Semi-Group Theory for the Stokes Operator with Navier-Type Boundary Conditions on L p -Spaces

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Abstract

In this article we consider the Stokes problem with Navier-type boundary conditions on a domain Ω, not necessarily simply connected. Since under these conditions the Stokes problem has a non trivial kernel, we also study the solutions lying in the orthogonal of that kernel. We prove the analyticity of several semigroups generated by the Stokes operator considered in different functional spaces. We obtain strong, weak and very weak solutions for the time dependent Stokes problem with the Navier-type boundary condition under different hypothesis on the initial data $\boldsymbol{u}_0$ and external force $\boldsymbol{f}$. Then, we study the fractional and pure imaginary powers of several operators related with our Stokes operators. Using the fractional powers, we prove maximal regularity results for the homogeneous Stokes problem. On the other hand, using the boundedness of the pure imaginary powers we deduce maximal $L^p - L^q$ regularity for the inhomoge neous Stokes problem.

Contents

Key words and phrases: Stokes operator, Navier boundary conditions, Analytical semi-group, fractional powers

1 Introduction

We consider in a bounded cylindrical domain, $\Omega \times (0,T)$ the linearised evolution Navier-Stokes problem

$$\begin{cases} \frac{\partial \boldsymbol{u}}{\partial t} - \Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f}, \quad \mathrm{div}\, \boldsymbol{u} = 0 & \text{in} \quad \Omega \times (0,T), \\ \boldsymbol{u}(0) = \boldsymbol{u}_0 & \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^3$, not necessarily simply connected, whose boundary Γ is of class $C^{2,1}$. Problem (1.1) describes the motion of a viscous incompressible fluid in Ω. The velocity of motion is denoted by $\boldsymbol{u}$ and the associated pressure by π. Given data are the external force $\boldsymbol{f}$ and the initial velocity $\boldsymbol{u}_0$.

Stokes and Navier-Stokes equations are often studied with Dirichlet boundary conditions

$$\boldsymbol{u} = \boldsymbol{0} \qquad \text{on } \Gamma,$$

when the boundary Γ represents a fixed wall. This condition was formulated by G. Stokes [71] in 1845, but as stated in [62] this condition is not always realistic since it doesn't reflect necessarily the behaviour of the fluid on or near the boundary.

Even before, H. Navier [56] suggested in 1827 alternative boundary conditions more precisely a type of slip boundary conditions with friction on the wall based on a proportionality between the tangential components of the normal dynamic tensor and the velocity

$$\boldsymbol{u}\cdot\boldsymbol{n}=0,\qquad 2\,\nu\,[\mathbb{D}\boldsymbol{u}\cdot\boldsymbol{n}]_{\boldsymbol{\tau}}+\alpha\,\boldsymbol{u}_{\boldsymbol{\tau}}=0\qquad\text{on }\ \Gamma\times(0,T), \tag{1.2}$$

where ν is the viscosity and $\alpha\geq 0$ is the coefficient of friction and $\mathbb{D}\boldsymbol{u}=\frac{1}{2}(\nabla\boldsymbol{u}+\nabla\boldsymbol{u}^T)$ denotes the deformation tensor associated to the velocity field $\boldsymbol{u}$. These Navier boundary conditions allows the fluid to slip and measure the friction on the wall. Observe that, formally, if α tends to infinity, the tangential component of the velocity will vanish and we recover the non slip boundary condition $\boldsymbol{u}=\boldsymbol{0}$ on Γ.

An interesting particular arises when the coefficient of friction α is zero. This corresponds to a Navier-slip boundary condition without friction. This condition has been considered in particular in the mathematical literature on flows near rough walls [6, 20, 21, 22, 45, 46]. We also mention that in the case of flat boundary and when $\alpha=0$ the second condition in (1.2) can be replaced by another boundary condition involving the vorticity

$$\boldsymbol{u}\cdot\boldsymbol{n}=0,\qquad \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n}=\boldsymbol{0}\ \text{ on }\qquad\Gamma\times(0,T). \tag{1.3}$$

We call them Navier-type boundary conditions. For a discussion on the No-Slip boundary condition in the physics literature we refer to [51] and the references therein.

The relation between Navier conditions on rough boundary and the Dirichlet boundary condition boundary is studied by Casado in [25, 26].

In this paper we study the Stokes operator with the Navier-type boundary conditions (1.3). Our goal is to obtain a semi-group theory for the Stokes operator with Navier-type boundary conditions as it already exists for other boundary conditions like Dirichlet and Robin. For instance K. Abe & Y. Giga [1], W. Borchers & T. Miyakawa [18, 19], R. Farwig & H. Sohr [35], Y. Giga [39, 40], Y. Giga & H. Sohr [42, 43], J. Saal [59], Y. Shibata & R. Shimada [63], V. A. Solonnikov [68, 69, 70]).

In what follows, if we do not state otherwise, Ω will be considered as an open bounded domain of $\mathbb{R}^3$ of class $C^{2,1}$. In some situation we suppose that Ω is of class $C^{1,1}$ in the case where the regularity is sufficient for the proof. Then a unit normal vector to the boundary can be defined almost everywhere it will be denoted by $\boldsymbol{n}$. The generic point in Ω is denoted by $\boldsymbol{x}=(x_1,\,x_2,\,x_3)$.

We do not assume that Ω is simply-connected neither that its boundary Γ is connected but we suppose that they satisfy the following condition (see [9] for instance): **Condition H**: there exist J connected open surfaces Σ_j, $1\leq j\leq J$, called "cuts", contained in Ω, such that each surface Σ_j is an open subset of a smooth manifold, the boundary of Σ_j is contained in Γ. The intersection $\overline{\Sigma}_i\cap\overline{\Sigma}_j$ is empty for $i\neq j$ and finally the open set $\Omega^\circ=\Omega\backslash\cup_{j=1}^J\Sigma_j$ is simply connected and pseudo-$C^{1,1}$. We denote by Γ_i, $0\leq i\leq I$, the connected component of Γ, Γ_0 being the boundary of the only unbounded connected component of $\mathbb{R}^3\backslash\overline{\Omega}$. We also fix a smooth open set ϑ with a connected boundary (a ball, for instance), such that $\overline{\Omega}$ is contained in ϑ, and we denote by Ω_i, $0\leq i\leq I$, the connected component of $\vartheta\backslash\overline{\Omega}$ with boundary Γ_i ($\Gamma_0\cup\partial\vartheta$ for $i=0$), (see figure above).

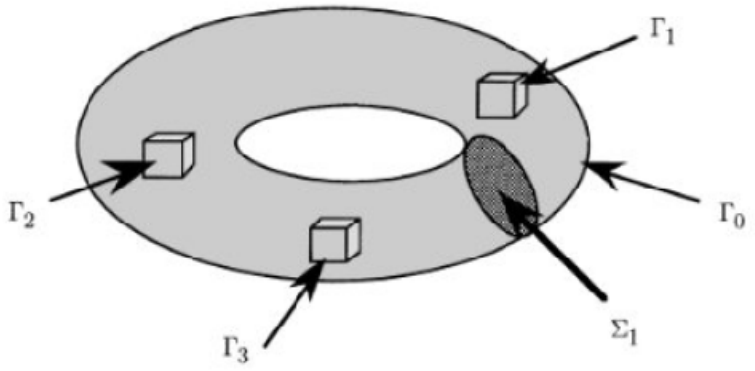

Figure 1: The domain Ω

We denote by $[\cdot]_j$ the jump of a function over Σ_j, *i.e.* the difference of the traces for $1 \leq j \leq J$. In all this article $L^p(\Omega)$ and $W^{s,p}(\Omega)$ denote the usual Lebesgue and Sobolev spaces for $s \in \mathbb{R}$ and $p \geq 1$ (cf. [2], [24]). We denote $\mathcal{D}(\Omega)$ the space of functions indefinitely differentiable and with compact support in Ω and by $\mathcal{D}'(\Omega)$ its dual space. We recall that $W^{-1,p}(\Omega)$ is the dual space of $W_0^{1,p'}(\Omega)$ for $p \in [1, \infty)$ (cf. [2], [24]). For any open connected surface Σ contained in $\overline{\Omega}$ the space $W^{s,p}(\Sigma)$, for $s \in (0,1)$ and $p > 1$ is the Sobolev space:

$$W^{s,p}(\Sigma) = \left\{ u \in L^p(\Sigma);\ \int_\Sigma \int_\Sigma \frac{|u(x) - u(y)|^p}{|x-y|^{2+sp}} dx\, dy < \infty \right\}$$

equipped with the natural norm.

1.1 Stokes problem with flux.

When Ω is not simply-connected, the Stokes operator with boundary condition (1.3) has a non trivial kernel $\boldsymbol{K}_\tau(\Omega)$ contained in all the $\boldsymbol{L}^r$ spaces for $r \in (1, \infty)$. This kernel is independent of r, it has been proved to be of finite dimension $J \geq 1$ (cf. [9], for $p = 2$ and [13, Corollary 4.1] for $p \in (1, \infty)$) and it is spanned by the function $\widetilde{\mathbf{grad}}\, q_j^\tau$, $1 \leq j \leq J$, where q_j^τ is the unique solution up to an additive constant to Problem (3.5) below. Note that, for any function q in $W^{1,p}(\Omega^\circ)$, $\mathbf{grad}\, q$ is the gradient of q in the sense of distribution in $\boldsymbol{\mathcal{D}}'(\Omega^\circ)$, it belongs to $\boldsymbol{L}^p(\Omega^\circ)$ and therefore can be extended to $\boldsymbol{L}^p(\Omega)$. In order to distinguish this extension from the gradient of q in $\boldsymbol{\mathcal{D}}'(\Omega^\circ)$ we denote it by $\widetilde{\mathbf{grad}}\, q$. On the other hand, it was proved in [13], see also [4], that when Ω satisfies Condition H, then, for any function $\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$, divergent free and such that $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on Γ, to satisfy

$$\forall\ \boldsymbol{v} \in \boldsymbol{K}_\tau(\Omega), \quad \int_\Omega \boldsymbol{u} \cdot \overline{\boldsymbol{v}}\, \mathrm{d}\, x = 0, \tag{1.4}$$

is equivalent to the condition

$$\langle \boldsymbol{u}(t) \cdot \boldsymbol{n}\,,\, 1 \rangle_{\Sigma_j} \,=\, 0, \quad 1 \le j \le J, \quad 0 \le t \le \infty, \tag{1.5}$$

with $\langle \cdot\,.\,. \cdot \rangle_{\Sigma_j}$ the duality product between the Sobolev space $\boldsymbol{W}^{\frac{1}{p},p'}(\Sigma_j)$ and its dual space $\boldsymbol{W}^{-\frac{1}{p},p}(\Sigma_j)$ where p' is the conjugate of p: $\frac{1}{p} + \frac{1}{p'} = 1$.

We will refer to the problem (1.1), (1.3), (1.5) as Stokes problem with flux condition. By the equivalence mentioned above, the addition of the extra boundary condition (1.5) makes the Stokes operator invertible with bounded and compact inverse on the space of L^p functions that are divergent free and satisfy $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on Γ.

1.2 Three types of solutions: strong, weak and very weak.

All along this paper we are interested in three different types of solutions for each of the two problems (1.1), (1.3) and (1.1), (1.3), (1.5) defined above. The first, that we call strong solutions, are solutions $\boldsymbol{u}$ that belong to $L^p(0,T,\boldsymbol{L}^q(\Omega))$ type spaces. The second, called weak solutions, are solutions (in a suitable sense) $\boldsymbol{u}(t)$ that may be writen for a.e. $t > 0$, as $\boldsymbol{u}(t) = \boldsymbol{v}(t) + \nabla w(t)$ where $\boldsymbol{v}(t) \in L^p(0,T;\boldsymbol{L}^q(\Omega)))$ and $w \in L^p(0,T;L^q(\Omega))$. The third and last, called very weak, are solutions $\boldsymbol{u}(t)$ that may be decomposed as before but where now $w \in L^p(0,T;W^{-1,\,q}(\Omega))$.

The concept of very weak solutions was introduced by Lions and Magenes in [52]. Later on, Amann considered this type of solutions in a series of articles [7, 8] in the setting of Besov spaces. More recently this concept was modified by R. Farwig, G.P. Galdi and H. Sohr in [31, 32, 33], R. Farwig and H. Kozono in [34], R. Farwig and H. Sohr in [36] and G.P. Galdi and CHR. Simader in [37] to a setting in classical L^p-spaces. This concept has also been generalized by K. Schumacher [60] to a setting in a weighted Lebesgue and Bessel potential spaces using arbitrary Muckenhoupt weights. The concept of very weak solutions is strongly based on duality arguments for strong solutions. Therefore the boundary regularity required in this theory is the same as for strong solutions.

1.3 Analytic semigroups.

In that general setting, we study first the existence of analytic semigroups generated by the Stokes operators, defined on different functional spaces both for the problem (1.1), (1.3) and for (1.1), (1.3), (1.5).

On the one hand, we consider Stokes operators defined on the three different spaces $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ (cf. Subsection 3.1), $[\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)]'_{\sigma,\tau}$ and $[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}$ (cf. Subsection 2.1 for precise definitions of these spaces). They lead respectively to some strong, weak and very weak solutions of (1.1), (1.3). Similarly, we consider three Stokes operators with flux, defined respectively on $\boldsymbol{X}_p$, $\boldsymbol{Y}_p$ and $\boldsymbol{Z}_p$ (cf. (5.2), (5.8), (5.10)) in Section 5), that lead to several solutions of (1.1), (1.3), (1.5).

In the first main result of this work, we prove that each of these six operators generates an analytic semigroup on the corresponding functional space. More precisely:

Theorem 1.1.
(i) The Stokes operators with Navier-type boundary conditions, A_p, B_p and C_p, generate a bounded analytic semi-group on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, $[\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)]'_{\sigma,\tau}$ and $[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}$ respectively for all $1 < p < \infty$.
(ii) The Stokes operator with Navier-type boundary conditions and flux condition A'_p, B'_p and C'_p generate a bounded analytic semi-group on $\boldsymbol{X}_p$, $\boldsymbol{Y}_p$ and $\boldsymbol{Z}_p$ respectively, for all $1 < p < \infty$.

The proof of Theorem 1.1 uses a classical approach and starts with the study of the resolvent of the Stokes operator and Stokes operator with flux conditions, both with boundary conditions (1.3). A key observation is that the Stokes operator with Navier-type boundary conditions, with and without flux conditions are equal to the Laplace operator with Navier-type boundary conditions.

For this reason the study of the Stokes operator is reduced to that of the three operators denoted A_p, B_p and C_p, defined on the spaces $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, $[\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)]'_{\sigma,\tau}$ and $[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}$ and whose resolvent sets are given by the solutions of the system

$$\left\{\begin{array}{rll} \lambda\boldsymbol{u} - \Delta\boldsymbol{u} \,=\, \boldsymbol{f},\ \mathrm{div}\,\boldsymbol{u} = 0 & & \text{in } \Omega, \\ \boldsymbol{u}\cdot\boldsymbol{n} = 0,\ \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n} = \mathbf{0} & & \text{on } \Gamma, \end{array}\right. \tag{1.6}$$

where $\lambda \in \mathbb{C}^*$ such that $\mathrm{Re}\,\lambda \geq 0$ and f belonging respectively to $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, $[\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)]'_{\sigma,\tau}$ and $[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}$. Similarly, the problem for the Stokes operator with flux conditions is reduced to the study of the three operators denoted A'_p, B'_p and C'_p, defined respectively on $\boldsymbol{X}_p$, $\boldsymbol{Y}_p$ and $\boldsymbol{Z}_p$ and whose resolvent sets are given by the solutions of the problem:

$$\left\{\begin{array}{rll} \lambda\boldsymbol{u} - \Delta\boldsymbol{u} \,=\, \boldsymbol{f},\ \mathrm{div}\,\boldsymbol{u} = 0 & & \text{in } \Omega, \\ \boldsymbol{u}\cdot\boldsymbol{n} = 0,\ \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n} = \mathbf{0} & & \text{on } \Gamma, \\ \langle\boldsymbol{u}\cdot\boldsymbol{n}, 1\rangle_{\Sigma_j} \,=\, 0,\ 1 \leq j \leq J. & & \end{array}\right. \tag{1.7}$$

where $\lambda \in \mathbb{C}^*$ such that $\mathrm{Re}\,\lambda \geq 0$.

We prove the existence of strong solutions of (1.6) satisfying the resolvent estimate

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \,\leq\, \frac{C(\Omega,p)}{|\lambda|}\,\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}. \tag{1.8}$$

For $p = 2$ one has estimate (1.8) in a sector $\lambda \in \Sigma_\varepsilon$ for a fixed $\varepsilon \in]0,\pi[$ where:

$$\Sigma_\varepsilon = \{\lambda \in \mathbb{C}^*;\ |\arg\lambda| \leq \pi - \varepsilon\}, \quad \text{with} \quad \mathbb{C}^* = \mathbb{C}\setminus\{0\}.$$

We also show the existence of weak and very weak solutions and prove estimates like (1.8) for the norms of $[\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)]'_{\sigma,\tau}$ and $[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}$. We obtain similar results for the operators A'_p, B'_p and C'_p.

There exists several results in the literature, on the *analyticity of the Stokes semi-group with Dirichlet boundary condition in L^p-spaces.* This question was already studied by V. A. Solonnikov in [68]. In that work, the author proves the resolvent estimate (1.8)

for $|\arg\lambda| \leq \delta + \pi/2$ where $\delta \geq 0$ is small. To derive this estimate [68] follows an idea of Sobolevskii [67] (see the proof [68, Theorem 5.2]). New proofs and extension of the result of [68] have been proved by Giga [39], Sohr and Farwig [35] and others.

In bounded domains the resolvent of the Stokes operator with Dirichlet boundary condition has been studied by Giga in [39]. Using the theory of pseudo-differential operators, the results in [39] extends those in [68] in two directions. First, the resolvent estimate (1.8) is proved for larger set of values of λ. More precisely the estimate (1.8) is proved in [39] for all λ in the sector Σ_ε for any $\varepsilon > 0$. Second, in [39] the resolvent of the Stokes operator is obtained explicitly and this enables him to describe the domains of fractional powers of the Stokes operator with Dirichlet boundary condition.

In exterior domains, Giga and Sohr [42] approximate the resolvent of the Stokes operator with Dirichlet boundary condition with the resolvent of the Stokes operator in the entire space to prove this analyticity.

Later on, Farwig and Sohr [35] investigated the resolvent of the Stokes operator with Dirichlet boundary conditions when $\operatorname{div} \boldsymbol{u} \neq 0$ in Ω. Their results include bounded and unbounded domains, for the whole and the half space the proof rests on multiplier technique. The problem is also investigated for bended half spaces and for cones by using perturbation criterion and referring to the half space problem.

More recently, the analyticity of the Stokes semi-group with Dirichlet boundary condition is studied in spaces of bounded functions by Abe and Giga [1] using a different approach. One of the keys to prove their result is the estimate:

$$\|N(\boldsymbol{u}, \pi)\|_{\boldsymbol{L}^\infty(\Omega\times]0,T_0[)} \leq C \, \|\boldsymbol{u}_0\|_{\boldsymbol{L}^\infty(\Omega)}$$

where:

$$N(\boldsymbol{u}, \pi)(x,t) = |\boldsymbol{u}(x,t)| + t^{1/2} \, |\nabla \boldsymbol{u}(x,t)| + t \, |\nabla^2 \boldsymbol{u}(x,t)| + t \, |\partial_t \boldsymbol{u}(x,t)| + |\nabla \pi(x,t)|.$$

This estimate is obtained by means of a blow-up argument, often used in the study of non linear elliptic and parabolic equations.

The resolvent of the Stokes operator is also studied with *Robin boundary conditions* by Saal [59], Shibata and Shimada [63]. In [59], Saal proved that the Stokes operator with Robin boundary conditions is sectorial and admits an $\boldsymbol{H}^\infty$-calculus on $\boldsymbol{L}^p_{\sigma,\tau}(\mathbb{R}^3_+)$. The strategy for proving these results is firstly to construct an explicit solution for the associated Stokes resolvent problem. Next, the required resolvent estimates to conclude that such an operator is sectorial are obtained by using the rotation invariance in $(n-1)$-dimensions of large parts of the constructed solution formula, followed by using the known bounded $\boldsymbol{H}^\infty$-calculus for the Poisson operator $(-\Delta_{\mathbb{R}^2})^{1/2}$ on $\boldsymbol{L}^p(\mathbb{R}^2)$ and performing further computations. Shibata and Shimada proved in [63] a generalized resolvent estimate for the St okes equ ations with non-homogeneous Robin boundary conditions and divergence condition in $\boldsymbol{L}^p$-framework in a bounded or exterior domain by extending the argument of Farwig and Shor [35]. So that, their approaches in [63] is different from Saal [59] and rather close to that in [35].

Concerning the *Navier-type boundary conditions*, Miyakawa [55] shows that the Laplacian with the Navier-type boundary conditions (1.3) on $\boldsymbol{L}^p(\Omega)$ leaves the space $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$

invariant and hence generates a holomorphic semi-group on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ when the domain Ω is of class C^∞. Mitrea and Monniaux [53] have studied the resolvent of the Stokes operator with Navier-type boundary conditions in Lipschitz domains and proved estimate (1.8) using differential forms on Lipschitz sub-domains of a smooth compact Riemannian manifold. In addition, when the boundary of Ω is sufficiently smooth, estimates of type (1.8) are proved using that the boundary conditions (1.3) are regular elliptic (e.g. [72]) and the so called "Agmon trick" (e.g. [3]). In [38] the authors proved that the Stokes operator with Navier-type boundary conditions admits a bounded $\mathcal{H}^\infty$-calculus in the case where the domain Ω is simply connected and this has many consequences in the associated parabolic problem. In [5] the authors proved the analyticity of the semi-group generated by the Stokes operator with these boundary conditions on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$. For this reason they established estimate (1.8) using a formula involving the boundary conditions (1.3) and that, for every $p \geq 2$ and for every $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ such that $\Delta \boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$ one has

$$
\begin{aligned}
-\int_\Omega |\boldsymbol{u}|^{p-2}\Delta\boldsymbol{u}\cdot\overline{\boldsymbol{u}}\,\mathrm{d}\,x \;=\; & \int_\Omega |\boldsymbol{u}|^{p-2}|\nabla\boldsymbol{u}|^2\,\mathrm{d}\,x \;+\; 4\,\frac{p-2}{p^2}\int_\Omega \Big|\nabla|\boldsymbol{u}|^{p/2}\Big|^2\,\mathrm{d}\,x \\
& +\;(p-2)\,i\sum_{k=1}^{3}\int_\Omega |\boldsymbol{u}|^{p-4}\,\mathrm{Re}\Big(\frac{\partial\,\boldsymbol{u}}{\partial x_k}\cdot\overline{\boldsymbol{u}}\Big)\mathrm{Im}\Big(\frac{\partial\,\boldsymbol{u}}{\partial x_k}\cdot\overline{\boldsymbol{u}}\Big)\,\mathrm{d}\,x \;-\; \Big\langle\frac{\partial\,\boldsymbol{u}}{\partial\boldsymbol{n}},\,|\boldsymbol{u}|^{p-2}\boldsymbol{u}\Big\rangle_\Gamma,
\end{aligned}
$$

where $\langle .\,,\,.\rangle_\Gamma$ is the anti-duality between the Sobolev space $\boldsymbol{W}^{1/p,p'}(\Gamma)$ and its dual space $\boldsymbol{W}^{-1/p,p}(\Gamma)$.

In this paper, we prove this resolvent estimate for the norms of $[\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)]'_{\sigma,\tau}$ and $[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}$ using a duality argument. The next step after establishing the analyticity of the semi-group is to solve the time dependent Stokes Problem (1.1) with the Navier-type boundary conditions (1.3).

Using that the Stokes semi-group with Navier-type boundary condition is holomorphic in $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, Miyakawa [55] studied the fractional powers of the Stokes operator and their domains. This allows him to consider the Navier-Stokes problem with the corresponding boundary condition and to prove a local in time existence and uniqueness results of strong solution to the Problem for an initial data in $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ and under some regularity assumptions on the external force $\boldsymbol{f}$. Existence and uniqueness of solutions for the Stokes system with Navier-type boundary conditions has been proved by Yudovich [76] in a two dimensional, simply connected bounded domain. These two-dimensional results are based on the fact that the vorticity is scalar and satisfies the maximum principle. However this technique can not be extended to the three-dimensional case since the standard maximum principle for the vorticity fails. On the other hand Mitrea and Monniaux [54] have employed the Fujita-Kato approach and proved the existence of a local mild solution to Problem (1.1) and (1.3).

1.4 Existence, uniqueness and maximal regularity of solutions.

With the analyticity of the different semi-groups in hand we can solve the time dependent Stokes Problem (1.1), (1.3) and Stokes Problem with flux condition (1.1), (1.3), (1.5).

We first deduce of course existence and uniqueness of several types of solutions, using the classical semi group theory. But one of our main goals is also to obtain maximal regularity results in each of these three cases. To this end, following classical arguments (cf. in particular [43]), we are led to study the fractional and pure imaginary powers of the operators $I + L$ and L' where $L = A_p$ (resp. $L = B_p$ and resp. $L = C_p$) is the Stokes operator with Navier boundary condition on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$) (resp. $[\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)]'_{\sigma,\tau}$ and resp. $[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}$). We denote by $L' = A'_p, B'_p, C'_p$ the corresponding operators with the supplementary condition on the fluxes.

1.4.1 The non homogeneous problem.

Consider first the non-homogeneous problems. When the external force $\boldsymbol{f}$ belongs to $L^q(0,T;\, \boldsymbol{L}^p_{\sigma,\tau}(\Omega))$ it is known that the unique solution $\boldsymbol{u}$ of Problem (1.1), (1.3) satisfies $\boldsymbol{u} \in C([0,\, T]\,;\, \boldsymbol{L}^p_{\sigma,\tau}(\Omega))$ for $T < \infty$ (cf. [57]). For such $\boldsymbol{f}$ the analyticity of the semi-group is not sufficient to obtain a solution $\boldsymbol{u}$ satisfying what is called the maximal $L^p - L^q$ regularity property, *i.e.*

$$\boldsymbol{u} \in L^q(0,T;\, \boldsymbol{W}^{2,p}(\Omega)), \qquad \frac{\partial \boldsymbol{u}}{\partial t} \in L^q(0,T;\, \boldsymbol{L}^p_{\sigma,\tau}(\Omega)).$$

In order to have that property, one possibility is to impose further regularity on $\boldsymbol{f}$, such that local Hölder continuity (see [57]). The *maximal L^p-regularity for the Stokes system with Dirichlet boundary conditions* was first studied by Solonnikov [68] when $0 < T < \infty$. Solonnikov [68] constructed a solution $(\boldsymbol{u}, \pi)$ of (1.1) in $\Omega \times [0,\, T)$ satisfying the L^p estimate

$$\int_0^T \Big\| \frac{\partial \boldsymbol{u}}{\partial t} \Big\|^p_{\boldsymbol{L}^p(\Omega)} \,\mathrm{d}\, t \,+\, \int_0^T \|\nabla^2 \boldsymbol{u}(t)\|^p_{\boldsymbol{L}^p(\Omega)} \,\mathrm{d}\, t \,+\, \int_0^T \|\nabla \pi(t)\|^p_{\boldsymbol{L}^p(\Omega)} \mathrm{d}\, t \\ \leq\, C(T,\Omega,p) \int_0^T \|\boldsymbol{f}(t)\|^p_{\boldsymbol{L}^p(\Omega)} \,\mathrm{d}\, t,$$

where the matrix $\nabla^2 \boldsymbol{u} = (\partial_i \partial_j \boldsymbol{u})_{i,j=1,2,3}$ is the matrix of the second order derivatives of $\boldsymbol{u}$. When Ω is not bounded Solonnikov's estimate is not global in time because $C(T,\Omega,p)$ may tend to infinity as $T \to \infty$. His approach is based on methods in the theory of potentials. Later on, Giga and Sohr [43] strengthened Solonnikov's result in two directions. First their estimate is global in time, *i.e.* the above constant is independent of T. Second, the integral norms that they used may have different exponent p, q in space and time. To derive such global $L^p - L^q$ estimate for the Stokes system with Dirichlet boundary conditions [43] use the boundedness of the pure imaginary power of the Stokes operator. More precisely they use and extend an abstract perturbation result developed by Dore and Venni [29].

Following the same strategy as in [43] we prove maximal regularity for the inhomogeneous Stokes problems by studying the pure imaginary powers of $I + L$ and L' for $L = A_p, B_p, C_p$. Among the earliest works on the boundedness of complex and pure imaginary powers of elliptic operators we refer to the work of R. Seeley [61]. In this

work Seeley proved that an elliptic operator A_B whose domain is defined by well posed boundary conditions has bounded complex and imaginary powers in L^p satisfying the estimate

$$\forall\ x \leq 0, \quad \forall\ y \in \mathbb{R}, \qquad \|(A_B)^{x+iy}\|_{\mathcal{L}(L^p(\Omega))} \leq\ C_p e^{\gamma|y|},$$

for some constant C_p and γ.

Maximal $L^p - L^q$ regularity for the Stokes problem with homogeneous *Robin boundary conditions* in $\mathbb{R}^3_+$ is obtained in Saal [59] from the boundedness of the pure imaginary powers. However, the same approach is not applicable to the non-homogeneous boundary condition case and for this reason Shimada [64] didn't follow Saal's arguments. Shimada [64] derive the maximal $L^q - L^p$ regularity for the Stokes problem with non-homogeneous Robin boundary conditions by applying Weis's operator-valued Fourier multiplier theorem to the concrete representation formulas of solutions to the Stokes problem.

Estimates of the imaginary powers of the Stokes operator with Dirichlet boundary condition have been proved in [40, 42, 43]. That result is proved in [40] using the theory of pseudo-differential operators. When $\Omega = \mathbb{R}^3$, this boundedness is proved in [42] using Fourier transform and multiplier theorem. Furthermore, in the case of an exterior domain the desired estimate is obtained in [42] by comparing the pure imaginary powers of the Stokes operator with the corresponding powers of the Stokes operator in $\mathbb{R}^3$. Finally, in the half space such theorem for the Stokes operator with Dirichlet conditions is obtained in [43] using the results in [18].

In our case, the boundedness of the imaginary powers of $I + L$ and of $I + L'$ with $L = A_p, B_p, C_p$ essentially follows from previous results in [38]. The boundedness of the imaginary powers of A'_p, B'_p, C'_p is then obtained using a scaling argument and passage to the limit following [43, Theorem A1].

Using these properties it is then possible to prove the second main result of this article, about the existence, uniqueness and maximal regularity of strong, weak and very weak solutions of the non homogeneous Stokes problem with flux (1.1), (1.3), (1.5). We only state in this Introduction the result for the strong solutions (cf. Theorem 7.14). Similar results hold for the weak and very weak solutions of the Stokes problem with flux (cf. Remark 7.15):

Theorem 1.2 (Strong Solutions for the inhomogeneous Stokes Problem with flux)**.** *Let $T \in (0, \infty]$, $1 < p, q < \infty$. For all $\boldsymbol{f} \in L^q(0, T;\, \boldsymbol{X}_p)$, there exists a unique solution $\boldsymbol{u}$ of (7.51) such that*

$$\boldsymbol{u} \in L^q(0, T_0;\, \mathbf{D}(A'_p)), \quad T_0 \leq T \ \text{ if } T < \infty \ \text{ and } \ T_0 < T \ \text{ if } T = \infty, \tag{1.9}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} \in L^q(0, T;\, \boldsymbol{X}_p) \tag{1.10}$$

$$\int_0^T \Big\| \frac{\partial \boldsymbol{u}}{\partial t} \Big\|^q_{\boldsymbol{L}^p(\Omega)} \,\mathrm{d}\, t \,+\, \int_0^T \|\Delta \boldsymbol{u}(t)\|^q_{\boldsymbol{L}^p(\Omega)} \,\mathrm{d}\, t \leq C(p, q, \Omega) \int_0^T \|\boldsymbol{f}(t)\|^q_{\boldsymbol{L}^p(\Omega)} \,\mathrm{d}\, t. \tag{1.11}$$

and such that $(\boldsymbol{u}, \pi)$ is a solution of the inhomogeneous Stokes Problem (1.1), (1.3) , (1.5) for all $\pi \in \mathbb{R}$.

1.4.2 The homogeneous problem.

In the homogeneous case, the Stokes Problem (1.1), (1.3) is equivalent to the problem (7.1). With an initial data $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, the analyticity of the semi-group generated by A_p gives a unique solution u of (7.1) satisfying $\boldsymbol{u} \in C^k(]0\,,\,\infty[\,,\,\mathbf{D}(A_p^\ell))$, for all $k \in \mathbb{N}$, for all $\ell \in \mathbb{N}^*$ (see Theorem 7.1 below). This function is a weak solution of the Stokes problem (1.1), (1.3) and satisfies (cf. Corollary 7.2)

$$\forall\, 1 \le q < 2,\ \forall\, T < \infty, \quad \boldsymbol{u} \in L^q(0,T;\, \boldsymbol{W}^{1,p}(\Omega)) \ \text{ and } \ \frac{\partial \boldsymbol{u}}{\partial t} \in L^q(0,T;\, [\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)]').$$

We also prove the existence of very weak solution for the homogeneous Stokes Problem (1.1), (1.3) when the initial data is less regular and belongs to the dual space $[\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)]'_{\sigma,\tau}$. In this case the solution $\boldsymbol{u}$ satisfy (see Theorem 7.7)

$$\forall\, 1 \le q < 2,\ \forall\, T < \infty, \quad \boldsymbol{u} \in L^q(0,T;\, \boldsymbol{L}^p(\Omega)) \ \text{ and } \ \frac{\partial \boldsymbol{u}}{\partial t} \in L^q(0,T;\, [\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}).$$

In order to obtain the $L^p - L^q$ estimates for the solution to the homogeneous Stokes problem (1.1), (1.3) with an initial data $\boldsymbol{u}_0 \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ we study the fractional powers of A'_p. We characterize the domain $\mathbf{D}((A'_p)^{\frac{1}{2}})$ and we prove that $\mathbf{D}((A'_p)^{\frac{1}{2}}) = \boldsymbol{V}^p_{\sigma,\tau}(\Omega)$, where $\boldsymbol{V}^p_{\sigma,\tau}(\Omega)$ is given by (6.22). This yields an equivalence of the two norms $\|(A'_p)^{\frac{1}{2}}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}$ and $\|\mathbf{curl}\,\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}$. We also prove an embedding of Sobolev type for the domain of the fractional powers of the Stokes operator $\mathbf{D}((A'_p)^\alpha)$, $\alpha \in \mathbb{R}^*_+$ such that $0 < \alpha < 3/2p$.

This is similar to previous results by Borchers and Miyakawa in [18, 19], Giga and Sohr [40, 42, 43] where the fractional powers of the Stokes operator with Dirichlet boundary conditions A is studied. They have proved that $\mathbf{D}(A^{1/2}) = \boldsymbol{W}^{1,p}_0(\Omega) \cap \boldsymbol{L}^p_\sigma(\Omega)$ and the equivalence of the two norms $\|A^{1/2}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}$ and $\|\nabla \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}$ for every $\boldsymbol{u} \in \mathbf{D}(A^{1/2})$.They also proved the Sobolev embedding of the domain $D(A^\alpha)$ into $L^q(\Omega)$ for the Stokes operator with Dirichlet boundary conditions.

Using the fractional powers of A'_p we prove our third main result:

Theorem 1.3. *Let $1 < p \le q < \infty$, $\boldsymbol{u}_0 \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ and*

$$\widetilde{\boldsymbol{u}}_0 = \boldsymbol{u}_0 - \boldsymbol{w}_0, \tag{1.12}$$

$$\boldsymbol{w}_0 = \sum_{j=1}^{J} \langle \boldsymbol{u}_0 \cdot \boldsymbol{n}\,,\, 1\rangle_{\Sigma_j} \widetilde{\mathbf{grad}}\, q^\tau_j. \tag{1.13}$$

Then the homogeneous Problem (1.1), (1.3) *has a unique solution $\boldsymbol{u}$ satisfying*

$$\boldsymbol{u} \in C([0,\,+\infty[,\, \boldsymbol{L}^p_{\sigma,\tau}(\Omega)) \cap C(]0,\,+\infty[,\, \mathbf{D}(A_p)) \cap C^1(]0,\,+\infty[,\, \boldsymbol{L}^p_{\sigma,\tau}(\Omega)), \tag{1.14}$$

$$\boldsymbol{u} \in C^k(]0,\,+\infty[,\, \mathbf{D}(A_p^\ell)), \qquad \forall\, k,\, \ell \in \mathbb{N}. \tag{1.15}$$

Moreover, for all $q \in [p, \infty)$, and for all integers $m, n \in \mathbb{N}$, such that $m + n > 0$, there exists constants $M > 0$ and $\mu > 0$, such that the solution $\boldsymbol{u}$ satisfies the estimates:

$$\|\boldsymbol{u}(t) - \boldsymbol{w}_0\|_{\boldsymbol{L}^q(\Omega)} \leq C\, e^{-\mu\, t}\, t^{-3/2(1/p-1/q)} \|\widetilde{\boldsymbol{u}}_0\|_{\boldsymbol{L}^p(\Omega)}, \tag{1.16}$$

$$\|\mathbf{curl}\, \boldsymbol{u}(t)\|_{\boldsymbol{L}^q(\Omega)} \leq M\, e^{-\mu t}\, t^{-3/2(1/p-1/q)-1/2} \|\widetilde{\boldsymbol{u}}_0\|_{\boldsymbol{L}^p(\Omega)} \tag{1.17}$$

and

$$\Big\| \frac{\partial^m}{\partial t^m} \Delta^n \boldsymbol{u}(t) \Big\|_{\boldsymbol{L}^q(\Omega)} \leq M\, e^{-\mu t}\, t^{-(m+n)-3/2(1/p-1/q)} \|\widetilde{\boldsymbol{u}}_0\|_{\boldsymbol{L}^p(\Omega)}. \tag{1.18}$$

The result condition (1.1), (1.3), (1.5) with an initial data in $\boldsymbol{X}_p$ is given in Theorem 7.4. In order to keep a reasonable size for this paper, we have not included the study of the fractional powers of the operators B_p, C_p, B'_p and C'_p. Therefore, there are no regularity results for the weak and very weak solutions for the homogeneous problem. Nevertheless, the general theory of analytic semigroups applied to the semigroups generated by B_p, C_p, B'_p and C'_p provide existence and uniqueness results of solutions to problems (1.1), (1.3) and (1.1), (1.3), (1.5) for initial data $\boldsymbol{u}_0$ with less regularity (cf. Theorem 7.6 for the Stokes problem and Theorem 7.11 for the Stokes problem with flux conditions).

When $\Omega = \mathbb{R}^3$, Kato [49] shows that estimate (1.16) follows directly from the corresponding estimates for the heat semi-group. In the half space, Borchers and Miyakawa [18, 19] deduced estimate (1.16) for the Stokes semi-group with Dirichlet boundary condition from Ukai's formula [74]. In a bounded domain Giga [41] derives this estimate for the Stokes semi-group with Dirichlet boundary conditions from the inequality

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^q(\Omega)} \leq C \|A^{\alpha/2} \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}, \text{ with } \alpha = 3(1/p - 1/q) \tag{1.19}$$

which can be obtained directly from the usual Sobolev inequality for the Laplacian and from the fact that in the case of bounded domains

$$\|\Delta^{\alpha/2} \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq C \|A^{\alpha/2} \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \tag{1.20}$$

for every regular function $\boldsymbol{u}$, for every $\alpha > 0$ and for every $1 < p < \infty$ (see [40]). In the case of exterior domains Giga and Sohr follow in [42] the same procedure as in the case of bounded domains but with limitations with respect to the values of p and q, because in this case the inequality (1.20) still hold true but for limited values p and q. We note also that in exterior domain Borchers and Miyakawa [19] prove the same result as [42] but using (1.19). More recently Coulhon and Lamberton [27] proved the estimate (1.16) by showing that some properties of the Stokes semi-group with Dirichlet boundary condition can be obtained by a simple transfer of the properties of the heat semi-group.

1.5 Plan of the paper.

This paper is organized as follows. In Section 2 we give the functional framework and some preliminary results at the basis of our proofs. In Section 3 we define the three

different Stokes operators with Navier-type boundary conditions, and prove some of their properties. In Section 4 we prove that the operators introduced in Section 3 generate bounded analytic semi-groups. Section 5 is devoted to Stokes operators with Navier-type boundary conditions and flux conditions. We introduce three operators of that kind and prove that they generate analytic semigroups. We prove in Section 6 several results on the pure imaginary and fractional powers of several operators. Then, in Section 7, we solve the Stokes problem and the Stokes problem with flux under different assumptions on the initial data $\boldsymbol{u}_0$ and the function $\boldsymbol{f}$.

2 Notations and preliminary results

2.1 Functional framework

In this subsection we review some basic notations, definitions and functional framework which are essential in our work. Vector fields, matrix fields and their corresponding spaces defined on Ω will be denoted by bold character. The functions treated here are complex valued functions. We will use also the symbol σ to represent a set of divergence free functions and the symbol τ when the normal component on the boundary is vanish. In other words if $\boldsymbol{E}$ is a subspace of $\boldsymbol{\mathcal{D}}'(\Omega)$, then

$$\boldsymbol{E}_\sigma \,=\, \big\{\boldsymbol{v}\in\boldsymbol{E};\ \operatorname{div}\boldsymbol{v}\,=\,0\ \text{ in }\Omega\big\}$$

and

$$\boldsymbol{E}_\tau \,=\, \big\{\boldsymbol{v}\in\boldsymbol{E};\ \ \boldsymbol{v}\cdot\boldsymbol{n}\,=\,0\text{ on }\Gamma\big\}.$$

Now, we introduce some functional spaces. Let $\boldsymbol{L}^p(\Omega)$ denotes the usual vector valued $\boldsymbol{L}^p$-space over Ω. Let us define the spaces:

$$\boldsymbol{H}^p(\mathbf{curl},\Omega) \,=\, \big\{\boldsymbol{v}\in\boldsymbol{L}^p(\Omega);\ \ \mathbf{curl}\,\boldsymbol{v}\in\boldsymbol{L}^p(\Omega)\big\},$$

$$\boldsymbol{H}^p(\operatorname{div},\Omega) \,=\, \big\{\boldsymbol{v}\in\boldsymbol{L}^p(\Omega);\ \ \operatorname{div}\boldsymbol{v}\in\boldsymbol{L}^p(\Omega)\big\},$$

$$\boldsymbol{X}^p(\Omega) \,=\, \boldsymbol{H}^p(\mathbf{curl},\Omega)\cap\boldsymbol{H}^p(\operatorname{div},\Omega),$$

equipped with the graph norm. Thanks to [13] we know that $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\boldsymbol{H}^p(\mathbf{curl},\Omega)$, $\boldsymbol{H}^p(\operatorname{div},\Omega)$ and $\boldsymbol{X}^p(\Omega)$.
We also define the subspaces:

$$\boldsymbol{H}^p_0(\mathbf{curl},\Omega) \,=\, \big\{\boldsymbol{v}\in\boldsymbol{H}^p(\mathbf{curl},\Omega);\ \ \boldsymbol{v}\times\boldsymbol{n}\,=\,\mathbf{0}\text{ on }\Gamma\big\},$$

$$\boldsymbol{H}^p_0(\operatorname{div},\Omega) \,=\, \big\{\boldsymbol{v}\in\boldsymbol{H}^p(\operatorname{div},\Omega);\ \ \boldsymbol{v}\cdot\boldsymbol{n}\,=\,0\text{ on }\Gamma\big\},$$

$$\boldsymbol{X}^p_N(\Omega) \,=\, \big\{\boldsymbol{v}\in\boldsymbol{X}^p(\Omega);\ \ \boldsymbol{v}\times\boldsymbol{n}\,=\,\mathbf{0}\text{ on }\Gamma\big\},$$

$$\boldsymbol{X}^p_\tau(\Omega) \,=\, \big\{\boldsymbol{v}\in\boldsymbol{X}^p(\Omega);\ \ \boldsymbol{v}\cdot\boldsymbol{n}\,=\,0\text{ on }\Gamma\big\}$$

and

$$\boldsymbol{X}^p_0(\Omega) \,=\, \boldsymbol{X}^p_N(\Omega)\cap\boldsymbol{X}^p_\tau(\Omega).$$

We have denoted by $\boldsymbol{v} \times \boldsymbol{n}$ (respectively by $\boldsymbol{v} \cdot \boldsymbol{n}$) the tangential (respectively normal) boundary value of $\boldsymbol{v}$ defined in $\boldsymbol{W}^{-1/p,p}(\Gamma)$ (respectively in $W^{-1/p,p}(\Gamma)$) as soon as $\boldsymbol{v}$ belongs to $\boldsymbol{H}^p(\mathbf{curl}, \Omega)$ (respectively to $\boldsymbol{H}^p(\mathrm{div}, \Omega)$). More precisely, any function $\boldsymbol{v}$ in $\boldsymbol{H}^p(\mathbf{curl}, \Omega)$ (respectively in $\boldsymbol{H}^p(\mathrm{div}, \Omega)$) has a tangential (respectively normal) trace $\boldsymbol{v} \times \boldsymbol{n}$ (respectively $\boldsymbol{v} \cdot \boldsymbol{n}$) in $\boldsymbol{W}^{-1/p,p}(\Gamma)$ (respectively in $W^{-1/p,p}(\Gamma)$) defined by:

$$\forall\, \boldsymbol{\varphi} \in \boldsymbol{W}^{1,p'}(\Omega),\ \ \langle \boldsymbol{v} \times \boldsymbol{n},\, \boldsymbol{\varphi} \rangle_\Gamma \,=\, \int_\Omega \mathbf{curl}\, \boldsymbol{v} \cdot \overline{\boldsymbol{\varphi}}\, \mathrm{d}\, x \,-\, \int_\Omega \boldsymbol{v} \cdot \mathbf{curl}\, \overline{\boldsymbol{\varphi}}\, \mathrm{d}\, x \tag{2.1}$$

and

$$\forall\, \varphi \in W^{1,p'}(\Omega),\ \ \langle \boldsymbol{v} \cdot \boldsymbol{n},\, \varphi \rangle_\Gamma \,=\, \int_\Omega \boldsymbol{v} \cdot \mathbf{grad}\, \overline{\varphi}\, \mathrm{d}\, x \,+\, \int_\Omega \mathrm{div}\, \boldsymbol{v}\, \overline{\varphi}\, \mathrm{d}\, x, \tag{2.2}$$

where $\langle ., . \rangle_\Gamma$ is the anti-duality between $\boldsymbol{W}^{-1/p,p}(\Gamma)$ and $\boldsymbol{W}^{1/p,p'}(\Gamma)$ in (2.1) and between $W^{-1/p,p}(\Gamma)$ and $W^{1/p,p'}(\Gamma)$ in (2.2). Thanks to [13] we know that $\boldsymbol{\mathcal{D}}(\Omega)$ is dense in $\boldsymbol{H}_0^p(\mathbf{curl}, \Omega)$ and in $\boldsymbol{H}_0^p(\mathrm{div}, \Omega)$. We denote by $[\boldsymbol{H}_0^p(\mathbf{curl}, \Omega)]'$ and $[\boldsymbol{H}_0^p(\mathrm{div}, \Omega)]'$ the dual spaces of $\boldsymbol{H}_0^p(\mathbf{curl}, \Omega)$ and $\boldsymbol{H}_0^p(\mathrm{div}, \Omega)$ respectively. Notice that we can characterize these dual spaces as follows: A distribution $\boldsymbol{f}$ belongs to $[\boldsymbol{H}_0^p(\mathbf{curl}, \Omega)]'$ if and only if there exist functions functions $\boldsymbol{\psi} \in \boldsymbol{L}^{p'}(\Omega)$ and $\boldsymbol{\xi} \in \boldsymbol{L}^{p'}(\Omega)$, such that $\boldsymbol{f} \,=\, \boldsymbol{\psi} \,+\, \mathbf{curl}\, \boldsymbol{\xi}$. Moreover one has

$$\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^p(\mathbf{curl},\Omega)]'} \,=\, \inf_{\boldsymbol{f} = \boldsymbol{\psi} + \mathbf{curl}\, \boldsymbol{\xi}} \max\left(\|\boldsymbol{\psi}\|_{\boldsymbol{L}^{p'}(\Omega)},\, \|\boldsymbol{\xi}\|_{\boldsymbol{L}^{p'}(\Omega)}\right).$$

Similarly, a distribution $\boldsymbol{f}$ belongs to $[\boldsymbol{H}_0^p(\mathrm{div}, \Omega)]'$ if and only if there exist $\boldsymbol{\psi} \in \boldsymbol{L}^{p'}(\Omega)$ and $\chi \in L^{p'}(\Omega)$ such that $\boldsymbol{f} \,=\, \boldsymbol{\psi} \,+\, \mathbf{grad}\, \chi$ and

$$\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^p(\mathrm{div},\Omega)]'} \,=\, \inf_{\boldsymbol{f} = \boldsymbol{\psi} + \mathbf{grad}\, \chi} \max\left(\|\boldsymbol{\psi}\|_{\boldsymbol{L}^{p'}(\Omega)}\,,\, \|\chi\|_{L^{p'}(\Omega)}\right).$$

Finally we consider the space

$$\boldsymbol{T}^p(\Omega) = \left\{\boldsymbol{v} \in \boldsymbol{H}_0^p(\mathrm{div}, \Omega);\ \ \mathrm{div}\, \boldsymbol{v} \in W_0^{1,p}(\Omega)\right\}, \tag{2.3}$$

equipped with the graph norm. Thanks to [12, Lemma 4.11, Lemma 4.12] we know that $\boldsymbol{\mathcal{D}}(\Omega)$ is dense in $\boldsymbol{T}^p(\Omega)$ and a distribution $\boldsymbol{f} \in (\boldsymbol{T}^p(\Omega))'$ if and only if there exists a function $\boldsymbol{\psi} \in \boldsymbol{L}^{p'}(\Omega)$ and a function $\chi \in W^{-1,p'}(\Omega)$ such that $\boldsymbol{f} = \boldsymbol{\psi} \,+\, \nabla\chi$.

2.2 Preliminary results

In this subsection, we review some known results which are essential in our work. First, we recall that the vector-valued Laplace operator of a vector field $\boldsymbol{v} \,=\, (v_1, v_2, v_3)$ is equivalently defined by

$$\Delta\, \boldsymbol{v} = \mathbf{grad}\, (\mathrm{div}\, \boldsymbol{v}) - \mathbf{curl}\, \mathbf{curl}\, \boldsymbol{v}.$$

Next, we review some Sobolev embeddings (see [13]):

Lemma 2.1. *The spaces $\boldsymbol{X}_N^p(\Omega)$ and $\boldsymbol{X}_\tau^p(\Omega)$ defined above are continuously embedded in $\boldsymbol{W}^{1,p}(\Omega)$.*

Consider now the spaces

$$\boldsymbol{X}^{2,p}(\Omega) = \left\{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \operatorname{div}\boldsymbol{v} \in W^{1,p}(\Omega),\ \mathbf{curl}\,\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega) \text{ and } \boldsymbol{v}\cdot\boldsymbol{n} \in W^{1-1/p,p}(\Gamma)\right\} \tag{2.4}$$

and

$$\boldsymbol{Y}^{2,p}(\Omega) = \left\{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \operatorname{div}\boldsymbol{v} \in W^{1,p}(\Omega),\ \mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega) \text{ and } \boldsymbol{v}\times\boldsymbol{n} \in \boldsymbol{W}^{1-1/p,p}(\Gamma)\right\}.$$

Lemma 2.2. *The spaces $\boldsymbol{X}^{2,p}(\Omega)$ and $\boldsymbol{Y}^{2,p}(\Omega)$ are continuously embedded in $\boldsymbol{W}^{2,p}(\Omega)$.*

Consider now the space

$$\boldsymbol{E}^{\,p}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{W}^{\,1,p}(\Omega);\ \Delta\boldsymbol{v} \in [\boldsymbol{H}_0^{\,p'}(\operatorname{div},\Omega)]'\},$$

which is a Banach space for the norm:

$$\|\ \boldsymbol{v}\ \|_{\boldsymbol{E}^{\,p}(\Omega)} = \|\ \boldsymbol{v}\ \|_{\boldsymbol{W}^{\,1,p}(\Omega)} + \|\ \Delta\boldsymbol{v}\ \|_{[\boldsymbol{H}_0^{\,p'}(\operatorname{div},\Omega)]'}\ .$$

Thanks to [12, Lemma 4.1] we know that $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\boldsymbol{E}^{\,p}(\Omega)$. Moreover we have the following Lemma (see [12, Corollary 4.2]):

Lemma 2.3. *The linear mapping $\gamma : \boldsymbol{v} \longrightarrow \mathbf{curl}\,\boldsymbol{v}\times\boldsymbol{n}$ defined on $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ can be extended to a linear and continuous mapping*

$$\gamma : \boldsymbol{E}^p(\Omega) \longrightarrow \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma).$$

Moreover, we have the Green formula: for any $\boldsymbol{v} \in \boldsymbol{E}^p(\Omega)$ and $\boldsymbol{\varphi} \in \boldsymbol{X}_\tau^{p'}(\Omega)$ such that $\operatorname{div}\boldsymbol{\varphi} = 0$ in Ω.

$$-\langle\Delta\boldsymbol{v},\boldsymbol{\varphi}\rangle_\Omega = \int_\Omega \mathbf{curl}\,\boldsymbol{v}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} - \langle\mathbf{curl}\boldsymbol{v}\times\boldsymbol{n},\boldsymbol{\varphi}\rangle_\Gamma.$$

where $\langle .,.\rangle_\Gamma$ denotes the anti-duality between $\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ and $\boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)$ and $\langle .,.\rangle_\Omega$ denotes the anti-duality between $[\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)]'$ and $\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)$.

Next we consider the space

$$\boldsymbol{H}^p(\Delta,\Omega) = \left\{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \ \Delta\boldsymbol{v} \in (\boldsymbol{T}^{p'}(\Omega))'\right\},$$

which is a Banach space for the graph norm. Thanks to [12, Lemma 4.13, Lemma 4.14] we know that

Proposition 2.4. *The space $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\boldsymbol{H}^p(\Delta,\Omega)$. Moreover for every $\boldsymbol{v}$ in $\boldsymbol{H}^p(\Delta,\Omega)$ the trace $\mathbf{curl}\,\boldsymbol{v}\times\boldsymbol{n}$ exists and belongs to $\boldsymbol{W}^{-1-1/p,p}(\Gamma)$. In addition we have the Green formula: for all $\boldsymbol{v} \in \boldsymbol{H}^p(\Delta,\Omega)$ and for all $\boldsymbol{\varphi} \in \boldsymbol{W}^{2,p}(\Omega)$ such that $\operatorname{div}\boldsymbol{\varphi} = \boldsymbol{\varphi}\cdot\boldsymbol{n} = 0$ on Γ and $\mathbf{curl}\,\boldsymbol{\varphi}\times\boldsymbol{n} = \mathbf{0}$ on Γ:*

$$\langle\Delta\boldsymbol{v}\,,\,\boldsymbol{\varphi}\rangle_{(\boldsymbol{T}^{p'}(\Omega))'\times\boldsymbol{T}^{p'}(\Omega)} = \int_\Omega \boldsymbol{v}\cdot\Delta\overline{\boldsymbol{\varphi}}\,dx + \langle\mathbf{curl}\,\boldsymbol{v}\times\boldsymbol{n}\,,\,\boldsymbol{\varphi}\rangle_\Gamma, \tag{2.5}$$

where $\langle .\,,\,.\rangle_\Gamma = \langle .\,,\,.\rangle_{\boldsymbol{W}^{-1-1/p,p}(\Gamma)\times\boldsymbol{W}^{1+1/p,p'}(\Gamma)}$.

Next we consider the problem:

$$\operatorname{div}(\mathbf{grad}\,\pi - \boldsymbol{f}) = 0 \qquad \text{in } \Omega, \qquad (\mathbf{grad}\,\pi - \boldsymbol{f})\cdot\boldsymbol{n} = 0 \qquad \text{on } \Gamma. \tag{2.6}$$

We recall the following lemma concerning the weak Neumann problem without giving the proof (see [65] for point i) and [10] for points ii) and iii)).

Lemma 2.5. *(i) Let $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$, the Problem (2.6) has a unique solution $\pi \in W^{1,p}(\Omega)/\mathbb{R}$ satisfying the estimate*

$$\|\mathbf{grad}\,\pi\|_{\boldsymbol{L}^p(\Omega)} \le C_1(\Omega)\,\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)},$$

for some constant $C_1(\Omega) > 0$.

***(ii)** Let $\boldsymbol{f} \in [\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)]'$, the Problem (2.6) has a unique solution $\pi \in L^p(\Omega)/\mathbb{R}$ satisfying the estimate*

$$\|\pi\|_{L^p(\Omega)/\mathbb{R}} \le C_2(\Omega,p)\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)]'},$$

***(iii)** Let $\boldsymbol{f} \in (\boldsymbol{T}^{p'}(\Omega))'$, where $\boldsymbol{T}^p(\Omega)$ is given by (2.3). The Problem (2.6) has a unique solution $\pi \in W^{-1,p}(\Omega)/\mathbb{R}$ satisfying the estimate*

$$\|\pi\|_{W^{-1,p}(\Omega)/\mathbb{R}} \le C(\Omega,p)\,\|\boldsymbol{f}\|_{(\boldsymbol{T}^{p'}(\Omega))'}.$$

2.3 Some Properties of sectorial and non-negative operators

This subsection is devoted to the definitions and some relevant properties of sectorial and non-negative operators very useful in our work. In all this subsection X denotes a Banach space and $\mathcal{A} : D(\mathcal{A}) \subset X \mapsto X$ is a closed linear operator. $D(\mathcal{A})$ is the domain of $\mathcal{A}$, it is equipped with the graph norm and form with this norm a Banach space.

Let $0 \le \theta < \pi/2$ and let Σ_θ be the sector

$$\Sigma_\theta = \Big\{\lambda \in \mathbb{C}^*;\ |\arg\lambda| < \pi - \theta\Big\}.$$

Thanks to [30, Chapter 2, page 96], we know that a linear densely defined operator $\mathcal{A}$ is sectorial if there exists a constant $M > 0$ and $0 \le \theta < \pi/2$ such that

$$\forall\,\lambda \in \Sigma_\theta, \qquad \|R(\lambda,\,\mathcal{A})\|_{\mathcal{L}(X)} \le \frac{M}{|\lambda|}, \tag{2.7}$$

where $R(\lambda,\,\mathcal{A}) = (\lambda I - \mathcal{A})^{-1}$. This means that the resolvent of a sectorial operator contains a sector Σ_θ for some $0 \le \theta < \pi/2$ and for every $\lambda \in \Sigma_\theta$ one has estimate (2.7).

Moreover, the authors give in [30] a necessary and sufficient condition for an operator $\mathcal{A}$ to generates a bounded analytic semi-group. In fact, according to [30, Chapter 2, Theorem 4.6, page 101], an operator $\mathcal{A}$ generates a bounded analytic semi-group if and only if it is sectorial in the sense of (2.7).

Nevertheless, it is not always easy to prove that an operator $\mathcal{A}$ is sectorial in the sense (2.7). Although, Yosida proved in [75] that it is suffices to prove (2.7) in the half plane $\{\lambda \in \mathbb{C}^*;\, \mathrm{Re}\,\lambda \geq 0\}$. This result is stated in [14, Chapter 1, Theorem 3.2, page 30] and proved by K. Yosida.

Proposition 2.6. *Let $\mathcal{A} : \mathrm{D}(\mathcal{A}) \subseteq X \longmapsto X$ be a linear densely defined operator and $M > 0$ such that*

$$\forall\, \lambda \in \mathbb{C}^*,\ \ \mathrm{Re}\,\lambda \geq 0, \qquad \|R(\lambda,\, \mathcal{A})\|_{\mathcal{L}(X)} \,\leq\, \frac{M}{|\lambda|}.$$

Then $\mathcal{A}$ is sectorial in the sense of (2.7)*.*

Proof. Thanks to Yosida [75, Chapter VIII, Theorem 1, page 211] we know that $\rho(\mathcal{A})$ is an open subset of $\mathbb{C}$ and for all $\lambda_0 \in \rho(\mathcal{A})$, the disc of center λ_0 and radius $|\lambda_0|/M$ is contained in $\rho(\mathcal{A})$. In particular, for every $r > 0$, the open disks with center $\pm\, i\, r$ and radius $|r|/M$ is contained in $\rho(\mathcal{A})$. The union of such disks and of the half plane $\{\lambda \in \mathbb{C};\, \mathrm{Re}\,\lambda \geq 0\}$ contains the sector

$$\Big\{\lambda \in \mathbb{C}^*;\ |\arg\lambda| < \pi \,-\, \arctan(M)\Big\},$$

hence it contains the sector

$$S \,=\, \Big\{\lambda \in \mathbb{C}^*;\ |\arg\lambda| < \pi \,-\, \arctan(2\,M)\Big\}.$$

If $\lambda \in S$ and $\mathrm{Re}\,\lambda < 0$, we write λ in the form $\lambda \,=\, \pm\, i\, r - (\theta\, r)/(2\,M)$ for some $\theta \in (0, 1)$. Thanks to [**?**, Chapter 4, formula 1.2, page 239]e know that

$$R(\lambda,\, \mathcal{A}) \,=\, R(\pm\, i\, r,\, \mathcal{A})\big[I \,+\, (\lambda \,\mp\, i\, r)R(\pm\, i\, r,\, \mathcal{A})\big]^{-1}.$$

We can easily verify that $\big\|\big[I \,+\, (\lambda \,\mp\, i\, r)R(\pm\, i\, r,\, \mathcal{A})\big]^{-1}\big\|_{\mathcal{L}(X)} \,\leq\, 2.$

Next, observe that $|\lambda| \,=\, \sqrt{r^2 \,+\, \frac{\theta^2\, r^2}{4\, M^2}} \,=\, r\, \frac{\sqrt{4\, M^2 + \theta^2}}{2\, M}$. Then

$$\|R(\lambda,\, \mathcal{A})\| \,\leq\, \frac{2\, M}{r} \,\leq\, \frac{2\, M\, \frac{\sqrt{4\, M^2 + \theta^2}}{2\, M}}{r\, \frac{\sqrt{4\, M^2 + \theta^2}}{2\, M}} \,\leq\, \frac{\sqrt{4\, M^2 \,+\, 1}}{|\lambda|}.$$

Now if $\lambda \in S$ such that $\mathrm{Re}\,\lambda \,\geq\, 0$ then thanks to our assumption one has

$$\|R(\lambda,\, \mathcal{A})\|_{\mathcal{L}(X)} \,\leq\, \frac{M}{|\lambda|} \tag{2.8}$$

which ends the proof. □

Remark 2.7. Proposition 2.6 means that there exists an angle $0 < \theta_0 < \pi/2$ such that the resolvent set of the operator $\mathcal{A}$ contains the sector

$$\Sigma_{\theta_0} = \big\{\lambda \in \mathbb{C}^*;\ \ |\arg\lambda| \leq \pi - \theta_0\big\}$$

where estimate (2.8) is satisfied.

Next we recall some definitions and properties concerning the fractional powers of a non-negative operator. We start by the following definition.

Definition 2.8. *An operator $\mathcal{A}$ is said to be a non-negative operator if its resolvent set contains all negative real numbers and*

$$\sup_{t>0} t\,\|(t\,I+\mathcal{A})^{-1}\|_{\mathcal{L}(X)} < \infty.$$

For a non-negative operator $\mathcal{A}$ it is possible to define its complex power $\mathcal{A}^z$ for every $z \in \mathbb{C}$ as a densely defined closed linear operator in the closed subspace $X_{\mathcal{A}} = \overline{D(\mathcal{A})} \cap \overline{R(\mathcal{A})}$ in X. Here $D(\mathcal{A})$ and $R(\mathcal{A})$ denote, respectively, the domain and the range of $\mathcal{A}$. Observe that, if both $D(\mathcal{A})$ and $R(\mathcal{A})$ are dense in X, then $X_{\mathcal{A}} = X$. We refer to [50, 73] for the definition and some relevant properties of the complex power of a non-negative operator.

For a non-negative bounded operator whose inverse $\mathcal{A}^{-1}$ exists and it is bounded (*i.e.* $0 \in \rho(\mathcal{A})$), the complex power $\mathcal{A}^z$ can be defined for all $z \in \mathbb{C}$ by the means of the Dunford integral ([75]):

$$\mathcal{A}^z f = \frac{1}{2\,\pi\,i}\int_{\Gamma_\theta} (-\lambda)^z\,(\lambda\,I+\mathcal{A})^{-1} f \mathrm{d}\,\lambda, \tag{2.9}$$

where Γ_θ runs in the resolvent set of $-\mathcal{A}$ from $\infty\, e^{i(\theta-\pi)}$ to zero and from zero to $\infty\, e^{i(\pi-\theta)}$, $0<\theta<\pi/2$ in $\mathbb{C}$ avoiding the non negative real axis. The branch of $(-\lambda)^z$ is taken so that $\mathrm{Re}((-\lambda)^z) > 0$ for $\lambda < 0$. It is proved by Triebel [73] that when the operator $\mathcal{A}$ is of bounded inverse, the complex powers $\mathcal{A}^z$ for $\mathrm{Re}\, z > 0$ are isomorphisms from $D(\mathcal{A}^z)$ to $X_{\mathcal{A}}$.

The following property plays an important role in the study of the abstract inhomogeneous Cauchy-Problem and give us more regularity for the solutions (see [43]).

Definition 2.9. *Let $\theta \geq 0$ and $K \geq 1$. A non-negative operator $\mathcal{A}$ belongs to $E_K^\theta(X)$ if $\mathcal{A}^{is} \in \mathcal{L}(X_{\mathcal{A}})$ for every $s \in \mathbb{R}$ and its norm in $\mathcal{L}(X_{\mathcal{A}})$ satisfies the estimate*

$$\|\mathcal{A}^{is}\|_{\mathcal{L}(X_{\mathcal{A}})} \leq K\, e^{\theta\,|s|}. \tag{2.10}$$

If in addition $D(\mathcal{A})$ and $R(\mathcal{A})$ are dense in X, we say that $\mathcal{A} \in \mathcal{E}_K^\theta(X)$.

We note that, these spaces $E_K^\theta(X)$ and $\mathcal{E}_K^\theta(X)$ were introduced by Dore and Venni [29], Giga and Sohr [43] in the abstract perturbation theory.

When $-\mathcal{A}$ is the infinitesimal generator of a bounded analytic semi-group $(T(t))_{t\geq 0}$, the following proposition is proved by Komatsu (see [50, Theorem 12.1] for instance)

Proposition 2.10. *Let $-\mathcal{A}$ be the infinitesimal generator of a bounded analytic semi-group $(T(t))_{t\geq 0}$. For any complex number α such that $\mathrm{Re}\,\alpha > 0$ one has*

$$\forall t>0, \qquad \|\mathcal{A}^\alpha T(t)\|_{\mathcal{L}(X)} \leq C\, t^{-\mathrm{Re}\,\alpha}. \tag{2.11}$$

The following lemma is proved by Komatsu (see [50]) and plays an important role in the study of the domains of fractional powers of the Stokes operator with Navier-type boundary conditions.

Lemma 2.11. *Let $\mathcal{A}$ be a non-negative closed linear operator. If* $\mathrm{Re}\alpha > 0$ *the domain $D((\nu\, I + \mathcal{A})^\alpha)$ doesn't depend on $\nu \geq 0$ and coincides with $D((\mu\, I + \mathcal{A})^\alpha)$ for $\mu \geq 0$. In other words*

$$\forall\, \mu,\, \nu > 0, \qquad D(\mathcal{A}^\alpha) \,=\, D((\mu\, I + \mathcal{A})^\alpha) \,=\, D((\nu\, I + \mathcal{A})^\alpha).$$

Finally, let $\mathcal{A}$ be a non-negative operator such that $0 \in \rho(\mathcal{A})$. The boundedness of $\mathcal{A}^{is}$, $s \in \mathbb{R}$ allows us to determine the domain of definition of $D(\mathcal{A}^\alpha)$, for complex number α satisfying $\mathrm{Re}\,\alpha > 0$ using complex interpolation. The following result is due to [73]

Theorem 2.12. *Let $\mathcal{A}$ be a non-negative operator with bounded inverse. We suppose that there exist two positive numbers ε and C such that $\mathcal{A}^{is}$ is bounded for $-\varepsilon \leq s \leq \varepsilon$ and $\|\mathcal{A}^{is}\|_{\mathcal{L}(X_A)} \leq C$. If α is a complex number such that $0 < \mathrm{Re}\,\alpha < \infty$ and $0 < \theta < 1$ then*

$$[X\,,\, D(\mathcal{A}^\alpha)]_\theta \,=\, D(\mathcal{A}^{\alpha\theta}).$$

2.4 Some auxiliary results on ζ-convexity.

In order to prove maximal $L^p - L^q$ regularity properties for the solutions of the inhomogeneous Stokes problem, we use the property of ζ-convexity of Banach spaces. This property has already proved to be useful in the same context (cf. [43]). For further readings on ζ-convex Banach spaces we refer to [23, 58].

The ζ convex property may be defined as follows:

Definition 2.13. *A Banach space X is ζ-convex if there is a symmetric biconvex function ζ on $X \times X$ such that $\zeta(0,0) > 0$ and*

$$\forall\, x, y \in X,\ \ \|x\|_X \geq 1, \qquad \zeta(x,y) \leq \|x + y\|_X. \tag{2.12}$$

For this and equivalent definitions see Theorem 1 and Theorem 2 in [58].

The ζ-convexity property is stronger than uniform convexity or reflexivity. It has been proved in Proposition 3 of [58] that for any Ω open domain of $\mathbb{R}^3$ the space $L^p(\Omega)$ is ζ-convex if and only if $1 < p < \infty$.

The following property of ζ-convex spaces is needed in the following. Since its proof is elementary we shall skip it.

Proposition 2.14. *Every closed subspace of a ζ-convex space is ζ-convex.*

On the other hand, the following characterization of ζ-convex spaces in terms of the Hilbert transform is proved in [23] (cf. Theorem 3.3 in Section 3 and Section 2). See also [58] (Theorem 1 and Theorem 2):

Theorem 2.15. *A Banach space X is ζ-convex if and only if, for some $s \in (1,\infty)$, the truncated Hilbert transform*

$$(H_\varepsilon f)(t) \,=\, \frac{1}{\pi}\int_{|\tau|>\varepsilon} \frac{f(t-\tau)}{\tau}\,\mathrm{d}\tau$$

converges as $\varepsilon \to 0$, for almost all $t \in \mathbb{R}$, for all $f \in L^s(\mathbb{R};\, X)$, and there is a constant $C = C(s,X)$ independent of f such that

$$\|Hf\|_{L^s(\mathbb{R},\,X)} \,\le\, C\,\|f\|_{L^s(\mathbb{R};\,X)},$$

where $(Hf)(t) = \lim_{\varepsilon\to 0}(H_\varepsilon f)(t)$.

Using Theorem 2.15 we prove the following proposition and show the ζ-convexity of the dual spaces $[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'$ and $[\boldsymbol{T}^{p'}(\Omega)]'$.

Proposition 2.16. *Let $1 < p < \infty$, the dual spaces $[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'$ and $[\boldsymbol{T}^{p'}(\Omega)]'$ are ζ-convex Banach spaces.*

Proof. We will only write the proof of the ζ-convexity of $[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'$ because the proof of the ζ-convexity of $[\boldsymbol{T}^{p'}(\Omega)]'$ is similar. Let $\boldsymbol{f} \in L^s(\mathbb{R};\, [\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]')$, then for almost all $t \in \mathbb{R}$, there exists $\boldsymbol{\psi}(t) \in \boldsymbol{L}^p(\Omega)$ and $\chi(t) \in L^p(\Omega)$ such that

$$\boldsymbol{f}(t) \,=\, \boldsymbol{\psi}(t) \,+\, \nabla\chi(t), \qquad \|\boldsymbol{f}(t)\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'} \,=\, \max(\|\boldsymbol{\psi}(t)\|_{\boldsymbol{L}^p(\Omega)},\, \|\chi(t)\|_{L^p(\Omega)}).$$

Since $\boldsymbol{f} \in L^s(\mathbb{R};\, [\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]')$, it is clear that $\boldsymbol{\psi} \in L^s(\mathbb{R};\, \boldsymbol{L}^p(\Omega))$ and $\chi \in L^s(\mathbb{R}; L^p(\Omega))$. On the other hand we can easily verify that

$$(H_\varepsilon \boldsymbol{f})(t) \,=\, (H_\varepsilon \boldsymbol{\psi})(t) \,+\, \nabla(H_\varepsilon\chi)(t).$$

Next, since $\boldsymbol{L}^p(\Omega)$ (respectively $L^p(\Omega)$) is ζ-convex then $(H_\varepsilon\boldsymbol{\psi})(t)$ (respectively $(H_\varepsilon\chi)(t)$) converges as $\varepsilon \to 0$ to $H\boldsymbol{\psi}(t)$ (respectively to $H\chi(t)$). Moreover we have the estimate

$$\|H\boldsymbol{\psi}(t)\|_{L^s(\mathbb{R};\,\boldsymbol{L}^p(\Omega))} \,\le\, C(s,\Omega,p)\,\|\boldsymbol{\psi}\|_{L^s(\mathbb{R};\,\boldsymbol{L}^p(\Omega))}$$

and

$$\|H\chi(t)\|_{L^s(\mathbb{R};\,L^p(\Omega))} \,\le\, C(s,\Omega,p)\,\|\psi\|_{L^s(\mathbb{R};\,L^p(\Omega))}$$

This means that $(H_\varepsilon\boldsymbol{f})(t)$ converges as $\varepsilon \to 0$ to $H\boldsymbol{f}(t) \,=\, H\boldsymbol{\psi}(t) \,+\, \nabla\, H\chi(t)$. Moreover we have the estimate

$$\|H\boldsymbol{f}(t)\|_{L^s(\mathbb{R};\,[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]')} \,\le\, C(s,\Omega,p)\,\|\boldsymbol{f}\|_{L^s(\mathbb{R};\,[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]')},$$

which ends the proof. □

3 The Stokes operator

The main object of this section is to introduce the different Stokes operators with Navier-type boundary conditions that we need in order to solve the Stokes problem for the different types of initial data $\boldsymbol{u}_0$ and external forces $\boldsymbol{f}$ that we want to consider. For the sake of comparison we also recall the definition of the Stokes operator with Dirichlet boundary conditions.

3.1 The Stokes operator with Dirichlet boundary conditions

We consider the space

$$\boldsymbol{L}^p_{\sigma,\tau}(\Omega) \,=\, \Big\{\boldsymbol{f} \in \boldsymbol{L}^p(\Omega);\ \mathrm{div}\,\boldsymbol{f} = 0 \ \text{ in } \ \Omega,\ \ \boldsymbol{f}\cdot\boldsymbol{n} = 0 \ \text{ on } \ \Gamma\Big\}.$$

Endowed the $L^p(\Omega)$ norm, it is a Banach space. We also define

$$\boldsymbol{V}^p_0(\Omega) \,=\, \big\{\boldsymbol{v} \in \boldsymbol{W}^{1,p}_0(\Omega);\, \mathrm{div}\,\boldsymbol{v} = 0 \ \text{ in } \Omega\big\}$$

which is a Banach space for the norm of $\boldsymbol{W}^{1,p}(\Omega)$. For every $\boldsymbol{u} \in \boldsymbol{V}^p_0(\Omega)$ we define the Stokes operator with Dirichlet boundary condition by

$$\forall\, \boldsymbol{v} \in \boldsymbol{V}^{p'}_0(\Omega), \qquad \langle A\boldsymbol{u}\,,\, \boldsymbol{v}\rangle_{(\boldsymbol{V}^{p'}_0(\Omega))'\times\boldsymbol{V}^{p'}_0(\Omega)} \,=\, \int_\Omega \nabla\boldsymbol{u} : \nabla\overline{\boldsymbol{v}}\,\mathrm{d}\,x.$$

Notice that, we can also define the Stokes operator with Dirichlet boundary condition by

$$A \,:\, \mathbf{D}(A) \subset \boldsymbol{L}^p_{\sigma,\tau}(\Omega) \longrightarrow \boldsymbol{L}^p_{\sigma,\tau}(\Omega),$$

where $\mathbf{D}(A) \,=\, \boldsymbol{W}^{2,p}(\Omega) \cap \boldsymbol{W}^{1,p}_0(\Omega) \cap \boldsymbol{L}^p_\sigma(\Omega)$ and $A \,=\, -\,P\Delta$. We recall that

$$P : \boldsymbol{L}^p(\Omega) \longrightarrow \boldsymbol{L}^p_{\sigma,\tau}(\Omega) \tag{3.1}$$

is the Helmholtz projection defined by,

$$\forall \boldsymbol{f} \in \boldsymbol{L}^p(\Omega), \qquad P\boldsymbol{f} \,=\, \boldsymbol{f} \,-\, \mathbf{grad}\,\pi, \tag{3.2}$$

where π is the unique solution of Problem (2.6). This means that, the Stokes operator is defined by :

$$\boldsymbol{u} \in \mathbf{D}(A), \qquad A\boldsymbol{u} \,=\, -P\Delta\boldsymbol{u} \,=\, -\Delta\boldsymbol{u} \,+\mathbf{grad}\,\pi,$$

where π is the unique solution up to an additive constant of the problem

$$\mathrm{div}(\mathbf{grad}\,\pi \,-\, \Delta\boldsymbol{u}) = 0 \qquad \text{in } \Omega, \qquad (\mathbf{grad}\,\pi \,-\, \Delta\boldsymbol{u})\cdot\boldsymbol{n} = 0 \qquad \text{on } \ \Gamma. \tag{3.3}$$

3.2 The Stokes operator with Navier-type boundary conditions

In this Section we consider three different Stokes operators with Navier type boundary conditions.

When Ω is not simply-connected, the Stokes operator with boundary condition (1.3) has a non trivial kernel included in all the L^p spaces for $p \in (1, \infty)$. It may be caracterised as follows:

$$\boldsymbol{K}_\tau(\Omega) \,=\, \big\{\boldsymbol{v} \in \boldsymbol{X}^p_\tau(\Omega);\ \operatorname{div} \boldsymbol{v} = 0,\ \mathbf{curl}\, \boldsymbol{v} = \mathbf{0} \text{ in } \Omega\big\}. \tag{3.4}$$

It has been proved that his kernel is actually independent of p (cf. [9], for $p = 2$ and [13] for $p \in (1, \infty)$), is of finite dimension $J \geq 1$ and spanned by the functions $\widetilde{\mathbf{grad}}\, q^\tau_j$, $1 \leq j \leq J$, (see [9, proposition 3.14]). For all $1 \leq j \leq J$, the function $\widetilde{\mathbf{grad}}\, q^\tau_j$ is the extension by continuity of $\mathbf{grad}\, q^\tau_j$ to Ω, with q^τ_j is the unique solution up to an additive constant of the problem:

$$\left\{\begin{array}{rl} -\Delta q^\tau_j = 0 & \text{in } \Omega^\circ, \\ \partial_n q^\tau_j = 0 & \text{on } \Gamma, \\ \Big[q^\tau_j\Big]_k = \text{constant}, & 1 \leq k \leq J, \\ \Big[\partial_n q^\tau_j\Big]_k = 0; & 1 \leq k \leq J, \\ \langle \partial_n q^\tau_j\,,\, 1\rangle_{\Sigma_k} = \delta_{jk}, & 1 \leq k \leq J. \end{array}\right. \tag{3.5}$$

We recall that, for all $1 \leq j \leq J$, the product $\langle \cdot\, . \cdot \rangle_{\Sigma_j}$ is the duality product between $\boldsymbol{W}^{-\frac{1}{p},p}(\Sigma_j)$ and $\boldsymbol{W}^{1-\frac{1}{p'},p'}(\Sigma_j)$.

3.2.1 The Stokes operator with Navier-type conditions on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$

Consider the space

$$\boldsymbol{X}^p_{\sigma,\tau}(\Omega) \,=\, \Big\{\boldsymbol{v} \in \boldsymbol{X}^p_\tau(\Omega);\ \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega\Big\}, \tag{3.6}$$

which is a Banach space for the norm $\boldsymbol{X}^p(\Omega)$. We recall that $\boldsymbol{X}^p_{\sigma,\tau}(\Omega)$ is a closed subspace of $\boldsymbol{X}^p_\tau(\Omega)$ and on $\boldsymbol{X}^p_{\sigma,\tau}(\Omega)$ the norm of $\boldsymbol{X}^p_\tau(\Omega)$ is equivalent to the norm of $\boldsymbol{W}^{1,p}(\Omega)$.

Let $\boldsymbol{u} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ be fixed and consider the mapping

$$\begin{array}{rccl} A_p\boldsymbol{u} & : & \boldsymbol{W} & \longrightarrow \mathbb{C} \\ & & \boldsymbol{v} & \longmapsto -\displaystyle\int_\Omega \boldsymbol{u} \cdot \Delta\overline{\boldsymbol{v}}\, \mathrm{d}\, x, \end{array}$$

where

$$\boldsymbol{W} = \boldsymbol{X}^{p'}_{\sigma,\tau}(\Omega) \cap \boldsymbol{W}^{2,p'}(\Omega).$$

It is clear that $A_p \in \mathcal{L}(\boldsymbol{L}^p_{\sigma,\tau}(\Omega), \boldsymbol{W}')$ and thanks to de Rham's Lemma there exists $\pi \in W^{-1,p}(\Omega)$ such that

$$A_p\boldsymbol{u} + \Delta\boldsymbol{u} = \nabla\pi \qquad \text{in} \quad \Omega.$$

Now suppose that $\boldsymbol{u} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ and $A_p\boldsymbol{u} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$. Since $\Delta\boldsymbol{u} = -A_p\boldsymbol{u} + \nabla\pi$, then thanks to [12, Lemma 4.14] $\mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{n} \in \boldsymbol{W}^{-1-1/p,p}(\Gamma)$. Moreover if we suppose that $\mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{n} = \mathbf{0}$ on Γ then $(\boldsymbol{u}, \pi) \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega) \times W^{-1,p}(\Omega)$ is a solution of the problem

$$\begin{cases} -\Delta\boldsymbol{u} + \nabla\pi = A_p\boldsymbol{u}, & \operatorname{div}\boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u}\cdot\boldsymbol{n} = 0, & \mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} & \text{on } \Gamma. \end{cases}$$

As a result using the regularity of the Stokes Problem [12, Theorem 4.8] one has $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$.

The operator $A_p : \mathbf{D}(A_p) \subset \boldsymbol{L}^p_{\sigma,\tau}(\Omega) \longrightarrow \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ is a linear operator with

$$\mathbf{D}(A_p) = \Big\{\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega);\ \operatorname{div}\boldsymbol{u} = 0 \text{ in } \Omega,\ \boldsymbol{u}\cdot\boldsymbol{n} = 0,\ \mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} \text{ on } \Gamma\Big\}, \tag{3.7}$$

provided that Ω is of class $C^{2.1}$ (cf. [5]). Moreover

$$\forall\ \boldsymbol{u} \in \mathbf{D}(A_p), \qquad A_p\boldsymbol{u} = -\Delta\boldsymbol{u} + \mathbf{grad}\,\pi, \tag{3.8}$$

where π is the unique solution up to an additive constant of the problem

$$\operatorname{div}(\mathbf{grad}\,\pi - \Delta\boldsymbol{u}) = 0 \qquad \text{in } \Omega, \qquad (\mathbf{grad}\,\pi - \Delta\boldsymbol{u})\cdot\boldsymbol{n} = 0 \qquad \text{on } \Gamma.$$

Observe that for all $\boldsymbol{u} \in \mathbf{D}(A_p)$ and for all $\boldsymbol{v} \in \boldsymbol{X}^{p'}_{\sigma,\tau}(\Omega)$ one has

$$\int_\Omega A_p\,\boldsymbol{u}\cdot\overline{\boldsymbol{v}}\,\mathrm{d}\,x = \int_\Omega \mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\overline{\boldsymbol{v}}\,\mathrm{d}\,x.$$

It easily follows that $(A_p)^* = A_{p'}$.

Notice also that for all $1 < p, q < \infty$ and $\boldsymbol{u} \in \mathbf{D}(A_p) \cap \mathbf{D}(A_q)$, $A_p\boldsymbol{u} = A_q\boldsymbol{u}$. We also recall the following propositions, see [5, Proposition 3.1] for the proof.

Proposition 3.1. *For all $\boldsymbol{u} \in \mathbf{D}(A_p)$, $A_p\boldsymbol{u} = -\Delta\boldsymbol{u}$.*

Remark 3.2. Unlike the Stokes operator with Dirichlet boundary condition, we observe that here the pressure is constant, while with Dirichlet boundary condition the pressure cannot be a constant since it is the solution of the Problem (3.3).

In the rest of this paper we will consider the Stokes operator with Navier-type boundary conditions (1.3). We end this section by the following propositions (see [5, Proposition 3.2, Proposition 3.3] for the proof):

Proposition 3.3. *The space $\mathbf{D}(A_p)$ is dense in $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$.*

Remark 3.4. (i) Notice that, thanks to Lemmas 2.1 and 2.2, since Ω is of class $C^{2,1}$ we have

$$\forall\,\boldsymbol{u} \in \mathbf{D}(A_p), \qquad \|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} \simeq \|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} + \|\Delta\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}.$$

(ii) We recall that, thanks to [12, Proposition 4.7], since Ω is of class $C^{2,1}$, for all $\boldsymbol{u} \in \mathbf{D}(A_p)$ such that $\langle\boldsymbol{u}\cdot\boldsymbol{n}\,,\,1\rangle_{\Sigma_j} = 0$, $1 \le j \le J$ we have

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} \simeq \|\Delta\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}.$$

Proposition 3.5. *Suppose that Ω is not simply connected. The range $R(A_p)$ of the Stokes operator is not dense in $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$.*

Proof. Since the domain Ω is not simply connected, the dimension of the kernel $\boldsymbol{K}_\tau(\Omega)$ of the Stokes operator $A_{p'}$ on $\boldsymbol{L}^{p'}_{\sigma,\tau}(\Omega)$ is finite and greater than or equal to 1. Suppose then that the range $R(A_p)$ is dense in $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$. Using the fact that $(A_p)^* = A_{p'}$ and that

$$\boldsymbol{L}^p_{\sigma,\tau}(\Omega) = \overline{R(A_p)} = [Ker(A_{p'})]^\perp,$$

where

$$Ker(A_{p'}) = \Big\{\boldsymbol{v} \in \mathbf{D}(A_{p'});\ A_{p'}\boldsymbol{v} = \mathbf{0} \text{ in } \Omega\Big\},$$

and

$$[Ker(A_{p'})]^\perp = \Big\{\boldsymbol{f} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega);\ \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} = 0,\ \forall \boldsymbol{v} \in Ker(A_{p'})\Big\}.$$

we obtain that

$$Ker(A_{p'}) = \boldsymbol{K}_\tau(\Omega) = \{0\},$$

which is a contradiction. □

3.2.2 The Stokes operator with Navier-type conditions on $[\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)]'_{\sigma,\tau}$

Consider now the space:

$$\boldsymbol{E} = \{\boldsymbol{f} \in [\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)]';\ \mathrm{div}\,\boldsymbol{f} \in L^p(\Omega)\},$$

which is a Banach space with the norm

$$\|\boldsymbol{f}\|_{\boldsymbol{E}} = \|\boldsymbol{f}\|_{[\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)]'} + \|\mathrm{div}\,\boldsymbol{f}\|_{L^p(\Omega)}. \tag{3.9}$$

We introduce also the following space:

$$\mathcal{D}(\overline{\Omega}) = \{v_{|\Omega};\ v \in \mathcal{D}(\mathbb{R}^3)\}.$$

Lemma 3.6. *The space $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\boldsymbol{E}$.*

Proof. Let $\boldsymbol{\ell} \in \boldsymbol{E}'$ such that $\langle \boldsymbol{\ell}\,,\,\boldsymbol{v}\rangle_{\boldsymbol{E}'\times\boldsymbol{E}} = 0$ for all $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}(\overline{\Omega})$ and let us show that $\boldsymbol{\ell}$ is null in $\boldsymbol{E}$. We know that there exists a function $\boldsymbol{u}$ in $\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)$ and a function χ in $L^{p'}(\Omega)$ such that for all $\boldsymbol{f}$ in $\boldsymbol{E}$ one has:

$$\langle \boldsymbol{\ell}\,,\,\boldsymbol{f}\rangle_{\boldsymbol{E}'\times\boldsymbol{E}} = \langle \boldsymbol{f}\,,\,\boldsymbol{u}\rangle_{[\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)]'\times\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)} + \int_\Omega \mathrm{div}\,\boldsymbol{f}\,\overline{\chi}\,\mathrm{d}\,x. \tag{3.10}$$

We denote by $\widetilde{\boldsymbol{u}}$ and $\widetilde{\chi}$ the extension of $\boldsymbol{u}$ and χ by zero to $\mathbb{R}^3$. As a result for every $\boldsymbol{f} \in \boldsymbol{\mathcal{D}}(\mathbb{R}^3)$ one has

$$\langle \boldsymbol{f}\,,\,\widetilde{\boldsymbol{u}}\rangle_{[\boldsymbol{H}^{p'}_0(\mathrm{div},\mathbb{R}^3)]'\times\boldsymbol{H}^{p'}_0(\mathrm{div},\mathbb{R}^3)} + \int_{\mathbb{R}^3} \mathrm{div}\,\boldsymbol{f}\,\overline{\widetilde{\chi}}\,\mathrm{d}\,x = 0.$$

Then $\widetilde{\boldsymbol{u}} = \nabla\,\widetilde{\chi}$ and $\boldsymbol{u} = \nabla\,\chi$. This means that $\widetilde{\chi} \in L^{p'}(\mathbb{R}^3)$ and $\nabla\,\widetilde{\chi} \in \boldsymbol{H}_0^{p'}(\mathrm{div},\,\mathbb{R}^3)$. Then $\widetilde{\chi} \in W^{2,p'}(\mathbb{R}^3)$ and $\chi \in W_0^{2,p'}(\Omega)$. Now since $\mathcal{D}(\Omega)$ dense in $W_0^{2,p'}(\Omega)$ there exists a sequence $(\chi_k)_k$ in $\mathcal{D}(\Omega)$ that converges to χ in $W^{2,p'}(\Omega)$. Finally for all $\boldsymbol{f} \in \boldsymbol{E}$ one has:

$$\begin{aligned}
\langle \boldsymbol{\ell}\,,\,\boldsymbol{f}\rangle_{\boldsymbol{E}'\times\boldsymbol{E}} &= \langle \boldsymbol{f}\,,\,\boldsymbol{u}\rangle_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'\times\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)} + \int_\Omega \mathrm{div}\,\boldsymbol{f}\,\overline{\chi}\,\mathrm{d}\,x.\\
&= \lim_{k\to+\infty}\Big[\langle \boldsymbol{f}\,,\,\nabla\,\chi_k\rangle_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'\times\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)} + \int_\Omega \mathrm{div}\,\boldsymbol{f}\,\overline{\chi_k}\,\mathrm{d}\,x.\Big]\\
&= 0.
\end{aligned}$$

□

The following Corollary gives us the normal trace of a function $\boldsymbol{f}$ in $\boldsymbol{E}$.

Corollary 3.7. *The linear mapping $\gamma\,:\,\boldsymbol{f}\longmapsto\boldsymbol{f}\cdot\boldsymbol{n}$ defined on $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ can be extended to a linear continuous mapping still denoted by $\gamma\,:\,\boldsymbol{E}\longrightarrow W^{-1-1/p,p}(\Gamma)$. Moreover we have the following Green formula: for all $\boldsymbol{f}\in\boldsymbol{E}$ and for all $\chi\in W^{2,p'}(\Omega)$ such that $\frac{\partial\chi}{\partial\boldsymbol{n}}=0$ on Γ,*

$$\int_\Omega(\mathrm{div}\,\boldsymbol{f})\,\overline{\chi}\,d\,x\;=\;-\,\langle\boldsymbol{f}\,,\,\nabla\,\chi\rangle_\Omega\;+\;\langle\boldsymbol{f}\cdot\boldsymbol{n}\,,\,\chi\rangle_\Gamma, \tag{3.11}$$

where $\langle\cdot\,,\,\cdot\rangle_\Omega=\langle\cdot\,,\,\cdot\rangle_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'\times\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)}$ and $\langle\cdot\,,\,\cdot\rangle_\Gamma\;=\;\langle\cdot\,,\,\cdot\rangle_{W^{-1-1/p,p}(\Gamma)\times W^{1+1/p,p'}(\Gamma)}$.

Now we consider the space

$$[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'_{\sigma,\tau}\;=\;\big\{\boldsymbol{f}\in[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]';\;\mathrm{div}\,\boldsymbol{f}=0\text{ in }\Omega,\;\boldsymbol{f}\cdot\boldsymbol{n}=0\text{ on }\Gamma\big\}. \tag{3.12}$$

We define the operator

$$B_p\;:\;\mathbf{D}(B_p)\subset[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'_{\sigma,\tau}\longmapsto[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'_{\sigma,\tau},$$

by

$$\forall\,\boldsymbol{u}\in\mathbf{D}(B_p),\qquad B_p\,\boldsymbol{u}=-\Delta\boldsymbol{u}\qquad\text{in }\Omega. \tag{3.13}$$

The domain of B_p is given by

$$\begin{aligned}\mathbf{D}(B_p)\;=\big\{\boldsymbol{u}\in\boldsymbol{W}^{1,p}(\Omega);\;\Delta\boldsymbol{u}\in[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'\;\mathrm{div}\,\boldsymbol{u}=0\text{ in }\Omega,\\ \boldsymbol{u}\cdot\boldsymbol{n}\;=\;0,\;\;\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n}\;=\;\mathbf{0}\text{ on }\Gamma\big\}.\end{aligned} \tag{3.14}$$

Remark 3.8. The operator B_p is the extension of the Stokes operator to $[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'_{\sigma,\tau}$.

Proposition 3.9. *The space $\boldsymbol{\mathcal{D}}_\sigma(\Omega)$ is dense in $[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'_{\sigma,\tau}$.*

Proof. Let $\boldsymbol{\ell}$ be a linear form on $[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'_{\sigma,\tau}$ such that $\boldsymbol{\ell}$ vanishes on $\boldsymbol{\mathcal{D}}_\sigma(\Omega)$ and let us show that $\boldsymbol{\ell}$ is null on $[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'_{\sigma,\tau}$. Thanks to the Hahn-Banach theorem, $\boldsymbol{\ell}$ can be extended to a linear continuous form on $[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'$ denoted by $\widetilde{\boldsymbol{\ell}}$. Moreover

$$\forall\,\boldsymbol{f}\in[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'_{\sigma,\tau},\qquad\boldsymbol{\ell}(\boldsymbol{f})\;=\;\langle\widetilde{\boldsymbol{\ell}}\,,\,\boldsymbol{f}\rangle_{\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)\times[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'}.$$

Since $\boldsymbol{\ell}$ vanishes on $\boldsymbol{\mathcal{D}}_\sigma(\Omega)$ then thanks to De-Rham lemma there exists a function $\pi \in W^{2,p'}(\Omega)$ such that $\frac{\partial \pi}{\partial \boldsymbol{n}} = 0$ on Γ and $\widetilde{\boldsymbol{\ell}} = \nabla \pi$ in Ω. Now let $\boldsymbol{f} \in [\boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega)]'_{\sigma,\tau}$ then by Corollary 3.7 we have

$$\begin{aligned} \boldsymbol{\ell}(\boldsymbol{f}) &= \langle \boldsymbol{f}\,,\, \nabla\pi \rangle_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]' \times \boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)} \\ &= -\int_\Omega (\mathrm{div}\, \boldsymbol{f})\, \overline{\pi}\, \mathrm{d}\, x \,+\, \langle \boldsymbol{f}\cdot\boldsymbol{n}\,,\, \pi\rangle_\Gamma \\ &= 0. \end{aligned}$$

□

As a result of Proposition 3.9 we deduce the density of the domain of the operator B_p.

Corollary 3.10. *The operator B_p is a densely defined operator.*

3.2.3 The Stokes operator with Navier-type conditions on $[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}$

Consider the space

$$\boldsymbol{G} \,=\, \big\{ \boldsymbol{f} \in (\boldsymbol{T}^{p'}(\Omega))';\ \mathrm{div}\, \boldsymbol{f} \in L^p(\Omega) \big\},$$

equipped with the graph norm. We skip the proof of the following lemma because it is similar to the proof of Lemma 3.6:

Lemma 3.11. *The space $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\boldsymbol{G}$.*

As in the previous Subsection The following Corollary gives the normal trace of functions in $\boldsymbol{G}$.

Corollary 3.12. *The linear mapping $\gamma : \boldsymbol{f} \longmapsto \boldsymbol{f}\cdot\boldsymbol{n}$ defined on $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ can be extended to a linear continuous mapping still denoted by $\gamma : \boldsymbol{G} \longrightarrow W^{-2-1/p,p}(\Gamma)$. Moreover we have the following Green formula: for all $\boldsymbol{f} \in \boldsymbol{G}$ and for all $\chi \in W^{3,p'}(\Omega)$ such that $\frac{\partial \chi}{\partial \boldsymbol{n}} = 0$ on Γ and $\Delta\chi = 0$ on Γ,*

$$\int_\Omega (\mathrm{div}\, \boldsymbol{f})\, \overline{\chi}\, d\, x \,=\, -\,\langle \boldsymbol{f}\,,\, \nabla\,\chi\rangle_{(\boldsymbol{T}^{p'}(\Omega))' \times \boldsymbol{T}^{p'}(\Omega)} \,+\, \langle \boldsymbol{f}\cdot\boldsymbol{n}\,,\, \chi\rangle_\Gamma. \tag{3.15}$$

We recall that $\langle .\,,\, .\rangle_\Gamma \,=\, \langle .\,,\, .\rangle_{W^{-2-1/p,p}(\Gamma)\times W^{2+1/p,p'}(\Gamma)}$.

Now we consider the space

$$[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau} \,=\, \big\{ \boldsymbol{f} \in (\boldsymbol{T}^{p'}(\Omega))';\ \mathrm{div}\, \boldsymbol{f} = 0 \text{ in } \Omega,\ \boldsymbol{f}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma \big\}. \tag{3.16}$$

Next, we consider the operator:

$$C_p \,:\, \mathbf{D}(C_p) \subset [\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau} \longrightarrow [\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau},$$

defined by

$$\forall\, \boldsymbol{u} \in \mathbf{D}(C_p), \qquad C_p\, \boldsymbol{u} = -\Delta \boldsymbol{u} \qquad \text{in}\Omega. \tag{3.17}$$

The domain of C_p is given by

$$\mathbf{D}(C_p) = \big\{\boldsymbol{u} \in \boldsymbol{L}^p(\Omega);\ \Delta\boldsymbol{u} \in (\boldsymbol{T}^{p'}(\Omega))',\ \operatorname{div}\boldsymbol{u} = 0 \text{ in } \Omega,\ \boldsymbol{u}\cdot\boldsymbol{n} = 0,\ \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n} = \mathbf{0} \text{ on } \Gamma\big\}. \tag{3.18}$$

Remark 3.13. The operator C_p is the extension of the stokes operator to $[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}$.

We skip the proof of the following proposition because it is similar to the proof of Proposition 3.9:

Proposition 3.14. *The space $\boldsymbol{\mathcal{D}}_\sigma(\Omega)$ is dense in $[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}$.*

4 Analyticity results

In this section we will state our main result and its proof. We will prove that the Stokes operator with Navier-type boundary conditions (1.3) generates a bounded analytic semi-group on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, $[\boldsymbol{H}^{p'}_0(\operatorname{div},\Omega)]'_{\sigma,\tau}$ and $[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}$ respectively for all $1 < p < \infty$.

4.1 Analyticity on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$

In this subsection, we review the main results of [5] concerning the analyticity of the semi-group generated by the Stokes operator with Navier-type boundary conditions A_p on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, (see [5] for the proof).

4.1.1 The Hilbertian case

The results of [5] on the resolvent of the Stokes operator are obtained considering the problem (1.6), that we recall here:

$$\begin{cases} \lambda\boldsymbol{u} - \Delta\boldsymbol{u} = \boldsymbol{f},\ \operatorname{div}\boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u}\cdot\boldsymbol{n} = 0,\ \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n} = \mathbf{0} & \text{on } \Gamma, \end{cases} \tag{4.1}$$

where $\boldsymbol{f} \in \boldsymbol{L}^2_{\sigma,\tau}(\Omega)$ and $\lambda \in \Sigma_\varepsilon$.

Remark 4.1. Observe that, Problem (4.1) is equivalent to the problem

$$\begin{cases} \lambda\boldsymbol{u} - \Delta\boldsymbol{u} = \boldsymbol{f}, & \text{in } \Omega, \\ \boldsymbol{u}\cdot\boldsymbol{n} = 0, \quad \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n} = \mathbf{0} & \text{on } \Gamma. \end{cases} \tag{4.2}$$

Let $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ be the unique solution of Problem (4.2) and set $\operatorname{div}\boldsymbol{u} = \chi$. It is clear that $\lambda\chi - \Delta\chi = 0$ in Ω. Moreover, since $\boldsymbol{f}\cdot\boldsymbol{n} = 0$ and $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on Γ then $\Delta\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on Γ. Notice also that the condition $\mathbf{curl}\boldsymbol{u}\times\boldsymbol{n} = \mathbf{0}$ on Γ implies that $\mathbf{curl}\,\mathbf{curl}\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on Γ. Finally since $\Delta\boldsymbol{u} = \mathbf{grad}(\operatorname{div}\boldsymbol{u}) - \mathbf{curl}\,\mathbf{curl}\boldsymbol{u}$ one gets $\frac{\partial\chi}{\partial\boldsymbol{n}} = 0$ on Γ. Thus $\chi = 0$ in Ω and the result is proved.

We have the following theorem, for the proof see [5, Theorem 4.3].

Theorem 4.2. *Let $\varepsilon \in]0,\pi[$ be fixed, $\boldsymbol{f} \in \boldsymbol{L}^2_{\sigma,\tau}(\Omega)$ and $\lambda \in \Sigma_\varepsilon$.*
(i) *The Problem (4.1) has a unique solution $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$.*
(ii) *There exist a constant $C'_\varepsilon > 0$ independent of $\boldsymbol{f}$ and λ such that the solution $\boldsymbol{u}$ satisfies the estimates*

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \leq \frac{C'_\varepsilon}{|\lambda|}\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} \tag{4.3}$$

and

$$\|\mathbf{curl}\,\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \leq \frac{C'_\varepsilon}{\sqrt{|\lambda|}}\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}. \tag{4.4}$$

(iii) *The solution $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega)$ and satisfies the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)} \leq \frac{C(\Omega,\lambda,\varepsilon)}{|\lambda|}\,\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}, \tag{4.5}$$

where $C(\Omega,\lambda,\varepsilon) = C(\Omega)(C'_\varepsilon + 1)(|\lambda| + 1)$.

Remark 4.3. We note that for $\lambda > 0$ the constant C'_ε is equal to 1 and we recover the m-accretive property of the stokes operator on $\boldsymbol{L}^2_{\sigma,\tau}(\Omega)$.

Remark 4.4. Consider the sesqui-linear form:

$$\forall\, \boldsymbol{u},\, \boldsymbol{v} \in \boldsymbol{X}^2_{\sigma,\tau}(\Omega), \qquad a(\boldsymbol{u},\boldsymbol{v}) \,=\, \int_\Omega \mathbf{curl}\,\boldsymbol{u} \cdot \mathbf{curl}\,\overline{\boldsymbol{v}}\,\mathrm{d}\,x. \tag{4.6}$$

If Ω is simply connected, we know that (see [9, Corollary 3.16]) for all $\boldsymbol{v} \in \boldsymbol{X}^2_{\sigma,\tau}(\Omega)$ one has

$$\|\boldsymbol{v}\|_{\boldsymbol{X}^2(\Omega)} \leq C\,\|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}. \tag{4.7}$$

As a result, the sesqui-linear form a is coercive and we can apply Lax-Milgram Lemma to find solution to the problem: find $\boldsymbol{u} \in \boldsymbol{X}^2_{\sigma,\tau}(\Omega)$ such that for all $\boldsymbol{v} \in \boldsymbol{X}^2_{\sigma,\tau}(\Omega)$

$$a(\boldsymbol{u},\boldsymbol{v}) \,=\, \int_\Omega \boldsymbol{f} \cdot \overline{\boldsymbol{v}}\,\mathrm{d}\,x,$$

where $\boldsymbol{f} \in \boldsymbol{L}^2_{\sigma,\tau}(\Omega)$. This means that the operator $A_2 : \mathbf{D}(A_2) \subset \boldsymbol{L}^2_{\sigma,\tau}(\Omega) \longrightarrow \boldsymbol{L}^2_{\sigma,\tau}(\Omega)$ is bijective.

Now, if Ω is multiply-connected, the inequality (4.7) is false because the kernel $K_\tau(\Omega)$ of the Stokes operator with Navier-type boundary conditions is not trivial (cf. [9]). It is also proved in [9], that for all $\boldsymbol{v} \in \boldsymbol{X}^2_\tau(\Omega)$ we have instead the following Poincaré-type inequality:

$$\|\boldsymbol{v}\|_{\boldsymbol{X}^2_\tau(\Omega)} \leq C_2(\Omega)(\|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} \,+\, \|\mathrm{div}\,\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} \,+\, \sum_{j=1}^{J} |\langle \boldsymbol{v}\cdot\boldsymbol{n}\,,\,1\rangle_{\Sigma_j}|). \tag{4.8}$$

As a consequence of Theorem 4.2 we have the following theorem

Theorem 4.5. *The operator $-A_2$ generates a bounded analytic semi-group on $\boldsymbol{L}^2_{\sigma,\tau}(\Omega)$.*

Remark 4.6. We recall that the restriction of an analytic semi-group to the non negative real axis is a C_0 semi-group. Thanks to Remark 4.3 the restriction of our analytic semi-group to the real axis gives a C_0 semi-group of contraction.

The following proposition gives the eigenvalues of the Stokes operator. We will see later that the following proposition allows us to obtain an explicit form for the unique solution of the homogeneous Stokes Problem (7.2) as a linear combination of the eigenfunctions of the Stokes operator.

Proposition 4.7. *There exists a sequence of functions $(\boldsymbol{z}_k)_k \subset \mathbf{D}(A_2)$ and an increasing sequence of real numbers $(\lambda_k)_k$ such that $\lambda_k \geq 0$, $\lambda_k \to +\infty$ as $k \to +\infty$ and*

$$\forall\, \boldsymbol{v} \in \boldsymbol{X}^2_\tau(\Omega), \qquad \int_\Omega \mathbf{curl}\, \boldsymbol{z}_k \cdot \mathbf{curl}\, \overline{\boldsymbol{v}}\, \mathrm{d}\, x \;=\; \lambda_k \int_\Omega \boldsymbol{z}_k \cdot \overline{\boldsymbol{v}}\, \mathrm{d}\, x.$$

In other words, $(\lambda_k)_k$ are the eigenvalues of the Stokes operator and $(\boldsymbol{z}_k)_k$ are the associated eigenfunctions.

Proof. Consider the operator

$$\Lambda \;:\; \boldsymbol{L}^2_{\sigma,\tau}(\Omega) \longrightarrow \mathbf{D}(A_2) \hookrightarrow \boldsymbol{L}^2_{\sigma,\tau}(\Omega)$$

$$\boldsymbol{f} \longmapsto \boldsymbol{u}$$

where $\boldsymbol{u}$ is the unique solution of the problem

$$\left\{ \begin{array}{lll} \boldsymbol{u} + A_2 \boldsymbol{u} = \boldsymbol{f}, & \mathrm{div}\, \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0, & \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} & \text{on } \Gamma. \end{array} \right.$$

Thanks to Theorem 4.2, we know that Λ is a bounded linear operator from $\boldsymbol{L}^2_{\sigma,\tau}(\Omega)$ into itself. Moreover, thanks to Lemma 2.1 and the compact embedding of $\boldsymbol{H}^1(\Omega)$ in $\boldsymbol{L}^2(\Omega)$, the canonical embedding $\mathbf{D}(A_2) \hookrightarrow \boldsymbol{L}^2_{\sigma,\tau}(\Omega)$ is compact. Equivalently, the operator Λ is compact from $\boldsymbol{L}^2_{\sigma,\tau}(\Omega)$ into itself. Moreover we can easily verify that this operator is also a self adjoint operator. Thus $\boldsymbol{L}^2_{\sigma,\tau}(\Omega)$ has a Hilbertian basis formed from the eigenvectors of the operator Λ. Then, there exists a sequence of real numbers $(\mu_k)_{k\geqslant 0}$ and eigenfunctions $(\boldsymbol{z}_k)_{k\geqslant 0}$ such that $\Lambda \boldsymbol{z}_k = \mu_k\, \boldsymbol{z}_k$ and $\mu_k \longrightarrow 0$ as $k \to +\infty$. This means that $-\mu_k\, \Delta \boldsymbol{z}_k + \mu_k\, \boldsymbol{z}_k = \boldsymbol{z}_k$. Note that $0 < \mu_k \leq 1$. As a result $A_2\, \boldsymbol{z}_k = \lambda_k\, \boldsymbol{z}_k$, where $\lambda_k = \frac{1}{\mu_k} - 1$ and $\lambda_k \longrightarrow +\infty$ as $k \to +\infty$. In conclusion $(\boldsymbol{z}_k)_k$ is a sequence of eigenfunctions of the Stokes operator associated to the eigenvalues $(\lambda_k)_k$. □

Remark 4.8. As a consequence of Proposition 4.7, $\boldsymbol{L}^2_{\sigma,\tau}(\Omega)$ can be written in the form

$$\boldsymbol{L}^2_{\sigma,\tau}(\Omega) \;=\; \mathbf{Ker} A_2 \bigoplus_{k=1}^{+\infty} \mathbf{Ker}(\lambda_k\, I \,-\, A_2).$$

In other words, any vector $\boldsymbol{v} \in \boldsymbol{L}^2_{\sigma,\tau}(\Omega)$ can be written in the form

$$\boldsymbol{v} \,=\, \sum_{k=1}^{J} \alpha_k \widetilde{\mathbf{grad}}\, q_k^\tau \,+\, \sum_{k=1}^{+\infty} \beta_k \boldsymbol{z}_k,$$

where $(\widetilde{\mathbf{grad}}\, q_k^\tau)_{1\le k\le J}$ is a basis for $\mathbf{ker}\, A_2 \,=\, \boldsymbol{K}^2_\tau(\Omega)$ and $\forall\, k \in \mathbb{N},\ \ \boldsymbol{z}_k \in \mathbf{ker}\,(\lambda_k\, I - A_2)$. We recall that J is the dimension of $\mathbf{ker}\, A_2 \,=\, \boldsymbol{K}^2_\tau(\Omega)$, (see [9]).

As described above, when Ω is simply-connected, $\boldsymbol{K}^2_\tau(\Omega) \,=\, \{\mathbf{0}\}$, $\lambda_0 \,=\, 0$ is not an eigenvalue and the Stokes operator is bijective from $\mathbf{D}(A_2)$ into $\boldsymbol{L}^2_{\sigma,\tau}(\Omega)$ with bounded and compact inverse. In this case,

$$\boldsymbol{L}^2_{\sigma,\tau}(\Omega) \,=\, \bigoplus_{k=1}^{+\infty} \mathbf{Ker}(\lambda_k\, I \,-\, A_2),$$

where $(\lambda_k)_{k\ge1}$ are the eigenvalues of the Stokes operator and $(\boldsymbol{z_k})_k$ are the eigenfunctions associated to eigenvalues $(\lambda_k)_{k\ge1}$. Moreover, the sequence $(\lambda_k)_{k\ge1}$ is an increasing sequence of positive real numbers and the first eigenvalue λ_1 is equal to $\frac{1}{C_2(\Omega)}$ where $C_2(\Omega)$ is the constant that comes from the Poincaré-type inequality (4.8).

4.1.2 L^p-theory

This subsection extends Theorem 4.2 to every $1 < p < \infty$. Theorem 4.9 gives the well posedness of the resolvent Problem (4.1) in $\boldsymbol{L}^p(\Omega)$, while Theorem 4.10 extends estimates (4.3-4.5) to all $1 < p < \infty$ (see [5, Theorem 4.8, Theorem 4.11] for the proof).

Theorem 4.9. *Let $\lambda \in \Sigma_\varepsilon$, with $0 < \varepsilon < \pi/2$, and let $\boldsymbol{f} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$. The Problem (4.1) has a solution $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$. Moreover, this solution is unique in $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$.*

Theorem 4.10. *Let $\lambda \in \mathbb{C}^*$ such that $\mathrm{Re}\,\lambda \,\ge\, 0$, let $1 < p < \infty$, $\boldsymbol{f} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ and let $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ be the unique solution of Problem (4.1). Then $\boldsymbol{u}$ satisfies the estimates*

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \,\le\, \frac{\kappa_1(\Omega,p)}{|\lambda|}\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}, \tag{4.9}$$

$$\|\mathbf{curl}\,\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \,\le\, \frac{\kappa_2(\Omega,p)}{\sqrt{|\lambda|}}\,\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \tag{4.10}$$

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} \,\le\, \kappa_3(\Omega,p)\,\frac{1+|\lambda|}{|\lambda|}\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}. \tag{4.11}$$

where $\kappa_i(\Omega,p)$, $i = 1,2,3$ are positive constants independent of λ and $\boldsymbol{f}$.

As a result we have the following theorem (see [5, Theorem 4.12] for the proof)

Theorem 4.11. *The operator $-A_p$ generates a bounded analytic semi-group on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ for all $1 < p < \infty$.*

Remark 4.12. Consider the two problems:

$$\left\{\begin{array}{ccccc} \lambda \boldsymbol{u} - \Delta \boldsymbol{u} = \boldsymbol{f}, & \operatorname{div} \boldsymbol{u} = 0 & \text{in} & \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} & & \text{on} & \Gamma \end{array}\right. \tag{4.12}$$

and

$$\left\{\begin{array}{ccccc} \lambda \boldsymbol{u} - \Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f}, & \operatorname{div} \boldsymbol{u} = 0 & \text{in} & \Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0, & [\mathbb{D}\boldsymbol{u} \cdot \boldsymbol{n}]_{\boldsymbol{\tau}} = 0 & \text{on} & \Gamma, \end{array}\right. \tag{4.13}$$

where $\lambda \in \mathbb{C}^*$ is such that $\operatorname{Re} \lambda \geq 0$ and $\boldsymbol{f} \in \boldsymbol{L}^p_\sigma(\Omega)$ (respectively $\boldsymbol{f} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$).

In two forthcoming papers we study the two Problems (4.12) and (4.13). Proceeding in a similar way as in [5] we prove that these two Problems have a unique solution $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ (respectively $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times W^{1,p}(\Omega)/\mathbb{R}$) that satisfy the estimate

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq \frac{C(\Omega, p)}{|\lambda|} \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}.$$

Moreover when Ω is of class $C^{2,1}$, we have $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$. This means that the Laplace operator with normal boundary conditions and the Stokes operator with Navier boundary condition generate a bounded analytic semi-group on $\boldsymbol{L}^p_\sigma(\Omega)$ and $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ respectively .

This analyticity allows us to solve the time dependent Stokes Problem with normal boundary condition and pressure boundary condition:

$$\left\{\begin{array}{ccccc} \frac{\partial \boldsymbol{u}}{\partial t} - \Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f}, & \operatorname{div} \boldsymbol{u} = 0 & \text{in} & \Omega \times (0,T), \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0}, & \pi = 0 & \text{on} & \Gamma \times (0,T), \\ & \boldsymbol{u}(0) = \boldsymbol{u}_0 & \text{in} & \Omega, \end{array}\right. \tag{4.14}$$

as well as the time dependent Stokes Problem (1.1) with Navier-boundary condition (1.2) for a given $\boldsymbol{f} \in L^q(0,T; \boldsymbol{L}^p(\Omega))$ and $\boldsymbol{u}_0 \in \boldsymbol{L}^p_\sigma(\Omega)$ (respectively $\boldsymbol{u}_0 \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$).

4.2 Analyticity on $[\boldsymbol{H}^{p'}_0(\operatorname{div}, \Omega)]'_{\sigma,\tau}$

This subsection is devoted to the analyticity of the semi-group generated by the Stokes operator on $[\boldsymbol{H}^{p'}_0(\operatorname{div}, \Omega)]'_{\sigma,\tau}$. This analyticity allows us to obtain the weak solution to the Problem (1.1) with the boundary condition (1.3).

To this end we consider the problem:

$$\left\{\begin{array}{ccccc} \lambda \boldsymbol{u} - \Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f}, & \operatorname{div} \boldsymbol{u} = 0 & \text{in} & \Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0, & \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} & \text{on} & \Gamma, \end{array}\right. \tag{4.15}$$

where $\lambda \in \mathbb{C}^*$ such that $\operatorname{Re} \lambda \geq 0$ and $\boldsymbol{f} \in [\boldsymbol{H}^{p'}_0(\operatorname{div}, \Omega)]'$. The following theorem gives the existence and uniqueness of solution to Problem (4.15):

Theorem 4.13. *Let $\lambda \in \mathbb{C}^*$ such that $\operatorname{Re} \lambda \geq 0$ and let $\boldsymbol{f} \in [\boldsymbol{H}^{p'}_0(\operatorname{div}, \Omega)]'$. The Problem (4.15) has a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$ satisfying*

$$\|\boldsymbol{u}\|_{[\boldsymbol{H}^{p'}_0(\operatorname{div},\Omega)]'} \leq \frac{C(\Omega, p)}{|\lambda|} \|\boldsymbol{f}\|_{[\boldsymbol{H}^{p'}_0(\operatorname{div},\Omega)]'} \tag{4.16}$$

for some constant $C(\Omega, p) > 0$ independent of λ and $\boldsymbol{f}$.

Proof. **(i)** For the existence of solutions for Problem (4.15) we proceed in the same way as in [12, Theorem 4.4], Theorem 4.2 and Theorem 4.9.
(ii) To prove estimate (4.16) we proceed as follows: Consider the problem:

$$\left\{\begin{array}{lll} \lambda \boldsymbol{v} - \Delta \boldsymbol{v} + \nabla\theta = \boldsymbol{F}, & \operatorname{div} \boldsymbol{v} = 0 & \text{in} \quad \Omega, \\ \boldsymbol{v}\cdot\boldsymbol{n} = 0, & \mathbf{curl}\, \boldsymbol{v}\times\boldsymbol{n} = \boldsymbol{0} & \text{on} \quad \Gamma, \end{array}\right. \tag{4.17}$$

where $\boldsymbol{F} \in \boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)$ and $\lambda \in \mathbb{C}^*$ such that $\operatorname{Re}\lambda \geq 0$. Thanks to Lemma 2.5 there exists a unique up to an additive function $\theta \in W^{1,p'}(\Omega)/\mathbb{R}$ solution of

$$\operatorname{div}(\nabla\theta - \boldsymbol{F}) = 0 \qquad \text{in } \Omega \qquad (\nabla\theta - \boldsymbol{F})\cdot\boldsymbol{n} = 0 \qquad \text{on } \Gamma.$$

Moreover the function θ satisfies the estimate

$$\|\nabla\theta\|_{\boldsymbol{L}^{p'}(\Omega)} \leq C(\Omega,p')\,\|\boldsymbol{F}\|_{\boldsymbol{L}^{p'}(\Omega)}.$$

As a result, thanks to Theorem 4.9 and Theorem 4.10, Problem (4.17) has a unique solution $(\boldsymbol{v},\theta) \in \boldsymbol{W}^{1,p'}(\Omega)\times W^{1,p'}(\Omega)/\mathbb{R}$ that satisfies the estimate

$$\|\boldsymbol{v}\|_{\boldsymbol{L}^{p'}(\Omega)} \leq \frac{C(\Omega,p')}{|\lambda|}\,\|\boldsymbol{F}\|_{\boldsymbol{L}^{p'}(\Omega)}.$$

Thus

$$\|\boldsymbol{v}\|_{\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)} \leq \frac{C(\Omega,p')}{|\lambda|}\,\|\boldsymbol{F}\|_{\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)}.$$

Now let $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{1,p}(\Omega)\times \boldsymbol{L}^p(\Omega)/\mathbb{R}$ be the solution of Problem (4.15), then by using (3.11) we have:

$$\begin{aligned} \|\boldsymbol{u}\|_{[\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)]'} &= \sup_{\boldsymbol{F}\in\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega),\boldsymbol{F}\neq 0} \frac{|\langle \boldsymbol{u}\,,\,\boldsymbol{F}\rangle_\Omega|}{\|\boldsymbol{F}\|_{\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)}} \\ &= \sup_{\boldsymbol{F}\in\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega),\boldsymbol{F}\neq 0} \frac{|\langle \boldsymbol{u}\,,\,\lambda\,\boldsymbol{v} - \Delta\boldsymbol{v} - \nabla\theta\rangle_\Omega|}{\|\boldsymbol{F}\|_{\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)}} \\ &= \sup_{\boldsymbol{F}\in\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega),\boldsymbol{F}\neq 0} \frac{|\langle \lambda\,\boldsymbol{u} - \Delta\boldsymbol{u} - \nabla\pi\,,\,\boldsymbol{v}\rangle_\Omega|}{\|\boldsymbol{F}\|_{\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)}} \\ &= \sup_{\boldsymbol{F}\in\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega),\boldsymbol{F}\neq 0} \frac{|\langle \boldsymbol{f}\,,\,\boldsymbol{v}\rangle_\Omega|}{\|\boldsymbol{F}\|_{\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)}} \\ &\leq \frac{C(\Omega,p')}{|\lambda|}\,\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)]'}, \end{aligned}$$

which is estimate (4.16). □

As consequence of Theorem 4.13 we have the following corollary

Corollary 4.14. *Let $\lambda \in \mathbb{C}^*$ such that $\operatorname{Re}\lambda \geq 0$ and let $\boldsymbol{f} \in [\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)]'$ such that $\operatorname{div}\boldsymbol{f} = 0$ in Ω and $\boldsymbol{f}\cdot\boldsymbol{n} = 0$ on Γ. The Problem (4.1) has a unique solution $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ satisfying the estimate (4.16).*

Next, using Proposition 2.6, one gets the analyticity of the semi-group generated by the operator B_p:

Theorem 4.15. *The operator $-B_p$ generates a bounded analytic semi-group on the space $[\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)]'_{\sigma,\tau}$.*

4.3 Analyticity on $[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}$

In this subsection we prove the analyticity of the semi-group generated by the Stokes operator on $[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}$. Using this property we will show the existence of very weak solutions to the Problem (1.1) with the Navier-type boundary condition (1.3). The method and arguments in this Section are very similar to those of the previous one.

The following theorem gives the very weak solution to Problem (4.15).

Theorem 4.16. *Let $\lambda \in \mathbb{C}^*$ such that $\operatorname{Re}\lambda \geq 0$ and let $\boldsymbol{f} \in (\boldsymbol{T}^{p'}(\Omega))'$ then the Problem (4.15) has a unique solution $(\boldsymbol{u},\pi) \in \boldsymbol{L}^p(\Omega) \times W^{-1,p}(\Omega)/\mathbb{R}$. Moreover we have the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq \frac{C(\Omega,p)}{|\lambda|}\,\|\boldsymbol{f}\|_{(\boldsymbol{T}^{p'}(\Omega))'}, \tag{4.18}$$

for some constant $C(\Omega,p) > 0$ independent of λ and $\boldsymbol{f}$.

Proof. **(i)** Thanks to the Green formula (2.5) and to [12, Theorem 4.15] we can easily verify that Problem (4.15) is equivalent to the problem: Find $\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$ such that for all $\boldsymbol{\varphi} \in \mathbf{D}(A_{p'})$ (given by (3.7)) and for all $q \in W^{1,p'}(\Omega)$

$$\begin{array}{c} \lambda \int_\Omega \boldsymbol{u}\cdot\overline{\boldsymbol{\varphi}}\,\mathrm{d}\,x \,-\, \int_\Omega \boldsymbol{u}\cdot\Delta\overline{\boldsymbol{\varphi}}\,\mathrm{d}\,x = \langle \boldsymbol{f},\,\boldsymbol{\varphi}\rangle_{(\boldsymbol{T}^{p'}(\Omega))'\times\boldsymbol{T}^{p'}(\Omega)} \\ \int_\Omega \boldsymbol{u}\cdot\nabla\overline{q}\,\mathrm{d}\,x = 0. \end{array} \tag{4.19}$$

Notice that we recuperate the pressure using the De-Rham argument: if $\boldsymbol{F} \in \boldsymbol{W}^{-2,p}(\Omega)$, the dual of the space $\boldsymbol{W}_0^{2,p'}(\Omega)$, verifying $\langle \boldsymbol{F}\,,\,\boldsymbol{v}\rangle_{\boldsymbol{\mathcal{D}'}(\Omega)\times\boldsymbol{\mathcal{D}}(\boldsymbol{\Omega})} = 0$, for all $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}_\sigma(\Omega)$ then there exists $\chi \in W^{-1,p}(\Omega)$ such that $\boldsymbol{F} = \nabla\,\chi$.

(ii) Let us now solve (4.19). As in the proof of Theorem 4.13, we know that for all $\boldsymbol{F} \in \boldsymbol{L}^{p'}(\Omega)$ the problem:

$$\left\{\begin{array}{ccccc} \lambda\boldsymbol{\varphi} - \Delta\boldsymbol{\varphi} - \nabla\theta = \boldsymbol{F}, & & \operatorname{div}\boldsymbol{\varphi} = 0 & \text{in} & \Omega, \\ \boldsymbol{\varphi}\cdot\boldsymbol{n} = 0, & & \mathbf{curl}\,\boldsymbol{\varphi}\times\boldsymbol{n} = \mathbf{0} & \text{on} & \Gamma, \end{array}\right. \tag{4.20}$$

has a unique solution $(\boldsymbol{\varphi},\theta) \in \mathbf{D}(A_{p'}) \times W^{1,p'}(\Omega)/\mathbb{R}$ that satisfies the estimate

$$\|\boldsymbol{\varphi}\|_{\boldsymbol{L}^{p'}(\Omega)} \leq \frac{C(\Omega,p')}{|\lambda|}\,\|\boldsymbol{F}\|_{\boldsymbol{L}^{p'}(\Omega)}.$$

Now the following linear mapping:

$$\begin{aligned} L \;\; : \;\; & \boldsymbol{L}^{p'}(\Omega) \longrightarrow \mathbb{C} \\ & \boldsymbol{F} \longmapsto \langle \boldsymbol{f}\,,\,\boldsymbol{\varphi}\rangle_{(\boldsymbol{T}^{p'}(\Omega))'\times\boldsymbol{T}^{p'}(\Omega)}, \end{aligned}$$

where $\boldsymbol{\varphi}$ is the unique solution of Problem (4.20), satisfies

$$|L(\boldsymbol{F})| \;\leq\; \|\boldsymbol{f}\|_{(\boldsymbol{T}^{p'}(\Omega))'}\|\boldsymbol{\varphi}\|_{\boldsymbol{L}^{p'}(\Omega)} \;\leq\; \frac{C(\Omega,p')}{|\lambda|}\,\|\boldsymbol{f}\|_{(\boldsymbol{T}^{p'}(\Omega))'}\|\boldsymbol{F}\|_{\boldsymbol{L}^{p'}(\Omega)}.$$

Then there exists a unique $\boldsymbol{u}\in\boldsymbol{L}^{p}(\Omega)$ such that

$$L(\boldsymbol{F}) \;=\; \int_{\Omega}\boldsymbol{u}\cdot\overline{\boldsymbol{F}}\,\mathrm{d}\,x \;=\; \langle \boldsymbol{f}\,,\,\boldsymbol{\varphi}\rangle_{(\boldsymbol{T}^{p'}(\Omega))'\times\boldsymbol{T}^{p'}(\Omega)}$$

and satisfying the estimate (4.18). On other worlds $\boldsymbol{u}$ is the unique solution of Problem (4.19). □

As a consequence of Theorem 4.16 we deduce the existence and uniqueness of very weak solutions to Problem (4.1).

Corollary 4.17. *Let $\lambda\in\mathbb{C}^*$ such that $\operatorname{Re}\lambda\geq 0$ and let $\boldsymbol{f}\in(\boldsymbol{T}^{p'}(\Omega))'$ such that $\operatorname{div}\boldsymbol{f}=0$ in Ω and $\boldsymbol{f}\cdot\boldsymbol{n}=0$ on Γ. The Problem (4.1) has a unique solution $\boldsymbol{u}\in\boldsymbol{L}^{p}(\Omega)$ that satisfies the estimate (4.18).*

As described above, using Proposition 2.6 with $w=0$, we have the analyticity of the semi-group generated by the Stokes operator on $[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}$:

Theorem 4.18. *The operator $-C_p$ is a densely defined operator and it generates a bounded analytic semi-group on $[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}$.*

5 Stokes operator with flux boundary conditions

As we have already mentioned, the Stokes operator with Navier-type boundary conditions in a non simply connected domain has a non trivial finite dimensional kernel $\boldsymbol{K}_{\tau}(\Omega)$. It is then natural to study the Stokes problem on the orthogonal of that kernel. To this end we first consider the Stokes operator on that space. It turns out that, under the assumption of Condition H for the domain Ω, for a function $\boldsymbol{u}\in L^{p}_{\sigma,\tau}(\Omega)$, to be in the orthogonal of $\boldsymbol{K}_{\tau}(\Omega)$ is equivalent to the condition (1.5) (cf. [13], see also [4]). It is then equivalent for our purpose to consider the Stokes problem, with Navier-type boundary conditions, with the supplementary flux condition (1.5).

We then begin this section considering A'_p, the Stokes operator, with Navier-type boundary conditions, and with flux condition (1.5). Its resolvent set is given by the solutions of problem (1.7) that we may recall here:

$$\left\{\begin{array}{rll} \lambda\boldsymbol{u}-\Delta\boldsymbol{u} \;=\; \boldsymbol{f}, & \operatorname{div}\boldsymbol{u}=0 & \text{in }\;\Omega, \\ \boldsymbol{u}\cdot\boldsymbol{n}=0, & \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n}=\boldsymbol{0} & \text{on }\;\Gamma, \\ \langle\boldsymbol{u}\cdot\boldsymbol{n},1\rangle_{\Sigma_j} \;=\; 0, & 1\leq j\leq J. & \end{array}\right. \tag{5.1}$$

The addition of the extra boundary condition on the cuts Σ_j, $1 \leq j \leq J$ makes the Stokes operator invertible on $L^p_{\sigma,\tau}(\Omega)$ with bounded and compact inverse.

Consider then the space

$$\boldsymbol{X}_p \,=\, \big\{\boldsymbol{f} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega);\;\; \int_\Omega \boldsymbol{f}\cdot\overline{\boldsymbol{v}}\,\mathrm{d}\,x = 0,\;\forall\;\boldsymbol{v}\in\boldsymbol{K}_\tau(\Omega)\big\}, \tag{5.2}$$

(not to confuse with the space $\boldsymbol{X}^p(\Omega)$ defined in the subsection 2.1). It worth noting that,

$$\forall\, 1<p<\infty, \quad \boldsymbol{L}^p_{\sigma,\tau}(\Omega) = \boldsymbol{K}_\tau(\Omega)\oplus\boldsymbol{X}_p \quad \text{and} \quad (\boldsymbol{X}_p)' = \boldsymbol{X}_{p'}. \tag{5.3}$$

Next, we define the operator $A'_p : \mathbf{D}(A'_p)\subset \boldsymbol{X}_p \longrightarrow \boldsymbol{X}_p$ by:

$$\mathbf{D}(A'_p) = \big\{\boldsymbol{u}\in\mathbf{D}(A_p);\;\langle \boldsymbol{u}\cdot\boldsymbol{n}\,,\,1\rangle_{\Sigma_j} = 0,\;1\leq j\leq J\big\} \tag{5.4}$$

and $A'_p\boldsymbol{u} = A_p\boldsymbol{u}$, for all $\boldsymbol{u}\in\mathbf{D}(A'_p)$. In other words, the operator A'_p is the restriction of the Stokes operator to the space $\boldsymbol{X}_p$. It is clear that when Ω is simply connected the Stokes operator A_p coincides with the operator A'_p.

Remark 5.1. Let $\boldsymbol{u}\in\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, we note that the condition

$$\forall\;\boldsymbol{v}\in\boldsymbol{K}_\tau(\Omega), \qquad \int_\Omega \boldsymbol{u}\cdot\overline{\boldsymbol{v}}\,\mathrm{d}\,x = 0, \tag{5.5}$$

is equivalent to the condition (see [13, Lemma 3.2, Corollary 3.4]):

$$\langle \boldsymbol{u}\cdot\boldsymbol{n}\,,\,1\rangle_{\Sigma_j} = 0,\;1\leq j\leq J. \tag{5.6}$$

We prove in the following proposition the density of the domain of the operator A'_p.

Proposition 5.2. *The operator A'_p is densely defined operator on X_p.*

Proof. Thanks to Remark 5.1 it is clear that $\mathbf{D}_p(A')\subset\boldsymbol{X}_p$. Moreover, using Lemma 2.3 we can easily verify that for all $\boldsymbol{v}\in\boldsymbol{K}_\tau(\Omega)$, $\int_\Omega \Delta\boldsymbol{u}\cdot\overline{\boldsymbol{v}}\,\mathrm{d}\,x = 0$. As a result $A'\boldsymbol{u}\in\boldsymbol{X}_p$ and A' is a well defined operator.

Now, for the density, let $\boldsymbol{w}\in\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ such that $\langle\boldsymbol{w}\cdot\boldsymbol{n}\,,\,1\rangle_{\Sigma_j}=0$ for all $1\leqslant j\leqslant J$. We know that there exists a sequence $(\boldsymbol{w}_k)_k$ in $\boldsymbol{\mathcal{D}}_\sigma(\Omega)$ such that $\boldsymbol{w}_k\longrightarrow\boldsymbol{w}$ in $\boldsymbol{L}^p(\Omega)$. As a consequence for all $1\leqslant j\leqslant J$, $\langle\boldsymbol{w}_k\cdot\boldsymbol{n}\,,\,1\rangle_{\Sigma_j}\longrightarrow\langle\boldsymbol{w}\cdot\boldsymbol{n}\,,\,1\rangle_{\Sigma_j}=0$, as $k\to+\infty$. Now for all $k\in\mathbb{N}$, setting $\widetilde{\boldsymbol{w}}_k = \boldsymbol{w}_k - \sum_{j=1}^J\langle\boldsymbol{w}_k\cdot\boldsymbol{n}\,,\,1\rangle_{\Sigma_j}\widetilde{\mathbf{grad}}\,q^\tau_j$. We can easily verify that $(\widetilde{\boldsymbol{w}}_k)_k$ is in $\mathbf{D}_p(A')$ and converges to $\boldsymbol{w}$ in $\boldsymbol{L}^p(\Omega)$. □

Next we study the resolvent of the operator A'_p and, to this end, consider the problem (5.1) for $\lambda\in\mathbb{C}$. The following results holds:

Theorem 5.3. *Let $\lambda\in\mathbb{C}$ such that $\mathrm{Re}\,\lambda\geq 0$ and $\boldsymbol{f}\in\boldsymbol{X}_p$. The problem (5.1) has a unique solution $\boldsymbol{u}\in\boldsymbol{W}^{1,p}(\Omega)$ that satisfies the estimates (4.9)-(4.10).*
In addition, the solution $\boldsymbol{u}$ belongs to $\boldsymbol{W}^{2,p}(\Omega)$ and satisfies the estimate

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} \leq\, C(\Omega,p)\,\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}, \tag{5.7}$$

where $C(\Omega,p)$ is independent of λ and $\boldsymbol{f}$.

This result is proved for $\lambda = 0$ in [12] (cf. Proposition 4.3). On the other hand, for $\lambda \in \mathbb{C}^*$, $\mathrm{Re}\,\lambda \geq 0$ a similar Theorem has been proved for the problem (4.1) in [5] (cf. Theorem 4.8 and Theorem 4.11). Since the proof of Theorem 5.3 is very similar it will be skipped.

The following theorem follows:

Theorem 5.4. *The operator $-A'_p$ generates a bounded analytic semi-group on $\boldsymbol{X}_p$ for all $1 < p < \infty$.*

Remark 5.5. Let $(S(t))_{t\geqslant 0}$ be the semi-group generated by $-A'_p$ on $\boldsymbol{X}_p$. We notice that $S(t) = T(t)_{|\boldsymbol{X}_p}$ where $(T(t))_{t\geq 0}$ is the analytic semi-group generated by the operator $-A_p$ on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$.

Remark 5.6. Thanks to Proposition 4.7 we conclude that the space $\boldsymbol{X}_2$ has a Hilbertian basis formed from the eigenfunctions of the operator A'_2. Moreover $\sigma(A_2) = \sigma(A'_2) \cup \{0\}$ and $\boldsymbol{X}_2 = \bigoplus_{k=1}^{+\infty} \mathbf{Ker}(\lambda_k I - A_2)$.

In a similar way, we now define and give some properties of the Stokes operators with flux boundary conditions defined on the subspaces of $[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'_{\sigma,\tau}$ and $[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}$. Since the proof of these properties are completely similar to those for the operator A'_p we do not write any detail.

(i) Consider the space

$$\boldsymbol{Y}_p = \Big\{ \boldsymbol{f} \in [\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'_{\sigma,\tau};\ \forall\, \boldsymbol{v} \in \boldsymbol{K}_\tau(\Omega),\ \ \langle \boldsymbol{f},\, \boldsymbol{v}\rangle_\Omega = 0 \Big\}, \tag{5.8}$$

where $\langle .\,,\, .\rangle_\Omega = \langle .\,,\, .\rangle_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]' \times \boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)}$.
We define the operator $B'_p : \mathbf{D}(B'_p) \subset \boldsymbol{Y}_p \longrightarrow \boldsymbol{Y}_p$ by:

$$\mathbf{D}(B'_p) = \big\{ \boldsymbol{u} \in \mathbf{D}(B_p);\ \langle \boldsymbol{u}\cdot\boldsymbol{n}\,,\, 1\rangle_{\Sigma_j} = 0,\ 1 \leq j \leq J \big\} \tag{5.9}$$

and $B'_p\boldsymbol{u} = B_p\boldsymbol{u}$, for all $\boldsymbol{u} \in \mathbf{D}(B'_p)$. We recall that $\mathbf{D}(B_p)$ is given by (3.14). Observe that, the operator B'_p is the restriction of the Stokes operator to the space $\boldsymbol{Y}_p$. It is clear that when Ω is simply connected the Stokes operator B_p coincides with the operator B'_p.

We easily verify that $\boldsymbol{f} \in \boldsymbol{Y}_p$ and for all $\lambda \in \mathbb{C}^*$ such that $\mathrm{Re}\lambda \geq 0$ the Problem (5.1) has a unique solution $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ satisfying the estimate (4.16). In other words, the operator B'_p is a well defined densely defined operator and $-B'_p$ generates a bounded analytic semi-group on $\boldsymbol{Y}_p$.

(ii) Consider the space

$$\boldsymbol{Z}_p = \Big\{ \boldsymbol{f} \in [\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau};\ \forall\, \boldsymbol{v} \in \boldsymbol{K}_\tau(\Omega),\ \ \langle \boldsymbol{f},\, \boldsymbol{v}\rangle_\Omega = 0 \Big\}, \tag{5.10}$$

where $\langle .\,,\, .\rangle_\Omega = \langle .\,,\, .\rangle_{[\boldsymbol{T}^{p'}(\Omega)]' \times \boldsymbol{T}^{p'}(\Omega)}$.
We define the operator $C'_p : \mathbf{D}(C'_p) \subset \boldsymbol{Z}_p \longrightarrow \boldsymbol{Z}_p$ by:

$$\mathbf{D}(C'_p) = \big\{ \boldsymbol{u} \in \mathbf{D}(C_p);\ \langle \boldsymbol{u}\cdot\boldsymbol{n}\,,\, 1\rangle_{\Sigma_j} = 0,\ 1 \leq j \leq J \big\} \tag{5.11}$$

and $C'_p\boldsymbol{u} = C_p\boldsymbol{u}$, for all $\boldsymbol{u} \in \mathbf{D}(C'_p)$. We recall that $\mathbf{D}(C_p)$ is given by (3.18). Notice that, the operator C'_p is the restriction of the Stokes operator to the space $\boldsymbol{Z}_p$. Similarly, when Ω is simply connected the Stokes operator C_p coincides with the operator C'_p.

We verify that $\boldsymbol{f} \in \boldsymbol{Z}_p$ and for all $\lambda \in \mathbb{C}^*$ such that $\mathrm{Re}\lambda \geq 0$ the Problem (5.1) has a unique solution $\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$ satisfying the estimate (4.18). In other words, the operator C'_p is a well defined densely defined operator and $-C'_p$ generates a bounded analytic semi-group on $\boldsymbol{Z}_p$.

6 Complex and fractional powers of the Stokes operator

In this section we are interested in the study of the complex and the fractional powers of the Stokes operators A_p and A'_p on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ and $\boldsymbol{X}_p$ respectively. Since theses operators generates bounded analytic semi-groups in their corresponding Banach spaces (see Theorems 4.11 and 5.4), they are in particular non-negative operators. It then follows from the results in [50] and in [73] that their powers A_p^α and $(A'_p)^\alpha$, $\alpha \in \mathbb{C}$, are well, densely defined and closed linear operators on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ and $\boldsymbol{X}_p$ with domain $\mathbf{D}(A_p^\alpha)$ and $\mathbf{D}((A'_p)^\alpha)$ respectively.

The purpose of this section is to prove some properties and estimates for these operators A_p^α and $(A'_p)^\alpha$. Since it will be needed, we also obtain in this section a result on the purely imaginary powers of $(I + A_p)$ that easily follows from previous results in [38].

The fractional powers of the Stokes operator with Dirichlet boundary conditions on a bounded domains are studied in detail in [40]. In that case the Stokes operator is bijective with bounded inverse. This is still true for the Stokes operator with Navier-type and flux boundary conditions A'_p but not true for the Stokes operator with Navier-type boundary conditions A_p.

6.1 Pure imaginary powers.

In this section we prove that the pure imaginary powers of the operators $(I + L)$ and L', for $L = A_p, B_p, C_p$, are bounded. The proofs are based on Lemma A2 in [43] and some results in [38] about the operator Δ_M defined as follows :

$$\Delta_M : \mathbf{D}(\Delta_M) \subset \boldsymbol{L}^p(\Omega) \longrightarrow \boldsymbol{L}^p(\Omega),$$

where

$$\mathbf{D}(\Delta_M) = \big\{\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega);\, \boldsymbol{u}\cdot\boldsymbol{n} = 0,\ \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n} = \mathbf{0} \text{ on } \Gamma\big\} \tag{6.1}$$

and

$$\forall\, \boldsymbol{u} \in \mathbf{D}(\Delta_M), \quad \Delta_M\boldsymbol{u} = \Delta\boldsymbol{u} \text{ in } \Omega. \tag{6.2}$$

As it was noticed in [38], (see also [5]):

$$\mathbf{D}(\Delta_M) \cap \boldsymbol{L}^p_{\sigma,\tau}(\Omega) = \mathbf{D}(A_p), \tag{6.3}$$

$$\forall\, \boldsymbol{u} \in \mathbf{D}(A_p), \quad A_p\boldsymbol{u} = -\Delta_M\,\boldsymbol{u} \quad \text{in } \Omega \tag{6.4}$$

and

$$R(\lambda,\, \Delta_M)(\boldsymbol{L}^p_{\sigma,\tau}(\Omega)) \subset \boldsymbol{L}^p_{\sigma,\tau}(\Omega). \tag{6.5}$$

Similarly we have:

$$\mathbf{D}(\Delta_M) \cap \boldsymbol{X}_p = \mathbf{D}(A'_p), \tag{6.6}$$

$$\forall\, \boldsymbol{u} \in \mathbf{D}(A'_p), \quad A_p \boldsymbol{u} = -\Delta_M\, \boldsymbol{u} \quad \text{in } \Omega \tag{6.7}$$

and

$$R(\lambda,\, \Delta_M)(\boldsymbol{X}_p) \subset \boldsymbol{X}_p. \tag{6.8}$$

Our first result in this section is the following:

Theorem 6.1. *There exists an angle $0 < \theta_0 < \pi/2$ and a constant $M > 0$ such that for all $s \in \mathbb{R}$ we have*

$$\|(I + A_p)^{i\,s}\|_{\mathcal{L}(\boldsymbol{L}^p_{\sigma,\tau}(\Omega))} \,\leq\, M\, e^{|s|\,\theta_0}. \tag{6.9}$$

Proof. Using Theorem 3.1 and Remark 3.2 in [38] with $\lambda = 1$, we deduce that $(I - \Delta_M)$ has a bounded $\mathcal{H}^\infty$- calculus on $\boldsymbol{L}^p(\Omega)$. Then, there exist an angle $0 < \theta_0 < \pi/2$ and a constant $M > 0$ such that for all $s \in \mathbb{R}$

$$\|(I - \Delta_M)^{i\,s}\|_{\mathcal{L}(\boldsymbol{L}^p(\Omega))} \,\leq\, M\, e^{|s|\,\theta_0}.$$

(For the definition of $\mathcal{H}^\infty$- calculus of an operator in a Banach space and its relation with the pure imaginary powers of this operator see [28, Section 2] for instance). Using now (6.3)-(6.5), the estimate (6.9) follows. □

Remark 6.2. The results in [38] are proved under the hypothesis that the domain Ω is bounded with a uniform C^3- boundary. On the other hand, it is well known in elliptic theory (cf. Grsivard [44]) that the same regularity results hold if the C^k regularity is replaced by the regularity $C^{k-1,1}$. Notice that this is precisely our hypothesis with $k = 3$.

In the following Proposition we prove that the pure imaginary powers of the operators $(I+B_p)$ and $(I+C_p)$ are bounded on $[\boldsymbol{H}^{p'}_0(\mathrm{div}, \Omega)]'_{\sigma,\tau}$ (given by (3.12)) and on $[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}$ (given by (3.16)) respectively. We recall that the operators B_p and C_p given by (3.13) and (3.17) respectively are the extensions of the Stokes operator to the spaces $[\boldsymbol{H}^{p'}_0(\mathrm{div}, \Omega)]'_{\sigma,\tau}$ and $[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}$ respectively.

Proposition 6.3. *There exists $0 < \theta_0 < \pi/2$ and a constant $C > 0$ such that for all $s \in \mathbb{R}$*

$$\|(I + B_p)^{i\,s}\|_{\mathcal{L}([\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)]'_{\sigma,\tau})} \,\leq\, C\, e^{|s|\,\theta_0} \tag{6.10}$$

and

$$\|(I + C_p)^{i\,s}\|_{\mathcal{L}([\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau})} \,\leq\, C\, e^{|s|\,\theta_0}. \tag{6.11}$$

Proof. We will prove estimate (6.10), estimate (6.11) follows in the same way . Consider the operator B_p defined in (3.13) and let $\boldsymbol{f} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$. Notice that

$$\begin{aligned}\|(I+B_p)^{i\,s}\boldsymbol{f}\|_{[\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)]'} = \|(I+A_p)^{i\,s}\boldsymbol{f}\|_{[\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)]'} &\leq \|(I+A_p)^{i\,s}\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \\ &\leq C\,e^{|s|\,\theta_0}\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}.\end{aligned}$$

This means that for all $s \in \mathbb{R},$ the operator $(I+B_p)^{i\,s}$ is bounded from $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ into $[\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)]'_{\sigma,\tau}$. Next, observe that

$$\boldsymbol{\mathcal{D}}_\sigma(\Omega) \subset \boldsymbol{L}^p_{\sigma,\tau}(\Omega) \subset [\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)]'_{\sigma,\tau}.$$

As a result, using the density of $\boldsymbol{\mathcal{D}}_\sigma(\Omega)$ in $[\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)]'_{\sigma,\tau}$ (see Proposition 3.9) and the Hahn-Banach theorem we can extend $(I+B_p)^{i\,s}$ to a bounded linear operator on $[\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)]'_{\sigma,\tau}$ and we deduce deduce estimate (6.10). □

We consider now the Stokes operators with flux condition A'_p, B'_p, C'_p on $\boldsymbol{X_p}$, $\boldsymbol{Y_p}$, $\boldsymbol{Z_p}$ respectively. Using that these operators are densely defined and invertible with bounded inverse, we prove that their pure imaginary powers are bounded in $\boldsymbol{X_p}$, $\boldsymbol{Y_p}$, $\boldsymbol{Z_p}$ respectively. To this end, we first show the following auxiliary Proposition for whose proof we use again the results of [38].

Proposition 6.4. *Suppose that Ω is strictly star shaped with respect to one of its points and let $1 < p < \infty$. There exist an angle θ_0 and a constant $M > 0$ such that, for all $s \in \mathbb{R}$:*

$$\|(A'_p)^{i\,s}\|_{\mathcal{L}(\boldsymbol{X}_p)} + \|(B'_p)^{i\,s}\|_{\mathcal{L}(\boldsymbol{Y}_p)} + \|(C'_p)^{i\,s}\|_{\mathcal{L}(\boldsymbol{Z}_p)} \leq M\,e^{|s|\,\theta_0}. \tag{6.12}$$

$$\Big\|\Big(\lambda\,I - \Delta_M\Big)^{i\,s}\Big\|_{\mathcal{L}(\boldsymbol{L}^p(\Omega))} \leq M\,e^{|s|\,\theta_0},\ \ \forall \lambda > 0. \tag{6.13}$$

Remark 6.5. It follows from the results in [38] that the imaginary powers of $(\lambda\,I - \Delta_M)$ are bounded in $\mathcal{L}(\boldsymbol{L}^p(\Omega))$. We explicitly write the estimate (6.13) in the statement of Proposition 6.4 in order to emphasize that the constants M and θ_0 are independent of $\lambda > 0$.

Proof. Estimate (6.12) is proved by showing that it holds separately for each of the three terms in its left hand side. Since the proof is the same for the three terms, we only write the details for $\|(A'_p)^{i\,s}\|_{\mathcal{L}(\boldsymbol{X}_p)}$.

The first step of the proof is to show the existence of constants $C > 0$ and $\theta_0 \in (0, \pi/2)$ such that:

$$\Big\|\Big(\frac{1}{\mu^2}\,I - \Delta_M\Big)^{i\,s}\Big\|_{\mathcal{L}(\boldsymbol{L}^p(\Omega))} \leq C\,e^{|s|\,\theta_0} \tag{6.14}$$

and

$$\Big\|\Big(\frac{1}{\mu^2}\,I + A'_p\Big)^{i\,s}\Big\|_{\mathcal{L}(\boldsymbol{X}_p)} \leq C\,e^{|s|\,\theta_0}, \tag{6.15}$$

for all $\mu > 0$.

Since Ω is strictly star shaped with respect to one of its points, then after translation in $\mathbb{R}^3$, we can suppose that this point is 0. It follows that for all $\mu > 1$ and $x \in \Omega$ we have $x/\mu \in \Omega$. The proof is based on the scaling transformation

$$\forall\, x \in \Omega, \qquad (S_\mu \boldsymbol{f})(x) = \boldsymbol{f}(x/\mu), \qquad \boldsymbol{f} \in \boldsymbol{L}^p(\Omega). \tag{6.16}$$

As in the proof of Theorem A1 in [43] we can easily verify that

$$-\mu^2 \Delta_M \;=\; S_\mu(-\Delta_M)\, S_\mu^{-1}, \qquad I - \mu^2 \Delta_M \;=\; S_\mu(I - \Delta_M)\, S_\mu^{-1}.$$

We recall that the operator Δ_M is defined by (6.1)-(6.2).
Similarly we can also verify that

$$\mu^2\, A'_p \;=\; S_\mu\, A'_p\, S_\mu^{-1}, \qquad I + \mu^2 A'_p \;=\; S_\mu(I + A'_p)\, S_\mu^{-1}.$$

As a result for all $z \in \mathbb{C}$ using (2.9) we have,

$$(I - \mu^2 \Delta_M)^z \;=\; S_\mu (I - \Delta_M)^z S_\mu^{-1} \quad \text{and} \quad (I + \mu^2\, A'_p)^z \;=\; S_\mu (I + A'_p)^z S_\mu^{-1}.$$

Thus for all $z \in \mathbb{C}$ we have

$$\|(I - \mu^2 \Delta_M)^z\|_{\mathcal{L}(\boldsymbol{L}^p(\Omega))} = \|S_\mu (I - \Delta_M)^z S_\mu^{-1}\|_{\mathcal{L}(\boldsymbol{L}^p(\Omega))} \le \|(I - \Delta_M)^z\|_{\mathcal{L}(\boldsymbol{L}^p(\Omega))}$$

and

$$\|(I + \mu^2\, A'_p)^z\|_{\mathcal{L}(\boldsymbol{X}_p)} = \|S_\mu (I + A'_p)^z S_\mu^{-1}\|_{\mathcal{L}(\boldsymbol{X}_p)} \le \|(I + A'_p)^z\|_{\mathcal{L}(\boldsymbol{X}_p)}.$$

Using Theorem 3.1 and Remark 3.2 in [38], respectively to Theorem 6.1, we deduce that there exist $0 < \theta_1,\, \theta_2 < \pi/2$ and constants $M_1,\, M_2 > 0$ such that :

$$\forall\, s \in \mathbb{R}, \quad \|(I - \mu^2 \Delta_M)^{i\,s}\|_{\mathcal{L}(\boldsymbol{L}^p(\Omega))} \;\le\; M_1\, e^{|s|\,\theta_1}, \tag{6.17}$$

and

$$\forall\, s \in \mathbb{R}, \quad \|(I + \mu^2\, A'_p)^{i\,s}\|_{\mathcal{L}(\boldsymbol{X}_p)} \;\le\; M_2\, e^{|s|\,\theta_2}, \tag{6.18}$$

where the constants M_1 in (6.17) and M_2 in (6.18) are independents of μ. Since

$$\Big(\frac{1}{\mu^2}\, I - \Delta_M\Big)^{i\,s} \;=\; \frac{1}{\mu^{2\,i\,s}}\, (I - \mu^2 \Delta_M)^{i\,s} \quad \text{and} \quad \Big(\frac{1}{\mu^2}\, I + A'_p\Big)^{i\,s} \;=\; \frac{1}{\mu^{2\,i\,s}}\, (I + \mu^2 A'_p)^{i\,s}$$

(6.14) and (6.15) follow.
Of course, (6.13) follows from (6.14). On the other hand, since by Proposition 4.3 in [12] and Proposition 5.2 above, the range and the domains of A'_p, B'_p and C'_p are dense in $\boldsymbol{X}_p$, $\boldsymbol{Y}_p$, $\boldsymbol{Z}_p$ respectively, we may apply Lemma A2 of [43] and obtain that, for all $\boldsymbol{f} \in \mathbf{D}(A'_p)$

$$\|(A'_p)^{i\,s} \boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \;=\; \lim_{\mu \to +\infty} \Big\| \Big(\frac{1}{\mu^2}\, I + A'_p\Big)^{i\,s} \boldsymbol{f} \Big\|_{\boldsymbol{L}^p(\Omega)}. \tag{6.19}$$

As a result we deduce from (6.15) and (6.19) that (6.12) holds for all $\boldsymbol{f} \in \mathbf{D}(A'_p)$. By the density of $\mathbf{D}(A'_p)$ in $\boldsymbol{X}_p$ (see Proposition 5.2) it then follows that (6.12) holds for all $\boldsymbol{f}$ in $\boldsymbol{X}_p$. □

For a general domain Ω of Class $C^{2,1}$, not necessarily strictly star shaped with respect to one of its points, we use that (see [17] for instance), a bounded Lipschitz-Continuous open set is the union of a finite number of star-shaped, Lipschitz-continuous open sets. The idea is then to apply the argument above to each of these sets in order to derive the desired result on the entire domain. However, the divergence-free condition of a function $\boldsymbol{f} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ is not preserved under the cut-off procedure and this process is non-trivial. This is done in the following Theorem.

Theorem 6.6. *There exist an angle $0 < \theta_0 < \pi/2$ and a constant $M > 0$ such that for all $s \in \mathbb{R}$ we have*

$$\|(A'_p)^{i\,s}\|_{\mathcal{L}(\boldsymbol{X}_p)} + \|(B'_p)^{i\,s}\|_{\mathcal{L}(\boldsymbol{Y}_p)} + \|(C'_p)^{i\,s}\|_{\mathcal{L}(\boldsymbol{Z}_p)} \leq M\, e^{|s|\,\theta_0} \tag{6.20}$$

Proof. As in the proof of Proposition 6.4, we prove Theorem 6.6 by showing that estimate (6.20) holds separately for each term in the left hand side. Since the proof is the same for the three terms, we only write the details for $\|(A'_p)^{i\,s}\|_{\mathcal{L}(\boldsymbol{X}_p)}$.

Let $(\Theta_j)_{j\in J}$ be an open covering of Ω by a finite number of star-shaped open sets and let us consider a partition of unity $(\varphi_j)_{j\in J}$ subordinated to the covering $(\Omega_j)_{j\in J}$ where for all $j \in J,\ \Omega_j = \Theta_j \cap \Omega$. This means that

$$\forall j \in J, \qquad \mathrm{Supp}\varphi_j \subset \Omega_j$$

and

$$\sum_{j\in J} \varphi_j = 1, \qquad \varphi_j \in \mathcal{D}(\Omega_j).$$

Let $\boldsymbol{f} \in \boldsymbol{X}_p$, then $\boldsymbol{f}$ can be written as

$$\boldsymbol{f} = \sum_{j\in J} \boldsymbol{f}_j, \qquad \forall j \in J, \quad \boldsymbol{f}_j = \varphi_j \boldsymbol{f}.$$

Notice that for all $j \in J,\ \boldsymbol{f}_j$ is not necessarily a divergence free function.

Let $\mu > 0$ and let $s \in \mathbb{R}$. From (6.6)-(6.8) we know that

$$\Big(\frac{1}{\mu^2} I + A'_p\Big)^{is} \boldsymbol{f} = \Big(\frac{1}{\mu^2} I - \Delta_M\Big)^{is} \boldsymbol{f} = \sum_{j\in J} \Big(\frac{1}{\mu^2} I - \Delta_M\Big)^{is} \boldsymbol{f}_j.$$

As a result, one has

$$\begin{aligned} \Big\|\Big(\frac{1}{\mu^2} I + A'_p\Big)^{is} \boldsymbol{f}\Big\|_{\boldsymbol{L}^p(\Omega)} &\leq \sum_{j\in J} \Big\|\Big(\frac{1}{\mu^2} I - \Delta_M\Big)^{is} \boldsymbol{f}_j\Big\|_{\boldsymbol{L}^p(\Omega)} \\ &= \sum_{j\in J} \Big\|\Big(\frac{1}{\mu^2} I - \Delta_M\Big)^{is} \boldsymbol{f}_j\Big\|_{\boldsymbol{L}^p(\Omega_j)} \end{aligned}$$

Since for all $j \in J,$ the domain Ω_j is strictly star shaped with respect to one of its points, then using (6.13) we have

$$\begin{aligned} \Big\|\Big(\frac{1}{\mu^2} I + A'_p\Big)^{is} \boldsymbol{f}\Big\|_{\boldsymbol{L}^p(\Omega)} &\leq e^{|s|\,\theta_0} \sum_{j\in J} C_j\, \|\boldsymbol{f}_j\|_{\boldsymbol{L}^p(\Omega_j)} \\ &\leq C(\Omega, p)\, e^{|s|\,\theta_0} \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \end{aligned}$$

with a constant $C(\Omega, p)$ independent of μ and $\boldsymbol{f}$. As a result one has

$$\Big\| \Big(\frac{1}{\mu^2}\, I + A'_p \Big)^{is} \Big\|_{\mathcal{L}(\boldsymbol{X}_p)} \,\leq\, C(\Omega, p)\, e^{|s|\,\theta_0}.$$

Thus as in the proof of Theorem 6.1, using [43, Lemma A2] we deduce that for all $\boldsymbol{f} \in \mathbf{D}(A'_p)$

$$\|(A'_p)^{i\,s} \boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \,=\, \lim_{\mu \to +\infty} \Big\| \Big(\frac{1}{\mu^2}\, I + A'_p \Big)^{i\,s} \boldsymbol{f} \Big\|_{\boldsymbol{L}^p(\Omega)}. \tag{6.21}$$

This means that (6.20) hold for all $\boldsymbol{f} \in \mathbf{D}(A'_p)$. Using the density of $\mathbf{D}(A'_p)$ in $\boldsymbol{X}_p$ we deduce our result in $\boldsymbol{X}_p$. □

Remark 6.7. Notice that estimate (6.15) is also true if we replace A'_p by A_p. However, since the range of A_p is not dense in $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ it is not possible to apply Lemma A2 of [43] to pass to the limit as $\mu \to \infty$.

6.2 Domains of fractional powers.

For all $\alpha \in \mathbb{R}$, the map $\boldsymbol{v} \longmapsto \|(A'_p)^{\alpha}\, \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}$ is a norm on $\mathbf{D}((A'_p)^{\alpha})$. This is due to the fact that (cf. [73, Theorem 1.15.2, part (e)]), the operator A'_p has a bounded inverse and thus for all $\alpha \in \mathbb{C}^*$, the operator $(A'_p)^{\alpha}$ is an isomorphism from $\mathbf{D}((A'_p)^{\alpha})$ to $\boldsymbol{X}_p$.

Consider the space

$$\boldsymbol{V}^p_{\sigma,\tau}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{X}^p_{\sigma,\tau}(\Omega);\, \langle \boldsymbol{v} \cdot \boldsymbol{n}\,,\, 1 \rangle_{\Sigma_j} = 0,\ 1 \leq j \leq J\}, \tag{6.22}$$

with $\boldsymbol{X}^p_{\sigma,\tau}(\Omega)$ is defined by (3.6). Thanks to the work of [12, 13] we know that, for all $\boldsymbol{v} \in \boldsymbol{V}^p_{\sigma,\tau}(\Omega)$ the norm of $\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)}$ is equivalent to the norm $\|\mathbf{curl}\, \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}$. The following theorem characterizes the domain of $(A'_p)^{\frac{1}{2}}$.

Theorem 6.8. *For all $1 < p < \infty$, $\mathbf{D}((A'_p)^{\frac{1}{2}}) = \boldsymbol{V}^p_{\sigma,\tau}(\Omega)$ with equivalent norms. Furthermore, for every $\boldsymbol{u} \in \mathbf{D}((A'_p)^{\frac{1}{2}})$, the norm $\|(A'_p)^{\frac{1}{2}} \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}$ is a norm on $\mathbf{D}((A'_p)^{\frac{1}{2}})$ which is equivalent to the norm $\|\mathbf{curl}\, \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}$. In other words, there exists two constants C_1 and C_2 such that for all $\boldsymbol{u} \in \mathbf{D}((A'_p)^{\frac{1}{2}})$*

$$\|(A'_p)^{\frac{1}{2}} \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq C_1 \|\mathbf{curl}\, \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq C_2 \|(A'_p)^{\frac{1}{2}} \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}.$$

Proof. The goal is to prove the relation

$$\big[\mathbf{D}(A'_p);\, \boldsymbol{X}_p\big]_{1/2} = \boldsymbol{V}^p_{\sigma,\tau}(\Omega).$$

Thanks to Theorem 6.6 we know that that the pure imaginary powers of A'_p are bounded on $\boldsymbol{X}_p$ and satisfy estimate (6.20). As a result thanks to Theorem 2.12 we have

$$\mathbf{D}((A'_p)^{\frac{1}{2}}) \,=\, \big[\mathbf{D}(A'_p);\, \boldsymbol{X}_p\big]_{1/2}.$$

Consider now a function $\boldsymbol{u} \in \mathbf{D}(A'_p)$, set $\boldsymbol{z} = \mathbf{curl}\,\boldsymbol{u}$ and $\boldsymbol{U} = (\boldsymbol{u}, \boldsymbol{z})$. It is clear that $\boldsymbol{z} \in \boldsymbol{H}^p_0(\mathbf{curl}, \Omega)$ and using Lemma 2.1 we deduce that $\boldsymbol{z} \in \boldsymbol{X}^p_N(\Omega) \hookrightarrow \boldsymbol{W}^{1,p}(\Omega)$ and $\boldsymbol{U} \in \boldsymbol{X}_p \times \boldsymbol{W}^{1,p}(\Omega)$. On the other hand if $\boldsymbol{u} \in \boldsymbol{X}_p$, thanks to [12, 13], we know that $\boldsymbol{U} \in \boldsymbol{X}_p \times [\boldsymbol{H}^p_0(\mathbf{curl}, \Omega)]' \hookrightarrow \boldsymbol{X}_p \times \boldsymbol{W}^{-1,p}(\Omega)$. Next let $\boldsymbol{u} \in \mathbf{D}((A'_p)^{\frac{1}{2}})$ then $\boldsymbol{U} \in \boldsymbol{X}_p \times [\boldsymbol{W}^{1,p}(\Omega), \boldsymbol{W}^{-1,p}(\Omega)]_{1/2} = \boldsymbol{X}_p \times \boldsymbol{L}^p(\Omega)$. Thus using Lemma 2.1 we deduce that $\boldsymbol{u} \in \boldsymbol{V}^p_{\sigma,\tau}(\Omega)$. This amount to say that

$$\mathbf{D}((A'_p)^{\frac{1}{2}}) \hookrightarrow \boldsymbol{V}^p_{\sigma,\tau}(\Omega). \tag{6.23}$$

It remains to prove the second inclusion. First we recall that $(\boldsymbol{X}_p)' = \boldsymbol{X}_{p'}$ and the adjoint operator $((A'_p)^{\frac{1}{2}})^*$ is equal to $(A'_{p'})^{\frac{1}{2}}$. Observe that thanks to [73, Theorem 1.15.2, part (e)], since A'_p has a bounded inverse, then for all $1 < p < \infty$, $(A'_p)^{\frac{1}{2}}$ is an isomorphism from $\mathbf{D}((A'_p)^{\frac{1}{2}})$ to $\boldsymbol{X}_p$. This means that for all $\boldsymbol{F} \in \boldsymbol{X}_{p'}$ there exists a unique $\boldsymbol{v} \in \mathbf{D}((A'_{p'})^{\frac{1}{2}})$ solution of

$$(A'_{p'})^{\frac{1}{2}}\boldsymbol{v} \,=\, \boldsymbol{F}. \tag{6.24}$$

As a result for all $\boldsymbol{u} \in \mathbf{D}(A'_p)$ we have

$$\begin{aligned}
\|(A'_p)^{\frac{1}{2}}\boldsymbol{u}\|_{\boldsymbol{X}_p} &= \sup_{\boldsymbol{F}\in\boldsymbol{X}_{p'},\,\boldsymbol{F}\neq\boldsymbol{0}} \frac{\left|\langle (A'_p)^{\frac{1}{2}}\boldsymbol{u}\,,\,\boldsymbol{F}\rangle_{\boldsymbol{X}_p\times\boldsymbol{X}_{p'}}\right|}{\|\boldsymbol{F}\|_{\boldsymbol{L}^{p'}(\Omega)}} \\
&= \sup_{\boldsymbol{F}\in\boldsymbol{X}_{p'},\,\boldsymbol{F}\neq\boldsymbol{0}} \frac{\left|\langle (A'_p)^{\frac{1}{2}}\boldsymbol{u}\,,\,(A'_{p'})^{\frac{1}{2}}\boldsymbol{v}\rangle_{\boldsymbol{X}_p\times\boldsymbol{X}_{p'}}\right|}{\|\boldsymbol{F}\|_{\boldsymbol{L}^{p'}(\Omega)}},
\end{aligned}$$

where $\boldsymbol{v}$ is the unique solution of (6.24) and $\boldsymbol{X}_{p'}$ is a closed subspace of $\boldsymbol{L}^{p'}(\Omega)$ equipped with the norm of $\boldsymbol{L}^{p'}(\Omega)$.

As a result,

$$\begin{aligned}
\|(A'_p)^{\frac{1}{2}}\boldsymbol{u}\|_{\boldsymbol{X}_p} &= \sup_{\boldsymbol{v}\in\mathbf{D}(A'^{1/2}_{p'}),\,\boldsymbol{v}\neq\boldsymbol{0}} \frac{\left|\langle A'_p\boldsymbol{u}\,,\,\boldsymbol{v}\rangle_{\boldsymbol{X}_p\times\boldsymbol{X}_{p'}}\right|}{\|(A'_{p'})^{\frac{1}{2}}\boldsymbol{v}\|_{\boldsymbol{L}^{p'}(\Omega)}} \\
&= \sup_{\boldsymbol{v}\in\mathbf{D}(A'^{1/2}_{p'}),\,\boldsymbol{v}\neq\boldsymbol{0}} \frac{\left|\int_\Omega \mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\overline{\boldsymbol{v}}\,\mathrm{d}\,x\right|}{\|(A'_{p'})^{\frac{1}{2}}\boldsymbol{v}\|_{\boldsymbol{L}^{p'}(\Omega)}} \\
&\leq C(\Omega,p)\,\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)}.
\end{aligned} \tag{6.25}$$

Now since $\mathbf{D}(A'_p)$ is dense in $\boldsymbol{V}^p_{\sigma,\tau}(\Omega)$ one gets inequality (6.25) for all $\boldsymbol{u} \in \boldsymbol{V}^p_{\sigma,\tau}(\Omega)$ and then

$$\boldsymbol{V}^p_{\sigma,\tau}(\Omega) \hookrightarrow \mathbf{D}((A'_p)^{\frac{1}{2}})$$

and the result is proved. □

The following proposition shows an embedding of Sobolev type for the domains of fractional powers of the Stokes operator with flux boundary conditions. This embedding give us the $\boldsymbol{L}^p - \boldsymbol{L}^q$ estimates for the corresponding homogeneous problem.

Proposition 6.9. *For all $1 < p < \infty$ and for all $0 < \alpha \leq 1$ we define $\beta = \max(\alpha, 1-\alpha)$ then*

$$\mathbf{D}((A'_p)^\alpha) \hookrightarrow \boldsymbol{L}^q(\Omega) \tag{6.26}$$

for all q such that:
(i) *For $1 < p < \frac{3}{2\beta}$, $q \in \left[p,\ \frac{3p}{3-2\beta p}\right]$.*
(ii) *For $p = \frac{3}{2\beta}$, $q \in [1,\ +\infty[$.*
(iii) *For $p > \frac{3}{2\beta}$, $q = +\infty$.*

Moreover for such q, the following estimate holds

$$\forall \boldsymbol{u} \in \mathbf{D}((A'_p)^\alpha), \qquad \|\boldsymbol{u}\|_{\boldsymbol{L}^q(\Omega)} \leq C(\Omega, p)\, \|(A'_p)^\alpha \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}. \tag{6.27}$$

Proof. As described in the proof of Theorem 6.8 we know that $\mathbf{D}((A'_p)^\alpha) = \left[\mathbf{D}(A'_p)\,;\, \boldsymbol{X}_p\right]_\alpha$. Moreover, we know that, $\left[\mathbf{D}(A'_p)\,;\, \boldsymbol{X}_p\right]_\alpha \hookrightarrow \left[\boldsymbol{W}^{2,p}(\Omega)\,;\, \boldsymbol{L}^p(\Omega)\right]_\alpha = \boldsymbol{W}^{2(1-\alpha),p}(\Omega)$. It is clear that for $0 < \alpha < 1/2$, we have $1 - \alpha > \alpha$ and

$$\mathbf{D}((A'_p)^\alpha) \hookrightarrow \boldsymbol{W}^{2\alpha,p}(\Omega).$$

Similarly, for $1/2 \leq \alpha \leq 1$, we have $\alpha \geq 1 - \alpha$ and

$$\mathbf{D}((A'_p)^\alpha) \hookrightarrow \mathbf{D}((A'_p)^{1-\alpha}) \hookrightarrow \boldsymbol{W}^{2\alpha,p}(\Omega).$$

Thus one has, for all $0 < \alpha \leq 1$

$$\mathbf{D}((A'_p)^\alpha) \hookrightarrow \mathbf{D}((A'_p)^{1-\alpha}) \hookrightarrow \boldsymbol{W}^{2\beta,p}(\Omega).$$

Now using the result of [2, Theorem 7.57] we deduce the Sobolev embedding (6.26) with p and q satisfying (i), (ii) and (iii). Finally, estimate (6.27) is a direct consequence of the Sobolev embedding (6.26), since $\mathbf{D}((A'_p)^\alpha)$ is equipped with the graph norm of the operator $(A'_p)^\alpha$. □

The following Corollary extends Proposition 6.9 to any real α such that $0 < \alpha < 3/2p$. This result is similar to the result of Borchers and Miyakawa [19] who proved the same result for the Stokes operator with Dirichlet boundary conditions in exterior domains for $1 < p < 3$.

Corollary 6.10. *for all $1 < p < \infty$ and for all $\alpha \in \mathbb{R}$ such that $0 < \alpha < 3/2p$ the following Sobolev embedding holds*

$$\mathbf{D}((A'_p)^\alpha) \hookrightarrow \boldsymbol{L}^q(\Omega), \qquad \frac{1}{q} = \frac{1}{p} - \frac{2\alpha}{3}. \tag{6.28}$$

Moreover for all $\boldsymbol{u} \in \mathbf{D}((A'_p)^\alpha)$ the following estimate holds

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^q(\Omega)} \leq C(\Omega, p)\, \|(A'_p)^\alpha \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}. \tag{6.29}$$

Proof. First observe that for $0 < \alpha < \min(1, 3/2p)$ the Sobolev embedding (6.28) is a consequence of Proposition 6.9 part (i). Next, for any real α such that $0 < \alpha < 3/2p$ we write $\alpha = k + \theta$, where k is a non negative integer and $0 < \theta < 1$.

Next we set

$$\frac{1}{q_0} = \frac{1}{p} - \frac{2\theta}{3} \qquad \text{and} \qquad \frac{1}{q_j} = \frac{1}{q_0} - \frac{2j}{3}, \quad j = 0, 1, ..., k. \tag{6.30}$$

It is clear that $\frac{1}{q_j} = \frac{1}{q_{j-1}} - \frac{2}{3}$ and that $q_k = q$. Moreover, by assumptions on p and α we have for $j = 0, 1, ..., k$, $\theta + j < 3/2p$. As a consequence of Proposition 6.9 part (i) it follows that

$$\mathbf{D}((A'_p)^\theta) \hookrightarrow \boldsymbol{L}^{q_0}(\Omega)$$

and for all $1 \leq j \leq k$

$$\mathbf{D}(A'_{q_{j-1}}) \hookrightarrow \boldsymbol{L}^{q_j}(\Omega).$$

It thus follows that for all $\boldsymbol{u} \in \mathbf{D}((A'_p)^\infty) = \cap_{m\in\mathbb{N}}\mathbf{D}((A'_p)^m)$

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^q(\Omega)} \leq \|A'_{q_{k-1}}\boldsymbol{u}\|_{\boldsymbol{L}^{q_{k-1}}(\Omega)} \leq ... \leq \|(A'_{q_0})^k\boldsymbol{u}\|_{\boldsymbol{L}^{q_0}(\Omega)} \leq \|(A'_p)^\alpha\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}. \tag{6.31}$$

By density of $\mathbf{D}((A'_p)^\infty)$ in $\mathbf{D}((A'_p)^\alpha)$ on gets the Sobolev embeddings (6.28) and estimate (6.31). Finally, estimate (6.29) is a direct consequence of (6.28). □

7 The time dependent Stokes problem

In this section we solve the time dependent Stokes Problem (1.1) with the boundary condition (1.3) using the semi-group theory. As described above, due to the boundary conditions (1.3) the Stokes operator coincides with the Laplace operator.

7.1 The homogeneous problem

Consider the problem:

$$\left\{\begin{array}{ccccc} \frac{\partial \boldsymbol{u}}{\partial t} - \Delta\boldsymbol{u} = \boldsymbol{0}, & \operatorname{div}\boldsymbol{u} = 0 & \text{in} & \Omega\times(0,T), \\ \boldsymbol{u}\cdot\boldsymbol{n} = 0, & \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{0} & \text{on} & \Gamma\times(0,T), \\ & \boldsymbol{u}(0) = \boldsymbol{u}_0 & \text{in} & \Omega. \end{array}\right. \tag{7.1}$$

Usually in the Problem (7.1) where figures the constraint $\operatorname{div}\boldsymbol{u} = 0$ in Ω, a gradient of pressure appears. However, thanks to our boundary conditions, the pressure is constant in our case. For this reason, the Problem (7.1) is equivalent to the homogeneous Stokes problem

$$\left\{\begin{array}{ccccc} \frac{\partial \boldsymbol{u}}{\partial t} - \Delta\boldsymbol{u} + \nabla\pi = \boldsymbol{0}, & \operatorname{div}\boldsymbol{u} = 0 & \text{in} & \Omega\times(0,T), \\ \boldsymbol{u}\cdot\boldsymbol{n} = 0, & \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{0} & \text{on} & \Gamma\times(0,T), \\ & \boldsymbol{u}(0) = \boldsymbol{u}_0 & \text{in} & \Omega. \end{array}\right. \tag{7.2}$$

We start with the following result for initial data in $L^p_{\sigma,\tau}(\Omega)$ that follows easily from the classiacl semi group theory for the operator A_p on the space $L^p_{\sigma,\tau}(\Omega)$.

Theorem 7.1. *Let $\boldsymbol{u}_0 \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, then Problem (7.1) has a unique solution $\boldsymbol{u}(t)$ satisfying*

$$\boldsymbol{u} \in C([0,\,+\infty[,\,\boldsymbol{L}^p_{\sigma,\tau}(\Omega)) \cap C(]0,\,+\infty[,\,\mathbf{D}(A_p)) \cap C^1(]0,\,+\infty[,\,\boldsymbol{L}^p_{\sigma,\tau}(\Omega)), \tag{7.3}$$

$$\boldsymbol{u} \in C^k(]0,\,+\infty[,\,\mathbf{D}(A_p^\ell)), \qquad \forall\, k \in \mathbb{N},\ \forall\, \ell \in \mathbb{N} \setminus \{0\}. \tag{7.4}$$

Moreover we have the estimates

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{L}^p(\Omega)} \leq\, C_1(\Omega,p)\, \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)} \tag{7.5}$$

$$\Big\|\frac{\partial \boldsymbol{u}(t)}{\partial t}\Big\|_{\boldsymbol{L}^p(\Omega)} \leq \frac{C_2(\Omega,p)}{t}\, \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}. \tag{7.6}$$

$$\|\mathbf{curl}\, \boldsymbol{u}(t)\|_{\boldsymbol{L}^p(\Omega)} \,\leq\, \frac{C_3(\Omega,p)}{\sqrt{t}}\, \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)} \tag{7.7}$$

and

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{W}^{2,p}(\Omega)} \,\leq\, C_4(\Omega,p)\, (1+\frac{1}{t})\, \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}, \tag{7.8}$$

for all $t > 0$.

Proof. Since the operator $-A_p$ generates a bounded analytic semi-group $(T(t))_{t\geq 0}$ on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, the Problem (7.1) has a unique solution $\boldsymbol{u}(t) = T(t)\,\boldsymbol{u}_0$. Thanks to [30, Chapter 2, Proposition 4.3] we know that $\|T(t)\|_{\mathcal{L}(\boldsymbol{L}^p_{\sigma,\tau}(\Omega))} \leq C_1(\Omega,p)$, where $C_1(\Omega,p) = M_1\, \kappa_1(\Omega,p)$ for some constant $M_1 > 0$. We recall that $\kappa_1(\Omega,p)$ is the constant in (4.9). As a result one has estimate (7.5). We also know thanks to [30, Chapter 2, Theorem 4.6] that this solution belongs to $\mathbf{D}(A_p)$ thus one has (7.3). Now using the fact that $T(t)\,\boldsymbol{u}_0 \in \mathbf{D}(A_p^\infty)$ and the same argument of [24, Chapitre 7, Theorem 7.5, Theorem 7.7] one gets the regularity (7.4). We recall that $\mathbf{D}(A_p^\infty) = \cap_{n\in\mathbb{N}} \mathbf{D}(A_p^n)$.

Moreover, thanks to [30, Chapter 2, Theorem 4.6, page 101] we know that

$$\|A_p T(t)\|_{\boldsymbol{L}^p(\Omega)} \leq\, \frac{C_2(\Omega,p)}{t},$$

where $C_2(\Omega,p) = M_2\, \kappa_1(\Omega,p)$ for some constant $M_2 > 0$, which gives us estimate (7.6). Next, to prove estimate (7.7) we proceed in the same way as in the proof of the estimate (4.10) (see [5, Theorem 4.11] for the proof).

Since the norm of $\boldsymbol{W}^{2,p}(\Omega)$ is equivalent to the graph norm of the Stokes operator A_p one has estimate (7.8). □

Estimates (7.5) and (7.7) allow to deduce the following Corollary:

Corollary 7.2 (Weak Solutions for the Stokes Problem)**.** *Let $\boldsymbol{u}_0 \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ and $\boldsymbol{u}$ be the unique solution of Problem (7.1) given by Theorem 7.1. Then $\boldsymbol{u}$ satisfies*

$$\forall\, 1 \leq q < 2, \qquad \boldsymbol{u} \in L^q(0,T;\, \boldsymbol{W}^{1,p}(\Omega)) \ \text{ and } \qquad \frac{\partial \boldsymbol{u}}{\partial t} \in L^q(0,T;\, [\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)]'), \tag{7.9}$$

for all $T > 0$.

Proof. Let $\boldsymbol{u}(t)$ be the unique solution of Problem (7.1). By hypothesis we know that $\boldsymbol{u}$ satisfies the estimates (7.5)-(7.8). Now thanks to Lemma 2.1 we know that

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{W}^{1,p}(\Omega)} \simeq \|\boldsymbol{u}(t)\|_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{curl}\,\boldsymbol{u}(t)\|_{\boldsymbol{L}^p(\Omega)}.$$

Thus one deduces directly that $\boldsymbol{u} \in L^q(0,T;\, \boldsymbol{W}^{1,p}(\Omega))$ for all $1 \leq q < 2$ and for all $0 < T < \infty$.
Next, let us prove that $\frac{\partial \boldsymbol{u}}{\partial t} \in L^q(0,T;\, [\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]')$, set

$$\widetilde{\boldsymbol{u}}(t) = \boldsymbol{u}(t) - \sum_{j=1}^{J} \langle \boldsymbol{u}(t)\cdot\boldsymbol{n}\,,\, 1\rangle_{\Sigma_j} \widetilde{\mathbf{grad}}\, q_j^\tau.$$

It is clear that $\boldsymbol{u}(t) = \widetilde{\boldsymbol{u}}(t) + \sum_{j=1}^{J} \langle \boldsymbol{u}(t)\cdot\boldsymbol{n}\,,\, 1\rangle_{\Sigma_j} \widetilde{\mathbf{grad}}\, q_j^\tau$. Moreover thanks to [12, Theorem 4.4] we know that

$$\|\Delta \boldsymbol{u}\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'} = \|\Delta \widetilde{\boldsymbol{u}}\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'} \simeq \|\widetilde{\boldsymbol{u}}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq \|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)}.$$

The last inequality comes from the fact (see [13, Lemma 4.2])

$$|\langle \boldsymbol{u}\cdot\boldsymbol{n}\,,\, 1\rangle_{\Sigma_j}| \leq C(\Omega,p)\, \|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}.$$

Thus $\frac{\partial \boldsymbol{u}}{\partial t} = \Delta \boldsymbol{u} \in L^q(0,T;\, [\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]')$ and the result is proved. □

We observe the following remark:

Remark 7.3. (i) In the Hilbertian case ($\boldsymbol{u}_0 \in \boldsymbol{L}^2_{\sigma,\tau}(\Omega)$), the properties (7.5)-(7.8) are immediate. We will prove estimate (7.7). Observe that, thanks to Propositon 4.7 and Remark 4.8, on $\boldsymbol{L}^2_{\sigma,\tau}(\Omega)$ we can express $\boldsymbol{u}(t)$ explicitly in the form

$$\boldsymbol{u}(t) = \sum_{j=1}^{J} \alpha_j\, \widetilde{\mathbf{grad}}\, q_j^\tau + \sum_{k=1}^{+\infty} \beta_k\, e^{-\lambda_k\, t}\, \boldsymbol{z}_k, \tag{7.10}$$

where

$$\alpha_j = \int_\Omega \boldsymbol{u}_0 \cdot \widetilde{\mathbf{grad}}\, \overline{q_j^\tau}\, \mathrm{d}\, x \qquad \text{and} \qquad \beta_k = \int_\Omega \boldsymbol{u}_0 \cdot \overline{\boldsymbol{z}_k}\, \mathrm{d}\, x.$$

As a result, using the fact that $A_2 \boldsymbol{z}_k = \lambda_k\, \boldsymbol{z}_k$ and the fact that

$$\int_\Omega |\mathbf{curl}\, \boldsymbol{z}_k|^2 \mathrm{d}\, x = \lambda_k\, \|\boldsymbol{z}_k\|^2_{\boldsymbol{L}^2(\Omega)} = \lambda_k$$

one has

$$\|\mathbf{curl}\, \boldsymbol{u}(t)\|^2_{\boldsymbol{L}^2(\Omega)} = \sum_{k=1}^{+\infty} \beta_k^2\, e^{-2\lambda_k\, t} \lambda_k.$$

Finally, since

$$\|\boldsymbol{u}_0\|^2_{\boldsymbol{L}^2(\Omega)} = \sum_{j=1}^{J} \alpha_j^2 \, + \, \sum_{k=1}^{+\infty} \beta_k^2$$

estimate (7.7) follows directly. Similarly one gets directly estimates (7.5)-(7.8). We recall that $(\boldsymbol{z}_k)_k$ are eigenvectors for the Stokes operator associated to the eigenvalues $(\lambda_k)_k$ and they form with $(\widetilde{\mathbf{grad}}\, q_j^\tau)_{1\le j\le J}$ an orthonormal basis for $\boldsymbol{L}^2_{\sigma,\tau}(\Omega)$.
(ii) For $p = 2$, the solution $\boldsymbol{u}$ satisfies (see [11, Theorem 6.4])

$$\boldsymbol{u} \in L^2(0,T;\, \boldsymbol{H}^1(\Omega)) \;\text{ and } \qquad \frac{\partial \boldsymbol{u}}{\partial t} \in L^2(0,T;\, [\boldsymbol{H}^2_0(\mathrm{div},\Omega)]') \tag{7.11}$$

and

$$\frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}\,\mathrm{t}}\,\|\boldsymbol{u}(t)\|^2_{\boldsymbol{L}^2(\Omega)} \, + \, \int_\Omega |\mathbf{curl}\,\boldsymbol{u}(t)|^2\,\mathrm{d}\,x \, = \, 0.$$

In other words for $p = 2$, Corollary 7.2 still holds true for $q = 2$ included.

We consider now the case where the initial data $\boldsymbol{u}_0 \in \boldsymbol{X}_p$, (see (5.2) for the definition of $\boldsymbol{X}_p$).

Theorem 7.4. *Suppose that $\boldsymbol{u}_0 \in \boldsymbol{X}_p$ and let $\boldsymbol{u}$ be the unique solution to Problem (7.1). Then $\boldsymbol{u}$ satisfies the following:*

$$\boldsymbol{u} \in C([0,\,+\infty[,\,\boldsymbol{X}_p) \cap C(]0,\,+\infty[,\,\mathbf{D}(A'_p)) \cap C^1(]0,\,+\infty[,\,\boldsymbol{X}_p), \tag{7.12}$$

$$\boldsymbol{u} \in C^k(]0,\,+\infty[,\,\mathbf{D}((A'_p)^\ell)), \qquad \forall\, k,\,\ell \in \mathbb{N}. \tag{7.13}$$

Moreover, for all $q \in [p,\infty)$, for all integers $m \ge 0$, $n \ge 0$ and for all $\mu \in (0,\lambda_1)$ there exists a constant $M > 0$ such that the solution $\boldsymbol{u}$ satisfies, for all $t > 0$:

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{L}^q(\Omega)} \;\le\; M\, e^{-\mu\, t}\, t^{-3/2(1/p-1/q)} \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}, \tag{7.14}$$

$$\|\mathbf{curl}\,\boldsymbol{u}(t)\|_{\boldsymbol{L}^q(\Omega)} \;\le\; M\, e^{-\mu\, t}\, t^{-3/2(1/p-1/q)-1/2} \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)} \tag{7.15}$$

and

$$\Big\|\frac{\partial^m}{\partial t^m}\Delta^n \boldsymbol{u}(t)\Big\|_{\boldsymbol{L}^q(\Omega)} \;\le\; M\, e^{-\mu\, t}\, t^{-(m+n)-3/2(1/p-1/q)} \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}, \tag{7.16}$$

where λ_1 is the first non zero eigenvalue of the Stokes operator defined above.

Proof. Applying the semi-group theory to the operator A'_p, one gets the existence and uniqueness of a solution to the homogeneous Stokes Problem (7.1) given by $\boldsymbol{v}(t) = T(t)_{|\boldsymbol{X}_p}\boldsymbol{u}_0$ and satisfying (7.12)-(7.13). We recall that $(T(t)_{|\boldsymbol{X}_p})_{t\ge 0}$ is the semi-group generated by the Stokes operator with flux boundary conditions on $\boldsymbol{X}_p$. Moreover, since $\boldsymbol{X}_p \subset \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, by the uniqueness of solution $\boldsymbol{u}$ in Theorem 7.1, we deduce that $\boldsymbol{v}(t) = \boldsymbol{u}(t) = T(t)\boldsymbol{u}_0$, the unique solution to Problem (7.1). Let us prove estimates (7.14)–(7.16). To this end observe first that, by Theorem 5.3 and to [12], we have:

$$S(-A'_p) = \sup\{\mathrm{Re}\,\lambda \in \sigma(-A'_p)\} = -\lambda_1 < 0.$$

As a result, thanks to [57, Chapitre 4, Theorem 4.3, page 118], there is a constant $M > 0$ such that for all $0 < \mu < \lambda_1$, $\|T(t)_{|\boldsymbol{X}_p}\|_{\mathcal{L}(\boldsymbol{X}_p)} \leq M\,\kappa_1(\Omega,p)\,e^{-\mu\,t}$.

The estimates (7.14)–(7.16) follow for the cases where $q = p$ and $m = 1, n = 0$ or $m = 0, n = 0, 1, 2$ using the classical semi-group theory.

Suppose that $p \neq q$, the proof is similar to the proof of [19, Corollary 4.6]. Let $s \in \mathbb{R}$ such that $\frac{3}{2}(\frac{1}{p} - \frac{1}{q}) < s < \frac{3}{2p}$ and set $\frac{1}{p_0} = \frac{1}{p} - \frac{2s}{3}$. It is clear that $p < q < p_0$. Let $\boldsymbol{u}(t)$ be the unique solution of Problem (7.1). Since for all $t > 0$, $\boldsymbol{u}(t) \in \mathbf{D}((A'_p)^\infty)$, then thanks to Corollary 6.10, $\boldsymbol{u}(t) \in \mathbf{D}((A'_p)^s) \hookrightarrow \boldsymbol{L}^{p_0}(\Omega)$. Now set $\alpha = \frac{1/p-1/q}{1/p-1/p_0} \in]0,1[$, we can easily verify that $\frac{1}{q} = \frac{\alpha}{p_0} + \frac{1-\alpha}{p}$. Thus $\boldsymbol{u}(t) \in \boldsymbol{L}^q(\Omega)$ and

$$\begin{aligned}
\|\boldsymbol{u}(t)\|_{\boldsymbol{L}^q(\Omega)} &\leq C\|\boldsymbol{u}(t)\|^{\alpha}_{\boldsymbol{L}^{p_0}(\Omega)}\|\boldsymbol{u}(t)\|^{1-\alpha}_{\boldsymbol{L}^p(\Omega)} \\
&\leq C\|(A'_p)^s T(t)\boldsymbol{u}_0\|^{\alpha}_{\boldsymbol{L}^p(\Omega)}\|T(t)\boldsymbol{u}_0\|^{1-\alpha}_{\boldsymbol{L}^p(\Omega)} \\
&\leq C\,e^{-\mu t}t^{-\alpha s}\|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}. \qquad (7.17)\\
&= C\,e^{-\mu t}t^{-3/2(1/p-1/q)}\|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}. \qquad (7.18)
\end{aligned}$$

Estimate (7.17) follows from the fact that, (cf. [57, Chapter 2, Theorem 6.13, page 76]),

$$\|(A'_p)^\alpha\,T(t)_{|\boldsymbol{X}_p}\|_{\mathcal{L}(\boldsymbol{X}_p)} \leq M\,\kappa_1(\Omega,p)\,\frac{e^{-\mu\,t}}{t^\alpha}. \qquad (7.19)$$

Next, let $\boldsymbol{u}_0 \in \boldsymbol{X}_p \cap \boldsymbol{X}_q$ then $\mathbf{curl}\,\boldsymbol{u}(t) \in \boldsymbol{L}^q(\Omega)$ and

$$\begin{aligned}
\|\mathbf{curl}\,\boldsymbol{u}(t)\|_{\boldsymbol{L}^q(\Omega)} \leq C\,\|(A'_q)^{\frac{1}{2}}\boldsymbol{u}(t)\|_{\boldsymbol{L}^q(\Omega)} &= \|(A'_q)^{\frac{1}{2}}\,T(t/2)T(t/2)\boldsymbol{u}_0\|_{\boldsymbol{L}^q(\Omega)} \\
&\leq C\,e^{-\mu t}t^{-1/2}\,\|T(t/2)\boldsymbol{u}_0\|_{\boldsymbol{L}^q(\Omega)} \\
&\leq C\,e^{-\mu t}t^{-1/2}\,t^{-3/2(1/p-1/q)}\|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}.
\end{aligned}$$

Now let $\boldsymbol{u}_0 \in \boldsymbol{X}_p$, using the density of $\boldsymbol{X}_p \cap \boldsymbol{X}_q$ in $\boldsymbol{X}_p$ we know that there exists a sequence $(\boldsymbol{u}_{0_m})_{m\geq 0}$ in $\boldsymbol{X}_p \cap \boldsymbol{X}_q$ that converges to $\boldsymbol{u}_0$ in $\boldsymbol{X}_p$. For all $m \in \mathbb{N}$ we set $\boldsymbol{u}_m(t) = T(t)\boldsymbol{u}_{0_m}$, as a result the sequences $(\boldsymbol{u}_m(t))_{m\geq 0}$ and $(\mathbf{curl}\,\boldsymbol{u}_m(t))_{m\geq 0}$ converges to $\boldsymbol{u}(t)$ and $\mathbf{curl}\,\boldsymbol{u}(t)$ respectively in $\boldsymbol{L}^p(\Omega)$, where $u(t) = T(t)\boldsymbol{u}_0$. On the other hand, for all $m, n \in \mathbb{N}$ one has

$$\|\mathbf{curl}(\boldsymbol{u}_n(t) - \boldsymbol{u}_m(t))\|_{\boldsymbol{L}^q(\Omega)} \leq C\,e^{-\mu t}t^{-1/2}\,t^{-3/2(1/p-1/q)}\|\boldsymbol{u}_{0_n} - \boldsymbol{u}_{0_m}\|_{\boldsymbol{L}^p(\Omega)}.$$

Thus $(\mathbf{curl}\,\boldsymbol{u}_m(t))_{m\geq 0}$ is a Cauchy sequence in $\boldsymbol{L}^q(\Omega)$ and converges to $\mathbf{curl}\,\boldsymbol{u}(t)$ in $\boldsymbol{L}^q(\Omega)$. This means that $\mathbf{curl}\,\boldsymbol{u}(t) \in \boldsymbol{L}^q(\Omega)$ and by passing to the limit as $m \to \infty$ one gets estimate (7.21).

Finally, using (7.12)-(7.13), we have for all $m, n \in \mathbb{N}$, $\frac{\partial^m}{\partial t^m}\Delta^n\boldsymbol{u} \in C^\infty((0,\infty), \mathbf{D}(A'_p))$. Thus $\frac{\partial^m}{\partial t^m}\Delta^n\boldsymbol{u}(t)$ belongs to $\boldsymbol{L}^q(\Omega)$ and

$$\Big\|\frac{\partial^m}{\partial t^m}\Delta^n\boldsymbol{u}(t)\Big\|_{\boldsymbol{L}^q(\Omega)} = \|(A'_p)^{(m+n)}\,T(t)\boldsymbol{u}_0\|_{\boldsymbol{L}^q(\Omega)} \leq C\,e^{-\mu t}t^{-(m+n)-3/2(1/p-1/q)}\|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}.$$

□

Using now the results of Theorem 7.4 we will extend estimates (7.5)-(7.8) and obtain the following $L^p - L^q$ estimates

Theorem 7.5. *Let $1 < p \leq q < \infty$ and $\boldsymbol{u}_0 \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$. The unique solution $\boldsymbol{u}$ to Problem (7.1) given by Theorem 7.1 belongs to $\boldsymbol{L}^q(\Omega)$ and satisfies, for all $t > 0$:*

$$\|\boldsymbol{u}(t) - \boldsymbol{w}_0\|_{\boldsymbol{L}^q(\Omega)} \leq C\, e^{-\mu\, t}\, t^{-3/2(1/p-1/q)} \|\widetilde{\boldsymbol{u}}_0\|_{\boldsymbol{L}^p(\Omega)}, \tag{7.20}$$

with $\boldsymbol{w}_0$ and $\widetilde{\boldsymbol{u}}_0$ are given by (1.13) *and* (1.12) *respectively. Moreover, the following estimates hold*

$$\|\mathbf{curl}\, \boldsymbol{u}(t)\|_{\boldsymbol{L}^q(\Omega)} \leq C\, e^{-\mu t} t^{-3/2(1/p-1/q)-1/2} \|\widetilde{\boldsymbol{u}}_0\|_{\boldsymbol{L}^p(\Omega)}, \tag{7.21}$$

$$\forall\, m, n \in \mathbb{N},\ m + n > 0, \quad \Big\| \frac{\partial^m}{\partial t^m} \Delta^n \boldsymbol{u}(t) \Big\|_{\boldsymbol{L}^q(\Omega)} \leq C\, e^{-\mu t} t^{-(m+n)-3/2(1/p-1/q)} \|\widetilde{\boldsymbol{u}}_0\|_{\boldsymbol{L}^p(\Omega)}. \tag{7.22}$$

Proof. By definition, $\boldsymbol{u}_0 = \boldsymbol{w}_0 + \widetilde{\boldsymbol{u}}_0$, with $\boldsymbol{w}_0 \in \boldsymbol{K}_\tau(\Omega)$ and $\widetilde{\boldsymbol{u}}_0 \in \boldsymbol{X}_p$. It follows that the unique solution to Problem (7.1) given by Theorem 7.1 can be written in the form

$$\boldsymbol{u}(t) = \boldsymbol{w}_0 + T(t)\widetilde{\boldsymbol{u}}_0, \tag{7.23}$$

where $T(t)\widetilde{\boldsymbol{u}}_0$ satisfies (7.12)-(7.16).

The case $p = q$ follows directly from Theorem 7.1, so let us suppose that $p \neq q$. The estimate (7.20) follows from (7.23) and (7.14).

Estimate (7.21) follows from (7.15) using that $\mathbf{curl}\, \boldsymbol{u}(t) = \mathbf{curl}\, \boldsymbol{w}_0 + \mathbf{curl}\, (T(t)\widetilde{\boldsymbol{u}}_0) = \mathbf{curl}\, (T(t)\widetilde{\boldsymbol{u}}_0)$.

Finally, for all $m, n \in \mathbb{N},$ such that $m + n > 0$ we have

$$\frac{\partial^m}{\partial t^m} \Delta^n \boldsymbol{u}(t) = A_p^{m+n} \boldsymbol{u}(t) = A_p^{m+n} \boldsymbol{w}_0 + A_p^{m+n} (T(t)\widetilde{\boldsymbol{u}}_0) = (A_p')^{m+n}\, (T(t)\widetilde{\boldsymbol{u}}_0).$$

As a result, using Theorem 7.4 one has estimate (7.22). □

Proof of Theorem 1.3. Theorem 1.3 immediately follows from Theorem 7.1 and Theorem 7.5. □

We may also use the analyticity of the semigroups generated by the operators B_p and C_p, proved in Section 4.2 and Section 4.3. We then deduce the following result, as we did in Theorem 7.1.

Theorem 7.6. (i) For all $\boldsymbol{u}_0 \in [\boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega)]'_{\sigma,\tau}$ the Problem (7.1) has a unique solution $\boldsymbol{u}$ satisfying

$$\boldsymbol{u} \in C([0, +\infty[, [\boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega)]'_{\sigma,\tau}) \cap C(]0, +\infty[, \mathbf{D}(B_p)) \cap C^1(]0, +\infty[, [\boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega)]'_{\sigma,\tau}), \tag{7.24}$$

$$\boldsymbol{u} \in C^k(]0, +\infty[, \mathbf{D}(B_p^\ell)), \qquad \forall\, k \in \mathbb{N},\ \forall\, \ell \in \mathbb{N}^*. \tag{7.25}$$

Moreover, for all $t > 0$:

$$\|\boldsymbol{u}(t)\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'} \leq C(\Omega,p)\,\|\boldsymbol{u}_0\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'}, \tag{7.26}$$

$$\Big\|\frac{\partial \boldsymbol{u}(t)}{\partial t}\Big\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'} \leq \frac{C(\Omega,p)}{t}\,\|\boldsymbol{u}_0\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'} \tag{7.27}$$

and

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C(\Omega,p)\,(1+\frac{1}{t})\|\boldsymbol{u}_0\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'}. \tag{7.28}$$

(ii) For every $\boldsymbol{u}_0 \in [\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}$ the Problem (7.1) has a unique solution $\boldsymbol{u}$ satisfying

$$\boldsymbol{u} \in C([0,\,+\infty[,\,[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}) \cap C(]0,\,+\infty[,\,\mathbf{D}(C_p)) \cap C^1(]0,\,+\infty[,\,[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}), \tag{7.29}$$

$$\boldsymbol{u} \in C^k(]0,\,+\infty[,\,\mathbf{D}(C_p^\ell)), \qquad \forall\, k \in \mathbb{N},\ \forall\, \ell \in \mathbb{N}^*. \tag{7.30}$$

Moreover, for all $t > 0$:

$$\|\boldsymbol{u}(t)\|_{[\boldsymbol{T}^{p'}(\Omega)]'} \leq C(\Omega,p)\,\|\boldsymbol{u}_0\|_{[\boldsymbol{T}^{p'}(\Omega)]'}, \tag{7.31}$$

$$\Big\|\frac{\partial \boldsymbol{u}(t)}{\partial t}\Big\|_{[\boldsymbol{T}^{p'}(\Omega)]'} \leq \frac{C(\Omega,p)}{t}\,\|\boldsymbol{u}_0\|_{[\boldsymbol{T}^{p'}(\Omega)]'}. \tag{7.32}$$

and

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{L}^p(\Omega)} \leq C(\Omega,p)\,(1+\frac{1}{t})\,\|\boldsymbol{u}_0\|_{[\boldsymbol{T}^{p'}(\Omega)]'}. \tag{7.33}$$

In the same way as we deduced Corollary 7.2, we deduce the following Corollary from Theorem 7.6.

Corollary 7.7 (Very weak solutions for the homogeneous Stokes Problem)**.** *Let $\boldsymbol{u}_0 \in [\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'_{\sigma,\tau}$, $T < \infty$ and let $\boldsymbol{u}$ be the unique solution of Problem (7.1) given by Theorem 7.6, (i). Then $\boldsymbol{u}$ satisfies*

$$\forall\, q \in [1,2), \qquad \boldsymbol{u} \in L^q(0,T;\,\boldsymbol{L}^p(\Omega)) \ \text{ and } \qquad \frac{\partial \boldsymbol{u}}{\partial t} \in L^q(0,T;\,[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}). \tag{7.34}$$

Proof. Using the semi-group theory we know that the solution $\boldsymbol{u}(t) \in \boldsymbol{W}^{1,p}(\Omega)$ for all $t > 0$. As a result, using the interpolation inequality we have

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{L}^p(\Omega)} \leq C(\Omega,p)\,\|\boldsymbol{u}(t)\|^{1/2}_{\boldsymbol{W}^{1,p}(\Omega)}\|\boldsymbol{u}(t)\|^{1/2}_{\boldsymbol{W}^{-1,p}(\Omega)}. \tag{7.35}$$

On the other hand, thanks to Corollary 4.14 we know that

$$\begin{aligned}\|\boldsymbol{u}(t)\|_{\boldsymbol{W}^{1,p}(\Omega)} &\simeq \|\boldsymbol{u}(t)\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'} + \|\Delta \boldsymbol{u}(t)\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'} \\ &\leq \big(1+\frac{1}{t}\big)\,\|\boldsymbol{u}_0\|_{\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'}.\end{aligned} \tag{7.36}$$

Moreover, thanks to the continuous embeddings $[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]' \hookrightarrow \boldsymbol{W}^{-1,p}(\Omega)$ and to the semi-group theory we have

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{W}^{-1,p}(\Omega)} \le C(\Omega,p)\,\|\boldsymbol{u}(t)\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'} \le C(\Omega,p)\,\|\boldsymbol{u}_0\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'}. \tag{7.37}$$

As a result, putting together (7.35), (7.36) and (7.37) one gets

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{L}^p(\Omega)} \le C(\Omega,p)\left(1+\frac{1}{t}\right)^{1/2}\|\boldsymbol{u}_0\|_{\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'}.$$

Thus, for every $T<\infty$ and for every $1\le q<2$, $\boldsymbol{u}\in L^q(0,T;\,\boldsymbol{L}^p(\Omega))$.

It remains to prove that $\frac{\partial \boldsymbol{u}}{\partial t}\in L^q(0,T;\,[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau})$. We proceed in a similar way as in the proof of Corollary 7.2. We set

$$\widetilde{\boldsymbol{u}}(t) = \boldsymbol{u}(t) - \sum_{j=1}^{J}\langle \boldsymbol{u}(t)\cdot\boldsymbol{n}\,,\,1\rangle_{\Sigma_j}\widetilde{\mathbf{grad}}\,q_j^\tau.$$

It is clear that $\boldsymbol{u}(t) = \widetilde{\boldsymbol{u}}(t) + \sum_{j=1}^{J}\langle \boldsymbol{u}(t)\cdot\boldsymbol{n}\,,\,1\rangle_{\Sigma_j}\widetilde{\mathbf{grad}}\,q_j^\tau$. Moreover thanks to [12, Theorem 4.15] we know that

$$\|\Delta\boldsymbol{u}\|_{[\boldsymbol{T}^{p'}(\Omega)]'} = \|\Delta\widetilde{\boldsymbol{u}}\|_{[\boldsymbol{T}^{p'}(\Omega)]'} \simeq \|\widetilde{\boldsymbol{u}}\|_{\boldsymbol{L}^p(\Omega)} \le \|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}.$$

The last inequality comes from the fact (see [13, Lemma 4.2])

$$|\langle \boldsymbol{u}\cdot\boldsymbol{n}\,,\,1\rangle_{\Sigma_j}| \le C(\Omega,p)\,\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}.$$

Thus $\frac{\partial \boldsymbol{u}}{\partial t} = \Delta\boldsymbol{u}\in L^q(0,T;\,[\boldsymbol{T}^{p'}(\Omega)]')$ and the result is proved. □

We present now the remaining results for the homogeneous Stokes system with flux conditions. As it was said in the Introduction, they are very similar, although with some differences, to those for the problem without flux condition that are described just above. As for the proofs, they are also very similar and actually simpler to those without flux condition, reason for which we will not give all of them in detail.

Remark 7.8. By (7.12), the function $\boldsymbol{u}$ that is obtained in Theorem 7.4 solves Problem (7.1) and also satisfies condition (1.5). Then, for all $\pi\in\mathbb{R}$, $(\boldsymbol{u},\pi)$ is a solution of the Stokes problem with flux (1.1), (1.3), (1.5).

Remark 7.9. Notice that the decay rates in the estimates (7.14)–(7.16) for the solution $\boldsymbol{u}(t)$ are exponential, and not algebraic as in (7.5)-(7.8) of Theorem 7.1.

Remark 7.10. For $p=2$, the solution $\boldsymbol{u}$ can be written explicitly in the form

$$\boldsymbol{u}(t) = \sum_{k=1}^{+\infty}\beta_k\,e^{-\lambda_k t}\,\boldsymbol{z}_k, \qquad \beta_k = \int_\Omega \boldsymbol{u}_0\cdot\overline{\boldsymbol{z}_k}\,\mathrm{d}\,x$$

and the exponential decay with respect to time can be obtained directly. Moreover, contrary to the case $p \neq 2$ one has

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{L}^2(\Omega)} \leq e^{-\lambda_1 t} \|\boldsymbol{u}_0\|_{\boldsymbol{L}^2(\Omega)}. \tag{7.38}$$

It is clear that estimate (7.38) yields a faster decay rate than (7.14). We recall that λ_1 is the first eigenvalue for the operator A'_p and it is equal to $\frac{1}{C_2(\Omega)}$ where $C_2(\Omega)$ is the constant of the Poincaré-type inequality (4.8).

In our next Theorem we consider initial data $\boldsymbol{u}_0$ belonging to $\boldsymbol{Y}_p$ and to $\boldsymbol{Z}_p$.

Theorem 7.11. (i) For all $\boldsymbol{u}_0 \in \boldsymbol{Y}_p$ the Problem (7.1) has a unique solution $\boldsymbol{u}$ satisfying

$$\boldsymbol{u} \in C([0, +\infty[, \boldsymbol{Y}_p) \cap C(]0, +\infty[, \mathbf{D}(B'_p)) \cap C^1(]0, +\infty[, \boldsymbol{Y}_p), \tag{7.39}$$

$$\boldsymbol{u} \in C^k(]0, +\infty[, \mathbf{D}(B'^{\ell}_p)), \qquad \forall\, k,\, \ell \in \mathbb{N}. \tag{7.40}$$

Moreover there exists a constant $C(\Omega, p)$ and a constant $\mu > 0$, such that, for all $t > 0$:

$$\|\boldsymbol{u}(t)\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega)]'} \leq C(\Omega, p)\, e^{-\mu t} \|\boldsymbol{u}_0\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega)]'}, \tag{7.41}$$

$$\Big\|\frac{\partial \boldsymbol{u}(t)}{\partial t}\Big\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega)]'} \leq C(\Omega, p)\, \frac{e^{-\mu t}}{t} \|\boldsymbol{u}_0\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega)]'} \tag{7.42}$$

and

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C(\Omega, p)\, \frac{e^{-\mu t}}{t} \|\boldsymbol{u}_0\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega)]'}. \tag{7.43}$$

(ii) For all $\boldsymbol{u}_0 \in \boldsymbol{Z}_p$ the Problem (7.1) has a unique solution $\boldsymbol{u}$ satisfying

$$\boldsymbol{u} \in C([0, +\infty[, \boldsymbol{Z}_p) \cap C(]0, +\infty[, \mathbf{D}(C'_p)) \cap C^1(]0, +\infty[, \boldsymbol{Z}_p), \tag{7.44}$$

$$\boldsymbol{u} \in C^k(]0, +\infty[, \mathbf{D}(C'^{\ell}_p)), \qquad \forall\, k,\, \ell \in \mathbb{N}. \tag{7.45}$$

Moreover there exists a constant $C(\Omega, p)$ and a constant $\mu > 0$, such that, for all $t > 0$:

$$\|\boldsymbol{u}(t)\|_{[\boldsymbol{T}^{p'}(\Omega)]'} \leq C(\Omega, p)\, e^{-\mu t} \|\boldsymbol{u}_0\|_{[\boldsymbol{T}^{p'}(\Omega)]'}, \tag{7.46}$$

$$\Big\|\frac{\partial \boldsymbol{u}(t)}{\partial t}\Big\|_{[\boldsymbol{T}^{p'}(\Omega)]'} \leq C(\Omega, p)\, \frac{e^{-\mu t}}{t} \|\boldsymbol{u}_0\|_{[\boldsymbol{T}^{p'}(\Omega)]'} \tag{7.47}$$

and

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{L}^p(\Omega)} \leq C(\Omega, p)\, \frac{e^{-\mu t}}{t} \|\boldsymbol{u}_0\|_{[\boldsymbol{T}^{p'}(\Omega)]'}. \tag{7.48}$$

Proof. The theorem follows by the classical semigroup theory applied to the analytic semigroups generated by the operators B'_p and C'_p. □

Remark 7.12. By (7.39) and (7.45), the functions $\boldsymbol{u}$ obtained in Theorem 7.11 solve Problem (7.1) and satisfy condition (1.5). Then, for all $\pi \in \mathbb{R}$, $(\boldsymbol{u}, \pi)$ is a solution of the Stokes problem with flux (1.1), (1.3), (1.5).

7.2 The inhomogeneous problem

Given the Cauchy-Problem:

$$\begin{cases} \frac{\partial u}{\partial t} + \mathcal{A}\, u(t) = f(t) \quad 0 \le t \le T \\ \qquad u(0) = 0, \end{cases} \tag{7.49}$$

where $-\mathcal{A}$ is the infinitesimal generator of an analytic semi-group on a Banach space X and $f \in L^p(0,T;\, X)$, the analyticity of $-\mathcal{A}$ is not enough in general to ensure that solutions to Problem (7.49) satisfy

$$u \in W^{1,p}(0,T;\, X) \cap L^p(0,T;\, D(\mathcal{A})). \tag{7.50}$$

Although it is enough when X is a Hilbert space, (see [16, 66] for instance), in general it is necessary to impose some further regularity condition on f such as Hölder continuity, (see [57] for instance). However, using the concept of ζ-convexity and a perturbation argument, the existence of a solution to Problem (7.49) satisfying (7.50), when the pure imaginary powers of $\mathcal{A}$ satisfy estimate (2.10) is proved in [29, 43]. Moreover, [43, Theorem 2.1] extends [29, Theorem 3.2] in two directions: First, the operator $\mathcal{A}$ may not have bounded inverse and second, the maximal interval of time T may be infinite. In the case of a Hilbert space it was proved in [47, 48] that the pure imaginary powers of a maximal accretive operator are bounded and satisfy estimates of type (2.10).

For the sake of completeness we state the following theorem that is proved in [43] (cf. Theorem 2.1).

Theorem 7.13. *Let X be a ζ-convex Banach space. Assume that $0 < T \le \infty$, $1 < p < \infty$ and that $\mathcal{A} \in \mathcal{E}_K^\theta(X)$ for some $K \ge 1$, $0 \le \theta < \pi/2$ and $\mathcal{E}_K^\theta(X)$ as in Definition 2.9. Then for every $f \in L^p(0,T;\, X)$ there exists a unique solution $\boldsymbol{u}$ of the Cauchy-Problem (7.49) satisfying the properties:*

$$u \in L^p(0,T_0;\, D(\mathcal{A})), \quad T_0 \le T \ \text{ if } T < \infty \ \text{ and } \ T_0 < T \ \text{ if } T = \infty,$$

$$\frac{\partial u}{\partial t} \in L^p(0,T;\, X)$$

and

$$\int_0^T \left\| \frac{\partial u}{\partial t} \right\|_X^p \mathrm{d}\, t \,+\, \int_0^T \|\mathcal{A}u(t)\|_X^p \,\mathrm{d}\, t \,\le\, C \int_0^T \|f(t)\|_X^p \,\mathrm{d}\, t$$

with $C = C(p,\theta,K,X)$ independent of f and T.

Let us consider now the non homogeneous Problem:

$$\begin{cases} \frac{\partial \boldsymbol{u}}{\partial t} - \Delta \boldsymbol{u} = \boldsymbol{f}, & \operatorname{div} \boldsymbol{u} = 0 & \text{in} & \Omega \times (0,T), \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0, & \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} & \text{on} & \Gamma \times (0,T), \\ & \langle \boldsymbol{u}(t) \cdot \boldsymbol{n}\,,\, 1 \rangle_{\Sigma_j} = 0, & 1 \le j \le J, & t \in (0,T), \\ & \boldsymbol{u}(0) = \boldsymbol{0} & \text{in} & \Omega, \end{cases} \tag{7.51}$$

where $\boldsymbol{f} \in L^q(0,T;\, \boldsymbol{X}_p)$ and $1 < p, q < \infty$.

We treat now the Stokes problem with flux condition (1.1), (1.3), (1.5). Since the system (7.51)–(1.5) is equivalent to the Stokes Problem with flux condition (1.1), (1.3), (1.5), we deduce in that way the existence and maximal regularity of strong, weak and very weak solution for the Stokes Problem with flux condition (1.1), (1.3), (1.5).

Theorem 7.14 (Strong Solutions for the inhomogeneous Stokes Problem with flux)**.** *Let $T \in (0, \infty]$, $1 < p, q < \infty$. For all $\boldsymbol{f} \in L^q(0, T;\, \boldsymbol{X}_p)$, there exists a unique solution $\boldsymbol{u}$ of (7.51) such that*

$$\boldsymbol{u} \in L^q(0, T_0;\, \mathbf{D}(A'_p)), \quad T_0 \leq T \ \text{ if } T < \infty \ \text{ and } \ T_0 < T \ \text{ if } T = \infty, \tag{7.52}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} \in L^q(0, T;\, \boldsymbol{X}_p) \tag{7.53}$$

$$\int_0^T \Big\| \frac{\partial \boldsymbol{u}}{\partial t} \Big\|^q_{\boldsymbol{L}^p(\Omega)} \,\mathrm{d}\, t + \int_0^T \|\Delta \boldsymbol{u}(t)\|^q_{\boldsymbol{L}^p(\Omega)} \,\mathrm{d}\, t \leq C(p, q, \Omega) \int_0^T \|\boldsymbol{f}(t)\|^q_{\boldsymbol{L}^p(\Omega)} \,\mathrm{d}\, t. \tag{7.54}$$

and such that $(\boldsymbol{u}, \pi)$ is a solution of the inhomogeneous Stokes Problem (1.1), (1.3) , (1.5) for all $\pi \in \mathbb{R}$.

Proof. The space $\boldsymbol{X}_p$ is ζ-convex, and by Theorem 6.6 the pure imaginary powers of the operators A'_p are bounded in $\boldsymbol{X}_p$. It is then possible to apply Theorem 7.13 to the operator A'_p itself in $\boldsymbol{X}_p$ and Theorem 7.14 follows. □

Remark 7.15. The spaces $\boldsymbol{Y}_p$ and $\boldsymbol{Z}_p$ are also ζ-convex, and by Theorem 6.6 the pure imaginary powers of the operators B'_p and C'_p are bounded in $\boldsymbol{Y}_p$ and $\boldsymbol{Z}_p$ respectively. It is then possible to apply Theorem 7.13 to the operator B'_p and C'_p in $\boldsymbol{Y}_p$ and $\boldsymbol{Z}_p$. We obtain in this way the existence, uniqueness and maxial regularity of weak and very weak solutions for the Stokes problem with flux condition (1.1), (1.3) , (1.5). The corresponding Theorems are very similar to Theorem 7.14 and we do not write their statements in detail.

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