\
\partial_t\psi+\zeta+\dfrac{\varepsilon}{2}|\nabla\psi|^2-\varepsilon\mu\dfrac{(\frac{1}{\mu}\mathcal{G}_\mu[\varepsilon\zeta,\beta b]\psi+\nabla(\varepsilon\zeta)\cdot\nabla\psi)^2}{2(1+\varepsilon^2\mu|\nabla\zeta|^2)}=-\dfrac{1}{\mathrm{Bo}}\dfrac{\kappa(\varepsilon\sqrt{\mu}\zeta)}{\varepsilon\sqrt{\mu}}.
\end{array}\right. \tag{13}
$$

In the event that no presumption is made on the nonlinearity parameter defined above, a shallow water asymptotic regime ($\mu\ll 1$) is identified. Formally, this regime leads at second order $O(\mu^2)$ to a large amplitude model ($\mu\ll 1,\varepsilon\backsim 1$) called the Green-Naghdi system. A rigorous justification on the well-posedness of the standard Green-Naghdi equations was given by several works such as [17, 4, 5] in $1D$ and $2D$ with flat and non-flat bottoms ($\beta=0$, $\beta\neq 0$) respectively where a solution was constructed with a standard Picard iterative scheme so that there is no loss of regularity of the solution with respect to the initial condition, unlike a Nash-Moser scheme made in [21] for $2D$ case. The aim of this paper is to derive the $2D$ extended Green-Naghdi system for non-flat bottom of order three with respect to the shallowness parameter μ (dispersive term) (see [8, 9, 27] for the extended flat bottom system) applying the general method used in [22] for $O(\mu^2)$ approximation for uneven bottom topography represented by

$$
\left\{\begin{array}{l}
\partial_t\zeta+\nabla\cdot(hv)=0,\\
\big(h+\mu\mathcal{T}[h]+\mu^2\mathfrak{T}[h]\big)\partial_t v+h\nabla\zeta+\varepsilon h(v\cdot\nabla)v+\varepsilon\mu Q_1[U]v\\
\qquad\qquad +\varepsilon\mu\beta B_1[U]v+\varepsilon\mu\beta^2B_2[U]v+\varepsilon\mu^2Q_2[U]v+\varepsilon\mu^2\beta B_3[U]v+\varepsilon\mu^2\beta^2B_4[U]v=O(\mu^3),
\end{array}\right. \tag{14}
$$

where $v=(v_1,v_2)^T$, $U=(\zeta,v)^T$ and $h(t,X)=1+\varepsilon\zeta(t,X)-\beta b(X)$, denoting by

$$
\mathcal{T}[h,\beta b]w=-\frac{1}{3}\nabla(h^3\nabla\cdot w)+\frac{\beta}{2}\big[\nabla(h^2\nabla b\cdot w)-h^2\nabla\cdot w\nabla b\big]+\beta^2h(\nabla b\cdot w)\nabla b,
$$

$$\mathfrak{T}[h,\beta b]w=-\frac{1}{45}\nabla\Big(\nabla\cdot\big(h^5\nabla(\nabla\cdot w)\big)\Big)+\frac{1}{24}\beta\nabla\Big(\nabla\cdot\big(h^4\nabla(\nabla b\cdot w)\big)\Big)+\frac{1}{24}\beta\nabla\big(\nabla\cdot w\nabla\cdot(h^4\nabla b)\big)$$
$$-\frac{1}{24}\beta\nabla\cdot\big(h^4\nabla(\nabla\cdot w)\big)\nabla b+\frac{1}{12}\beta^2\nabla\big(h^3\nabla\cdot w(\nabla b\nabla b)\big)\big)+\frac{1}{12}\beta^2\nabla\big(h^3\nabla b\nabla(\nabla b\cdot w)\big)$$
$$+\frac{1}{12}\beta^2\nabla\cdot\big(h^3\nabla\cdot w\nabla b\big)\nabla b+\frac{1}{12}\beta^2\nabla\cdot\big(h^3\nabla(\nabla b\cdot w)\big)\nabla b,$$

where the non-topographical terms represented by $Q_1[U]$, $Q_2[U]$ as follows

$$Q_1[U]v=-\frac{1}{3}\nabla\Big(h^3\big((v\cdot\nabla)(\nabla\cdot v)-(\nabla\cdot v)^2\big)\Big),$$

$$Q_2[U]v=-\frac{1}{45}\nabla\Big[\nabla\cdot\Big\{h^5\big(\nabla^2(\nabla\cdot v)\big)v-5h^5(\nabla\cdot v)\nabla(\nabla\cdot v)+\nabla h^5\times\big(v\times\nabla(\nabla\cdot v)\big)\Big\}\Big]$$
$$+\frac{2}{45}\nabla\Big(h^5\big(\nabla(\nabla\cdot v)\big)^2\Big)+\frac{1}{45}\nabla\cdot\Big(h^5\nabla(\nabla\cdot v)\Big)\nabla(\nabla\cdot v)+\frac{1}{90}h^5\nabla\big\{\big(\nabla(\nabla\cdot v)\big)^2\big\},$$

while the purely-topographical terms are introduced by $B_1[U]$, $B_2[U]$, $B_3[U]$, $B_4[U]$ as follows

$$B_1[U]v=\frac{1}{2}\nabla\big(h^2(v\cdot\nabla)^2b\big)-\frac{1}{2}h^2\big((v\cdot\nabla)(\nabla\cdot v)-(\nabla\cdot v)^2\big)\nabla b,\qquad B_2[U]v=h\big((v\cdot\nabla)^2b\big)\nabla b,$$

and

$$B_3[U]v=+\frac{1}{24}\nabla\Big\{\nabla\cdot\Big(h^4\nabla^2(\nabla b\cdot v)v+h^4(\nabla\cdot v\nabla\cdot\nabla b)v-4h^4(\nabla\cdot v)^2\nabla b$$
$$-4h^4\nabla\cdot v\nabla(\nabla b\cdot v)+\nabla\cdot v\nabla h^4\times(v\times\nabla b)+\nabla h^4\times\big(v\times\nabla(\nabla b\cdot v)\big)\Big)\Big\}$$
$$+\frac{1}{48}\nabla b\times\big(\nabla h^4\times\nabla(\nabla\cdot v)^2\big)-\frac{1}{48}h^4\nabla\cdot\nabla b\nabla(\nabla\cdot v)^2-\frac{1}{24}\nabla\cdot\big(h^4\nabla(\nabla b\cdot v)\big)\nabla(\nabla\cdot v)$$
$$-\frac{1}{24}h^4\nabla^2(\nabla\cdot v)\nabla(\nabla b\cdot v)+\frac{1}{24}\nabla(\nabla\cdot v)\times\big(\nabla(\nabla b\cdot v)\times\nabla h^4\big)-\frac{1}{6}\nabla\big(h^4\nabla(\nabla\cdot v)\nabla(\nabla b\cdot v)\big)$$
$$+\frac{1}{24}\nabla\Big((\nabla h^4\cdot v)\big(\nabla b\nabla(\nabla\cdot v)\big)\Big)-\frac{1}{24}\nabla\cdot\Big\{h^4\big(\nabla^2(\nabla\cdot v)v-2\nabla(\nabla\cdot v)^2\big)+\nabla h^4\times\big(v\times\nabla(\nabla\cdot v)\big)\Big\}\nabla b,$$

with

$$B_4[U]v=\frac{1}{12}\nabla\Big\{h^3\nabla\cdot\Big(\nabla b\times\big(v\times\nabla(\nabla b\cdot v)\big)\Big)+h^3\nabla\cdot\big(\nabla\cdot v\nabla b\times(v\times\nabla b)\big)+h^3(\nabla b\cdot v)\nabla^2(\nabla b\cdot v)$$
$$+h^3(\nabla b\cdot v)\nabla\cdot(\nabla\cdot v\nabla b)+2h^3\big(\nabla(\nabla b\cdot v)\big)^2-2h^3(\nabla\cdot v)^2\nabla b\nabla b\Big\}$$
$$+\frac{1}{12}\nabla\cdot\big(h^3\nabla\cdot v\nabla b\big)\nabla(\nabla b\cdot v)+\frac{1}{12}\nabla\cdot\big(h^3\nabla(\nabla b\cdot v)\big)\nabla(\nabla b\cdot v)+\frac{1}{12}h^3\nabla\cdot v\nabla\big(\nabla b\nabla(\nabla b\cdot v)\big)$$
$$+\frac{1}{24}h^3(\nabla\cdot v)^2\nabla(\nabla b\nabla b)+\frac{1}{24}h^3\nabla\Big\{\big(\nabla(\nabla b\cdot v)\big)^2\Big\}$$
$$+\frac{1}{12}\nabla\cdot\Big\{-3h^3(\nabla\cdot v)^2\nabla b+\nabla\cdot v\nabla h^3\times(v\times\nabla b)+h^3v\nabla\cdot(\nabla\cdot v\nabla b)\Big\}\nabla b$$
$$+\frac{1}{12}\nabla\cdot\Big\{h^3\nabla^2(\nabla b\cdot v)v-3h^3\nabla\cdot v\nabla(\nabla b\cdot v)+\nabla h^3\times\big(v\times\nabla(\nabla b\cdot v)\big)\Big\}\nabla b.$$

To our knowledge, the existence of terms of order μ^2 which makes the analysis more difficult, has not been yet derived or analyzed especially in the case when the bottom is not flat. The construction of solution ($1D$ case) with a standard Picard iterative scheme as in [17, 4, 5, 27] can not be achieved without considering the effect of a small surface tension that smooth the way in controlling the energy estimates. Our objective here is to demonstrate that it is additionally conceivable to utilize such an iterative scheme to study the well-posedness of the extended $1D$ Green-Naghdi equations with surface tension, $O(\mu^3)$, and $\beta\neq 0$.

1.1. **Organization of the paper.** The aim of this paper is to derive and study the extended approximation of the full water waves problem in the case of an uneven bottom topography with an error of order μ^3. First of all, in Section 2, we derive the extended $2D$ Green-Naghdi system for non-flat bottom. Then in Section 3.1 some preliminary results are given. The well-posedness of the modified system is stated in Section 3.2 then proved in Section 3.3 .

1.2. **Notation.** We denote by $C(\lambda_1, \lambda_2, ...)$ a constant depending on the parameters λ_1, λ_2, ... and *whose dependence on the λ_j is always assumed to be nondecreasing.*
The notation $a \lesssim b$ means that $a \leq Cb$, for some non-negative constant C whose exact expression is of no importance (*in particular, it is independent of the small parameters involved*). Also, the notation $a \vee b$ stands for the maximum between a and b.
Let p be any constant with $1 \leq p < \infty$ and denote $L^p = L^p(\mathbb{R}^d)$ the space of all Lebesgue-measurable functions f with the standard norm $|f|_{L^p} = \big(\int_{\mathbb{R}^d} |f(X)|^p dX \big)^{1/p} < \infty$.
When $p = 2$, we denote the norm $|\cdot|_{L^2}$ simply by $|\cdot|_2$. The inner product of any functions f_1 and f_2 in the Hilbert space $L^2(\mathbb{R}^d)$ is denoted by $(f_1, f_2) = \int_{\mathbb{R}^d} f_1(X) f_2(X) dX$.
The space $L^\infty = L^\infty(\mathbb{R}^d)$ consists of all essentially bounded, Lebesgue-measurable functions f with the norm $|f|_{L^\infty} = \operatorname{ess\,sup} |f(X)| < \infty$.
We denote by $W^{1,\infty} = W^{1,\infty}(\mathbb{R}^d) = \big\{ f \in L^\infty, \nabla f \in (L^\infty)^d \big\}$ endowed with its canonical norm.
For any real constant s, $H^s = H^s(\mathbb{R}^d)$ denotes the Sobolev space of all tempered distributions f with the norm $|f|_{H^s} = |\Lambda^s f|_2 < \infty$, where Λ is the pseudo-differential operator $\Lambda^s = (1 - \partial_x^2)^{s/2}$.
For any functions $u = u(X,t)$ and $v(X,t)$ defined on $\mathbb{R}^d \times [0,T)$ with $T > 0$, we denote the inner product, the L^p-norm and especially the L^2-norm, as well as the Sobolev norm, with respect to the spatial variable X, by $(u, v) = (u(\cdot,t), v(\cdot,t))$, $|u|_{L^p} = |u(\cdot,t)|_{L^p}$, $|u|_{L^2} = |u(\cdot,t)|_{L^2}$, and $|u|_{H^s} = |u(\cdot,t)|_{H^s}$, respectively.
Let $C^k(\mathbb{R}^d)$ denote the space of k-times continuously differentiable functions and $C_0^\infty(\mathbb{R}^d)$ denote the space of infinitely differentiable functions, with compact support in $\mathbb{R}^d$.
We also denote by $C_b^\infty(\mathbb{R}^d)$ the space of infinitely differentiable functions that are bounded together with all their derivatives.
Let $f(X,t)$ be a vector field defined on $\mathbb{R}^d \times [0,\infty)$ of the independent variable $X = (x_1, x_2, ..., x_d) \in \mathbb{R}^d$; its partial derivative with respect to x_k is denoted by $\partial_{x_k} f = f_{x_k}$ for $1 \leq k \leq d$. This allows us to define the *gradient* of f, and we denote it $\nabla f = (\partial_{x_1} f_1, \partial_{x_2} f_2, ..., \partial_{x_d} f_d) \in \mathbb{R}^d$. Also we call *divergence* of f the scalar denoted $\nabla \cdot f = \sum_{i=1}^d \partial_{x_i} f_i$ and when $d = 3$, we call the *curl* of f the vector denoted $\nabla \times f = (\partial_{x_2} f_3 - \partial_{x_3} f_2, \partial_{x_3} f_1 - \partial_{x_1} f_3, \partial_{x_1} f_2 - \partial_{x_2} f_1)^T$.
For any closed operator T defined on a Banach space Y of functions, the commutator $[T, f]$ is defined by $[T, f]g = T(fg) - fT(g)$ with f, g and fg belonging to the domain of T.

2. Derivation of the uneven Extended $2D$ Green-Naghdi system

To derive the Green-Naghdi equations ($2D$ case), we introduce the depth averaged horizontal velocity

$$v(t,X) = \frac{1}{h(t,X)} \int_{-1+\beta b(X)}^{\varepsilon\zeta(t,X)} \nabla\varphi(t,X,z)\, dz, \qquad \text{with} \qquad h(t,X) = 1 + \varepsilon\zeta(t,X) - \beta b(X)\,. \tag{15}$$

The first equation of the Green-Naghdi system (14) $\partial_t \zeta + \nabla \cdot (hv) = 0$ which exactly coincides with the first equation of (13) stems from a clear outcome of Green's identity or by a straightforward calculation and rearranging terms using (9). Now as in [22], in order to derive the evolution equation on v, the key point is to obtain an asymptotic expansion $\nabla\psi$ with respect to μ and in terms of v and ζ. Since $\mu \ll 1$, we look for an asymptotic expansion of φ under the form

$$\varphi_{app}(t,X,z) = \varphi_0 + \mu\varphi_1 + \mu^2\varphi_2 + ... + \mu^N\varphi_N = \sum_{j=0}^N \mu^j \varphi_j\,. \tag{16}$$

Plugging expression (16) into the boundary value problem (12) and after dropping all terms of order $O(\mu^{N+1})$ one gets

$$\forall\, j = 0, 1, ..., N \qquad \partial_z^2 \varphi_j = -\partial_x^2 \varphi_{j-1} - \partial_y^2 \varphi_{j-1}\,, \tag{17}$$

with the convention $\varphi_{-1} = 0$ by definition and the boundary condition

$$\forall\, j = 0, 1, ..., N \qquad \begin{cases} -\beta\nabla b \nabla\varphi_{j-1} + \partial_z\varphi_j = 0 \quad at \quad z = -1 + \beta b, \\ (\varphi_j)_{|_{z=\varepsilon\zeta}} = \delta_{0,j}\psi, \end{cases} \tag{18}$$

where $\delta_{0,j}=1$ if $j=0$ and zero otherwise. Solving the ODE (17) with (18) yields to the following possible choices

$$\varphi_0(t,X,z)=\psi(t,X), \tag{19}$$

$$\varphi_1(t,X,z)=(z-\varepsilon\zeta)\big(-\frac{1}{2}(z+\varepsilon\zeta)-1+\beta b\big)\nabla\cdot(\nabla\psi)+\beta(z-\varepsilon\zeta)\nabla b\cdot\nabla\psi, \tag{20}$$

$$\begin{aligned}\varphi_2(t,X,z)&=(z-\varepsilon\zeta)\beta\nabla b\cdot(\nabla\varphi_1)_{|_{z=-1+\beta b}}+\frac{1}{2}\big((z+1-\beta b)^2-h^2\big)(\varepsilon\nabla\zeta)\big(-\varepsilon\nabla\zeta+2(\beta\nabla b)\big)\Delta\psi\\
&-2\Big[\frac{1}{2}\big((z+1-\beta b)^2-h^2\big)h(\varepsilon\nabla\zeta)+\frac{1}{2}\big(\frac{1}{3}(z-\varepsilon\zeta)^3-(z-\varepsilon\zeta)h^2\big)(\beta\nabla b)\Big]\nabla(\Delta\psi)\\
&-\Big[\frac{1}{2}\big((z+1-\beta b)^2-h^2\big)h\nabla\cdot(\varepsilon\nabla\zeta)+\frac{1}{2}\big(\frac{1}{3}(z-\varepsilon\zeta)^3-(z-\varepsilon\zeta)h^2\big)\nabla\cdot(\beta\nabla b)\Big]\Delta\psi\\
&+\Big[\frac{1}{24}\big(z^4-(\varepsilon\zeta)^4\big)-\frac{1}{6}(-1+\beta b)^3(z-\varepsilon\zeta)-\frac{(\varepsilon\zeta)^2}{4}\big((z+1-\beta b)^2-h^2\big)\\
&\qquad\qquad-\frac{1}{2}\big(\frac{1}{3}(z-\varepsilon\zeta)^3-h^2(z-\varepsilon\zeta)\big)(-1+\beta b)\Big]\nabla\cdot\Big(\nabla\big(\Delta\psi\big)\Big)\\
&+\big((z+1-\beta b)^2-h^2\big)(\varepsilon\nabla\zeta)\nabla\big(\beta\nabla b\cdot\nabla\psi\big)+\frac{1}{2}\big((z+1-\beta b)^2-h^2\big)\nabla\cdot(\varepsilon\nabla\zeta)\beta\nabla b\cdot\nabla\psi\\
&-\frac{1}{2}\big(\frac{1}{3}(z-\varepsilon\zeta)^3-(z-\varepsilon\zeta)h^2\big)\nabla\cdot\big(\nabla(\beta\nabla b\cdot\nabla\psi)\big).\end{aligned} \tag{21}$$

So the horizontal component of the velocity in the fluid domain is given by

$$V(t,X,z)=\nabla\varphi_{app}=\nabla\varphi_0(t,X,z)+\mu\nabla\varphi_1(t,X,z)+\mu^2\nabla\varphi_2(t,X,z)+O(\mu^3). \tag{22}$$

The averaged velocity is thus given by

$$v(t,X)=\nabla\psi+\frac{\mu}{h}\int_{-1+\beta b(X)}^{\varepsilon\zeta(t,X)}\nabla\varphi_1 dz+\frac{\mu^2}{h}\int_{-1+\beta b(X)}^{\varepsilon\zeta(t,X)}\nabla\varphi_2 dz+O(\mu^3). \tag{23}$$

As in [22], we have

$$\int_{-1+\beta b(X)}^{\varepsilon\zeta(t,X)}\nabla\varphi_1\,dz=\mathcal{T}[h,\beta b]\nabla\psi=-\frac{1}{3}\nabla(h^3\Delta\psi)+\frac{\beta}{2}[\nabla(h^2\nabla b\cdot\nabla\psi)-h^2\nabla b\Delta\psi]+\beta^2h\nabla b\nabla b\cdot\nabla\psi. \tag{24}$$

In order to compute $J_2[h,\beta b]w=\displaystyle\int_{-1+\beta b(X)}^{\varepsilon\zeta(t,X)}\nabla\varphi_2\,dz$, denote by $w=\nabla\psi$ (noting that w is independent of z). We first commute $\nabla\varphi_2$, then using the expression of $\nabla\varphi_1$ we may write

$$\begin{aligned}&\nabla\big[(z-\varepsilon\zeta)\beta\nabla b\cdot(\nabla\varphi_1)_{|_{z=1-\beta b}}\big]=-\beta h\nabla\cdot w(\nabla b\cdot\nabla h)\nabla h-\beta^2h\nabla\cdot w(\nabla h\nabla b)\nabla b+\beta(z-\varepsilon\zeta)\nabla\big(h\nabla\cdot w(\nabla h\nabla b)\big)\\
&-\frac{1}{2}\beta h^2\nabla h\big(\nabla b\nabla(\nabla\cdot w)\big)-\frac{1}{2}\beta^2h^2\big(\nabla b\nabla(\nabla\cdot w)\big)\nabla b+\frac{1}{2}(z-\varepsilon\zeta)\beta\nabla\Big(h^2\big(\nabla b\nabla(\nabla\cdot w)\big)\Big)+\beta^2\nabla h\big((\nabla h\nabla b)(\nabla b\cdot w)\big)\\
&+\beta^3\big((\nabla h\nabla b)(\nabla\cdot w)\big)\nabla b-(z-\varepsilon\zeta)\beta^2\nabla\big((\nabla h\nabla b)(\nabla b\cdot w)\big)+\beta^3\nabla h\big((\nabla b\nabla b)(\nabla b\cdot w)\big)+\beta^4\big((\nabla b\nabla b)(\nabla b\cdot w)\big)\nabla b\\
&-(z-\varepsilon\zeta)\beta^3\nabla\big((\nabla b\nabla b)(\nabla b\cdot w)\big)+\beta^2h\nabla h\big(\nabla b\nabla(\nabla\cdot w)\big)+\beta^3h\big(\nabla b\nabla(\nabla b\cdot w)\big)\nabla b-(z-\varepsilon\zeta)\beta^2\nabla\Big(h\nabla\big(\nabla b\nabla(\nabla b\cdot w)\big)\Big).\end{aligned}$$

Now, we evaluate the following integrals needed for the rest of this section

$$\int_{-1+\beta b(X)}^{\varepsilon\zeta(t,X)}(z-\varepsilon\zeta)\,dz=-\frac{1}{2}h^2,\qquad\int_{-1+\beta b(X)}^{\varepsilon\zeta(t,X)}\big[(z+1-\beta b)\beta\nabla b+h\nabla h\big]\,dz=h^2\nabla h+\frac{1}{2}h^2\nabla b,$$

$$\int_{-1+\beta b(X)}^{\varepsilon\zeta(t,X)}\big[\frac{1}{3}(z-\varepsilon\zeta)^3-(z-\varepsilon\zeta)h^2\big]\,dz=\frac{5}{12}h^4,\qquad\int_{-1+\beta b(X)}^{\varepsilon\zeta(t,X)}\big[(z+1-\beta b)^2-h^2\big]\,dz=-\frac{2}{3}h^2\nabla h,$$

$$\int_{-1+\beta b(X)}^{\varepsilon\zeta(t,X)}\big[(z-\varepsilon\zeta)^2\varepsilon\nabla\zeta-\varepsilon h^2\nabla\zeta+2(z-\varepsilon\zeta)h\nabla h\big]\,dz=-\frac{5}{3}h^2\nabla h-\frac{2}{3}h^2\nabla b,$$

$$\int_{-1+\beta b(X)}^{\varepsilon\zeta(t,X)}f_1(z)\,dz=\frac{2}{15}h^5,\qquad\int_{-1+\beta b(X)}^{\varepsilon\zeta(t,X)}f_2(z)\,dz=\frac{2}{15}\nabla h^5+\frac{5}{24}h^4\beta\nabla b,$$

where

$$f_1(z)=\frac{1}{24}\big(z^4-(\varepsilon\zeta)^4\big)-\frac{1}{6}(-1+\beta b)^3(z-\varepsilon\zeta)-\frac{(\varepsilon\zeta)^2}{4}\big((z+1-\beta b)^2-h^2\big)-\frac{1}{2}\big(\frac{1}{3}(z-\varepsilon\zeta)^3-h^2(z-\varepsilon\zeta)\big)(-1+\beta b),$$

$$\begin{aligned}f_2(z)=&-\frac{1}{6}(\varepsilon\zeta)^3\varepsilon\nabla\zeta-\frac{1}{2}(-1+\beta b)^2(z-\varepsilon\zeta)\beta\nabla b+\frac{1}{6}(-1+\beta b)^3\varepsilon\zeta-\frac{(\varepsilon\zeta)}{2}\varepsilon\nabla\zeta\big((z+1-\beta b)^2-h^2\big)\\&+\frac{(\varepsilon\zeta)^2}{2}\big((z+1-\beta b)\beta\nabla b+h\nabla h\big)+\frac{1}{2}\big((z-\varepsilon\zeta)^2-h^2\big)(-1+\beta b)\varepsilon\nabla\zeta+(z-\varepsilon\zeta)h\nabla h(-1+\beta b)\\&-\frac{1}{2}\big(\frac{1}{3}(z-\varepsilon\zeta)^3-(z-\varepsilon\zeta)h^2\big)\beta\nabla b.\end{aligned}$$

The non-topographical expressions (*i.e.* setting $\beta=0$) of $J_2[h,\beta b]w=\displaystyle\int_{-1+\beta b(X)}^{\varepsilon\zeta(t,X)}\nabla\varphi_2\,dz$ are equal to the following two factorized terms

$$\frac{2}{15}\nabla\Big(\nabla\cdot\big(h^5\nabla(\nabla\cdot w)\big)\Big)+\frac{1}{3}\nabla\big(h^3\nabla\cdot(h\nabla h)\nabla\cdot w\big).$$

The purely-topographical expression (*i.e.* setting $\beta\neq 0$) of $J_2[h,\beta b]w$ are separated into four categories where each one of them is multiplied by β, β^2, β^3, β^4 respectively.
The β-contributions are

$$\begin{aligned}&-\frac{1}{2}\nabla(h^3\nabla\cdot w(\nabla h\nabla b)\big)+\frac{1}{8}\nabla\big(h^4\nabla\cdot w\nabla\cdot(\nabla b)\big)-\frac{2}{3}\nabla\big(h^3\nabla(\nabla b\cdot w)\nabla h\big)-\frac{1}{3}\nabla\big(h^3\nabla\cdot(\nabla h)(\nabla b\cdot w)\big)\\&-\frac{5}{24}\nabla\Big(h^4\nabla\cdot\big(\nabla(\nabla b\cdot w)\big)\Big)+\frac{5}{24}h^4\nabla\cdot\big(\nabla(\nabla\cdot w)\big)\nabla b+\frac{1}{2}\big(\nabla\cdot w\nabla\cdot(\nabla h)\big)\nabla b+\frac{1}{2}h^2\nabla\cdot w(\nabla h\nabla h)\nabla b\\&+h^3\big(\nabla h\nabla(\nabla\cdot w)\big)\nabla b=T_1+T_2+...+T_9.\end{aligned}$$

The β^2-contributions are

$$\begin{aligned}&\frac{1}{2}\nabla\big(h^2(\nabla h\nabla b)(\nabla b\cdot w)\big)-\frac{1}{6}\nabla\big(h^3\nabla b\nabla(\nabla b\cdot w)\big)-\frac{1}{3}\nabla\big(h^3\nabla\cdot w(\nabla b\nabla b)\big)-\frac{1}{3}\nabla\big(h^3(\nabla b\cdot w)\nabla\cdot(\nabla b)\big)\\&-h^2\nabla\cdot w(\nabla h\nabla b)\nabla b-\frac{1}{6}h^3\big(\nabla b\nabla(\nabla\cdot w)\big)\nabla b+\frac{1}{6}h^3\big(\nabla\cdot w\nabla\cdot(\nabla b)\big)\nabla b-h^2\big(\nabla h\nabla(\nabla b\cdot w)\big)\nabla b\\&-\frac{1}{2}h^2\big((\nabla b\cdot w)\nabla\cdot(\nabla h)\big)\nabla b-\frac{1}{3}\nabla\cdot\big(\nabla(\nabla b\cdot w)\big)\nabla b=P_1+P_2+...+P_{10}.\end{aligned}$$

The β^3-contributions are

$$\begin{aligned}&h\big((\nabla h\nabla b)(\nabla b\cdot w)\big)\nabla b+h\big((\nabla b\nabla b)(\nabla b\cdot w)\big)\nabla h-\frac{1}{2}h^2\nabla\cdot w(\nabla b\nabla b)\nabla b+\frac{1}{2}h^2\nabla\big((\nabla b\nabla b)(\nabla b\cdot w)\big)\\&-\frac{1}{2}h^2\big((\nabla b\cdot w)\nabla\cdot(\nabla b)\big)\nabla b.\end{aligned}$$

Lastly, the only term of order β^4 is $h(\nabla b\cdot w)(\nabla b\nabla b)\nabla b$.
In the same sense, the expressions of $\mathcal{T}[h,\beta b]\big(\frac{1}{h}\mathcal{T}[h,\beta b]w\big)$ will be divided into topographical (*i.e* set $\beta=0$) and non-topographical terms .
The only non-topographical terms are the following

$$\frac{1}{9}\nabla\Big(h^3\nabla\cdot\big(\frac{1}{h}\nabla(h^3\nabla\cdot w)\big)\Big)=\frac{1}{9}\nabla\Big(\nabla\cdot\big(h^5\nabla(\nabla\cdot w)\big)\Big)+\frac{1}{3}\nabla\big(h^3\nabla\cdot(h\nabla h)\nabla\cdot w\big).$$

The purely-topographical expression of $\mathcal{T}[h,\beta b]\big(\frac{1}{h}\mathcal{T}[h,\beta b]w\big)$ will be separated into four categories where each one of them is multiplied by β, β^2, β^3, β^4 respectively.
The β-contributions are

$$\begin{aligned}&-\frac{1}{6}\nabla\big(h\nabla b\nabla(h^3\nabla\cdot w)\big)-\frac{1}{3}\nabla\big(h^3(\nabla b\cdot w)\nabla\cdot(\nabla h)\big)-\frac{1}{2}\nabla\big(h^3\nabla h\nabla(\nabla b\cdot w)\big)-\frac{1}{6}\nabla\Big(h^4\nabla\cdot\big(\nabla(\nabla b\cdot w)\big)\Big)\\&+\frac{1}{6}\nabla\big(h^3\nabla\cdot(h\nabla\cdot w\nabla b)\big)-\frac{1}{6}\big(\nabla h\nabla(h^3\nabla\cdot w)\big)\nabla b+\frac{1}{6}h\nabla\cdot\big(\nabla(h^3\nabla\cdot w)\big)\nabla b=T_1'+T_2'+...+T_7'.\end{aligned}$$

The β^2-contributions are

$$+\frac{1}{4}\nabla\Big(h\nabla b\nabla\big(h^2(\nabla b\cdot w)\big)\Big)-\frac{1}{4}\nabla\big(h^3\nabla\cdot w(\nabla b\nabla b)\big)-\frac{1}{3}\nabla\Big(h^3\nabla\cdot\big((\nabla b\cdot w)\nabla b\big)\Big)$$
$$-\frac{1}{3}\big(\nabla b\nabla(h^3\nabla\cdot w)\big)\nabla b-\frac{1}{2}h^2\nabla\cdot\big((\nabla b\cdot w)\nabla h\big)\nabla b-\frac{1}{4}h^2\nabla\cdot\big(h\nabla(\nabla b\cdot w)\big)\nabla b+\frac{1}{4}h^2\nabla\cdot(h\nabla\cdot w\nabla b)\nabla b$$
$$=P_1'+P_2'+...+P_7'.$$

The β^3-contributions are

$$\frac{1}{2}\Big(\nabla b\nabla\big(h^2(\nabla b\cdot w)\big)\Big)\nabla b-\frac{1}{2}h^2\nabla\cdot w(\nabla b\nabla b)\nabla b+\frac{1}{2}\nabla\big(h^2(\nabla b\nabla b)(\nabla b\cdot w)\big)-\frac{1}{2}h^2\nabla\cdot\big((\nabla b\cdot w)\nabla b\big)\nabla b.$$

Lastly, the only term of order β^4 is $h(\nabla b\cdot w)(\nabla b\nabla b)\nabla b$.

Now, we will find the non-topographical and the topographical expressions of the term $\mathcal{T}[h,\beta b]\big(\frac{1}{h}\mathcal{T}[h,\beta b]w\big)-J_2[h,\beta b]w$ using the previous results .
The non-topographical expression can be factorized in the following term

$$-\frac{1}{45}\nabla\Big(\nabla\cdot\big(h^5\nabla(\nabla\cdot v)\big)\Big).$$

The purely-topographical expression of $\mathcal{T}[h,\beta b]\big(\frac{1}{h}\mathcal{T}[h,\beta b]w\big)-J_2[h,\beta b]w$ will be separated into four categories where each one of them is multiplied by β, β^2, β^3, β^4 respectively.
The β-contributions are

$$T_1'+...+T_5'-T_1-...-T_5=\frac{1}{24}\nabla\Big(\nabla\cdot\big(h^4\nabla(\nabla b\cdot w)\big)\Big)+\frac{1}{24}\nabla\big(\nabla\cdot w\nabla\cdot(h^4\nabla b)\big),$$
$$T_6'+T_7'-T_6-...-T_9=-\frac{1}{24}\nabla\cdot\big(h^4\nabla(\nabla\cdot w)\big)\nabla b.$$

The β^2-contributions are

$$P_1'+P_2'+P_3'-P_1-...-P_4=\frac{1}{12}\nabla\big(h^3\nabla\cdot w(\nabla b\nabla b)\big)\big)+\frac{1}{12}\nabla\big(h^3\nabla b\nabla(\nabla b\cdot w)\big),$$
$$P_4'+...+P_7'-P_5-...-P_{10}=\frac{1}{12}\nabla\cdot\big(h^3\nabla\cdot w\nabla b\big)\nabla b+\frac{1}{12}\nabla\cdot\big(h^3\nabla(\nabla b\cdot w)\big)\nabla b.$$

The β^3 and β^4 terms will eliminate each other. Thus, we have

$$v=\nabla\psi-\frac{\mu}{h}\mathcal{T}[h,\beta b]\nabla\psi+\frac{\mu^2}{h}J_2[h,\beta b]\nabla\psi+O(\mu^3),\tag{25}$$

but

$$\nabla\psi=v+\frac{\mu}{h}\mathcal{T}[h,\beta b]v+\frac{\mu^2}{h}\Big[\mathcal{T}[h,\beta b]\big(\frac{1}{h}\mathcal{T}[h,\beta b]v\big)-J_2[h,\beta b]v\Big].\tag{26}$$

Therefore, we obtain

$$\nabla\psi=v+\frac{\mu}{h}\mathcal{T}[h,\beta b]v+\frac{\mu^2}{h}\mathfrak{T}[h,\beta b]v+O(\mu^3),\tag{27}$$

where

$$\mathcal{T}[h,\beta b]w=-\frac{1}{3}\nabla(h^3\nabla\cdot w)+\frac{\beta}{2}\big[\nabla(h^2\nabla b\cdot w)-h^2\nabla\cdot w\nabla b\big]+\beta^2h(\nabla b\cdot w)\nabla b,\tag{28}$$

and

$$\mathfrak{T}[h,\beta b]w=-\frac{1}{45}\nabla\Big(\nabla\cdot\big(h^5\nabla(\nabla\cdot w)\big)\Big)+\frac{1}{24}\beta\nabla\Big(\nabla\cdot\big(h^4\nabla(\nabla b\cdot w)\big)\Big)+\frac{1}{24}\beta\nabla\big(\nabla\cdot w\nabla\cdot(h^4\nabla b)\big)\tag{29}$$
$$-\frac{1}{24}\beta\nabla\cdot\big(h^4\nabla(\nabla\cdot w)\big)\nabla b+\frac{1}{12}\beta^2\nabla\big(h^3\nabla\cdot w(\nabla b\nabla b)\big)\big)+\frac{1}{12}\beta^2\nabla\big(h^3\nabla b\nabla(\nabla b\cdot w)\big)+\frac{1}{12}\beta^2\nabla\cdot\big(h^3\nabla\cdot w\nabla b\big)\nabla b$$
$$+\frac{1}{12}\beta^2\nabla\cdot\big(h^3\nabla(\nabla b\cdot w)\big)\nabla b.$$

Now, in order to derive the extended Green-Naghdi equations for non-flat bottom without surface tension (i.e. $\sigma = 0$), we will take the gradient of the second equation of (13) then multiply it by h, and replace $\nabla\psi$ by its expression (27) and $\frac{1}{\mu}\mathcal{G}[\varepsilon\zeta, \beta b]\psi$ by $\nabla\cdot(hv) = \nabla h\cdot v + h\nabla\cdot v$ in the resulting equations. Moreover, we drop the $O(\mu^3)$ terms and we use the following vector triple products and the vector identities

$$u\times(\nu\times\omega) = (u\cdot\omega)\nu - (u\cdot\nu)\omega\ , \tag{30}$$

$$\nabla\times(\nabla G) = 0 \qquad and \qquad \nabla\times(GF) = G\nabla\times F + \nabla G\times F, \tag{31}$$

where G is a differentiable scalar function and u,ν,ω,F are differentiable vector fields. Finally, after capturing the information above we obtain the extended $2D$ Green-Naghdi system for an uneven bottom topography ($\beta \neq 0$) with an error of order μ^3 represented by

$$\begin{cases} \partial_t\zeta + \nabla\cdot(hv) = 0, \\ \big(h + \mu\mathcal{T}[h] + \mu^2\mathfrak{T}[h]\big)\partial_t v + h\nabla\zeta + \varepsilon h(v\cdot\nabla)v + \varepsilon\mu Q_1[U]v \\ \qquad\qquad +\varepsilon\mu\beta B_1[U]v + \varepsilon\mu\beta^2 B_2[U]v + \varepsilon\mu^2 Q_2[U]v + \varepsilon\mu^2\beta B_3[U]v + \varepsilon\mu^2\beta^2 B_4[U]v = O(\mu^3), \end{cases} \tag{32}$$

where $v = (v_1, v_2)^T$, $U = (\zeta, v)^T$ and $h(t, X) = 1 + \varepsilon\zeta(t, X) - \beta b(X)$, denoting by

$$\mathcal{T}[h, \beta b]w = -\frac{1}{3}\nabla(h^3\nabla\cdot w) + \frac{\beta}{2}\big[\nabla(h^2\nabla b\cdot w) - h^2\nabla\cdot w\nabla b\big] + \beta^2 h(\nabla b\cdot w)\nabla b,$$

$$\begin{aligned}\mathfrak{T}[h, \beta b]w = &-\frac{1}{45}\nabla\Big(\nabla\cdot\big(h^5\nabla(\nabla\cdot w)\big)\Big) + \frac{1}{24}\beta\nabla\Big(\nabla\cdot\big(h^4\nabla(\nabla b\cdot w)\big)\Big) + \frac{1}{24}\beta\nabla\big(\nabla\cdot w\nabla\cdot(h^4\nabla b)\big) \\ &- \frac{1}{24}\beta\nabla\cdot\big(h^4\nabla(\nabla\cdot w)\big)\nabla b + \frac{1}{12}\beta^2\nabla\big(h^3\nabla\cdot w(\nabla b\nabla b)\big)\big) + \frac{1}{12}\beta^2\nabla\big(h^3\nabla b\nabla(\nabla b\cdot w)\big) \\ &+ \frac{1}{12}\beta^2\nabla\cdot\big(h^3\nabla\cdot w\nabla b\big)\nabla b + \frac{1}{12}\beta^2\nabla\cdot\big(h^3\nabla(\nabla b\cdot w)\big)\nabla b,\end{aligned}$$

where the non-topographical terms represented by $Q_1[U]$, $Q_2[U]$ as follows

$$Q_1[U]v = -\frac{1}{3}\nabla\Big(h^3\big((v\cdot\nabla)(\nabla\cdot v) - (\nabla\cdot v)^2\big)\Big),$$

$$\begin{aligned}Q_2[U]v = &-\frac{1}{45}\nabla\Big[\nabla\cdot\Big\{h^5\big(\nabla^2(\nabla\cdot v)\big)v - 5h^5(\nabla\cdot v)\nabla(\nabla\cdot v) + \nabla h^5\times\big(v\times\nabla(\nabla\cdot v)\big)\Big\}\Big] \\ &+\frac{2}{45}\nabla\Big(h^5\big(\nabla(\nabla\cdot v)\big)^2\Big) + \frac{1}{45}\nabla\cdot\Big(h^5\nabla(\nabla\cdot v)\Big)\nabla(\nabla\cdot v) + \frac{1}{90}h^5\nabla\big\{\big(\nabla(\nabla\cdot v)\big)^2\big\},\end{aligned}$$

while the purely-topographical terms are introduced by $B_1[U]$, $B_2[U]$, $B_3[U]$, $B_4[U]$ as follows

$$B_1[U]v = \frac{1}{2}\nabla\big(h^2(v\cdot\nabla)^2 b\big) - \frac{1}{2}h^2\big((v\cdot\nabla)(\nabla\cdot v) - (\nabla\cdot v)^2\big)\nabla b, \qquad B_2[U]v = h\big((v\cdot\nabla)^2 b\big)\nabla b,$$

and

$$\begin{aligned}B_3[U]v = &+\frac{1}{24}\nabla\Big\{\nabla\cdot\Big(h^4\nabla^2(\nabla b\cdot v)v + h^4(\nabla\cdot v\nabla\cdot\nabla b)v - 4h^4(\nabla\cdot v)^2\nabla b \\ &\qquad - 4h^4\nabla\cdot v\nabla(\nabla b\cdot v) + \nabla\cdot v\nabla h^4\times(v\times\nabla b) + \nabla h^4\times\big(v\times\nabla(\nabla b\cdot v)\big)\Big)\Big\} \\ &+\frac{1}{48}\nabla b\times\big(\nabla h^4\times\nabla(\nabla\cdot v)^2\big) - \frac{1}{48}h^4\nabla\cdot\nabla b\nabla(\nabla\cdot v)^2 - \frac{1}{24}\nabla\cdot\big(h^4\nabla(\nabla b\cdot v)\big)\nabla(\nabla\cdot v) \\ &-\frac{1}{24}h^4\nabla^2(\nabla\cdot v)\nabla(\nabla b\cdot v) + \frac{1}{24}\nabla(\nabla\cdot v)\times\big(\nabla(\nabla b\cdot v)\times\nabla h^4\big) - \frac{1}{6}\nabla\big(h^4\nabla(\nabla\cdot v)\nabla(\nabla b\cdot v)\big) \\ &+\frac{1}{24}\nabla\Big((\nabla h^4\cdot v)\big(\nabla b\nabla(\nabla\cdot v)\big)\Big) - \frac{1}{24}\nabla\cdot\Big\{h^4\big(\nabla^2(\nabla\cdot v)v - 2\nabla(\nabla\cdot v)^2\big) + \nabla h^4\times\big(v\times\nabla(\nabla\cdot v)\big)\Big\}\nabla b,\end{aligned}$$

with

$$B_4[U]v = \frac{1}{12}\nabla\Big\{h^3\nabla\cdot\Big(\nabla b\times\big(v\times\nabla(\nabla b\cdot v)\big)\Big) + h^3\nabla\cdot\big(\nabla\cdot v\nabla b\times(v\times\nabla b)\big) + h^3(\nabla b\cdot v)\nabla^2(\nabla b\cdot v) \\ + h^3(\nabla b\cdot v)\nabla\cdot(\nabla\cdot v\nabla b) + 2h^3\big(\nabla(\nabla b\cdot v)\big)^2 - 2h^3(\nabla\cdot v)^2\nabla b\nabla b\Big\} \\ + \frac{1}{12}\nabla\cdot\big(h^3\nabla\cdot v\nabla b\big)\nabla(\nabla b\cdot v) + \frac{1}{12}\nabla\cdot\big(h^3\nabla(\nabla b\cdot v)\big)\nabla(\nabla b\cdot v) + \frac{1}{12}h^3\nabla\cdot v\nabla\big(\nabla b\nabla(\nabla b\cdot v)\big) \\ + \frac{1}{24}h^3(\nabla\cdot v)^2\nabla(\nabla b\nabla b) + \frac{1}{24}h^3\nabla\Big\{\big(\nabla(\nabla b\cdot v)\big)^2\Big\} \\ + \frac{1}{12}\nabla\cdot\Big\{-3h^3(\nabla\cdot v)^2\nabla b + \nabla\cdot v\nabla h^3\times(v\times\nabla b) + h^3v\nabla\cdot(\nabla\cdot v\nabla b)\Big\}\nabla b \\ + \frac{1}{12}\nabla\cdot\Big\{h^3\nabla^2(\nabla b\cdot v)v - 3h^3\nabla\cdot v\nabla(\nabla b\cdot v) + \nabla h^3\times\big(v\times\nabla(\nabla b\cdot v)\big)\Big\}\nabla b,$$

where the expression of Q_2 introduces the Laplacian operator $\nabla^2 = \nabla\cdot\nabla = \Delta$.
In the presence of surface tension ($\sigma \neq 0$), different strategies exist to deal with it in the water-wave problem such as [14, 25, 23, 24], the main contrast in our work is that the gradient of the capillary term $-\frac{1}{\mathrm{Bo}}\frac{\kappa(\varepsilon\sqrt{\mu}\zeta)}{\varepsilon\sqrt{\mu}}$ multiplied by h must be added to the right-hand side of the second equation in (32) .
Let us define the *rescaled Bond number* bo instead of the *classical Bond number* Bo, as follows

$$\mathrm{bo} = \mu\mathrm{Bo} = \frac{\rho g h_0^2}{\sigma} > 0,$$

where h_0 denotes the reference depth, ρ denotes the positive constant density of the fluid, g denotes the acceleration of gravity, and $\sigma > 0$ denotes the surface tension coefficient, so that $\mathrm{Bo}^{-1} = \mu\mathrm{bo}^{-1} = \mathrm{O}(\mu)$ and the capillary term that should be added becomes

$$-\frac{1}{\mathrm{Bo}}h\nabla\Big\{\frac{\kappa(\varepsilon\sqrt{\mu}\zeta)}{\varepsilon\sqrt{\mu}}\Big\} = \frac{1}{\mathrm{bo}}\mu h\nabla\Big(\nabla\cdot\big(\nabla\zeta\big)\Big) - \frac{1}{2\mathrm{bo}}\varepsilon^2\mu^2 h\nabla\Big(\nabla\cdot\big(|\nabla\zeta|^2\nabla\zeta\big)\Big) + O(\varepsilon^4\mu^3). \tag{33}$$

3. Well-Posedness of the Extended $1D$ Green-Naghdi system for an uneven bottom with surface tension

For one dimensional ($d = 1$) surfaces, the Green-Naghdi system (32) with surface tension can be rearranged after a few calculations and considering (33), one may write

$$\begin{cases} \partial_t\zeta + \partial_x(hv) = 0, \\ \big(h + \mu\mathcal{T}[h,\beta b] + \mu^2\mathfrak{T}[h,\beta b]\big)\partial_t v + h\partial_x\zeta + \varepsilon hvv_x + \varepsilon\mu Q_1[U]v + \varepsilon\mu\beta B_1[U]v \\ +\varepsilon\mu\beta^2 B_2[U]v + \varepsilon\mu^2 Q_2[U]v + \varepsilon\mu^2\beta B_3[U]v + \varepsilon\mu^2\beta^2 B_4[U]v = \dfrac{1}{\mathrm{bo}}\mu h\zeta_{xxx} + \varepsilon^2\mu^2\dfrac{1}{\mathrm{bo}}T[U]\zeta_x + O(\mu^3), \end{cases} \tag{34}$$

where $U = (\zeta, v)^T$ and denoting by $h = h(t,x) = 1 + \varepsilon\zeta(t,x) - \beta b(x)$ the total non-dimensional height of the liquid, with

$$\mathcal{T}[h,\beta b]v = -\frac{1}{3}\partial_x(h^3\partial_x v) + \frac{\beta}{2}\big[\partial_x(h^2 b_x v) - h^2 b_x v_x\big] + \beta^2 h b_x^2 v, \qquad T[U]\zeta_x = -\frac{1}{2}h\partial_x^2\big(\zeta_x^2\zeta_x\big),$$

$$\mathfrak{T}[h,\beta b]v = -\frac{1}{45}\partial_x^2\big(h^5\partial_x^2 v\big) + \frac{\beta}{24}\big[\partial_x\big(\partial_x(h^4 b_x)\partial_x v\big) + \partial_x^2\big(h^4\partial_x(b_x v)\big) - b_x\partial_x(h^4\partial_x^2 v)\big] \\ + \frac{\beta^2}{12}\big[2\partial_x(h^3 b_x^2\partial_x v) + \partial_x(h^3 b_x b_{xx} v) + 2b_x\partial_x(h^3 b_x\partial_x v) + b_x\partial_x(h^3 b_{xx} v)\big],$$

where the non-topographical terms are represented by $Q_1[U]$, $Q_2[U]$ as follows

$$Q_1[U]v = -\frac{1}{3}\partial_x\Big(h^3\big(vv_{xx} - v_x^2\big)\Big), \qquad Q_2[U]v = -\frac{1}{45}\partial_x\Big\{\partial_x\big(h^5(vv_{xxx} - 5v_x v_{xx})\big) - 3h^5(v_{xx})^2\Big\},$$

while the purely-topographical terms are introduced by $B_1[U]$, $B_2[U]$, $B_3[U]$, $B_4[U]$ as follows

$$B_1[U]v = \frac{1}{2}\big[\partial_x(h^2 b_{xx}v^2) + \partial_x(h^2 b_x v v_x) - h^2(vv_{xx} - v_x^2)b_x\big], \qquad B_2[U]v = h\big\{b_{xx}v^2 + b_x v v_x\big\}b_x,$$

$$B_3[U]v=\frac{1}{24}\partial_x^2\Big\{h^4\big(b_{xxx}v^2-b_{xx}vv_x+b_xvv_{xx}-8b_xv_x^2\big)\Big\}-\frac{1}{4}h^4b_{xx}v_xv_{xx}+\frac{1}{24}\partial_x\big(h_x^4b_xvv_{xx}\big)$$
$$-\frac{5}{24}\partial_x\big(h^4b_{xx}vv_{xx}\big)-\frac{1}{24}b_x\partial_x\Big\{h^4\big(vv_{xxx}+v_xv_{xx}\big)\Big\},$$

$$B_4[U]v=\frac{1}{12}\partial_x\Big\{h^3\big(b_{xxx}b_xv^2+2b_x^2vv_{xx}+10b_xb_{xx}vv_x+2b_x^2v_x^2+3b_{xx}^2v^2\big)\Big\}+\frac{1}{12}h^3\big(b_{xx}v+2b_xv_x\big)b_{xx}v_x$$
$$+\frac{1}{12}b_x\partial_x\Big\{h^3\big(b_{xxx}v^2+2b_xvv_{xx}-6b_xv_x^2\big)\Big\}.$$

Now, setting $\pm\varepsilon\mu\mathcal{T}[h,\beta b](vv_x)$ and $\pm\varepsilon\mu^2\mathfrak{T}[h,\beta b](vv_x)$ in the second equation of (34), one gets a new formulation under the form

(35)
$$\begin{cases}\partial_t\zeta+\partial_x(hv)=0,\\ \big(h+\mu\mathcal{T}[h,\beta b]+\mu^2\mathfrak{T}[h,\beta b]\big)\big(\partial_tv+\varepsilon vv_x\big)+h\partial_x\zeta+\varepsilon\mu Q_1[U]v+\varepsilon\mu\beta B_1[U]v\\ +\varepsilon\mu\beta^2B_2[U]v+\varepsilon\mu^2Q_2[U]v+\varepsilon\mu^2\beta B_3[U]v+\varepsilon\mu^2\beta^2B_4[U]v=\dfrac{1}{\mathrm{bo}}\mu h\zeta_{xxx}+\varepsilon^2\mu^2\dfrac{1}{\mathrm{bo}}T[U]\zeta_x+O(\mu^3),\end{cases}$$

where $U=(\zeta,v)^T$ and $h(t,x)=1+\varepsilon\zeta(t,x)-\beta b(x)$, and one may write the above expressions as follows

$$\mathcal{T}[h,\beta b]v=-\frac{1}{3}\partial_x(h^3\partial_xv)+\frac{\beta}{2}\big[\partial_x(h^2b_xv)-h^2b_xv_x\big]+\beta^2hb_x^2v,\qquad T[U]\zeta_x=-\frac{1}{2}h\partial_x^2\big(\zeta_x^2\zeta_x\big),$$

$$\mathfrak{T}[h,\beta b]v=-\frac{1}{45}\partial_x^2\big(h^5\partial_x^2v\big)+\frac{\beta}{24}\big[\partial_x\big(\partial_x(h^4b_x)\partial_xv\big)+\partial_x^2\big(h^4\partial_x(b_xv)\big)-b_x\partial_x(h^4\partial_x^2v)\big]$$
$$+\frac{\beta^2}{12}\big[2\partial_x(h^3b_x^2\partial_xv)+\partial_x(h^3b_xb_{xx}v)+2b_x\partial_x(h^3b_x\partial_xv)+b_x\partial_x(h^3b_{xx}v)\big],$$

where the non-topographical terms are represented by $Q_1[U]$, $Q_2[U]$ as follows

$$Q_1[U]v=\frac{2}{3}\partial_x\big(h^3v_x^2\big),\qquad Q_2[U]v=\frac{1}{45}\partial_x\Big\{8\partial_x\big(h^5v_xv_{xx}\big)+3h^5v_{xx}^2\Big\},$$

while the purely-topographical terms are introduced by $B_1[U]v=\frac{1}{2}\partial_x(h^2b_{xx}v^2)+h^2b_xv_x^2$, and $B_2[U]v=hb_{xx}b_xv^2$ with

$$B_3[U]v=\frac{1}{24}\partial_x^2\Big\{h^4\big(b_{xxx}v^2-2b_{xx}vv_x-9b_xv_x^2\big)\Big\}-\frac{1}{4}h^4b_{xx}v_xv_{xx}-\frac{1}{24}\partial_x\big(h_x^4b_xv_x^2\big)$$
$$-\frac{1}{24}\partial_x\big(h^4b_{xx}v_x^2\big)-\frac{1}{4}\partial_x\big(h^4b_{xx}vv_{xx}\big)+\frac{1}{12}b_x\partial_x\big(h^4v_xv_{xx}\big),$$

$$B_4[U]v=\frac{1}{12}\partial_x\Big\{h^3\big(b_{xxx}b_xv^2+9b_xb_{xx}vv_x+3b_{xx}^2v^2\big)\Big\}+\frac{1}{12}h^3\big(b_{xx}v+2b_xv_x\big)b_{xx}v_x$$
$$+\frac{1}{12}b_x\partial_x\Big\{h^3\big(b_{xxx}v^2-b_{xx}vv_x-8b_xv_x^2\big)\Big\}.$$

Presently, denote by $\Im+2\mu^2\mathfrak{T}[h,\beta b]=h+\mu\mathcal{T}[h,\beta b]+\mu^2\mathfrak{T}[h,\beta b]$ $\big(i.e.$ set $\pm\mu^2\mathfrak{T}[h,\beta b](\partial_tv+\varepsilon vv_x)$ in $(35)_2\big)$ and note that from $(34)_2$ ($i.e.$ applying a trick type ***BBM trick*** Benjamin-Bona-Mahony [31]) we have the following approximated equation

$$\partial_tv+\varepsilon v\partial_xv=-\partial_x\zeta+O(\mu).\tag{36}$$

Subsequently, substitute (36) in the acquired new term $2\mu^2\mathfrak{T}[h,\beta b](\partial_tv+\varepsilon vv_x)$ to get $-2\mu^2\mathfrak{T}[h,\beta b]\zeta_x$, then after some computations $\big($needed especially for a suitable specification of a new operator $\mathfrak{J}_{\mathrm{bo}}$ introduced in $(52)\big)$ we obtain one-dimensional extended Green-Naghdi system with surface tension rewritten into

(37)
$$\begin{cases}\partial_t\zeta+\partial_x(hv)=0,\\ \Im(\partial_tv+\varepsilon vv_x)+h\partial_x\zeta-\dfrac{\mu}{\mathrm{bo}}h\zeta_{xxx}+\dfrac{2}{45}\mu^2h\partial_x^2(h^4\zeta_{xxx})+\mu^2\mathcal{I}[h,\beta b]\zeta_x-\varepsilon^2\mu^2\dfrac{1}{\mathrm{bo}}T[U]\zeta_x+\varepsilon\mu\mathcal{Q}_1[U]v_x\\ +\varepsilon\mu\beta\mathcal{B}_1[U]v_x+\varepsilon\mu^2\mathcal{Q}_2[U]v_x+\varepsilon\mu^2\mathcal{Q}_3[U]v_x+\varepsilon\mu^2\beta\mathcal{B}_2[U]v_x+\varepsilon\mu^2\beta^2\mathcal{B}_3[U]v_x+\mathcal{R}[\mu,\varepsilon h,\beta b](U)=O(\mu^3),\end{cases}$$

where $U=(\zeta,v)^T$ and $h(t,x)=1+\varepsilon\zeta(t,x)-\beta b(x)$. Denoting by

$$\mathcal{T}[h,\beta b]v=-\frac{1}{3}\partial_x(h^3\partial_x v)+\frac{\beta}{2}\big[\partial_x(h^2b_xv)-h^2b_xv_x\big]+\beta^2hb_x^2v,\qquad T[U]\zeta_x=-\frac{1}{2}h\partial_x^2\big(\zeta_x^2\zeta_x\big),\tag{38}$$

$$\begin{aligned}\mathfrak{T}[h,\beta b]v=&-\frac{1}{45}\partial_x^2\big(h^5\partial_x^2v\big)+\frac{\beta}{24}\big[\partial_x\big(\partial_x(h^4b_x)\partial_xv\big)+\partial_x^2\big(h^4\partial_x(b_xv)\big)-b_x\partial_x(h^4\partial_x^2v)\big]\\&+\frac{\beta^2}{12}\big[2\partial_x(h^3b_x^2\partial_xv)+\partial_x(h^3b_xb_{xx}v)+2b_x\partial_x(h^3b_x\partial_xv)+b_x\partial_x(h^3b_{xx}v)\big],\end{aligned}\tag{39}$$

where the non-topographical terms are represented by $\mathcal{Q}_1[U]$, $\mathcal{Q}_2[U]$, $\mathcal{Q}_3[U]$ as follows

$$\mathcal{Q}_1[U]f=\frac{2}{3}\partial_x\big(h^3v_xf\big),\qquad \mathcal{Q}_2[U]f=\frac{8}{45}\partial_x^2\big(h^5v_{xx}f\big),\qquad \mathcal{Q}_3[U]f=\frac{1}{15}\partial_x\big(h^5v_{xx}\partial_xf\big),\tag{40}$$

with

$$\begin{aligned}\mathcal{I}[h,\beta b]f=&\frac{4}{45}h_x\partial_x(h^4\partial_x^2f)+\frac{2}{45}h^4h_{xx}\partial_x^2f-\frac{\beta}{12}\big[\partial_x\big(\partial_x(h^4b_x)\partial_xf\big)+\partial_x^2(h^4b_{xx}f)+\partial_x^2(h^4b_x\partial_xf)\\&-b_x\partial_x(h^4\partial_x^2f)\big]-\frac{\beta^2}{6}\big[2\partial_x(h^3b_x^2\partial_xf)+\partial_x(h^3b_xb_{xx}f)+2b_x\partial_x(h^3b_x\partial_xf)+b_x\partial_x(h^3b_{xx}f)\big],\end{aligned}\tag{41}$$

while the purely-topographical terms are introduced by $\mathcal{B}_1[U]f=h^2b_xv_xf$ and

$$\begin{aligned}\mathcal{B}_2[U]f=&-\frac{1}{24}\partial_x^2\Big\{h^4\big(2b_{xx}v+9b_xv_x\big)f\Big\}-\frac{1}{4}h^4b_{xx}v_{xx}f-\frac{1}{24}\partial_x\big(h_x^4b_xv_xf\big)\\&-\frac{1}{24}\partial_x\big(h^4b_{xx}v_xf\big)-\frac{1}{4}\partial_x\big(h^4b_{xx}v\partial_xf\big)+\frac{1}{12}b_x\partial_x\big(h^4v_{xx}f\big),\end{aligned}\tag{42}$$

$$\mathcal{B}_3[U]f=\frac{3}{4}\partial_x(h^3b_xb_{xx}vf)+\frac{1}{12}h^3\big(b_{xx}v+2b_xv_x\big)b_{xx}f-\frac{1}{12}b_x\partial_x\Big\{h^3\big(b_{xx}vf+8b_xv_xf\big)\Big\},\tag{43}$$

$$\begin{aligned}\mathcal{R}[\mu,\varepsilon h,\beta b](U)=&\frac{1}{2}\varepsilon\mu\beta\partial_x(h^2b_{xx}v^2)+\varepsilon\mu\beta^2hb_xb_{xx}v^2+\frac{1}{24}\varepsilon\mu^2\beta\partial_x^2(h^4b_{xxx}v^2)\\&+\frac{1}{12}\varepsilon\mu^2\beta^2\partial_x(h^3b_xb_{xxx}v^2)+\frac{1}{4}\varepsilon\mu^2\beta^2\partial_x(h^3b_{xx}^2v^2).\end{aligned}\tag{44}$$

Remark 1. *The interest of formulation (35) is that all the fifth order derivatives of v are factorized in $\big(h+\mu\mathcal{T}[h]+\mu^2\mathfrak{T}[h]\big)\big(\partial_tv+\varepsilon vv_x\big)$ which is of notable help. However, the benefit of formulation (37) is in replacing $h+\mu\mathcal{T}[h]+\mu^2\mathfrak{T}[h]$ by a new operator $\mathfrak{I}$ so that the coercivity condition of the bilinear form is satisfied when applying a Lax-Millgram theorem for the proof of the invertibility of $\mathfrak{I}$ (see Lemma 1). Actually, the linear dispersion relation of the system exhibits no singularity as opposed to the original extended model. On the other hand, one can replace the relation $\mathfrak{I}+2\mu^2\mathfrak{T}[h,\beta b]=h+\mu\mathcal{T}[h,\beta b]+\mu^2\mathfrak{T}[h,\beta b]$ which defines the new operator $\mathfrak{I}$ by an alternative one $\mathfrak{I}+(\alpha+1)\mu^2\mathfrak{T}[h,\beta b]=h+\mu\mathcal{T}[h,\beta b]+\mu^2\mathfrak{T}[h,\beta b]$ with an arbitrary real parameter $\alpha>0$. The special case $\alpha=1$ recovers the present definition. In this general setting, also the linear dispersion relation would not give rise to any singularity.*

Remark 2. *The reason of considering the effect of surface tension (see [1] for a brief physical relevance) is due to the betterment needed on the natural associated energy norm $|\cdot|_{Y^s}$ $\big($i.e. when no surface tension $(\mathrm{bo}^{-1}=0)\big)$ defined by*

$$|(\zeta,v)|_{Y^s}^2\quad=\quad|\zeta|_{H^s}^2+\mu^2|\zeta_{xx}|_{H^s}^2+|v|_{H^s}^2+\mu|v_x|_{H^s}^2+\mu^2|v_{xx}|_{H^s}^2.$$

Because the norm $|\cdot|_{Y^s}$ is not adequate for the proof of the energy estimate, see for instance the control of the term A_2+A_3 in the case when neglecting surface tension where one may write A_2+A_3 as follows

$$\begin{aligned}A_2+A_3=&-\big(\Lambda^sv,\underline{h}_x\Lambda^s\zeta\big)+\frac{2}{45}\mu^2\big(\underline{h}_{xx}\Lambda^sv_x,\underline{h}^4\Lambda^s\zeta_{xx}\big)-\frac{2}{45}\mu^2\big(\Lambda^s\zeta_{xx},\partial_x(\underline{h}^4\underline{h}_{xx}\Lambda^sv)\big)\\&-\frac{4}{45}\mu^2\big(\Lambda^s\zeta_{xx},\partial_x(\underline{h}^4\underline{h}_x\Lambda^sv_x)\big)-\frac{2}{45}\mu^2\big(\Lambda^s\zeta_{xx},\underline{h}_x^5\Lambda^sv_{xx}\big),\end{aligned}$$

so, in this case $\underline{h}_{xx}$ and $\underline{h}_{xxx}$ cannot be controlled by $E^s(\underline{U})$. Also, one can see terms A_4 and A_5 and then introducing the operator $\mathfrak{J}=1+\frac{2}{45}\mu^2\partial_x^2(h^4\partial_x^2\cdot)$ is not sufficient. Moreover, taking $\mathfrak{J}=1$ yields to the

definition of $|U|^2_{X^s} := |\zeta|^2_{H^s} + |v|^2_{H^s} + \mu|v_x|^2_{H^s} + \mu^2|v_{xx}|^2_{H^s}$, *for* $U = (\zeta, v)^T \in X^s$ *which it is also not sufficient especially in controlling the term produced after using a BBM trick. The addition of the quantities in* (33) *permits the definition of a new energy norm*

$$|(\zeta, v)|^2_{X^s} \quad = \quad |\zeta|^2_{H^s} + \frac{\mu}{\text{bo}}|\zeta_x|^2_{H^s} + \mu^2|\zeta_{xx}|^2_{H^s} + |v|^2_{H^s} + \mu|v_x|^2_{H^s} + \mu^2|v_{xx}|^2_{H^s}.$$

The second term in $|\cdot|_{X^s}$ *is absent from the natural energy* $|\cdot|_{Y^s}$ *and this will be a fundamental term for allowing a control of several unconvenient terms as we going to figure out in the next section. Moreover, the space* $\{\zeta \in H^{s+2}(\mathbb{R})\ ;\ |\zeta|^2_{H^s} + \mu^2|\zeta_{xx}|^2_{H^s} < \infty\}$ *is not equivalent to the Sobolev space* $H^{s+2}(\mathbb{R})$, *required in defining the energy space* X^s *of our problem* (*see Definition* 1) .

3.1. **Preliminary results.** Under the nonzero depth-condition

$$\exists \quad h_{min} > 0, \quad \inf_{x\in\mathbb{R}} h \geq h_{min} \text{ where } h(t,x) = 1 + \varepsilon\zeta(t,x) - \beta b(x), \tag{45}$$

We introduce the symmetric operator

$$\Im = h + \mu\mathcal{T}[h, \beta b] - \mu^2\mathfrak{T}[h, \beta b], \tag{46}$$

which plays an important role in the energy estimate and the local well-posedness of the extended Green-Naghdi system. We shall give an essential invertibility result on $\Im$ and specify some properties on its inverse $\Im^{-1}$ explained in the following lemmas .

Lemma 1. *Let* $b \in C^\infty_b(\mathbb{R})$ *and assume that* $\zeta(t,\cdot) \in L^\infty(\mathbb{R})$ *is a differentiable scalar function under the condition (45). Then, the operator*

$$\Im\colon H^2_\mu(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$$

is well defined, one-to-one and onto .

Proof. The proof of the invertibility of $\Im$ is a direct application on the Lax-Millgram theorem. We introduce first the space $H^2_\mu(\mathbb{R})$ endowed with the norm $|\cdot|_\mu$ defined as

$$H^2_\mu(\mathbb{R}) = \{v \in H^2(\mathbb{R})\ ;\ |v|^2_\mu = |v|^2_2 + \mu|v_x|^2_2 + \mu^2|v_{xx}|^2_2 < \infty\}, \tag{47}$$

where $|\cdot|_\mu$ is equivalent to $|\cdot|_{H^2}$ but not uniformly with respect to μ .
Let $f \in L^2(\mathbb{R})$. Consider the weak problem

$$\begin{cases} Find \quad v \in H^2_\mu(\mathbb{R}) \quad such\ that \\ a(v,u) = L(u) \qquad \forall u \in H^2_\mu(\mathbb{R}), \end{cases}$$

with $L(u) = (f,u)$ and the bilinear form $a(v,u) = (\Im v, u)$ which can be written as follows

$$a(v,u) = (hv,u) + \mu\Big(h\big(\frac{\sqrt{3}}{3}hv_x - \frac{\sqrt{3}}{2}\beta b_x v\big), \frac{\sqrt{3}}{3}hu_x - \frac{\sqrt{3}}{2}\beta b_x u\Big) + \frac{\mu\beta^2}{4}\big(hb_x v, b_x u\big) + \mu^2\Big(h\big(\frac{\sqrt{5}}{15}h^2v_{xx} - \frac{\sqrt{5}}{4}\beta hb_x v_x$$
$$- \frac{\sqrt{5}}{8}\beta hb_{xx}v\big), \frac{\sqrt{5}}{15}h^2u_{xx} - \frac{\sqrt{5}}{4}\beta hb_x u_x - \frac{\sqrt{5}}{8}\beta hb_{xx}u\Big) + \mu^2\beta^2\Big(h\big(\frac{\sqrt{3}}{12}hb_x v_x + \frac{\sqrt{3}}{24}hb_{xx}v\big), \frac{\sqrt{3}}{12}hb_x u_x + \frac{\sqrt{3}}{24}hb_{xx}u\Big).$$

It is easy to see that a and L are continous on $H^2_\mu(\mathbb{R}) \times H^2_\mu(\mathbb{R})$ and $H^2_\mu(\mathbb{R})$ respectively. In addition, using (45) we have

$$a(v,v) \geq h_{min}|v|^2_2 + \mu h_{min}\big|\frac{\sqrt{3}}{3}hv_x - \frac{\sqrt{3}}{2}\beta b_x v\big|^2_2 + \frac{\mu\beta^2}{4}|b_x v|^2_2 + \mu^2 h_{min}\big|\frac{\sqrt{5}}{15}h^2v_{xx} - \frac{\sqrt{5}}{4}\beta hb_x v_x - \frac{\sqrt{5}}{8}\beta hb_{xx}v\big|^2_2$$
$$+ \mu^2\beta^2 h_{min}\big|\frac{\sqrt{3}}{12}hb_x v_x + \frac{\sqrt{3}}{24}hb_{xx}v\big|^2_2,$$

but, one has

$$|v|^2_\mu \leq |v|^2_2 + \frac{3\mu}{h^2_{min}}\big|\frac{\sqrt{3}}{3}hv_x\big|^2_2 + \frac{45\mu^2}{h^4_{min}}\big|\frac{\sqrt{5}}{15}h^2v_{xx}\big|^2_2$$
$$\leq |v|^2_2 + \frac{18}{h^2_{min}}\mu\Big(\big|\frac{\sqrt{3}}{3}hv_x - \frac{\sqrt{3}}{2}\beta b_x v\big|^2_2 + \frac{\beta^2}{4}|b_x v|^2_2\Big) + \frac{90}{h^4_{min}}\mu^2\big|\frac{\sqrt{5}}{15}h^2v_{xx} - \frac{\sqrt{5}}{4}\beta hb_x v_x - \frac{\sqrt{5}}{8}\beta hb_{xx}v\big|^2_2$$
$$+ \frac{1350}{h^4_{min}}\mu^2\beta^2\big|\frac{\sqrt{3}}{12}hb_x v_x + \frac{\sqrt{3}}{24}hb_{xx}v\big|^2_2.$$

Therefore, after denoting $\mathfrak{M}_{h_{min}} = \max\Big\{1, \frac{18}{h_{min}^2}, \frac{90}{h_{min}^4}, \frac{1350}{h_{min}^4}\Big\}$ we deduce the coercivity condition on $H^2_\mu(\mathbb{R}) \times H^2_\mu(\mathbb{R})$ represented by the following inequality

$$a(v,v) \geq \frac{h_{min}}{\mathfrak{M}_{h_{min}}} |v|^2_\mu. \tag{48}$$

Hence by Lax-Millgram theorem, for every $f \in L^2(\mathbb{R})$, there exists a unique $v \in H^2_\mu(\mathbb{R})$ such that for all $u \in H^2_\mu(\mathbb{R})$, we have $a(v,u) = (\Im v, u) = L(u) = (f,u)$. Thus the result. □

Remark 3. *All through whatever is left of this paper, and for sake of simplicity, we will not attempt to show the dependence on the bottom parametrization $b \in C^\infty_b(\mathbb{R})$ in all the verifications. This should effectively be possible easily, but is of no attention in the proof of our required results.*

The following lemma gives functional properties to the operator $\Im^{-1}$.

Lemma 2. *Let $b \in C^\infty_b(\mathbb{R})$, $t_0 > \frac{1}{2}$ and $\zeta \in H^{t_0+1}(\mathbb{R})$ be such that (45) is satisfied. Then, we have the following*:

(i) $\forall\ 0 \leq s \leq t_0 + 1$,

$$|\Im^{-1} f|_{H^s} + \sqrt{\mu}|\partial_x \Im^{-1} f|_{H^s} + \mu|\partial_x^2 \Im^{-1} f|_{H^s} \leq C\big(\frac{1}{h_{min}}, |h-1|_{H^{t_0+1}}\big)|f|_{H^s}.$$

(ii) $\forall\ 0 \leq s \leq t_0 + 1$,

$$\sqrt{\mu}|\Im^{-1}\partial_x f|_{H^s} + \mu|\partial_x \Im^{-1}\partial_x f|_{H^s} + \mu\sqrt{\mu}|\partial_x^2 \Im^{-1}\partial_x f|_{H^s} \leq C\big(\frac{1}{h_{min}}, |h-1|_{H^{t_0+1}}\big)|f|_{H^s},$$

and

$$\mu|\Im^{-1}\partial_x^2 f|_{H^s} + \mu\sqrt{\mu}|\partial_x \Im^{-1}\partial_x^2 f|_{H^s} + \mu^2|\partial_x^2 \Im^{-1}\partial_x^2 f|_{H^s} \leq C\big(\frac{1}{h_{min}}, |h-1|_{H^{t_0+1}}\big)|f|_{H^s}.$$

(iii) $\forall\ s \geq t_0 + 1$,

$$\|\Im^{-1}\|_{H^s(\mathbb{R})\to H^s(\mathbb{R})} + \sqrt{\mu}\|\Im^{-1}\partial_x\|_{H^s(\mathbb{R})\to H^s(\mathbb{R})} + \mu\|\Im^{-1}\partial_x^2\|_{H^s(\mathbb{R})\to H^s(\mathbb{R})} \leq C_s,$$

with

$$\mu\|\partial_x \Im^{-1}\partial_x\|_{H^s(\mathbb{R})\to H^s(\mathbb{R})} + \mu\sqrt{\mu}\|\partial_x^2 \Im^{-1}\partial_x\|_{H^s(\mathbb{R})\to H^s(\mathbb{R})} \leq C_s,$$

and

$$\mu\sqrt{\mu}\|\partial_x \Im^{-1}\partial_x^2\|_{H^s(\mathbb{R})\to H^s(\mathbb{R})} + \mu^2\|\partial_x^2 \Im^{-1}\partial_x^2 f\|_{H^s(\mathbb{R})\to H^s(\mathbb{R})} \leq C_s.$$

where C_s is a constant depending on $1/h_{min}$, $|h-1|_{H^s}$ and independent of $(\varepsilon,\mu) \in (0,1)^2$.

Proof. The proof is adapted as in [4] for $1D$ Green-Naghdi equations (μ^2 order with $\beta \neq 0$). First of all, after noticing the following relations

$$\partial_x\big(\partial_x(h^4 b_x)v_x\big) = \partial_x^2(h^4 b_x v_x) - \partial_x(h^4 b_x v_{xx}), \qquad b_x\partial_x(h^4 v_{xx}) = \partial_x(h^4 b_x v_{xx}) - h^4 b_{xx} v_{xx},$$
$$b_x\partial_x(h^3 b_x v_x) = \partial_x(h^3 b_x^2 v_x) - h^3 b_x b_{xx} v_x, \qquad b_x\partial_x(h^3 b_{xx} v) = \partial_x(h^3 b_x b_{xx} v) - h^3 b_{xx}^2 v,$$

one can rewrite the expression of $\mathfrak{T}[h,\beta b]\cdot$ under the form

$$\begin{aligned}\mathfrak{T}[h,\beta b]\cdot = -\frac{1}{45}\partial_x^2\big(h^5\partial_x^2\,\cdot\big) + \frac{\beta}{24}\big[2\partial_x^2(h^4 b_x\partial_x^2\cdot) + \partial_x^2\big(h^4 b_{xx}(\cdot)\big) - 2\partial_x(h^4 b_x\partial_x^2\cdot) + h^4 b_{xx}\partial_x^2\,\cdot\,\big] \\ + \frac{\beta^2}{12}\big[4\partial_x(h^3 b_x^2\partial_x\cdot) + \partial_x\big(h^3 b_x b_{xx}(\cdot)\big) - 2h^3 b_x b_{xx}\partial_x(\cdot) - h^3 b_{xx}^2(\cdot)\big].\end{aligned} \tag{49}$$

Now, assume that $f \in H^s(\mathbb{R})$ and $u = \Im^{-1} f$, then $\Im u = f$. Apply Λ^s to both sides, then multiplying by $\Lambda^s u$ yields the following (knowing that $\Im\Lambda^s u = \Lambda^s u - [\Lambda^s, \Im]u$)

$$a(\Lambda^s u, \Lambda^s u) = (\widetilde{f}, \Lambda^s u) + \sqrt{\mu}(\partial_x\widetilde{g}, \Lambda^s u) + \mu(\partial_x^2\widetilde{p}, \Lambda^s u),$$

where $\widetilde{f}$, $\widetilde{g}$, and $\widetilde{p}$ are written as follows

$$\widetilde{f} = \Lambda^s f - [\Lambda^s, h]u + \frac{\mu\beta}{2}[\Lambda^s, h^2 b_x]u_x - \mu\beta^2[\Lambda^s, h b_x^2]u + \frac{\mu^2\beta}{24}[\Lambda^s, h^4 b_{xx}]u_{xx} - \frac{\mu^2\beta^2}{6}[\Lambda^s, h^3 b_x b_{xx}]u_x - \frac{\mu^2\beta^2}{12}[\Lambda^s, h^3 b_{xx}^2]u,$$

$$\widetilde{g}=\frac{\sqrt{\mu}}{3}[\Lambda^s,h^3]u_x-\frac{\sqrt{\mu}\beta}{2}[\Lambda^s,h^2b_x]u-\frac{\mu\sqrt{\mu}\beta}{12}[\Lambda^s,h^4b_x]u_{xx}+\frac{\mu\sqrt{\mu}\beta^2}{3}[\Lambda^s,h^3b_x^2]u_x+\frac{\mu\sqrt{\mu}\beta^2}{6}[\Lambda^s,h^3b_xb_{xx}]u,$$

$$\widetilde{p}=-\frac{\mu}{45}[\Lambda^s,h^5]u_{xx}+\frac{\mu\beta}{12}[\Lambda^sh^4b_x]u_x+\frac{\mu\beta}{24}[\Lambda^s,h^4b_{xx}]u.$$

Integrating by parts and using (48), we get

$$\frac{h_{min}}{\mathfrak{M}_{h_{min}}}|\Lambda^s u|_\mu\le|\widetilde{f}|_2+|\widetilde{g}|_2+|\widetilde{p}|_2.$$

Now, using the Kato-Pance commutator estimate (see Lemma 4.6 of [21]) we have

$$\big|[\Lambda^s,f]u\big|_2\lesssim|\nabla f|_{H^{t_0}}|u|_{H^{s-1}},\qquad\text{if } 0\le s\le t_0+1, \tag{50}$$

$$\big|[\Lambda^s,f]u\big|_2\lesssim|\nabla f|_{H^{s-1}}|u|_{H^{s-1}},\qquad\text{if } s\ge t_0+1. \tag{51}$$

One can deduce

$$|\widetilde{f}|_2+|\widetilde{g}|_2+|\widetilde{p}|_2\le|f|_{H^s}+C\big(|h-1|_{H^{t_0+1}}\big)|\Lambda^{s-1}u|_\mu.$$

Hence, the inequality (i) holds after a continuous induction on s. For the proof of (ii), one has to replace $u=\sqrt{\mu}\Im^{-1}\partial_x f$ and $u=\mu\Im^{-1}\partial_x^2 f$ for a second time. The general strategy is the same as in (i) noticing that Λ^s commutes with ∂_x, ∂_x^2. The only difference is in the expression of $\widetilde{f}$, $\widetilde{g}$, $\widetilde{p}$ when setting $u=\sqrt{\mu}\Im^{-1}\partial_x f$ and similarly when $u=\mu\Im^{-1}\partial_x^2 f$. The rest of the proof is as in [27]. □

3.2. **Linear analysis.** In order to rewrite the extended Green-Naghdi system for an uneven bottom with surface tension in a condensed form, we introduce a new operator symmetric $\mathfrak{J}_{\rm bo}$ as follows

$$U=(\zeta,v)^T,\qquad \mathfrak{J}_{\rm bo}=1-\frac{\mu}{\rm bo}\partial_x^2(\cdot)+\frac{2}{45}\mu^2\partial_x^2\big(h^4\partial_x^2\,\cdot\,\big),\qquad h(t,x)=1+\varepsilon\zeta(t,x)-\beta b(x). \tag{52}$$

The first equation in (37) can be written as

$$\partial_t\zeta+\varepsilon v\partial_x\zeta+h\partial_x v-\beta v\partial_x b=0.$$

For the second equation in (37) apply $\Im^{-1}$ to both sides, we get

$$\partial_t v+\varepsilon vv_x+\Im^{-1}\big(h\mathfrak{J}_{\rm bo}\zeta_x\big)+\mu^2\Im^{-1}\big(\mathcal{I}[h,\beta b]\zeta_x\big)-\varepsilon^2\mu^2\frac{1}{\rm bo}\Im^{-1}\big(T[U]\zeta_x\big)+\varepsilon\mu\Im^{-1}\big(\mathcal{Q}_1[U]v_x\big)+\varepsilon\mu\beta\Im^{-1}\big(\mathcal{B}_1[U]v_x\big)$$
$$+\varepsilon\mu^2\Im^{-1}\big(\mathcal{Q}_2[U]v_x\big)+\varepsilon\mu^2\Im^{-1}\big(\mathcal{Q}_3[U]v_x\big)+\varepsilon\mu^2\beta\Im^{-1}\big(\mathcal{B}_2[U]v_x\big)+\varepsilon\mu^2\beta^2\Im^{-1}\big(\mathcal{B}_3[U]v_x\big)+\Im^{-1}\big(\mathcal{R}[\mu,\varepsilon h,\beta b](U)\big)=O(\mu^3).$$

Hence the extended Green-Naghdi system ($\beta\neq0$) with surface tension can be written under the form

$$\partial_t U+A[U]\partial_x U+B(U)=0, \tag{53}$$

where

$$A[U]=\begin{pmatrix}\varepsilon v & h\\ \Im^{-1}\big(h\mathfrak{J}_{\rm bo}\,\cdot\,\big)+\mu^2\Im^{-1}\big(\mathcal{I}[h,\beta b]\,\cdot\,\big)-\varepsilon^2\mu^2\frac{1}{\rm bo}\Im^{-1}\big(T[U]\,\cdot\,\big) & \varepsilon v+\varepsilon\mu Q_b[U]\cdot+\varepsilon\mu^2Q_{bb}[U]\cdot\end{pmatrix},$$

with

$$Q_b[U]\cdot=\Im^{-1}\big(\mathcal{Q}_1[U]\,\cdot\,\big)+\Im^{-1}\big(\beta\mathcal{B}_1[U]\,\cdot\,\big)\qquad\text{and}\qquad Q_{bb}[U]\cdot=\Im^{-1}\big(\mathcal{Q}_2[U]\cdot+\mathcal{Q}_3[U]\cdot+\beta\mathcal{B}_2[U]\cdot+\beta^2\mathcal{B}_3[U]\,\cdot\,\big),$$

and

$$B(U)=\begin{pmatrix}-\beta b_x v\\ \Im^{-1}\big(\mathcal{R}[\mu,\varepsilon h,\beta b](U)\big)\end{pmatrix}. \tag{54}$$

Now, consider the linearized system around some reference state $\underline{U}=(\underline{\zeta},\underline{v})^T$

$$\begin{cases}\partial_t U+A[\underline{U}]\partial_x U+B(\underline{U})=0,\\ U_{|_{t=0}}=U_0.\end{cases} \tag{55}$$

The proof of the energy estimate which permits the convergence of an iterative scheme to construct a solution to the extended Green Naghdi system (37) with surface tension for the initial value problem (55) requires to define the X^s spaces, which are the energy spaces for this problem .

Definition 1. *For all $s \geq 0$ and $T > 0$, we denote by X^s the vector space $H^{s+2}(\mathbb{R}) \times H^{s+2}(\mathbb{R})$ endowed with the norm*

$$|U|^2_{X^s} \quad := \quad |\zeta|^2_{H^s} + \frac{\mu}{\mathrm{bo}}|\zeta_x|^2_{H^s} + \mu^2|\zeta_{xx}|^2_{H^s} + |v|^2_{H^s} + \mu|v_x|^2_{H^s} + \mu^2|v_{xx}|^2_{H^s},$$

while X^s_T stands for $C([0, \frac{T}{\varepsilon}]; X^s)$ endowed with its canonical norm.

First, recall that a pseudo-symmetrizer for $A[\underline{U}]$ is given by

$$S = \begin{pmatrix} \underline{\mathfrak{J}}_{\mathrm{bo}} & 0 \\ 0 & \underline{\mathfrak{I}} \end{pmatrix}, \tag{56}$$

with $\underline{h} = 1 + \varepsilon\underline{\zeta} - \beta b$, $\underline{\mathfrak{I}} = \underline{h} + \mu\underline{\mathcal{T}}[\underline{h}, \beta b] - \mu^2\underline{\mathfrak{T}}[\underline{h}, \beta b]$ and $\underline{\mathfrak{J}}_{\mathrm{bo}} = 1 - \frac{\mu}{\mathrm{bo}}\partial_x^2(\cdot) + \frac{2}{45}\mu^2\partial_x^2\big(\underline{h}^4\partial_x^2\,\cdot\,\big)$. A natural energy for the initial value problem (55) is suggested to be

$$E^s(U)^2 = (\Lambda^s U, S\Lambda^s U). \tag{57}$$

The connection between $E^s(U)$ and the X^s-norm is examined using the lemma below .

Lemma 3. *Let $b \in C^\infty_b(\mathbb{R})$, $s \geq 0$ and $\underline{\zeta} \in L^\infty(\mathbb{R})$. Under the depth-condition*

$$\exists \quad h_{min} > 0, \quad \inf_{x\in\mathbb{R}} h \geq h_{min}, \quad h(t,x) = 1 + \varepsilon\zeta(t,x) - \beta b(x), \tag{58}$$

$E^s(U)$ is uniformly equivalent to the $|\cdot|_{X^s}$-norm with respect to $(\mu, \varepsilon) \in (0,1)^2$

$$E^s(U) \leq C\big(\frac{1}{h_{min}}, |\underline{h}|_\infty\big)|U|_{X^s}, \qquad \text{and} \qquad |U|_{X^s} \leq C\big(h_{min}, \frac{1}{h_{min}}\big)E^s(U).$$

Proof. First note that, $E^s(U)^2 = (\Lambda^s U, S\Lambda^s U)$ with $S\Lambda^s U = (\underline{\mathfrak{J}}_{\mathrm{bo}}\Lambda^s\zeta, \underline{\mathfrak{I}}\Lambda^s v)$. Then we get

$$E^s(U)^2 = (\Lambda^s\zeta, \underline{\mathfrak{J}}_{\mathrm{bo}}\Lambda^s\zeta) + (\Lambda^s v, \underline{\mathfrak{I}}\Lambda^s v).$$

Using the expression of $\underline{\mathfrak{I}}$, $\underline{\mathfrak{J}}_{\mathrm{bo}}$, Lemma.1 and integrating by parts we get

$$E^s(U)^2 = (\Lambda^s\zeta, \Lambda^s\zeta) + \frac{\mu}{\mathrm{bo}}\big(\Lambda^s\zeta_x, \Lambda^s\zeta_x\big) + \frac{2}{45}\mu^2\big(\underline{h}^4\Lambda^s\zeta_{xx}, \Lambda^s\zeta_{xx}\big) + \underline{a}(\Lambda^s v, \Lambda^s v),$$

where

$$\underline{a}(v,u) = (hv,u) + \mu\Big(\underline{h}\big(\frac{\sqrt{3}}{3}\underline{h}v_x - \frac{\sqrt{3}}{2}\beta b_x v\big), \frac{\sqrt{3}}{3}\underline{h}u_x - \frac{\sqrt{3}}{2}\beta b_x u\Big) + \frac{\mu\beta^2}{4}\big(\underline{h}b_x v, b_x u\big) + \mu^2\Big(\underline{h}\big(\frac{\sqrt{5}}{15}\underline{h}^2 v_{xx} - \frac{\sqrt{5}}{4}\beta\underline{h}b_x v_x$$
$$- \frac{\sqrt{5}}{8}\beta\underline{h}b_{xx}v\big), \frac{\sqrt{5}}{15}\underline{h}^2 u_{xx} - \frac{\sqrt{5}}{4}\beta\underline{h}b_x u_x - \frac{\sqrt{5}}{8}\beta\underline{h}b_{xx}u\Big) + \mu^2\beta^2\Big(\underline{h}\big(\frac{\sqrt{3}}{12}\underline{h}b_x v_x + \frac{\sqrt{3}}{24}\underline{h}b_{xx}v\big), \frac{\sqrt{3}}{12}\underline{h}b_x u_x + \frac{\sqrt{3}}{24}\underline{h}b_{xx}u\Big).$$

Now, knowing that $b \in C^\infty_b(\mathbb{R}), \underline{\zeta} \in L^\infty(\mathbb{R})$, and the fact that the water depth is always bounded from below by a non-negative constant (45), with the help of the Cauchy-Schwartz inequality and the results of lemma1, we get the following two inequalities there exists m, M depending on h_{min} and $|\underline{h}|_\infty$ greater than zero such that $m \leq E^s(U) \leq M$. Hence the lemma is confirmed . □

The proof of the energy estimate is given in the following proposition .

Proposition 1. *Let $b \in C^\infty_b(\mathbb{R})$, $t_0 > \frac{1}{2}$, $s \geq t_0 + 1$. Let also $\underline{U} = (\underline{\zeta}, \underline{v})^T \in X^s_T$ be such that $\partial_t\underline{U} \in X^{s-1}_T$ and satisfying condition (45) on $[0, \frac{T}{\varepsilon\vee\beta}]$. Then, for all $U_0 \in X^s$ there exists a unique solution $U = (\zeta, v)^T \in X^s_T$ to (55) and for all $0 \leq t \leq \frac{T}{\varepsilon\vee\beta}$ satisfying*

$$E^s\big(U(t)\big) \leq \big(e^{(\varepsilon\vee\beta)\lambda_T t}\big)^{1/2}E^s(U_0) + (\varepsilon\vee\beta)\int_0^t \big(e^{(\varepsilon\vee\beta)\lambda_T(t-t')}\big)^{1/2}C(E^s(\underline{U})(t'))dt', \tag{59}$$

for some $\lambda_T = \lambda_T(\sup_{0\leq t\leq T/\varepsilon\vee\beta} E^s(\underline{U}(t)), \sup_{0\leq t\leq T/\varepsilon\vee\beta}|\partial_t\underline{h}(t)|_{L^\infty})$, where $\varepsilon\vee\beta = \max\{\varepsilon,\beta\} = \frac{\varepsilon+\beta+|\varepsilon-\beta|}{2}$.

Proof. For any $\lambda \in \mathbb{R}$, we compute

$$e^{(\varepsilon\vee\beta)\lambda t}\partial_t(e^{-(\varepsilon\vee\beta)\lambda t}E^s(U)^2) = -(\varepsilon\vee\beta)\lambda E^s(U)^2 + \partial_t(E^s(U)^2),$$

with

$$\partial_t(E^s(U)^2) = (\Lambda^s\zeta_t, \underline{\mathfrak{J}}_{\rm bo}\Lambda^s\zeta) + \big(\Lambda^s\zeta, \partial_t(\underline{\mathfrak{J}}_{\rm bo}\Lambda^s\zeta)\big) + (\Lambda^s v, \underline{\mathfrak{K}}\Lambda^s v_t) + \big(\Lambda^s v, \partial_t(\underline{\mathfrak{K}}\Lambda^s v)\big).$$

Using the fact that $\underline{\mathfrak{K}}$ and $\underline{\mathfrak{J}}_{\rm bo}$ are symmetric, in addition to the following identities

$$\partial_t(\underline{\mathfrak{K}}\Lambda^s v) = [\partial_t, \underline{\mathfrak{K}}]\Lambda^s v + \underline{\mathfrak{K}}\Lambda^s v_t, \qquad \partial_t(\underline{\mathfrak{J}}_{\rm bo}\Lambda^s\zeta) = [\partial_t, \underline{\mathfrak{J}}_{\rm bo}]\Lambda^s\zeta + \underline{\mathfrak{J}}_{\rm bo}\Lambda^s\zeta_t, \tag{60}$$

one gets after using (55) the following

$$\begin{aligned}\partial_t(E^s(U)^2) = -2\big(S\Lambda^s U, \big[\Lambda^s, A[\underline{U}]\big]\partial_x U\big) - 2\big(S\Lambda^s U, A[\underline{U}]\Lambda^s\partial_x U\big)\\ - 2\big(\Lambda^s B(\underline{U}), S\Lambda^s U\big) + \big(\Lambda^s\zeta, [\partial_t, \underline{\mathfrak{J}}_{\rm bo}]\Lambda^s\zeta\big) + \big(\Lambda^s v, [\partial_t, \underline{\mathfrak{K}}]\Lambda^s v\big).\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\frac{1}{2}e^{(\varepsilon\vee\beta)\lambda t}\partial_t(e^{-(\varepsilon\vee\beta)\lambda t}E^s(U)^2) = -\frac{(\varepsilon\vee\beta)\lambda}{2}E^s(U)^2 - \big(SA[\underline{U}]\Lambda^s\partial_x U, \Lambda^s U\big)\\ - \big(\big[\Lambda^s, A[\underline{U}]\big]\partial_x U, S\Lambda^s U\big) - \big(\Lambda^s B(\underline{U}), S\Lambda^s U\big) + \frac{1}{2}\big(\Lambda^s\zeta, [\partial_t, \underline{\mathfrak{J}}_{\rm bo}]\Lambda^s\zeta\big) + \frac{1}{2}(\Lambda^s v, [\partial_t, \underline{\mathfrak{K}}]\Lambda^s v).\end{aligned} \tag{61}$$

We will focus now on bounding from above the purely-topographical components of the r.h.s of (61), knowing that the non-topographical expressions have been controlled in [27].

• Estimation of $(SA[\underline{U}]\Lambda^s\partial_x U, \Lambda^s U)$. Denoting by $R[\underline{U}]\cdot = \varepsilon\mu\mathcal{Q}_1[\underline{U}]\cdot + \varepsilon\mu^2\mathcal{Q}_2[\underline{U}]\cdot + \varepsilon\mu^2\mathcal{Q}_3[\underline{U}]\cdot$, and remarking that

$$SA[\underline{U}] = \begin{pmatrix} \varepsilon\underline{\mathfrak{J}}_{\rm bo}(\underline{v}\cdot) & \underline{\mathfrak{J}}_{\rm bo}(\underline{h}\cdot) \\ \underline{h}\underline{\mathfrak{J}}_{\rm bo}\cdot + \mu^2\mathcal{I}[\underline{h}, \beta b]\cdot - \varepsilon^2\mu^2\frac{1}{\rm bo}T[\underline{U}]\cdot & \varepsilon\underline{\mathfrak{K}}(\underline{v}\cdot) + R[\underline{U}]\cdot + \varepsilon\mu\beta\mathcal{B}_1[\underline{U}]\cdot + \varepsilon\mu^2\beta\mathcal{B}_2[\underline{U}]\cdot + \varepsilon\mu^2\beta^2\mathcal{B}_3[\underline{U}]\cdot \end{pmatrix},$$

then, one has

$$\begin{aligned}\big(SA[\underline{U}]\Lambda^s\partial_x U, \Lambda^s U\big) &= \varepsilon\big(\underline{\mathfrak{J}}_{\rm bo}(\underline{v}\Lambda^s\zeta_x), \Lambda^s\zeta\big) + \big(\underline{\mathfrak{J}}_{\rm bo}(\underline{h}\Lambda^s v_x), \Lambda^s\zeta\big) + \big(\underline{h}\underline{\mathfrak{J}}_{\rm bo}\Lambda^s\zeta_x, \Lambda^s v\big) + \mu^2\big(\mathcal{I}[\underline{h}, \beta b]\Lambda^s\zeta_x, \Lambda^s v\big)\\ &\quad - \varepsilon^2\mu^2\frac{1}{\rm bo}\big(T[\underline{U}]\Lambda^s\zeta_x, \Lambda^s v\big) + \varepsilon\big(\underline{\mathfrak{K}}(\underline{v}\Lambda^s v_x), \Lambda^s v\big) + \big(R[\underline{U}]\Lambda^s v_x, \Lambda^s v\big)\\ &\quad + \varepsilon\mu\beta\big(\mathcal{B}_1[\underline{U}]\Lambda^s v_x, \Lambda^s v\big) + \varepsilon\mu^2\beta\big(\mathcal{B}_2[\underline{U}]\Lambda^s v_x, \Lambda^s v\big) + \varepsilon\mu^2\beta^2\big(\mathcal{B}_3[\underline{U}]\Lambda^s v_x, \Lambda^s v\big)\\ &= A_1 + A_2 + ... + A_{10}.\end{aligned}$$

One can see [27], so that the non-topographical terms A_1, A_2, A_3, A_5, A_7 were bounded from above by

$$\big|A_1 + A_2 + A_3 + A_5 + A_7\big| \leq (\varepsilon\vee\beta)C\big(|\underline{\zeta}|_{W^{1,\infty}}, |\underline{v}|_{W^{1,\infty}}, \sqrt{\frac{\mu}{\rm bo}}|\underline{\zeta}_x|_{H^s}, \sqrt{\mu}|\underline{v}_{xx}|_\infty, \sqrt{\rm bo}\big)E^s(U)^2.$$

To control A_4, an integration by parts yields to write

$$\begin{aligned}&A_4 = \frac{4}{45}\mu^2\big(\underline{h}_x\partial_x(\underline{h}^4\Lambda^s\zeta_{xxx}), \Lambda^s v\big) + \frac{2}{45}\mu^2\big(\underline{h}^4\underline{h}_{xx}\Lambda^s\zeta_{xxx}, \Lambda^s v\big) + \frac{1}{12}\mu^2\beta\big(\partial_x(\underline{h}^4 b_x)\Lambda^s\zeta_{xx}, \Lambda^s v_x\big)\\ &- \frac{1}{12}\mu^2\beta\big(\underline{h}^4 b_{xx}\Lambda^s\zeta_x, \Lambda^s v_{xx}\big) - \frac{1}{12}\mu^2\beta\big(\underline{h}^4 b_x\Lambda^s\zeta_{xx}, \Lambda^s v_{xx}\big) - \frac{1}{12}\mu^2\beta\big(\underline{h}^4\Lambda^s\zeta_{xxx}, \partial_x(b_x\Lambda^s v)\big) + \frac{1}{3}\mu^2\beta^2\big(\underline{h}^3 b_x^2\Lambda^s\zeta_{xx}, \Lambda^s v_x\big)\\ &+ \frac{1}{6}\mu^2\beta^2\big(\underline{h}^3 b_x b_{xx}\Lambda^s\zeta_x, \Lambda^s v_x\big) + \frac{1}{3}\mu^2\beta^2\big(\underline{h}^3 b_x\Lambda^s\zeta_{xx}, \partial_x(b_x\Lambda^s v_x)\big) + \frac{1}{6}\mu^2\beta^2\big(\underline{h}^3 b_{xx}\Lambda^s\zeta_x, \partial_x(b_x\Lambda^s v_x)\big)\\ &= A_{41} + A_{42} + ... + A_{4(10)}.\end{aligned}$$

For controlling A_{41} and A_{42} one can see [27], so we may write

$$\big|A_{41} + A_{42}\big| \leq (\varepsilon\vee\beta)C\big(|\underline{\zeta}|_{W^{1,\infty}}, \sqrt{\frac{\mu}{\rm bo}}|\underline{\zeta}_x|_{H^s}, \mu|\underline{\zeta}_{xx}|_{H^s}\big)E^s(U)^2.$$

Again, using integration by parts one can write

$$\big|A_{46}\big| = \frac{1}{12}\mu^2\beta\Big|\big(\Lambda^s\zeta_{xx}, \partial_x(\underline{h}^4 b_{xx}\Lambda^s v)\big) + \frac{1}{12}\mu^2\beta\big(\Lambda^s\zeta_{xx}, \partial_x(\underline{h}^4 b_x\Lambda^s v_x)\big)\Big| \leq \beta C\big(|\underline{\zeta}|_{W^{1,\infty}}\big)E^s(U)^2.$$

Therefore, we obtain

$$\big|A_4\big| \leq (\varepsilon \vee \beta) C\big(|\underline{\zeta}|_{W^{1,\infty}}, \sqrt{\frac{\mu}{\mathrm{bo}}}|\underline{\zeta}_x|_{H^s}, \mu|\underline{\zeta}_{xx}|_{H^s}\big) E^s(U)^2.$$

To control A_6 one should notice that the non-topographical terms are bounded from above in [27] by

$$(\varepsilon \vee \beta) C\big(|\underline{\zeta}|_{W^{1,\infty}}, |\underline{v}_x|_\infty, \sqrt{\mu}|\underline{v}_{xx}|_\infty\big) E^s(U)^2,$$

while the purely-topographical terms can be written as follows

$$\begin{aligned}
&-\frac{1}{2}\varepsilon\mu\beta\big(\underline{h}^2 b_x \underline{v}\Lambda^s v_x, \Lambda^s v_x\big) - \frac{1}{2}\varepsilon\mu\beta\big(\underline{h}^2 b_x \partial_x(\underline{v}\Lambda^s v_x), \Lambda^s v\big) + \varepsilon\mu\beta^2\big(\underline{h} b_x^2 \underline{v}\Lambda^s v_x, \Lambda^s v\big) + \frac{1}{24}\varepsilon\mu^2\beta\big((\underline{h}^4 b_x)_x \partial_x(\underline{v}\Lambda^s v_x), \Lambda^s v_x\big)\\
&-\frac{1}{24}\varepsilon\mu^2\beta\big(\underline{h}^4 \partial_x(b_x \underline{v}\Lambda^s v_x), \Lambda^s v_{xx}\big) - \frac{1}{24}\varepsilon\mu^2\beta\big(\underline{h}^4 \partial_x^2(\underline{v}\Lambda^s v_x), \partial_x(b_x\Lambda^s v)\big) + \frac{1}{6}\varepsilon\mu^2\beta^2\big(\underline{h}^3 b_x^2 \partial_x(\underline{v}\Lambda^s v_x), \Lambda^s v_x\big)\\
&+\frac{1}{12}\varepsilon\mu^2\beta^2\big(\underline{h}^3 b_x b_{xx} \underline{v}\Lambda^s v_x, \Lambda^s v_x\big) + \frac{1}{6}\varepsilon\mu^2\beta^2\big(\underline{h}^3 b_x \partial_x(\underline{v}\Lambda^s v_x), \partial_x(b_x\Lambda^s v)\big) + \frac{1}{6}\varepsilon\mu^2\beta^2\big(\underline{h}^3 b_{xx} \underline{v}\Lambda^s v_x, \partial_x(b_x\Lambda^s v)\big)\\
&= A_{61} + A_{62} + ... + A_{6(10)}.
\end{aligned}$$

Again, by integration by parts one has

$$|A_{66}| = \frac{1}{24}\varepsilon\mu^2\beta\Big|\big(\partial_x(\underline{v}\Lambda^s v_x), \partial_x(\underline{h}^4 b_{xx}\Lambda^s v)\big) + \big(\partial_x(\underline{v}\Lambda^s v_x), \partial_x(\underline{h}^4 b_x \Lambda^s v_x)\big)\Big| \leq \varepsilon\beta C\big(|\underline{\zeta}|_{W^{1,\infty}}, |\underline{v}|_{W^{1,\infty}}\big) E^s(U)^2.$$

Hence, we get

$$|A_6| \leq (\varepsilon \vee \beta) C\big(|\underline{\zeta}|_{W^{1,\infty}}, |\underline{v}|_{W^{1,\infty}}, \sqrt{\mu}|\underline{v}_{xx}|_\infty\big) E^s(U)^2.$$

Similarly, using the expressions of $\mathcal{B}_1[U]$, $\mathcal{B}_2[U]$, $\mathcal{B}_3[U]$ and integrations by parts one has

$$|A_8| \leq \varepsilon\beta C\big(|\underline{\zeta}|_\infty, |\underline{v}_x|_\infty\big) E^s(U)^2, \quad |A_9| \leq \varepsilon\beta C\big(|\underline{\zeta}|_{W^{1,\infty}}, |\underline{v}|_{W^{1,\infty}}, \sqrt{\mu}|\underline{v}_{xx}|_\infty\big) E^s(U)^2, \quad |A_{10}| \leq \varepsilon\beta C\big(|\underline{\zeta}|_\infty, |\underline{v}|_{W^{1,\infty}}\big) E^s(U)^2.$$

Therefore, we get

$$\big|\big(SA[\underline{U}]\Lambda^s\partial_x U, \Lambda^s U\big)\big| \leq (\varepsilon \vee \beta) C\big(|\zeta|_{W^{1,\infty}}, \sqrt{\frac{\mu}{\mathrm{bo}}}|\underline{\zeta}_x|_{H^s}, \mu|\underline{\zeta}_{xx}|_{H^s}, |\underline{v}|_{W^{1,\infty}}, \sqrt{\mu}|\underline{v}_{xx}|_\infty, \sqrt{\mathrm{bo}}\big) E^s(U)^2.$$

• Estimation of $\big(\big[\Lambda^s, A[\underline{U}]\big]\partial_x U, S\Lambda^s U\big)$. First of all, we have

$$\begin{aligned}
\big(\big[\Lambda^s, A[\underline{U}]\big]\partial_x U, S\Lambda^s U\big) &= \varepsilon\big([\Lambda^s, \underline{v}]\zeta_x, \mathfrak{J}_{\mathrm{bo}}\Lambda^s\zeta\big) + \varepsilon\big([\Lambda^s, \underline{h}]v_x, \mathfrak{J}_{\mathrm{bo}}\Lambda^s\zeta\big) + \big([\Lambda^s, \underline{\mathfrak{T}}^{-1}(\underline{h}\underline{\mathfrak{J}}_{\mathrm{bo}}\cdot)]\zeta_x, \underline{\mathfrak{T}}\Lambda^s v\big)\\
&\quad + \mu^2\big([\Lambda^s, \underline{\mathfrak{T}}^{-1}(\mathcal{I}[\underline{h}, \beta b]\cdot)]\zeta_x, \underline{\mathfrak{T}}\Lambda^s v\big) - \varepsilon^2\mu^2\frac{1}{\mathrm{bo}}\big([\Lambda^s, \underline{\mathfrak{T}}^{-1}(T[\underline{U}]\cdot)]\zeta_x, \underline{\mathfrak{T}}\Lambda^s v\big)\\
&\quad + \varepsilon\big([\Lambda^s, \underline{v}]v_x, \underline{\mathfrak{T}}\Lambda^s v\big) + \varepsilon\mu\big([\Lambda^s, \underline{\mathfrak{T}}^{-1}(Q_b[\underline{U}]\cdot)]v_x, \underline{\mathfrak{T}}\Lambda^s v\big) + \varepsilon\mu^2\big([\Lambda^s, \underline{\mathfrak{T}}^{-1}(Q_{bb}[\underline{U}]\cdot)]v_x, \underline{\mathfrak{T}}\Lambda^s v\big)\\
&\quad = D_1 + D_2 + ... + D_8.
\end{aligned}$$

To control D_1, D_2 we use the expression of $\underline{\mathfrak{J}}_{\mathrm{bo}}$, the commutator estimate (51), and the fact that

$$\partial_x^2[\Lambda^s, M]N = [\Lambda^s, M_{xx}]N + 2[\Lambda^s, M_x]N_x + [\Lambda^s, M]N_{xx}, \tag{62}$$

then, one can deduce as in [27] that

$$|D_1 + D_2| \leq (\varepsilon \vee \beta) C\big(|\underline{\zeta}|_\infty, |\underline{h} - 1|_{H^s}, \sqrt{\frac{\mu}{\mathrm{bo}}}|\underline{\zeta}_x|_{H^s}, \mu|\underline{\zeta}_{xx}|_{H^s}, |\underline{v}|_{H^s}, \mu|\underline{v}_{xx}|_{H^s}, \sqrt{\mathrm{bo}}\big) E^s(U)^2.$$

To control $D_3 + D_4$, remark that $\underline{\mathfrak{T}}$ is symmetric with

$$\underline{\mathfrak{T}}[\Lambda^s, \underline{\mathfrak{T}}^{-1}]\underline{h}\underline{\mathfrak{J}}_{\mathrm{bo}}\zeta_x = \underline{\mathfrak{T}}[\Lambda^s, \underline{\mathfrak{T}}^{-1}(\underline{h}\underline{\mathfrak{J}}_{\mathrm{bo}}\cdot)]\zeta_x - [\Lambda^s, \underline{h}\underline{\mathfrak{J}}_{\mathrm{bo}}]\zeta_x.$$

Moreover, since $[\Lambda^s, \underline{\mathfrak{T}}^{-1}] = -\underline{\mathfrak{T}}^{-1}[\Lambda^s, \underline{\mathfrak{T}}]\underline{\mathfrak{T}}^{-1}$, one gets

$$\underline{\mathfrak{T}}[\Lambda^s, \underline{\mathfrak{T}}^{-1}\ (\underline{h}\underline{\mathfrak{J}}_{\mathrm{bo}}\cdot)]\zeta_x = -[\Lambda^s, \underline{\mathfrak{T}}]\underline{\mathfrak{T}}^{-1}\underline{h}\underline{\mathfrak{J}}_{\mathrm{bo}}\zeta_x + [\Lambda^s, \underline{h}\underline{\mathfrak{J}}_{\mathrm{bo}}]\zeta_x.$$

By using the explicit expression of $\underline{\mathfrak{T}}$, integration by parts and the fact that

$$[\Lambda^s, \partial_x(M\partial_x\cdot)]N = \partial_x[\Lambda^s, M]N_x, \qquad [\Lambda^s, \partial_x^2(M\partial_x^2\cdot)]N = \partial_x^2[\Lambda^s, M]N_{xx}, \tag{63}$$

one can write

$$\begin{aligned}
D_3 + D_4 = &\big([\Lambda^s, \underline{\mathfrak{T}}]\underline{\mathfrak{T}}^{-1}(\underline{h}\underline{\mathfrak{J}}_{\mathrm{bo}}\zeta_x), \Lambda^s v\big) + \big([\Lambda^s, \underline{h}\underline{\mathfrak{J}}_{\mathrm{bo}}]\zeta_x, \Lambda^s v\big)\\
&+ \mu^2\big([\Lambda^s, \underline{\mathfrak{T}}]\underline{\mathfrak{T}}^{-1}(\mathcal{I}[\underline{h}, \beta b]\zeta_x), \Lambda^s v\big) + \mu^2\big([\Lambda^s, \mathcal{I}[\underline{h}, \beta b]]\zeta_x, \Lambda^s v\big).
\end{aligned}$$

Now, using the expression of $\underline{\mathfrak{J}}_{\rm bo}$ one has

$$\frac{2}{45}\mu^2\underline{h}\partial_x^2(\underline{h}^4\zeta_{xxx}) = 2\underline{\mathfrak{K}}\zeta_x - 2\underline{h}\zeta_x + \frac{\mu}{3}\partial_x(\underline{h}^3\zeta_{xx}) - \mu\beta\big[\partial_x(\underline{h}^2b_x\zeta_x) - \underline{h}^2b_x\zeta_{xx}\big] - 2\mu\beta^2\underline{h}b_x^2\zeta_x - \mu^2\mathcal{I}[\underline{h},\beta b]\zeta_x.$$

Then, we get

$$\underline{\mathfrak{K}}^{-1}(\underline{h}\underline{\mathfrak{J}}_{\rm bo}\zeta_x) + \mu^2\underline{\mathfrak{K}}^{-1}(\mathcal{I}[\underline{h},\beta b]\zeta_x) = 2\zeta_x - \underline{\mathfrak{K}}^{-1}(\underline{h}\zeta_x) - \frac{\mu}{\rm bo}\underline{\mathfrak{K}}^{-1}(\underline{h}\zeta_{xxx}) + \frac{2\mu}{3}\underline{\mathfrak{K}}^{-1}\partial_x(\underline{h}^3\zeta_{xx})$$
$$- \mu\beta\underline{\mathfrak{K}}^{-1}\big(\partial_x(\underline{h}^2b_x\zeta_x) - \underline{h}^2b_x\zeta_{xx} + 2\beta\underline{h}b_x^2\zeta_x\big),$$

which implies that

$$D_3 + D_4 = 2\big([\Lambda^s,\underline{\mathfrak{K}}]\zeta_x,\Lambda^sv\big) - \big([\Lambda^s,\underline{\mathfrak{K}}]\underline{\mathfrak{K}}^{-1}(\underline{h}\zeta_x),\Lambda^sv\big) + \frac{2\mu}{3}\big([\Lambda^s,\underline{\mathfrak{K}}]\underline{\mathfrak{K}}^{-1}\partial_x(\underline{h}^3\zeta_{xx}),\Lambda^sv\big) - \frac{\mu}{\rm bo}\big([\Lambda^s,\underline{\mathfrak{K}}]\underline{\mathfrak{K}}^{-1}(\underline{h}\zeta_{xxx}),\Lambda^sv\big)$$
$$+ \big([\Lambda^s,\underline{h}\underline{\mathfrak{J}}_{\rm bo}]\zeta_x,\Lambda^sv\big) - \mu\beta\big([\Lambda^s,\underline{\mathfrak{K}}]\underline{\mathfrak{K}}^{-1}\big(\partial_x(\underline{h}^2b_x\zeta_x) - \underline{h}^2b_x\zeta_{xx} + 2\beta\underline{h}b_x^2\zeta_x\big),\Lambda^sv\big) + \mu^2\big([\Lambda^s,\mathcal{I}[\underline{h},\beta b]]\zeta_x,\Lambda^sv\big)$$
$$= D_{341} + D_{342} + ... + D_{347}.$$

In order to facilitate our way in controlling $D_3 + D_4$, we may rewrite the expression of $\underline{\mathfrak{K}}$ using (49) under the form

$$\underline{\mathfrak{K}} = \underline{h} - \frac{1}{3}\mu\partial_x(\underline{h}^3\partial_x\cdot) + \frac{\mu\beta}{2}\partial_x(\underline{h}^2b_x(\cdot)) - \frac{\mu\beta}{2}\underline{h}^2b_x\partial_x\cdot + \mu\beta^2\underline{h}b_x^2(\cdot) + \frac{1}{45}\mu^2\partial_x^2\big(\underline{h}^5\partial_x^2\cdot\big) - \frac{\mu^2\beta}{24}\big[2\partial_x^2(\underline{h}^4b_x\partial_x^2\cdot)$$
$$+ \partial_x^2\big(\underline{h}^4b_{xx}(\cdot)\big) - 2\partial_x(\underline{h}^4b_x\partial_x^2\cdot) + \underline{h}^4b_{xx}\partial_x^2\cdot\big] - \frac{\mu^2\beta^2}{12}\big[4\partial_x(\underline{h}^3b_x^2\partial_x\cdot) + \partial_x\big(\underline{h}^3b_xb_{xx}(\cdot)\big) - 2\underline{h}^3b_xb_{xx}\partial_x\cdot - \underline{h}^3b_{xx}^2(\cdot)\big].$$

Now, using the explicit expression of $\underline{\mathfrak{K}}$, the commutator estimate (51), the identities (63) and the help of Lemma 2, we get

$$|D_{341}| \leq (\varepsilon\vee\beta)C\big(|\underline{h}-1|_{H^s},\sqrt{\rm bo}\big)E^s(U)^2, \qquad |D_{342}+D_{343}+D_{344}| \leq (\varepsilon\vee\beta)C\big(|\underline{h}-1|_{H^s},C_s\big)E^s(U)^2$$

For controlling D_{345}, one should use the explicit expression of $\mathfrak{J}_{\rm bo}$ and the fact that

$$[\Lambda^s,M\partial_x^{(\imath)}(N\partial_x^2\cdot)]P = [\Lambda^s,M\partial_x^{(\imath)}]N\partial_x^2P + M\partial_x^{(\imath)}[\Lambda^s,N]\partial_x^2P \qquad \text{with} \qquad \imath = \big\{1,2\big\}, \tag{64}$$

to write D_{345} as follows

$$D_{345} = \big([\Lambda^s,\underline{h}]\zeta_x,\Lambda^sv\big) - \frac{\mu}{\rm bo}\big([\Lambda^s,\underline{h}]\zeta_{xxx},\Lambda^sv\big) + \frac{2}{45}\mu^2\big([\Lambda^s,\underline{h}^5]\zeta_{xxx},\partial_x^2(\underline{h}\Lambda^sv)\big)$$
$$+ \frac{2}{45}\mu^2\big([\Lambda^s,\underline{h}\partial_x^2]\underline{h}^5\zeta_{xxx},\Lambda^sv\big) = D_{3451} + D_{3452} + D_{3453} + D_{3454},$$

where,

$$|D_{3451}+D_{3452}+D_{3453}| \leq (\varepsilon\vee\beta)C\big(|\underline{h}-1|_{H^s},|\underline{\zeta}|_\infty,\sqrt{\frac{\mu}{\rm bo}}|\underline{\zeta}_x|_{H^s},\sqrt{\rm bo}\big)E^s(U)^2.$$

For controlling D_{3454}, one should notice the fact that

$$[\Lambda^s,M\partial_x^2]N = \partial_x^2[\Lambda^s,M]N - 2\partial_x[\Lambda^s,\partial_xM]N + [\Lambda^s,\partial_x^2M]N = [\Lambda^s,M]\partial_x^2N, \tag{65}$$

then, we get

$$|D_{3454}| \leq (\varepsilon\vee\beta)C\big(|\underline{h}-1|_{H^s},\sqrt{\frac{\mu}{\rm bo}}|\underline{\zeta}_x|_{H^s},\mu|\underline{\zeta}_{xxx}|_{H^{s-1}},\sqrt{\rm bo}\big)E^s(U)^2.$$

Therefore, we obtain

$$|D_{345}| \leq (\varepsilon\vee\beta)C\big(|\underline{h}-1|_{H^s},|\underline{\zeta}|_\infty,\sqrt{\frac{\mu}{\rm bo}}|\underline{\zeta}_x|_{H^s},\mu|\underline{\zeta}_{xxx}|_{H^{s-1}},C_s,\sqrt{\rm bo}\big)E^s(U)^2.$$

For controlling D_{346}, by integration by parts and using the explicit expression of $\underline{\mathfrak{K}}$, the commutator estimate (51), the identities (63) and the help of Lemma 2, we get

$$|D_{346}| \leq \beta C\big(|\underline{h}-1|_{H^s},C_s\big)E^s(U)^2.$$

For controlling D_{347}, one should notice that the non-topographical terms are bounded from above we use integration by parts and the commutator (51) with the help of (64), (65) and noticing the fact that

$$[\Lambda^s,M\partial_x]N = \partial_x[\Lambda^s,M]N - [\Lambda^s,\partial_xM]N = [\Lambda^s,M]\partial_xN, \tag{66}$$

so one can see [27] to control the non-topographical terms by $(\varepsilon\vee\beta)C\big(|\underline{h}-1|_{H^s},\sqrt{\frac{\mu}{\mathrm{bo}}}|\underline{\zeta}_x|_{H^s},\mu|\underline{\zeta}_{xx}|_{H^s},\sqrt{\mathrm{bo}}\big)E^s(U)^2$. While the purely-topographical terms in D_{347} can be written after noticing that

$$[\Lambda^s,\partial_x(M\cdot)]N=\partial_x[\Lambda^s,M]N,\qquad\text{and}\qquad[\Lambda^s,\partial_x^2(M\cdot)]N=\partial_x^2[\Lambda^s,M]N, \tag{67}$$

then, by integartion by parts and (51) we have

$$\begin{aligned}\mu^2\beta\Big|&-\frac{1}{6}\big([\Lambda^s,\underline{h}^4b_x]\zeta_{xxx},\Lambda^sv_{xx}\big)-\frac{1}{12}\big([\Lambda^s,\underline{h}^4b_{xx}]\zeta_x,\Lambda^sv_{xx}\big)-\frac{1}{6}\big([\Lambda^s,\underline{h}^4b_x]\zeta_{xxx},\Lambda^sv_x\big)-\frac{1}{12}\big([\Lambda^s,\underline{h}^4b_{xx}]\zeta_{xxx},\Lambda^sv\big)\\
&-\frac{2}{3}\beta\big([\Lambda^s,\underline{h}^3b_x^2]\zeta_{xx},\Lambda^sv_x\big)+\frac{1}{6}\beta\big([\Lambda^s,\underline{h}^3b_xb_{xx}]\zeta_x,\Lambda^sv_x\big)+\frac{1}{3}\beta\big([\Lambda^s,\underline{h}^3b_xb_{xx}]\zeta_{xx},\Lambda^sv\big)+\frac{1}{6}\beta\big([\Lambda^s,\underline{h}^3b_{xx}^2]\zeta_x,\Lambda^sv\big)\Big|\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\leq(\varepsilon\vee\beta)C\big(|\underline{h}-1|_{H^s}\big)E^s(U)^2.\end{aligned}$$

Thus, after collecting the information above, we obtain

$$|D_3+D_4|\leq(\varepsilon\vee\beta)C\big(|\underline{h}-1|_{H^s},|\underline{\zeta}|_\infty,\sqrt{\frac{\mu}{\mathrm{bo}}}|\underline{\zeta}_x|_{H^s},\mu|\underline{\zeta}_{xxx}|_{H^{s-1}},C_s,\sqrt{\mathrm{bo}}\big)E^s(U)^2.$$

To control D_5, we first must notice the identities (67) and

$$[\Lambda^s,M\partial_x^{(\imath)}(N\cdot)]P=[\Lambda^s,M\partial_x^{(\imath)}]NP+M\partial_x^{(\imath)}[\Lambda^s,N]P=[\Lambda^s,M]\partial_x^{(\imath)}(NP)+M\partial_x^{(\imath)}[\Lambda^s,N]P\quad\text{where}\quad\imath=\big\{1,2\big\}, \tag{68}$$

so one may write as in D_3 using (63) with integration by parts, Lemma 2, with the help of the Kato-Pance commutator (51) we get

$$|D_5|\leq(\varepsilon\vee\beta)C\big(|\underline{h}-1|_{H^{s-1}},|\underline{\zeta}_x|_{H^{s-1}},\sqrt{\frac{\mu}{\mathrm{bo}}}|\underline{\zeta}_x|_{H^s},\frac{\mu}{\mathrm{bo}}|\underline{\zeta}_x|_{H^s}^2,\mu|\underline{\zeta}_{xx}|_{H^s},C_s\big)E^s(U)^2.$$

To control D_6, one can write after checking the expression of $\underline{\mathfrak{X}}$ and using (62) with integration by parts and the fact that $\partial_x[\Lambda^s,M]N=[\Lambda^s,M_x]N+[\Lambda^s,M]N_x$

$$\begin{aligned}|D_6|=\varepsilon\Big|&\big([\Lambda^s,\underline{v}]v_x,\underline{h}\Lambda^sv\big)+\frac{1}{3}\mu\big([\Lambda^s,\underline{v}_x]v_x,\underline{h}^3\Lambda^sv_x\big)+\frac{1}{3}\mu\big([\Lambda^s,\underline{v}]v_{xx},\underline{h}^3\Lambda^sv_x\big)-\frac{1}{2}\mu\beta\big([\Lambda^s,\underline{v}_x]v_x,\underline{h}^2b_x\Lambda^sv\big)\\
-\frac{1}{2}&\mu\beta\big([\Lambda^s,\underline{v}]v_{xx},\underline{h}^2b_x\Lambda^sv\big)+\mu\beta^2\big([\Lambda^s,\underline{v}]v_x,\underline{h}b_x^2\Lambda^sv\big)+\frac{1}{45}\mu^2\big([\Lambda^s,\underline{v}_{xx}]v_x,\underline{h}^5\Lambda^sv_{xx}\big)+\frac{2}{45}\mu^2\big([\Lambda^s,\underline{v}_x]v_{xx},\underline{h}^5\Lambda^sv_{xx}\big)\\
&+\frac{1}{45}\mu^2\big([\Lambda^s,\underline{v}]v_{xxx},\underline{h}^5\Lambda^sv_{xx}\big)-\frac{1}{12}\mu^2\beta\big([\Lambda^s,\underline{v}_{xx}]v_x,\underline{h}^4b_x\Lambda^sv_{xx}\big)-\frac{1}{12}\mu^2\beta\big([\Lambda^s,\underline{v}]v_{xxx},\underline{h}^4b_x\Lambda^sv_{xx}\big)\\
&-\frac{1}{6}\mu^2\beta\big([\Lambda^s,\underline{v}_x]v_{xx},\underline{h}^4b_x\Lambda^sv_{xx}\big)-\frac{1}{24}\mu^2\beta\big([\Lambda^s,\underline{v}_{xx}]v_x,\underline{h}^4b_{xx}\Lambda^sv\big)-\frac{1}{24}\mu^2\beta\big([\Lambda^s,\underline{v}]v_{xxx},\underline{h}^4b_{xx}\Lambda^sv\big)\\
&-\frac{1}{12}\mu^2\beta\big([\Lambda^s,\underline{v}_x]v_{xx},\underline{h}^4b_{xx}\Lambda^sv\big)-\frac{1}{12}\mu^2\beta\big([\Lambda^s,\underline{v}_x]v_x,\underline{h}^4b_x\Lambda^sv_{xx}\big)-\frac{1}{12}\mu^2\beta\big([\Lambda^s,\underline{v}]v_{xx},\underline{h}^4b_x\Lambda^sv_{xx}\big)\\
&-\frac{1}{24}\mu^2\beta\big([\Lambda^s,\underline{v}]v_x,\underline{h}^4b_{xx}\Lambda^sv_{xx}\big)+\frac{2}{3}\mu^2\beta^2\big([\Lambda^s,\underline{v}_x]v_x,\underline{h}^3b_x^2\Lambda^sv_x\big)+\frac{2}{3}\mu^2\beta^2\big([\Lambda^s,\underline{v}]v_{xx},\underline{h}^3b_x^2\Lambda^sv_x\big)\\
+\frac{1}{12}&\mu^2\beta^2\big([\Lambda^s,\underline{v}_x]v_x,\underline{h}^3b_xb_{xx}\Lambda^sv\big)+\frac{1}{12}\mu^2\beta^2\big([\Lambda^s,\underline{v}]v_{xx},\underline{h}^3b_xb_{xx}\Lambda^sv\big)+\frac{1}{6}\mu^2\beta^2\big([\Lambda^s,\underline{v}]v_x,\underline{h}^3b_xb_{xx}\Lambda^sv\big)\\
&+\frac{1}{12}\mu^2\beta^2\big([\Lambda^s,\underline{v}]v_x,\underline{h}^3b_{xx}^2\Lambda^sv\big)\Big|\leq(\varepsilon\vee\beta)C\big(|\underline{\zeta}|_\infty,|\underline{v}|_{H^s},\sqrt{\mu}|\underline{v}_{xx}|_{H^{s-1}},\mu|\underline{v}_{xxx}|_{H^{s-1}}\big)E^s(U)^2.\end{aligned}$$

To control D_7, one can realize D_3 and integrate by parts with the help of (63) to write

$$\begin{aligned}D_7=-\varepsilon\mu\big([\Lambda^s,\underline{\mathfrak{X}}]\underline{\mathfrak{X}}^{-1}\mathcal{Q}_1[\underline{U}]v_x,\Lambda^sv\big)+\varepsilon\mu\big([\Lambda^s,\mathcal{Q}_1[\underline{U}]]v_x,\Lambda^sv\big)\\
-\varepsilon\mu\beta\big([\Lambda^s,\underline{\mathfrak{X}}]\underline{\mathfrak{X}}^{-1}\mathcal{B}_1[\underline{U}]v_x,\Lambda^sv\big)+\varepsilon\mu\beta\big([\Lambda^s,\mathcal{B}_1[\underline{U}]]v_x,\Lambda^sv\big).\end{aligned}$$

Now, as above using the expressions of $\underline{\mathfrak{X}}$, $\mathcal{Q}_1[\underline{U}]$, $\mathcal{B}_1[\underline{U}]$ with the help of Lemma 2, the commutator (51), in addition to (63) and (67) one gets

$$|D_7|\leq(\varepsilon\vee\beta)C\big(|\underline{h}-1|_{H^s},|\underline{v}_x|_{H^{s-1}},\sqrt{\mu}|\underline{v}_{xx}|_{H^{s-1}},C_s\big)E^s(U)^2.$$

To control D_8, one can write by integration by parts

$$\begin{aligned}D_8 &= -\varepsilon\mu^2\big([\Lambda^s,\underline{\mathfrak{K}}]\underline{\mathfrak{K}}^{-1}\mathcal{Q}_2[\underline{U}]v_x,\Lambda^s v\big)+\varepsilon\mu^2\big([\Lambda^s,\mathcal{Q}_2[\underline{U}]]v_x,\Lambda^s v\big)-\varepsilon\mu^2\big([\Lambda^s,\underline{\mathfrak{K}}]\underline{\mathfrak{K}}^{-1}\mathcal{Q}_3[\underline{U}]v_x,\Lambda^s v\big)\\&\quad+\varepsilon\mu^2\big([\Lambda^s,\mathcal{Q}_3[\underline{U}]]v_x,\Lambda^s v\big)-\varepsilon\mu^2\beta\big([\Lambda^s,\underline{\mathfrak{K}}]\underline{\mathfrak{K}}^{-1}\mathcal{B}_2[\underline{U}]v_x,\Lambda^s v\big)+\varepsilon\mu^2\beta\big([\Lambda^s,\mathcal{B}_2[\underline{U}]]v_x,\Lambda^s v\big)\\&\quad-\varepsilon\mu^2\beta^2\big([\Lambda^s,\underline{\mathfrak{K}}]\underline{\mathfrak{K}}^{-1}\mathcal{B}_3[\underline{U}]v_x,\Lambda^s v\big)+\varepsilon\mu^2\beta^2\big([\Lambda^s,\mathcal{B}_3[\underline{U}]]v_x,\Lambda^s v\big).\end{aligned}$$

Now, as above using the expressions of $\underline{\mathfrak{K}}$, $\mathcal{Q}_2[\underline{U}]$, $\mathcal{Q}_3[\underline{U}]$, $\mathcal{B}_2[\underline{U}]$, $\mathcal{B}_3[\underline{U}]$ with the help of Lemma 2, the commutator (51), in addition to (63)-(67) one gets

$$|D_8|\leq(\varepsilon\vee\beta)C\big(|\underline{h}-1|_{H^s},|\underline{v}_x|_{H^{s-1}},\sqrt{\mu}|\underline{v}_{xx}|_{H^{s-1}},C_s\big)E^s(U)^2.$$

Eventually, as a conclusion, one gets

$$\begin{aligned}\big|\big([\Lambda^s,A[\underline{U}]]\partial_xU,S\Lambda^sU\big)\big|\leq\varepsilon C\big(|\underline{h}-1|_{H^s},|\underline{\zeta}|_{H^s},\sqrt{\frac{\mu}{\mathrm{bo}}}|\underline{\zeta}_x|_{H^s},\frac{\mu}{\mathrm{bo}}|\underline{\zeta}_x|^2_{H^s},\\ \mu|\underline{\zeta}_{xx}|_{H^s},|\underline{v}|_{H^s},\sqrt{\mu}|\underline{v}_x|_{H^s},\mu|\underline{v}_{xx}|_{H^s},C_s,\mathrm{bo}\big)E^s(U)^2.\end{aligned}$$

• Estimation of $\big(\Lambda^sB(\underline{U}),S\Lambda^sU\big)$. Recall that

$$B(\underline{U})=\begin{pmatrix}-\beta b_x\underline{v}\\ \underline{\mathfrak{K}}^{-1}\big(\mathcal{R}[\mu,\varepsilon\underline{h},\beta b](\underline{U})\big)\end{pmatrix},\tag{69}$$

where

$$\begin{aligned}\underline{\mathfrak{K}}&=\underline{h}-\frac{1}{3}\mu\partial_x(\underline{h}^3\partial_x\cdot)+\frac{\mu\beta}{2}\partial_x(\underline{h}^2b_x(\cdot))-\frac{\mu\beta}{2}\underline{h}^2b_x\partial_x\cdot+\mu\beta^2\underline{h}b_x^2(\cdot)+\frac{1}{45}\mu^2\partial_x^2\big(\underline{h}^5\partial_x^2\cdot\big)-\frac{\mu^2\beta}{24}\big[2\partial_x^2(\underline{h}^4b_x\partial_x^2\cdot)\\&+\partial_x^2\big(\underline{h}^4b_{xx}(\cdot)\big)-2\partial_x(\underline{h}^4b_x\partial_x^2\cdot)+\underline{h}^4b_{xx}\partial_x^2\cdot\big]-\frac{\mu^2\beta^2}{12}\big[4\partial_x(\underline{h}^3b_x^2\partial_x\cdot)+\partial_x\big(\underline{h}^3b_xb_{xx}(\cdot)\big)-2\underline{h}^3b_xb_{xx}\partial_x\cdot-\underline{h}^3b_{xx}^2(\cdot)\big],\end{aligned}$$

and

$$\begin{aligned}\mathcal{R}[\mu,\varepsilon\underline{h},\beta b](\underline{U})&=\frac{1}{2}\varepsilon\mu\beta\partial_x(\underline{h}^2b_{xx}v^2)+\varepsilon\mu\beta^2\underline{h}b_xb_{xx}\underline{v}^2+\frac{1}{24}\varepsilon\mu^2\beta\partial_x^2(\underline{h}^4b_{xxx}\underline{v}^2)\\&\quad+\frac{1}{12}\varepsilon\mu^2\beta^2\partial_x(\underline{h}^3b_xb_{xxx}\underline{v}^2)+\frac{1}{4}\varepsilon\mu^2\beta^2\partial_x(\underline{h}^3b_{xx}^2\underline{v}^2).\end{aligned}$$

As in D_3 one may write

$$\big(\Lambda^sB(\underline{U}),S\Lambda^sU\big)=-\beta\big(\Lambda^s(b_x\underline{v}),\underline{\mathfrak{J}}_{\mathrm{bo}}\Lambda^s\zeta\big)+\big(\Lambda^s\mathcal{R}[\mu,\varepsilon\underline{h},\beta b](\underline{U}),\Lambda^sv\big)-\big([\Lambda^s,\underline{\mathfrak{K}}]\underline{\mathfrak{K}}^{-1}\mathcal{R}[\mu,\varepsilon\underline{h},\beta b](\underline{U}),\Lambda^sv\big).$$

Now, using the expressions of $\underline{\mathfrak{K}}$, $\mathcal{R}[\mu,\varepsilon\underline{h},\beta b](\underline{U})$, $\underline{\mathfrak{J}}_{\mathrm{bo}}$ with the help of Lemma 2, the commutator estimate (51), in addition to (63) and (67) one gets

$$\big|\big(\Lambda^sB(\underline{U}),S\Lambda^sU\big)\big|\leq(\varepsilon\vee\beta)C\big(E^s(\underline{U})\big)E^s(U).$$

• Estimation of $\big(\Lambda^s\zeta,[\partial_t,\underline{\mathfrak{J}}_{\mathrm{bo}}]\Lambda^s\zeta\big)$. One can write after checking the expression of $\underline{\mathfrak{J}}_{\mathrm{bo}}$ and doing an integration by parts, that

$$\big|\big(\Lambda^s\zeta,[\partial_t,\underline{\mathfrak{J}}_{\mathrm{bo}}]\Lambda^s\zeta\big)\big|=\big|\frac{2}{45}\mu^2\big(\partial_t\underline{h}^4\Lambda^s\zeta_{xx},\Lambda^s\zeta_{xx}\big)\big|\leq(\varepsilon\vee\beta)C(|\partial_t\underline{h}|_\infty)E^s(U)^2.$$

• Estimation of $\big(\Lambda^sv,[\partial_t,\underline{\mathfrak{K}}]\Lambda^sv\big)$. Remarking that we have

$$[\partial_t,\underline{h}]\Lambda^sv=\partial_t\underline{h}\Lambda^sv\qquad[\partial_t,\partial_x(\underline{h}^3\partial_x\cdot)]\Lambda^sv=\partial_x(\partial_t\underline{h}^3\Lambda^sv_x)\qquad\text{and}\qquad[\partial_t,\partial_x^2(\underline{h}^5\partial_x^2\cdot)]\Lambda^sv=\partial_x^2(\partial_t\underline{h}^5\Lambda^sv_{xx}),$$

so one gets using integration by parts

$$\begin{aligned}\big|\big(\Lambda^sv,[\partial_t,\underline{\mathfrak{K}}]\Lambda^sv\big)\big|&=\big|\big(\partial_t\underline{h}\Lambda^sv,\Lambda^sv\big)+\frac{\mu}{3}\big(\partial_t\underline{h}^3\Lambda^sv_x,\Lambda^sv_x\big)\\&\quad+\frac{\mu^2}{45}\big(\partial_t\underline{h}^5\Lambda^sv_{xx},\Lambda^sv_{xx}\big)\big|\leq(\varepsilon\vee\beta)C(|\partial_t\underline{h}|_\infty,E^s(\underline{U}))E^s(U)^2.\end{aligned}$$

Gathering the information provided by the above estimates and using the fact that $H^s(\mathbb{R})$ is continuously embedded in $W^{1,\infty}(\mathbb{R})$, we get

$$\frac{1}{2}e^{(\varepsilon\vee\beta)\lambda t}\partial_t((\varepsilon\vee\beta)^{-\varepsilon\lambda t}E^s(U)^2)\leq(\varepsilon\vee\beta)\big(C(E^s(\underline{U}),|\partial_t\underline{h}|_{L^\infty})-\lambda\big)E^s(U)^2+(\varepsilon\vee\beta)C\big(E^s(\underline{U})\big)E^s(U).$$

Taking $\lambda = \lambda_T$ large enough (how large depending on $\sup_{t\in[0,\frac{T}{\varepsilon\vee\beta}]} C(E^s(\underline{U}), |\partial_t \underline{h}|_{L^\infty})$ to have the first term of the right hand side negative for all $t \in [0, \frac{T}{\varepsilon\vee\beta}]$, one deduces

$$\forall\, t \in \big[0, \frac{T}{\varepsilon \vee \beta}\big]\,, \qquad \frac{1}{2} e^{(\varepsilon\vee\beta)\lambda t}\partial_t\big(e^{-(\varepsilon\vee\beta)\lambda t}E^s(U)^2\big) \leq (\varepsilon\vee\beta)C\big(E^s(\underline{U})\big)E^s(U).$$

Integrating this differential inequality with the help of Grönwall's inequality yields therefore

$$\forall\, t \in \big[0, \frac{T}{\varepsilon \vee \beta}\big]\,, \qquad E^s\big(U(t)\big) \leq e^{\frac{(\varepsilon\vee\beta)}{2}\lambda_T t}E^s(U_0) + (\varepsilon\vee\beta)\int_0^t e^{\frac{(\varepsilon\vee\beta)}{2}\lambda_T(t-t')}C(E^s(\underline{U})(t'))dt',$$

which is the desired result. The existence and uniqueness of the solution (one can see [12] for general details) is a direct adaptation of the proof in Appendix A of [4] . □

3.3. **Main result.** The following theorem proves the main result in this paper, which is the well-posedness in $X^s = H^{s+2}(\mathbb{R}) \times H^{s+2}(\mathbb{R})$ for some $s > \frac{3}{2}$ of the extended ($\beta \neq 0$) system (37) with surface tension for a finite large time existence $t = O(\frac{1}{\varepsilon\vee\beta})$, realizing that if some littleness supposition is made on $\varepsilon \vee \beta$, at that point the presence time $t = O(\frac{1}{\varepsilon\vee\beta})$ ends up bigger.

Theorem 1. *Let $t_0 > \frac{1}{2}$, $s \geq t_0 + 1$. Let also $U_0 = (\zeta_0, v_0)^T \in X^s$ be such that (45) is satisfied. Then there exists a maximal $T_{max} > 0$, uniformly bounded from below with respect to $\varepsilon, \mu \in (0,1)$, such that the extended one-dimentional Green-Naghdi equations (37) with surface tension admits a unique solution $U = (\zeta, v)^T \in X^s_{T_{max}}$ with the initial condition $(\zeta_0, v_0)^T$ and preserving the nonvanishing depth condition (45) for any $t \in [0, \frac{T_{max}}{\varepsilon\vee\beta})$. In particular if $T_{max} < \infty$ one has*

$$|U(t,\cdot)|_{X^s} \longrightarrow \infty \quad as \quad t \longrightarrow \frac{T_{max}}{\varepsilon\vee\beta},$$

or

$$\inf_{\mathbb{R}} h(t,\cdot) = \inf_{\mathbb{R}} 1 + \varepsilon\zeta(t,\cdot) - \beta b(\cdot) \longrightarrow 0 \quad as \quad t \longrightarrow \frac{T_{max}}{\varepsilon\vee\beta}.$$

Proof. The proof is a straightforward readjustment of the proof of Theorem 7.3 in [32] or Theorem 1 in [4, 27] using the energy estimate proved in Proposition 1, which is itself an adaptation of the proof of the well-posedness of hyperbolic systems (see [26, 12] for general details), where no smallness assumption on the parameters ε, μ, β is required in the theorem. □

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