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RANK TWO ACM BUNDLES ON THE DEL PEZZO THREEFOLD WITH PICARD NUMBER 3

GIANFRANCO CASNATI, DANIELE FAENZI, FRANCESCO MALASPINA

ABSTRACT. A del Pezzo threefold F with maximal Picard number is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. In the present paper we completely classify locally free sheaves \mathcal{E} of rank 2 such that $h^i(F, \mathcal{E}(t)) = 0$ for $i = 1, 2$ and $t \in \mathbb{Z}$. Such a classification extends similar results proved by E. Arrondo and L. Costa regarding del Pezzo threefolds with Picard number 1.

1. INTRODUCTION

Let \mathbb{P}^N be the N -dimensional projective space over an algebraically closed field k of characteristic 0. A well-known theorem of Horrocks (see [38] and the references therein) states that a locally free sheaf \mathcal{E} on \mathbb{P}^N splits as direct sum of invertible sheaves if and only if it has no intermediate cohomology, i.e. $h^i(\mathbb{P}^N, \mathcal{E}(t)) = 0$ for $0 < i < N$ and $t \in \mathbb{Z}$.

It is thus natural to ask for the meaning of such a vanishing on other kind of algebraic varieties F . Such a vanishing makes sense only if a natural polarization is defined on F . For instance, if there is a natural embedding $F \subseteq \mathbb{P}^N$, then one can consider $\mathcal{O}_F(h) := \mathcal{O}_{\mathbb{P}^N}(1) \otimes \mathcal{O}_F$ and we can ask for locally free sheaves \mathcal{E} on F such that $H_*^i(F, \mathcal{E}) := \bigoplus_{t \in \mathbb{Z}} H^i(F, \mathcal{E}(th)) = 0$ for $0 < i < \dim(F)$ which are called arithmetically Cohen-Macaulay (aCM for short) bundles. We are particularly interested in characterizing indecomposable aCM bundles, i.e. bundles of rank $r \geq 2$ which do not split as sum of invertible sheaves. Among aCM bundles, there are bundles \mathcal{E} such that $H_*^0(F, \mathcal{E})$ has the highest possible number of generators in degree 0. After [40] such bundles are simply called Ulrich bundles. Ulrich bundles have many good properties, thus their description is of particular interest.

There are a lot of classical and recent papers devoted to the aforementioned topics. For example, in the case of smooth quadrics and Grassmannians, there are some classical result generalizing Horrocks' criterion by adding suitable vanishing to \mathcal{E} to force splitting (see [29], [1], [6], and [39]). More recently, several authors considered cohomological splitting criteria on hypersurfaces or Segre products, most notably for the first non-trivial case, namely bundles of rank 2 (see e.g. [30], [31], [32], [15], [33], [16], [14], [35], [36], [8]). Ulrich bundles (even of higher rank) have been recently object of deep inspection (see, e.g. [10], [11], [19], [20]).

Another case which deserves particular attention is that of Fano and del Pezzo n -folds. We recall that a smooth n -fold F is Fano if its anticanonical sheaf ω_F^{-1} is ample (see [28] for results about Fano and del Pezzo varieties). The greatest positive integer r such that $\omega_F \cong \mathcal{O}_F(-rh)$ for some ample class $h \in \text{Pic}(F)$ is called the index of F and one has $1 \leq r \leq n + 1$. If $r = n - 1$, then F is called del Pezzo. For such an F , the group $\text{Pic}(F)$ is torsion-free so that $\mathcal{O}_F(h)$ is uniquely determined. By definition, the degree of F is the integer $d := h^n$. In the case $n = 3$ several results about aCM bundles on Fano and del Pezzo threefolds can be found in [3], [5], [9], [12], [32], [22], [4] for particular values of d .

Let us restrict to the case of del Pezzo threefolds. It is well-known that $1 \leq d \leq 8$. When $d \geq 3$ the sheaf $\mathcal{O}_F(h)$ is actually very ample so h is the hyperplane class of a natural embedding

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$F \subseteq \mathbb{P}^{d+1}$. These varieties are classified (see again [28]). For $3 \leq d \leq 5$ and $d = 8$, the complete classification of aCM bundles of rank 2 on F can be found in [3] and [12] (where one can also find some generalization to the cases $d = 1, 2$).

A fundamental hypothesis in almost all the aforementioned papers is that the varieties always have Picard number $\rho(F) := \text{rk}(\text{Pic}(F)) = 1$. Indeed in this case both c_1 and c_2 can be handled as integral numbers.

The aim of the present paper is to give a complete classification of aCM bundle of rank 2 on $F := \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subseteq \mathbb{P}^7$. This case is at the opposite extreme. Indeed we have $\rho(F) = 3$ and all the other del Pezzo threefolds have Picard number at most 2.

To formulate our main result, we introduce a bit of notation now. In what follows we will denote by $A(X)$ the Chow ring of a variety X , so that $A^r(X)$ denotes the set of cycles of codimension r .

We have three different projections $\pi_i: F \rightarrow \mathbb{P}^1$ and we denote by h_i the pull-back in $A^1(F)$ of the class of a point in the i -th copy of \mathbb{P}^1 . The exterior product morphism $A(\mathbb{P}^1) \otimes A(\mathbb{P}^1) \otimes A(\mathbb{P}^1) \rightarrow A(F)$ is an isomorphism (see [24], Example 8.3.7), thus we finally obtain

$$A(F) \cong \mathbb{Z}[h_1, h_2, h_3]/(h_1^2, h_2^2, h_3^2).$$

In particular, if \mathcal{E} is any bundle of rank 2 on F , then we can write $c_1(\mathcal{E}) = \alpha_1 h_1 + \alpha_2 h_2 + \alpha_3 h_3$ for suitable integers $\alpha_1, \alpha_2, \alpha_3$. There is an obvious action of the symmetric group of order 3 on F that lifts to an analogous action on $A(F)$. Thanks to such an action, we can often restrict our attention to the case $\alpha_1 \leq \alpha_2 \leq \alpha_3$.

Our first main result is the following.

Theorem A. *Let \mathcal{E} be an indecomposable aCM bundle of rank 2 on F and let $c_1(\mathcal{E}) = \alpha_1 h_1 + \alpha_2 h_2 + \alpha_3 h_3$ with $\alpha_1 \leq \alpha_2 \leq \alpha_3$. Assume that $h^0(F, \mathcal{E}) \neq 0$ and $h^0(F, \mathcal{E}(-h)) = 0$ (we briefly say that \mathcal{E} is initialized). Then:*

- (1) *the zero locus $E := (s)_0$ of a general section $s \in H^0(F, \mathcal{E})$ has codimension 2 inside F ;*
- (2) *either $0 \leq \alpha_i \leq 2$, $i = 1, 2, 3$, or $c_1 = h_1 + 2h_2 + 3h_3$;*
- (3) *if $c_1 = 2h$, or $c_1 = h_1 + 2h_2 + 3h_3$, then \mathcal{E} is Ulrich.*

After a basic reminder on aCM bundles in Section 2, we give the proof of this result, which occupies Sections 3, 4 and 5.

In order to better understand the above statement we recall that in [3] the authors prove that for each indecomposable, initialized aCM bundle \mathcal{E} of rank 2 on del Pezzo threefolds of degree $d = 3, 4, 5$, the bound $0 \leq c_1 \leq 2$ holds. Such a bound remains true also in the cases $d = 1, 2$.

Thus, from such a viewpoint, the statement of Theorem A can be read as follows: either c_1 satisfies the *standard bound* “ $0 \leq c_1 \leq 2$ ”, or \mathcal{E} is a *sporadic bundle*, i.e. $c_1 = h_1 + 2h_2 + 3h_3$.

In Section 6 we provide a complete classification of bundles satisfying the standard bound. Then we examine the sporadic case in Section 7. Finally, we will answer to the natural question of determining which intermediate cases are actually admissible in Section 8. We summarize this second main result in the following simplified statement (see Theorems 6.1, 6.7, 7.3, 8.1 for an expanded and detailed statement).

Theorem B. *There exists an indecomposable and initialized aCM bundle \mathcal{E} of rank 2 on F with $c_1(\mathcal{E}) = \alpha_1 h_1 + \alpha_2 h_2 + \alpha_3 h_3$ with $\alpha_1 \leq \alpha_2 \leq \alpha_3$ if and only if $(\alpha_1, \alpha_2, \alpha_3)$ is one of the following:*

$$(0, 0, 0), \quad (2, 2, 2), \quad (1, 2, 3), \quad (0, 0, 1), \quad (1, 2, 2).$$

Moreover, denote by $E \subseteq F \subseteq \mathbb{P}^7$ the zero-locus of a general section of such an \mathcal{E} . Then:

- (1) *if $\alpha_1 = \alpha_2 = \alpha_3 = 0$, then E is a line and each such a curve on F can be obtained in this way;*
- (2) *if $\alpha_1 = \alpha_2 = \alpha_3 = 2$, then E is a smooth elliptic normal curve of degree 8 and each such a curve on F can be obtained in this way;*
- (3) *if $\alpha_i = i$, then E is a rational normal curve of degree 7 and each such a curve on F can be obtained in this way;*

- (4) if $\alpha_1 = \alpha_2 = 0$, $\alpha_3 = 1$, then E is a line and each such a curve on F can be obtained in this way;
 (5) if $\alpha_1 = 1$, $\alpha_2 = \alpha_3 = 2$, then E is a, possibly reducible, quintic with arithmetic genus 0.

In particular, for each line $E \subseteq F$ we show the existence of exactly four indecomposable non-isomorphic aCM bundles of rank 2 whose general section vanishes exactly along E .

The study of (semi)stability of aCM bundles of rank 2 on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and of the moduli spaces of such (semi)stable bundles are the object of the paper [13].

The case of del Pezzo threefolds with Picard number 2 will be the object of forthcoming papers.

Throughout the whole paper we refer to [28] and [27] for all the unmentioned definitions, notations and results.

We conclude this introduction by expressing our thanks to the referee and to M. Filip for their comments that allowed us to considerably improve the exposition of the paper.

2. SOME FACTS ON ACM LOCALLY FREE SHEAVES

Throughout the whole paper k will denote an algebraically closed field of characteristic 0. Let $X \subseteq \mathbb{P}^N$ be a subvariety, i.e. an integral closed subscheme defined over k . We set $\mathcal{O}_X(h) := \mathcal{O}_{\mathbb{P}^N}(h) \otimes \mathcal{O}_X$. We start this section by recalling two important definitions.

The variety $X \subseteq \mathbb{P}^N$ is called arithmetically Cohen–Macaulay (aCM for short) if and only if the natural restriction maps $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(t)) \rightarrow H^0(X, \mathcal{O}_X(th))$ are surjective and $h^i(X, \mathcal{O}_X(th)) = 0$, $1 \leq i \leq \dim(X) - 1$.

The variety $X \subseteq \mathbb{P}^N$ is called arithmetically Gorenstein (aG for short) if and only if it is aCM and α -subcanonical, i.e. its dualizing sheaf satisfies $\omega_X \cong \mathcal{O}_X(\alpha h)$ for some $\alpha \in \mathbb{Z}$.

In what follows F will denote an aCM, integral and smooth subvariety of \mathbb{P}^N of positive dimension n .

Definition 2.1. Let \mathcal{E} be a vector bundle on F . We say that \mathcal{E} is *arithmetically Cohen–Macaulay* (aCM for short) if $H_*^i(F, \mathcal{E}) = 0$ for each $i = 1, \dots, n - 1$.

If \mathcal{E} is an aCM bundle, then the minimal number of generators $m(\mathcal{E})$ of $H_*^0(F, \mathcal{E})$ as a module over the graded coordinate ring of F is $\text{rk}(\mathcal{E}) \deg(F)$ at most (e.g. see [10]). The aCM bundles for which the maximum is attained are worth of particular interest for several reasons: e.g. they are semistable and they form an open subset in their moduli space.

Such bundles are called Ulrich bundles in the sequel. We recall the following definition (see [11], Definition 2.1 and Lemma 2.2: see also [10], Definition 3.4 and which is slightly different).

Definition 2.2. Let \mathcal{E} be a vector bundle on F . We say that \mathcal{E} is *initialized* if

$$\min\{ t \in \mathbb{Z} \mid h^0(F, \mathcal{E}(th)) \neq 0 \} = 0.$$

We say that \mathcal{E} is *Ulrich* if it is initialized, aCM and $h^0(F, \mathcal{E}) = \text{rk}(\mathcal{E}) \deg(F)$.

Let \mathcal{E} be an Ulrich bundle. On one hand we know that $m(\mathcal{E}) = \text{rk}(\mathcal{E}) \deg(F)$. On the other hand the generators of $H^0(F, \mathcal{E})$ are minimal generators of $H_*^0(F, \mathcal{E})$ due to the vanishing of $H^0(F, \mathcal{E}(-h))$. We conclude that \mathcal{E} is necessarily globally generated. Several other results are known for Ulrich bundles (e.g. see [21], [10], [11], [19]).

Assume now $n = 3$ and let \mathcal{E} be a bundle on F . We denote by ω_i the Chern classes of the sheaf $\Omega_{F|k}^1$. In this case Riemann–Roch theorem is

$$(1) \quad \chi(\mathcal{E}) = -\frac{\text{rk}(\mathcal{E})}{24} \omega_1 \omega_2 + \frac{1}{6} (c_1(\mathcal{E})^3 - 3c_1(\mathcal{E})c_2(\mathcal{E}) + 3c_3(\mathcal{E})) - \frac{1}{4} (\omega_1 c_1(\mathcal{E})^2 - 2\omega_1 c_2(\mathcal{E})) + \frac{1}{12} (\omega_1^2 c_1(\mathcal{E}) + \omega_2 c_1(\mathcal{E})).$$

Assume $\text{rk}(\mathcal{E}) = 2$ and let s be a global section of \mathcal{E} . In general its zero-locus $(s)_0 \subseteq F$ is either empty or its codimension is at most 2. Thus, in this second case, we can always write $(s)_0 = E \cup D$

where E has codimension 2 (or it is empty) and D has pure codimension 1 (or it is empty). In particular $\mathcal{E}(-D)$ has a section vanishing on E , thus we can consider its Koszul complex

$$(2) \quad 0 \longrightarrow \mathcal{O}_F(D) \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_{E|F}(c_1 - D) \longrightarrow 0.$$

If $D = 0$, then E is locally complete intersection, being $\mathrm{rk}(\mathcal{E}) = 2$, hence it has no embedded components.

Moreover we also have the following exact sequence

$$(3) \quad 0 \longrightarrow \mathcal{I}_{E|F} \longrightarrow \mathcal{O}_F \longrightarrow \mathcal{O}_E \longrightarrow 0.$$

The above construction can be reversed. Indeed on a smooth threefold the following particular case of the more general Hartshorne–Serre correspondence holds (for further details about the statement in the general case see [41], [26], [2]).

Theorem 2.3. *Let $E \subseteq F$ be a local complete intersection subscheme of codimension 2 and \mathcal{L} an invertible sheaf on F such that $H^2(F, \mathcal{L}^\vee) = 0$. If $\det(\mathcal{N}_{E|F}) \cong \mathcal{L} \otimes \mathcal{O}_E$, then there exists a vector bundle \mathcal{E} of rank 2 on F such that:*

- (1) $\det(\mathcal{E}) \cong \mathcal{L}$;
- (2) \mathcal{E} has a section s such that E coincides with the zero locus $(s)_0$ of s .

Moreover, if $H^1(F, \mathcal{L}^\vee) = 0$, the above two conditions determine \mathcal{E} up to isomorphism.

From now on, F will be $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. We recall that the canonical sheaf satisfies $\omega_F \cong \mathcal{O}_F(-2h)$ where h is the hyperplane class on F given by the natural embedding $F \subseteq \mathbb{P}^7$. Since F is also aCM, it follows that it is also aG, thus a vector bundle \mathcal{E} on F is aCM if and only if the same holds for \mathcal{E}^\vee .

As pointed out in the introduction $A(F) \cong \mathbb{Z}[h_1, h_2, h_3]/(h_1^2, h_2^2, h_3^2)$: thus $\mathrm{Pic}(F)$ is freely generated by h_1, h_2, h_3 . We have $h = h_1 + h_2 + h_3$, $\deg(F) = h^3 = 6$, $\omega_1 = \omega_F = -2h$, $\omega_1\omega_2 = -24$ in Formula (1). More generally, for each $D \in \mathrm{Pic}(F)$, then there are $\delta_1, \delta_2, \delta_3 \in \mathbb{Z}$ such that $\mathcal{O}_F(D) \cong \mathcal{O}_F(\delta_1 h_1 + \delta_2 h_2 + \delta_3 h_3)$. The following Künneth's formulas will be repeatedly used without explicit mention throughout the paper

$$h^i(F, \mathcal{O}_F(D)) = \sum_{\substack{(\delta_1, \delta_2, \delta_3) \in \mathbb{N}^3, \\ \delta_1 + \delta_2 + \delta_3 = i}} h^{i_1}(F, \mathcal{O}_{\mathbb{P}^1}(\delta_1)) h^{i_2}(F, \mathcal{O}_{\mathbb{P}^1}(\delta_2)) h^{i_3}(F, \mathcal{O}_{\mathbb{P}^1}(\delta_3)).$$

We will write $D \geq 0$ to denote that D has sections, i.e. $\delta_j \geq 0$, $j = 1, 2, 3$.

In particular we immediately have the following result.

Lemma 2.4. *The initialized aCM bundles of rank 1 on F are:*

$$\begin{aligned} & \mathcal{O}_F, \quad \mathcal{O}_F(h_1), \quad \mathcal{O}_F(h_2), \quad \mathcal{O}_F(h_3), \\ & \mathcal{O}_F(h_1 + h_2), \quad \mathcal{O}_F(h_1 + h_3), \quad \mathcal{O}_F(h_1 + h_2), \\ & \mathcal{O}_F(2h_1 + h_2), \quad \mathcal{O}_F(2h_1 + h_3), \quad \mathcal{O}_F(h_1 + 2h_2), \\ & \mathcal{O}_F(2h_2 + h_3), \quad \mathcal{O}_F(h_1 + 2h_3), \quad \mathcal{O}_F(h_2 + 2h_3). \end{aligned}$$

Proof. Straightforward. □

We now turn our attention on rank 2 vector bundles \mathcal{E} on F . In what follows, we will denote its Chern classes by

$$\begin{aligned} c_1 & := c_1(\mathcal{E}) = \alpha_1 h_1 + \alpha_2 h_2 + \alpha_3 h_3, \\ c_2 & := c_2(\mathcal{E}) = \beta_1 h_2 h_3 + \beta_2 h_1 h_3 + \beta_3 h_1 h_2. \end{aligned}$$

3. A LOWER BOUND ON THE FIRST CHERN CLASS

In this section we will find a bound from below for the first Chern class c_1 of an indecomposable initialized aCM bundle \mathcal{E} of rank 2 on F .

The following lemma will be useful.

Lemma 3.1. *Let \mathcal{E} be an initialized aCM bundle of rank 2 on F . Then $\mathcal{E}^\vee(2h)$ is globally generated.*

Proof. We have $h^i(F, \mathcal{E}^\vee((2-i)h)) = h^{3-i}(F, \mathcal{E}((i-4)h)) = 0$, $i = 1, 2, 3$. It follows that \mathcal{E} is 2-regular in the sense of Castelnuovo–Mumford (see [37]), hence it follows the first assertion. \square

Remark 3.2. *An immediate consequence of Lemma 3.1 is that $4h - c_1 = c_1(\mathcal{E}^\vee(2h))$ is effective, hence $\alpha_i \leq 4$.*

If $\alpha_i \geq 3$, $i = 1, 2, 3$, then there would be an injective morphism $\mathcal{O}_F(2h - c_1) \rightarrow \mathcal{O}_F(-h)$, which would induce an injective morphism $H^0(F, \mathcal{E}(2h - c_1)) \rightarrow H^0(F, \mathcal{E}(-h))$. On the one hand we know that the target space is zero. On the other hand $H^0(F, \mathcal{E}(2h - c_1)) \neq 0$ since $\mathcal{E}(2h - c_1) \cong \mathcal{E}^\vee(2h)$ is globally generated, a contradiction. It follows that at least one of the α_i is at most 2.

We now concentrate our attention in the proof that each non-zero section of an indecomposable initialized aCM bundle of rank 2 vanishes on F exactly along a curve. We first check that its zero-locus is non-empty.

Lemma 3.3. *Let \mathcal{E} be an indecomposable initialized aCM bundle of rank 2 on F . Then the zero locus $(s)_0 \subseteq F$ of a section of \mathcal{E} is non-empty.*

Proof. Assume that $\alpha_1 \leq \alpha_2 \leq \alpha_3$. If $(s)_0 = \emptyset$, then sequence (2) becomes

$$0 \rightarrow \mathcal{O}_F \rightarrow \mathcal{E} \rightarrow \mathcal{O}_F(c_1) \rightarrow 0.$$

Such a sequence corresponds to an element of $\text{Ext}^1(\mathcal{O}_F(c_1), \mathcal{O}_F) = H^1(F, \mathcal{O}_F(-c_1))$. Since \mathcal{E} is indecomposable, it follows that the last space must be non-zero. Thus $\alpha_3 \geq 2$ and $\alpha_1 \leq \alpha_2 \leq 0$.

If either $\alpha_1 \leq -1$ or $\alpha_3 \geq 3$, then the cohomology of the above sequence twisted by $-2h$ gives

$$0 \rightarrow H^2(F, \mathcal{E}(-2h)) \rightarrow H^2(F, \mathcal{O}_F(c_1 - 2h)) \rightarrow H^3(F, \mathcal{O}_F(-2h)) \cong k.$$

Due to the hypothesis on c_1 we have $1 \leq h^2(F, \mathcal{E}(-2h))$, contradicting that \mathcal{E} is aCM. We conclude that $\alpha_1 = \alpha_2 = 0$, i.e. $c_1 = 2h_3$.

Since $h^1(F, \mathcal{O}_F(-2h_3)) = 1$ there exists exactly one non-trivial exact sequence of the form

$$0 \rightarrow \mathcal{O}_F(-2h_3) \rightarrow \mathcal{E}(-2h_3) \rightarrow \mathcal{O}_F \rightarrow 0.$$

Notice that we always have on F the pull-back via π_3^* of the Euler exact sequence, i.e.

$$0 \rightarrow \mathcal{O}_F(-2h_3) \rightarrow \mathcal{O}_F(-h_3) \oplus \mathcal{O}_F(-h_3) \rightarrow \mathcal{O}_F \rightarrow 0.$$

We conclude that the two sequences above are isomorphic, hence $\mathcal{E} \cong \mathcal{O}_F(h_3) \oplus \mathcal{O}_F(h_3)$. It follows that there are no indecomposable initialized aCM vector bundles of rank 2 on F with $c_1 = 2h_3$. \square

Lemma 3.3 implies that for each $s \in H^0(F, \mathcal{E})$ we have $(s)_0 = E \cup D$ where E has codimension 2 (or it is empty) and $D \in |\delta_1 h_1 + \delta_2 h_2 + \delta_3 h_3|$ has codimension 1 (or it is empty).

On the one hand, let us take the cohomology of Sequence (2) twisted by $\mathcal{O}_F(-h)$. Then the vanishing of $h^0(F, \mathcal{E}(-h))$ implies $h^0(F, \mathcal{O}_F(D - h)) = 0$. In particular we know that at least one of the δ_i is zero.

On the other hand, twisting the same sequence by $\mathcal{O}_F(2h - c_1)$, using the isomorphism $\mathcal{E}(-c_1) \cong \mathcal{E}^\vee$ and taking into account that $\mathcal{E}^\vee(2h)$ is globally generated (see Lemma 3.1), we obtain that $\mathcal{I}_{E|F}(2h - D)$ is globally generated too. Thus

$$0 \neq H^0(F, \mathcal{I}_{E|F}(2h - D)) \subseteq H^0(F, \mathcal{O}_F(2h - D)),$$

hence $\delta_i \leq 2$, $i = 1, 2, 3$. Thus, up to permutations,

$$(4) \quad (\delta_1, \delta_2, \delta_3) \in \{ (0, 0, 0), (0, 0, 1), (0, 1, 1), (0, 0, 2), (0, 1, 2), (0, 2, 2) \}.$$

We now prove that $c_1 - D$ is effective: notice that this implies the effectiveness of c_1 too. Assume $c_1 - D$ is non-effective on F : thanks to Sequence (3) we have

$$(5) \quad h^0(F, \mathcal{I}_{E|F}(c_1 - D)) \leq h^0(F, \mathcal{O}_F(c_1 - D)) = 0.$$

Taking into account that \mathcal{E} is aCM and $\delta_i \geq 0$, the cohomology of Sequence (2) yields also the vanishings

$$(6) \quad h^1(F, \mathcal{I}_{E|F}(c_1 - D)) = h^2(F, \mathcal{I}_{E|F}(c_1 - D)) = 0.$$

Twisting Sequence (2) by $\mathcal{O}_F(-h)$, the same argument also gives

$$(7) \quad h^0(F, \mathcal{I}_{E|F}(c_1 - D - h)) = h^1(F, \mathcal{I}_{E|F}(c_1 - D - h)) = h^2(F, \mathcal{I}_{E|F}(c_1 - D - h)) = 0.$$

Finally, the cohomology of Sequence (2) twisted by $\mathcal{O}_F(-2h)$ gives

$$0 \longrightarrow H^1(F, \mathcal{I}_{E|F}(c_1 - D - 2h)) \longrightarrow H^2(F, \mathcal{O}_F(D - 2h)) \longrightarrow 0$$

because \mathcal{E} is aCM. Let

$$\eta_1(D) := \begin{cases} 0 & \text{if } D \notin |2h_j|, \\ 1 & \text{if } D \in |2h_j|. \end{cases}$$

Due to the restrictions on the δ_i 's, we immediately have

$$(8) \quad h^1(F, \mathcal{I}_{E|F}(c_1 - D - 2h)) = h^2(F, \mathcal{O}_F(D - 2h)) = \eta_1(D).$$

We also have the exact sequence

$$\begin{aligned} 0 \longrightarrow H^2(F, \mathcal{I}_{E|F}(c_1 - D - 2h)) &\longrightarrow H^3(F, \mathcal{O}_F(D - 2h)) \longrightarrow \\ &\longrightarrow H^3(F, \mathcal{E}(-2h)) \longrightarrow H^3(F, \mathcal{I}_{E|F}(c_1 - D - 2h)) \longrightarrow 0. \end{aligned}$$

If $D \neq 0$, then $h^3(F, \mathcal{O}_F(D - 2h)) = 0$, thus $h^2(F, \mathcal{I}_{E|F}(c_1 - D - 2h)) = 0$ too. Assume $D = 0$: then $E \neq \emptyset$ (see Lemma 3.3), so that $h^0(F, \mathcal{I}_{E|F}) = 0$, hence

$$\begin{aligned} h^3(F, \mathcal{E}(-2h)) &= h^0(F, \mathcal{E}(-c_1)) = h^0(F, \mathcal{O}_F(-c_1)) = \\ &= h^3(F, \mathcal{O}_F(c_1 - 2h)) = h^3(F, \mathcal{O}_F(c_1 - D - 2h)). \end{aligned}$$

Sequence (3) twisted by $\mathcal{O}_F(c_1 - D - 2h)$, also yields $h^3(F, \mathcal{I}_{E|F}(c_1 - D - 2h)) = h^3(F, \mathcal{O}_F(c_1 - D - 2h))$. Thus the last map in the above sequence is an isomorphism. We conclude that if we define

$$\eta_2(D) := \begin{cases} 0 & \text{if } D \neq 0, \\ 1 & \text{if } D = 0, \end{cases}$$

then

$$(9) \quad h^2(F, \mathcal{I}_{E|F}(c_1 - D - 2h)) = h^3(F, \mathcal{O}_F(D - 2h)) = \eta_2(D).$$

Lemma 3.4. *Let \mathcal{E} be an indecomposable initialized aCM bundle of rank 2 on F . Assume that $s \in H^0(F, \mathcal{E})$ is such that $(s)_0 = E \cup D$ where E has codimension 2 (or it is empty) and D has codimension 1. If $c_1 - D$ is not effective, then $E = \emptyset$.*

Proof. Assume $E \neq \emptyset$, hence $\deg(E) \geq 1$. Let H be a general hyperplane in \mathbb{P}^7 . Define $S := F \cap H$ and $Z := E \cap H$, so that $\dim(Z) = 0$. We have the following exact sequence

$$(10) \quad 0 \longrightarrow \mathcal{I}_{E|F}(c_1 - D - h) \longrightarrow \mathcal{I}_{E|F}(c_1 - D) \longrightarrow \mathcal{I}_{Z|S}(c_1 - D) \longrightarrow 0.$$

The vanishing of the cohomologies of $\mathcal{I}_{E|F}(c_1 - D)$ and $\mathcal{I}_{E|F}(c_1 - D - h)$ (see Equalities (5), (6), (7)) implies

$$h^0(S, \mathcal{I}_{Z|S}(c_1 - D)) = h^1(S, \mathcal{I}_{Z|S}(c_1 - D)) = 0.$$

The above equalities and the cohomology of

$$(11) \quad 0 \longrightarrow \mathcal{I}_{Z|S}(c_1 - D) \longrightarrow \mathcal{O}_S(c_1 - D) \longrightarrow \mathcal{O}_Z \longrightarrow 0,$$

give $h^1(S, \mathcal{O}_S(c_1 - D)) = 0$ and

$$h^0(S, \mathcal{O}_S(c_1 - D)) = h^0(Z, \mathcal{O}_Z) = \deg(E) \geq 1.$$

Twisting Sequence (10) by $\mathcal{O}_F(-h)$, thanks to (8) and (9), a similar argument also gives

$$\begin{aligned} h^0(S, \mathcal{I}_{Z|S}(c_1 - D - h)) &= h^1(F, \mathcal{I}_{E|F}(c_1 - D - 2h)) = \eta_1(D), \\ h^1(S, \mathcal{I}_{Z|S}(c_1 - D - h)) &= h^2(F, \mathcal{I}_{E|F}(c_1 - D - 2h)) = \eta_2(D). \end{aligned}$$

The above equalities and the cohomology of Sequence (11) twisted by $\mathcal{O}_S(-h)$ yield

$$h^0(Z, \mathcal{O}_Z) = h^0(S, \mathcal{O}_S(c_1 - D - h)) - h^1(S, \mathcal{O}_S(c_1 - D - h)) - \eta_1(D) + \eta_2(D),$$

and $h^1(S, \mathcal{O}_S(c_1 - D - h)) \leq \eta_2(D)$. It follows

$$h^0(S, \mathcal{O}_S(c_1 - D - h)) - 1 \leq h^0(Z, \mathcal{O}_Z) \leq h^0(S, \mathcal{O}_S(c_1 - D - h)) + 1.$$

Set $c_1 - D := \epsilon_1 h_1 + \epsilon_2 h_2 + \epsilon_3 h_3$ and assume $\epsilon_1 \leq \epsilon_2 \leq \epsilon_3$: $\epsilon_1 \leq -1$ because $c_1 - D$ is assumed to be not effective. Thus the cohomology of

$$0 \longrightarrow \mathcal{O}_F(c_1 - D - h) \longrightarrow \mathcal{O}_F(c_1 - D) \longrightarrow \mathcal{O}_S(c_1 - D) \longrightarrow 0$$

and the vanishing $h^1(S, \mathcal{O}_S(c_1 - D)) = 0$ proved above, yield

$$h^0(S, \mathcal{O}_S(c_1 - D)) = h^1(F, \mathcal{O}_F(c_1 - D - h)) - h^1(F, \mathcal{O}_F(c_1 - D)).$$

Since $h^0(S, \mathcal{O}_S(c_1 - D)) \geq 1$, it follows that $h^1(F, \mathcal{O}_F(c_1 - D - h)) \geq 1$. In particular $1 \leq \epsilon_2 \leq \epsilon_3$, hence

$$h^0(F, \mathcal{O}_F(c_1 - D)) = \epsilon_1 \epsilon_2 + \epsilon_1 \epsilon_3 + \epsilon_2 \epsilon_3 + \epsilon_1 + \epsilon_2 + \epsilon_3 + 1.$$

Taking again the cohomology of the above sequence twisted by $\mathcal{O}_F(c_1 - D - h)$ we obtain the exact sequence (recall that $c_1 - D$ is not effective)

$$\begin{aligned} 0 \longrightarrow H^0(S, \mathcal{O}_S(c_1 - D - h)) \longrightarrow H^1(F, \mathcal{O}_F(c_1 - D - 2h)) \xrightarrow{\varphi} \\ \longrightarrow H^1(F, \mathcal{O}_F(c_1 - D - h)) \longrightarrow H^1(S, \mathcal{O}_S(c_1 - D - h)). \end{aligned}$$

If φ is not surjective then necessarily $1 \leq h^1(S, \mathcal{O}_S(c_1 - D - h)) \leq \eta_2(D) \leq 1$, thus equality must hold. In particular $D = 0$, $h^0(S, \mathcal{O}_S(c_1 - D - h)) = h^0(Z, \mathcal{O}_Z) \geq 1$ and

$$h^0(S, \mathcal{O}_S(c_1 - D - h)) = h^1(F, \mathcal{O}_F(c_1 - D - 2h)) - h^1(F, \mathcal{O}_F(c_1 - D - h)) + 1.$$

If $\epsilon_2 = 1$, then $h^1(F, \mathcal{O}_F(c_1 - D - 2h)) = 0$ and $h^1(F, \mathcal{O}_F(c_1 - D - h)) \geq 1$, hence $h^0(S, \mathcal{O}_S(c_1 - D - h)) \leq 0$, a contradiction. Thus $2 \leq \epsilon_2 \leq \epsilon_3$, hence

$$h^0(S, \mathcal{O}_S(c_1 - D - h)) = \epsilon_1 \epsilon_2 + \epsilon_1 \epsilon_3 + \epsilon_2 \epsilon_3 - \epsilon_1 - \epsilon_2 - \epsilon_3 + 2.$$

The equality $h^0(S, \mathcal{O}_S(c_1 - D)) = h^0(Z, \mathcal{O}_Z) = h^0(S, \mathcal{O}_S(c_1 - D - h))$ implies $2\epsilon_1 + 2\epsilon_2 + 2\epsilon_3 + 1 = 0$, which is not possible, because the ϵ_i 's are integers.

If φ is surjective, then

$$h^0(S, \mathcal{O}_S(c_1 - D - h)) = h^1(F, \mathcal{O}_F(c_1 - D - 2h)) - h^1(F, \mathcal{O}_F(c_1 - D - h)),$$

and we can easily again infer $2 \leq \epsilon_2 \leq \epsilon_3$, hence

$$h^0(S, \mathcal{O}_S(c_1 - D - h)) = \epsilon_1 \epsilon_2 + \epsilon_1 \epsilon_3 + \epsilon_2 \epsilon_3 - \epsilon_1 - \epsilon_2 - \epsilon_3 + 1.$$

If $h^0(Z, \mathcal{O}_Z) = h^0(S, \mathcal{O}_S(c_1 - D - h)) \pm 1$ we obtain $2\epsilon_1 + 2\epsilon_2 + 2\epsilon_3 \pm 1 = 0$ which is not possible.

If $h^0(Z, \mathcal{O}_Z) = h^0(S, \mathcal{O}_S(c_1 - D - h))$, then $\epsilon_1 + \epsilon_2 + \epsilon_3 = 0$, thus $\epsilon_1 = -\epsilon_2 - \epsilon_3$. Substituting in the expression of $h^0(S, \mathcal{O}_S(c_1 - D))$ we obtain

$$h^0(S, \mathcal{O}_S(c_1 - D)) = -\epsilon_2^2 - \epsilon_2 \epsilon_3 - \epsilon_3^2 + 1 \leq -11$$

an absurd.

We conclude that the assumption $E \neq \emptyset$ yields a contradiction. This completes the proof of the statement. \square

We are now ready to prove the claimed effectiveness of $c_1 - D$.

Proposition 3.5. *If \mathcal{E} is an indecomposable initialized aCM bundle of rank 2 on F , $s \in H^0(F, \mathcal{E})$ and D denotes the component of codimension 1 of $(s)_0$, if any, then $c_1 \geq c_1 - D \geq 0$.*

In particular if $c_1 = 0$, then the zero-locus of each section $s \in H^0(F, \mathcal{E})$ has codimension 2.

Proof. Assume again that $c_1 - D$ is non-effective, so that $E = \emptyset$ due to the above Lemma. Thus $\mathcal{I}_{E|F} \cong \mathcal{O}_F$, hence Sequence (2) becomes

$$(12) \quad 0 \longrightarrow \mathcal{O}_F(D) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_F(c_1 - D) \longrightarrow 0.$$

Thanks to Lemma 3.3 we know that D is non-zero. If $D \in |2h_2 + 2h_3|$, then the cohomology of Sequence (12) twisted by $\mathcal{O}_F(-2h)$ would give

$$0 = h^0(F, \mathcal{O}_F(c_1 - D)) \geq h^0(F, \mathcal{O}_F(c_1 - D - 2h)) = h^1(F, \mathcal{O}_F(D - 2h)) = 1,$$

a contradiction. If $D \in |2h_3|$, then $h^1(F, \mathcal{O}_F(c_1 - D - 2h)) = h^2(F, \mathcal{O}_F(D - 2h)) = 1$ contradicting the condition $\epsilon_1 \leq -1$.

In the remaining cases we know that $h^i(F, \mathcal{O}_F(D - h)) = h^i(F, \mathcal{O}_F(D - 2h)) = 0$, $i = 1, 2$. Taking the cohomology of Sequence (12) suitably twisted, we obtain $h^i(F, \mathcal{O}_F(c_1 - D - h)) = h^i(F, \mathcal{O}_F(c_1 - D - 2h)) = 0$ in the same range, because both \mathcal{E} and $\mathcal{O}_F(D)$ are aCM.

If Sequence (12) does not split, then

$$H^1(F, \mathcal{O}_F(2D - c_1)) = \text{Ext}_F^1(\mathcal{O}_F(c_1 - D), \mathcal{O}_F(D)) \neq 0.$$

Thus $\delta_j - \epsilon_j \leq -2$ for exactly one $j = 1, 2, 3$. Since $\delta_1 \geq 0$ and $\epsilon_1 \leq -1$ (as we assumed above) such a j is not 1. We can assume $\delta_3 - \epsilon_3 \leq -2$, hence we obtain $\epsilon_3 \geq 2$. If $\epsilon_2 \geq 1$, then $h^1(F, \mathcal{O}_F(c_1 - D - h)) \neq 0$. If $\epsilon_2 \leq 0$, then $h^2(F, \mathcal{O}_F(c_1 - D - 2h)) \neq 0$. In both cases we have a contradiction. \square

4. AN UPPER BOUND ON THE FIRST CHERN CLASS

In this section we will find a bound from above for the first Chern class c_1 of an indecomposable initialized aCM bundle \mathcal{E} of rank 2 on F . Thus $h^i(F, \mathcal{E}^\vee(-h)) = h^{3-i}(F, \mathcal{E}(-h)) = 0$ for $i \geq 1$. As pointed out in Lemma 3.3 the zero locus of a general section $s \in H^0(F, \mathcal{E})$ is non-empty. We will still decompose it as $(s)_0 = E \cup D$, where D is the divisorial part and E has pure codimension 2.

We have

$$h^0(F, \mathcal{I}_{E|F}(-D - h)) \leq h^0(F, \mathcal{O}_F(-D - h)) = 0$$

(see Sequence (3)). Thus the cohomology of Sequence (2) twisted by $\mathcal{O}_F(-c_1)$, the standard isomorphism $\mathcal{E}^\vee \cong \mathcal{E}(-c_1)$ and the effectiveness of $c_1 - D$ proved in Proposition 3.5 imply

$$\chi(\mathcal{E}^\vee(-h)) = h^0(F, \mathcal{E}^\vee(-h)) = h^0(F, \mathcal{O}_F(D - c_1 - h)) = 0.$$

Such an equality, Formula (1) and the equalities

$$\begin{aligned} c_1^3 &= 6\alpha_1\alpha_2\alpha_3, \\ c_1^2h &= 2(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3), \\ c_1h^2 &= 2(\alpha_1 + \alpha_2 + \alpha_3), \\ \omega_2c_1 &= 4(\alpha_1 + \alpha_2 + \alpha_3), \end{aligned}$$

finally imply

$$(13) \quad c_1c_2 = 2\alpha_1\alpha_2\alpha_3.$$

Since \mathcal{E} is initialized and aCM we also have $h^i(F, \mathcal{E}(-2h)) = 0$ for $i \leq 2$. Thus $h^i(F, \mathcal{E}^\vee) = h^{3-i}(F, \mathcal{E}(-2h)) = 0$, for $i \geq 1$. Let $D \neq 0$: $H^0(F, \mathcal{I}_{E|F})$ is contained in the kernel of the natural injective map $H^0(F, \mathcal{O}_F) \rightarrow H^0(E, \mathcal{O}_E)$, hence $h^0(F, \mathcal{I}_{E|F}) = h^0(F, \mathcal{I}_{E|F}(-D)) = 0$. If $D \neq 0$, then

$$h^0(F, \mathcal{I}_{E|F}(-D)) \leq h^0(F, \mathcal{O}_F(-D)) = 0.$$

We conclude that $h^0(F, \mathcal{I}_{E|F}(-D)) = 0$ without restrictions on D . Let

$$e(c_1, D) := \begin{cases} 0 & \text{if } D \neq c_1, \\ 1 & \text{if } D = c_1. \end{cases}$$

Since $(s)_0 \neq \emptyset$, it follows that

$$h^0(F, \mathcal{E}^\vee) = h^0(F, \mathcal{O}_F(D - c_1)) = e(c_1, D)$$

Combining as above Sequences (3), (2) and Formulas (1), (13) we finally obtain

$$(14) \quad hc_2 = \alpha_1\alpha_2\alpha_3 + (1 - \alpha_1)(1 - \alpha_2)(1 - \alpha_3) + 1 - e(c_1, D).$$

In what follows we will often combine the above equalities with the trivial ones

$$(15) \quad \begin{cases} \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 = c_1c_2 \\ \beta_1 + \beta_2 + \beta_3 = hc_2. \end{cases}$$

Finally, assume that $E \neq \emptyset$. Its class in $A^2(F)$ is $c_2(\mathcal{E}(-D)) = c_2 - c_1D + D^2$. Since $|h_i|$, $i = 1, 2, 3$ is base-point-free on F , $D \geq 0$ and $c_1 - D \geq 0$ (see Proposition 3.5), it follows that

$$(16) \quad \beta_i \geq \beta_i - h_i(c_1 - D)D = h_i c_2 - h_i(c_1 - D)D = h_i c_2(\mathcal{E}(-D)) \geq 0, \quad i = 1, 2, 3.$$

The numbers c_1c_2 and hc_2 will be very important in what follows.

Lemma 4.1. *Let \mathcal{E} be a vector bundle of rank 2 on F and $s \in H^0(F, \mathcal{E})$ a section such that $E := (s)_0$ has codimension 2. Then it has no embedded components and*

$$\deg(E) = hc_2, \quad p_a(E) = \frac{1}{2}c_1c_2 - hc_2 + 1.$$

Proof. We already noticed that if E has codimension 2, then it has no embedded components.

The assertion on $\deg(E)$ is immediate, because the class in $A^2(F)$ is c_2 . Let us compute $p_a(E)$. Adjunction formula on F yields $\omega_E \cong \omega_F \otimes \det(\mathcal{N}_{E|F})$. We have $\mathcal{N}_{E|F} \cong \mathcal{E} \otimes \mathcal{O}_E$, thus $\det(\mathcal{N}_{E|F}) \cong \mathcal{O}_E(c_1)$. We also have that $\omega_F \cong \mathcal{O}_F(-2h)$. Equating the degrees of the two members of the above formula we obtain also the second equality. \square

Now we turn our attention on the upper bound on c_1 .

Proposition 4.2. *If \mathcal{E} is an indecomposable initialized aCM bundle of rank 2 on F , then either $2h - c_1 \geq 0$ or $c_1 = h_1 + 2h_2 + 3h_3$ up to permutations of the h_i 's.*

Moreover, if either $c_1 = 2h$ or $c_1 = h_1 + 2h_2 + 3h_3$ up to permutations of the h_i 's, then \mathcal{E} is Ulrich, hence the zero-locus $E := (s)_0$ of a general section $s \in H^0(F, \mathcal{E})$ is a smooth curve.

Proof. We distinguish two cases according to whether \mathcal{E} is regular in the sense of Castelnuovo-Mumford or not.

In the latter case we infer $h^0(F, \mathcal{E}^\vee(h)) = h^3(F, \mathcal{E}(-3h)) \neq 0$ because \mathcal{E} is aCM. Thus, if $t \in \mathbb{Z}$ is such that $\mathcal{E}^\vee(th)$ is initialized, we know that $t \leq 1$. Since $\mathcal{E}^\vee(th)$ is obviously also aCM, we know that $2th - c_1 = c_1(\mathcal{E}^\vee(th))$ is effective, due to Proposition 3.5. We conclude that the same is true for $2h - c_1$, because $t \leq 1$.

Now we assume that \mathcal{E} is regular, hence it is globally generated. Thus the zero-locus $E := (s)_0$ of a general section $s \in H^0(F, \mathcal{E})$ is a smooth curve, i.e. the divisorial part D is 0. Let $\alpha_1 \leq \alpha_2 \leq \alpha_3$: we already know from Remark 3.2 that $0 \leq \alpha_1 \leq 2$ and $\alpha_2, \alpha_3 \leq 4$. We want to examine the cases $\alpha_3 = 3, 4$. In any case we have that $h^i(F, \mathcal{E}^\vee(h)) = h^{3-i}(F, \mathcal{E}(-3h)) = 0$, $i = 0, 1, 2, 3$, because \mathcal{E} is assumed aCM, regular and initialized. Combining the above remarks with equality (1), Formulas (13), (14) and the vanishing of $e(c_1, D)$ (recall that $D = 0$ and $c_1 \neq 0$), we finally obtain

$$0 = \chi(\mathcal{E}^\vee(h)) = 12 - 2\alpha_1 - 2\alpha_2 - 2\alpha_3.$$

Thus we have the following possibilities for $(\alpha_1, \alpha_2, \alpha_3)$ with $\alpha_3 \geq 3$: $(1, 2, 3)$, $(1, 1, 4)$, $(0, 3, 3)$, $(0, 2, 4)$.

If $c_1 = h_1 + 2h_2 + 3h_3$ up to permutations, then Formula (14) give $hc_2 = 7$ and $c_1c_2 = 12$. Substituting in Equality (1) we obtain $\chi(\mathcal{E}) = 12$. Since \mathcal{E} is aCM, it follows that $h^1(F, \mathcal{E}) =$

$h^2(F, \mathcal{E}) = 0$. Since \mathcal{E} is initialized, it follows that $h^3(F, \mathcal{E}) = h^0(F, \mathcal{E}(-2h - c_1)) = 0$. We conclude that $h^0(F, \mathcal{E}) = 12$ and \mathcal{E} turns out to be Ulrich. Similarly, one can also prove that \mathcal{E} is Ulrich in all the other cases.

We now prove that if $c_1 = \alpha_1 h_1 + \alpha_2 h_2 + \alpha_3 h_3$ is either $h_1 + h_2 + 4h_3$ or $2h_2 + 4h_3$ or $3h_2 + 3h_3$ up to permutations, then \mathcal{E} splits as a sum of invertible sheaves.

Let $c_1 = h_1 + h_2 + 4h_3$. Then we know that $\mathcal{E}^\vee(2h)$ is Ulrich (see Lemma 2.4 of [11]): in particular it is aCM and initialized and we have $c_1(\mathcal{E}^\vee(2h)) = 3h_1 + 3h_2$.

Hence it suffices to examine only the two last cases $(0, 3, 3)$, $(0, 2, 4)$ where $\alpha_1 = 0$. Recall that E is a smooth curve. Thus the System (15) and the Inequalities (16) with $D = 0$ force $c_2 = ch_2 h_3$ where $c := \alpha_2 \alpha_3 - 4$: due to the restrictions on α_2 and α_3 we have that c is either 4, or 5.

Lemma (4.1) implies $\deg(E) = c$ and $p_a(E) = 1 - c \leq -3$. Since E is smooth, it follows that it is necessarily the non-connected union of c pairwise skew lines whose class in $A^2(F)$ is $h_2 h_3$. In particular

$$(17) \quad \mathcal{O}_E(a_1 h_1 + a_2 h_2 + a_3 h_3) \cong \mathcal{O}_{\mathbb{P}^1}(a_1)^{\oplus c}.$$

Recalling the possible values of c_1 we have $h^0(F, \mathcal{O}_F(c_1 - h_2 - 2h_3)) = 6$. On the one hand, the cohomology of Sequence (3) twisted by $\mathcal{O}_F(c_1 - h_2 - 2h_3)$ gives rise to a monomorphism

$$\vartheta: H^0(F, \mathcal{I}_{E|F}(c_1 - h_2 - 2h_3)) \rightarrow H^0(F, \mathcal{O}_F(c_1 - h_2 - 2h_3)).$$

We fix three points p_1, p_2, p_3 on three different lines in E . We have that $c_1 - h_2 - 2h_3$ is either $h_2 + 2h_3$ or $2h_2 + h_3$.

Let $V_i \subseteq H^0(F, \mathcal{O}_F(c_1 - h_2 - 2h_3))$ be the subspace of sections corresponding to divisors through the points p_1, \dots, p_i . We have $V := V_3 \subseteq V_2 \subseteq V_1 \subseteq H^0(F, \mathcal{O}_F(c_1 - h_2 - 2h_3))$. On the one hand, at each step the dimension can go down at most by 1, hence $\dim(V) \geq h^0(F, \mathcal{O}_F(c_1 - h_2 - 2h_3)) - 3 = 3$. On the other hand, using the fact that $\mathcal{O}_F(h_i)$ is globally generated, it is easy to check that the above inclusions are all strict, thus we also have $\dim(V) \leq 3$. We conclude that $\dim(V) = 3$.

We trivially have that $H^0(F, \mathcal{I}_{E|F}(c_1 - h_2 - 2h_3)) \cong \text{im}(\vartheta) \subseteq V$, thus $h^0(F, \mathcal{I}_{E|F}(c_1 - h_2 - 2h_3)) \leq 3$. Since $h^0(E, \mathcal{O}_E(c_1 - h_2 - 2h_3)) = c$ due to the Isomorphism (17) above, it follows that $h^0(F, \mathcal{I}_{E|F}(c_1 - h_2 - 2h_3)) \geq 6 - c$.

The cohomology of Sequence (2) for s twisted by $\mathcal{O}_F(-h_2 - 2h_3)$ thus yields

$$3 \geq h^0(F, \mathcal{E}(-h_2 - 2h_3)) = h^0(F, \mathcal{I}_{E|F}(c_1 - h_2 - 2h_3)) \geq 6 - c \geq 1.$$

We have that $c_1(\mathcal{E}(-h_2 - 2h_3)) = c_1 - 2h_2 - 4h_3$ and $c_2(\mathcal{E}(-h_2 - 2h_3)) = 0$.

Let $\sigma \in H^0(F, \mathcal{E}(-h_2 - 2h_3))$ be a general section. As usual we can write $(\sigma)_0 = \Theta \cup \Xi$ where Θ has codimension 2 (or it is empty) and Ξ is a, possibly 0, effective divisor.

The cohomology of Sequence (3) yields

$$h^0(F, \mathcal{I}_{\Theta|F}(c_1 - 2h_2 - 4h_3 - \Xi)) \leq h^0(F, \mathcal{O}_F(c_1 - 2h_2 - 4h_3 - \Xi))$$

which is zero because c_1 is either $2h_2 + 4h_3$ or $3h_2 + 3h_3$ and Ξ is effective. The cohomology of Sequence (2) for σ implies $h^0(F, \mathcal{O}_F(\Xi)) = h^0(F, \mathcal{E}(-h_2 - 2h_3)) \leq 3$. We deduce that $\Xi \in |\xi h_i|$ for some integer $0 \leq \xi \leq 2$. Moreover \mathcal{E} is initialized, thus we immediately infer from the monomorphism $\mathcal{O}_F(\Xi) \rightarrow \mathcal{E}(-h_2 - 2h_3)$ that $\Xi \in |\xi h_i|$ where $i \neq 1$. It follows that

$$c_2(\mathcal{E}(-h_2 - 2h_3 - \Xi)) = c_2 - c_1(h_2 + 2h_3 + \Xi) + (h_2 + 2h_3 + \Xi)^2 = \begin{cases} 0 & \text{if } \alpha_2 = 2, \alpha_3 = 4, \\ -\xi h_2 h_3 & \text{if } \alpha_2 = \alpha_3 = 3, \Xi = \xi h_3, \\ \xi h_2 h_3 & \text{if } \alpha_2 = \alpha_3 = 3, \Xi = \xi h_2. \end{cases}$$

Recall that $\deg(\Theta) = c_2(\mathcal{E}(-h_2 - 2h_3 - \Xi))h$. In the first case, $\deg(\Theta) = 0$, hence $\Theta = \emptyset$. In the second case, $\deg(\Theta) = -\xi \leq 0$, hence $\xi = 0$: in particular $\Xi = 0$ and $\Theta = \emptyset$. In the third case Sequence (2) for σ becomes

$$0 \longrightarrow \mathcal{O}_F(\xi h_2) \longrightarrow \mathcal{E}(-h_2 - 2h_3) \longrightarrow \mathcal{I}_{\Theta|F}((1 - \xi)h_2 - h_3) \longrightarrow 0.$$

Taking the cohomology of the above sequence twisted by $\mathcal{O}_F(-2h_1 - h_2)$, we would obtain that $h^1(F, \mathcal{E}(-2h)) \geq h^1(F, \mathcal{O}_F(-2h_1 + (\xi - 1)h_2))$, because $h^0(F, \mathcal{I}_{\Theta|F}(-2h_1 - \xi h_2 - h_3)) = 0$. Thus

$\xi = 0$ because \mathcal{E} is aCM. Since $\deg(\Theta) = 0$, it again follows that $\Theta = \emptyset$. Thus, in any case Sequence (2) for σ becomes

$$0 \longrightarrow \mathcal{O}_F(\Xi) \longrightarrow \mathcal{E}(-h_2 - 2h_3) \longrightarrow \mathcal{O}_F(c_1 - 2h_2 - 4h_3 - \Xi) \longrightarrow 0.$$

The above sequence corresponds to an element of

$$\mathrm{Ext}_F^1(\mathcal{O}_F(c_1 - 2h_2 - 4h_3 - \Xi), \mathcal{O}_F(\Xi)) \cong H^1(F, \mathcal{O}_F(2h_2 + 4h_3 - c_1 + 2\Xi)).$$

The last space is zero, due to the possible values of c_1 and the corresponding values of ξ . We conclude that \mathcal{E} would be decomposable, a contradiction. \square

5. THE PROOF OF THE THEOREM A

In this section we will complete the proof of Theorem A stated in the introduction. It only remains to show that if \mathcal{E} is an indecomposable, initialized, aCM bundle, then its general section vanishes exactly along a curve (see Proposition 3.5).

We already checked in the previous sections (see Propositions 3.5 and 4.2) that either $c_1 \geq 0$ and $2h - c_1 \geq 0$ or $c_1 = h_1 + 2h_2 + 3h_3$ up to permutations of the h_i 's for such kind of bundles and that the general section vanishes exactly along a curve when either $c_1 = 0$, or $2h - c_1 = 0$, or $c_1 = h_1 + 2h_2 + 3h_3$ up to permutations of the h_i 's.

It follows that we can restrict our attention to the remaining cases. If we assume $\alpha_1 \leq \alpha_2 \leq \alpha_3$, such cases satisfy $\alpha_1 \leq 1 \leq \alpha_3$.

Lemma 5.1. *Let \mathcal{E} be an indecomposable initialized aCM bundle of rank 2 on F whose general section $s \in H^0(F, \mathcal{E})$ satisfies $(s)_0 = E \cup D$ where E has codimension 2 (or it is empty) and $D \in |\delta_1 h_1 + \delta_2 h_2 + \delta_3 h_3|$ is non-zero. Then:*

(1) up to permutations

$$(\delta_1, \delta_2, \delta_3) \in \{ (0, 0, 1), (0, 1, 1), (0, 0, 2), (0, 1, 2) \};$$

(2) \mathcal{E} is not globally generated;

(3) $E \neq \emptyset$.

Proof. We already know that

$$(\delta_1, \delta_2, \delta_3) \in \{ (0, 0, 0), (0, 0, 1), (0, 1, 1), (0, 0, 2), (0, 1, 2), (0, 2, 2) \}$$

(see Relation (4)). Since $D \neq 0$, it follows that $(\delta_1, \delta_2, \delta_3) \neq (0, 0, 0)$. Let $D \in |2h_2 + 2h_3|$ and look at the cohomologies of Sequences (2) and (3) respectively twisted by $\mathcal{O}_F(-2h)$ and $\mathcal{O}_F(c_1 - D - 2h)$. On the one hand, taking into account that $h^0(F, \mathcal{E}(-2h)) = h^1(F, \mathcal{E}(-2h)) = 0$ by hypothesis we obtain

$$1 = h^1(F, \mathcal{O}_F(D - 2h)) = h^0(F, \mathcal{I}_{E|F}(c_1 - D - 2h)) \leq h^0(F, \mathcal{O}_F(c_1 - D - 2h)).$$

On the other hand we also know that the last dimension is zero, due to the restrictions $D \geq 0$, $2h - c_1 \geq 0$ and $c_1 \neq 2h$. It follows that equality $(\delta_1, \delta_2, \delta_3) = (0, 2, 2)$ cannot occur.

If \mathcal{E} would be globally generated, then the general section $s \in H^0(F, \mathcal{E})$ should vanish on a curve, contradicting the hypothesis.

Finally, assume that $E = \emptyset$. Thus $\mathcal{I}_{E|F}(c_1 - D) \cong \mathcal{O}_F(c_1 - D)$ in Sequence (2). We obviously have that $\mathcal{O}_F(D)$ is globally generated and $h^1(F, \mathcal{O}_F(D)) = 0$. Moreover $\mathcal{O}_F(c_1 - D)$ is globally generated too by Proposition 3.5. We conclude that \mathcal{E} is globally generated too thanks to Sequence (2) contradicting what we just proved above. \square

From now on in this section we will assume that \mathcal{E} satisfies the hypothesis of the above lemma, hence we can assume $E \neq \emptyset$. We want to classify all the possible cases. To this purpose we recall that E is in the class $c_2(\mathcal{E}(-D)) = c_2 - c_1 D + D^2 \in A^2(F)$. Once the values of $(\alpha_1, \alpha_2, \alpha_3)$ and

$(\delta_1, \delta_2, \delta_3)$ are fixed, the class c_2 can be computed thanks to System (15) and Formulas (13), (14). Moreover we have

$$\begin{aligned} c_1 D &= (\delta_2 \alpha_3 + \delta_3 \alpha_2) h_2 h_3 + (\delta_1 \alpha_3 + \delta_3 \alpha_1) h_1 h_3 + (\delta_1 \alpha_2 + \delta_2 \alpha_1) h_1 h_2, \\ D^2 &= 2(\delta_2 \delta_3 h_2 h_3 + \delta_1 \delta_3 h_1 h_3 + \delta_1 \delta_2 h_1 h_2). \end{aligned}$$

If $D \neq 0$, then Inequalities (16) give further restrictions on the β_i 's.

For example let $(\alpha_1, \alpha_2, \alpha_3) = (1, 2, 2)$ and $(\delta_1, \delta_2, \delta_3) = (0, 1, 2)$. In this case $e(c_1, D) = 0$, hence System (15) is

$$\begin{cases} \beta_1 + 2\beta_2 + 2\beta_3 = 8 \\ \beta_1 + \beta_2 + \beta_3 = 5. \end{cases}$$

Inequalities (16) imply $\beta_i \geq 0$, $i = 1, 2, 3$, thus the only possible solutions $(\beta_1, \beta_2, \beta_3)$ of the above system are:

$$(2, 1, 2), \quad (2, 2, 1), \quad (2, 0, 3), \quad (2, 3, 0).$$

Using the expressions above for $c_1 D$ and D^2 , an immediate computation shows that the corresponding values of $c_2(\mathcal{E}(-D))$ are $-h_1 h_3 + h_1 h_2$, 0 , $-2h_1 h_3 + 2h_1 h_2$, $h_1 h_3 - h_1 h_2$ respectively. Taking into account that $c_2(\mathcal{E}(-D))$ is the class in $A^2(F)$ of a non-empty scheme of pure codimension 2, namely E , it follows that none of the above cases are admissible.

By examining along the same lines the all the possible cases, we obtain 53 cases up to permutations of the h_i 's. We list all these cases in the following table.

$(\alpha_1, \alpha_2, \alpha_3)$	$(\delta_1, \delta_2, \delta_3)$	$e(c_1, D)$	$(\beta_1, \beta_2, \beta_3)$	class of E in $A^2(F)$
(0, 0, 1)	(0, 0, 1)	1	(0, 0, 0)	0
(0, 0, 2)	(0, 0, 1)	0	(0, 0, 0)	0
(0, 1, 1)	(0, 0, 1)	0	(1, 0, 0)	0
(0, 1, 1)	(0, 1, 0)	0	(1, 0, 0)	0
(0, 1, 1)	(0, 1, 1)	1	(0, 0, 0)	0
(0, 1, 2)	(0, 0, 1)	0	(1, 0, 0)	0
(0, 1, 2)	(0, 1, 0)	0	(1, 0, 0)	$-h_2 h_3$
(0, 1, 2)	(0, 1, 1)	0	(1, 0, 0)	0
(0, 1, 2)	(0, 1, 2)	1	(0, 0, 0)	0
(0, 2, 2)	(0, 0, 1)	0	(2, 0, 0)	0
(0, 2, 2)	(0, 0, 2)	0	(2, 0, 0)	$-2h_2 h_3$
(0, 2, 2)	(0, 1, 1)	0	(2, 0, 0)	0
(0, 2, 2)	(0, 1, 2)	0	(2, 0, 0)	0
(1, 1, 1)	(0, 0, 1)	0	(0, 1, 1)	$-h_2 h_3 + h_1 h_2$
(1, 1, 1)	(0, 0, 1)	0	(1, 1, 0)	0
(1, 1, 1)	(0, 1, 1)	0	(0, 1, 1)	0
(1, 1, 1)	(0, 1, 1)	0	(1, 0, 1)	$h_2 h_3 - h_1 h_3$
(1, 1, 2)	(0, 0, 1)	0	(1, 1, 1)	$h_1 h_3$
(1, 1, 2)	(0, 0, 1)	0	(0, 2, 1)	$-h_2 h_3 + h_1 h_3$
(1, 1, 2)	(0, 1, 0)	0	(0, 1, 1)	$-h_2 h_3 + h_1 h_3$
(1, 1, 2)	(0, 1, 0)	0	(0, 2, 1)	0
(1, 1, 2)	(0, 1, 1)	0	(1, 1, 1)	0
(1, 1, 2)	(0, 1, 1)	0	(0, 2, 1)	$-h_2 h_3 + h_1 h_3$
(1, 1, 2)	(1, 0, 1)	0	(0, 2, 1)	$-h_2 h_3 + h_1 h_3$
(1, 1, 2)	(1, 1, 0)	0	(0, 2, 1)	$-2h_2 h_3 + h_1 h_2$
(1, 1, 2)	(0, 1, 2)	0	(1, 1, 1)	$h_2 h_3 - h_1 h_3$
(1, 1, 2)	(0, 1, 2)	0	(0, 2, 1)	0

$(\alpha_1, \alpha_2, \alpha_3)$	$(\delta_1, \delta_2, \delta_3)$	$e(c_1, D)$	$(\beta_1, \beta_2, \beta_3)$	class of E in $A^2(F)$
(1, 2, 2)	(0, 0, 1)	0	(2, 1, 2)	$2h_1h_3$
(1, 2, 2)	(0, 0, 1)	0	(2, 2, 1)	$h_1h_3 + h_1h_2$
(1, 2, 2)	(0, 0, 1)	0	(2, 0, 3)	$-h_1h_3 + 3h_1h_2$
(1, 2, 2)	(0, 0, 1)	0	(2, 3, 0)	$2h_1h_3$
(1, 2, 2)	(1, 0, 0)	0	(2, 1, 2)	$2h_2h_3 - h_1h_3$
(1, 2, 2)	(1, 0, 0)	0	(2, 2, 1)	$2h_2h_3 - h_1h_2$
(1, 2, 2)	(1, 0, 0)	0	(2, 0, 3)	$2h_2h_3 - 2h_1h_3 + h_1h_2$
(1, 2, 2)	(0, 0, 2)	0	(2, 1, 2)	$-2h_2h_3 - h_1h_3 + 2h_1h_2$
(1, 2, 2)	(0, 0, 2)	0	(2, 2, 1)	$-2h_2h_3 + h_1h_2$
(1, 2, 2)	(0, 0, 2)	0	(2, 0, 3)	$-2h_2h_3 - 2h_1h_3 + 3h_1h_2$
(1, 2, 2)	(0, 0, 2)	0	(2, 3, 0)	$-2h_2h_3 + h_1h_3$
(1, 2, 2)	(0, 1, 1)	0	(2, 2, 1)	h_1h_2
(1, 2, 2)	(0, 1, 1)	0	(2, 3, 0)	$2h_1h_3 - h_1h_2$
(1, 2, 2)	(1, 0, 1)	0	(2, 1, 2)	0
(1, 2, 2)	(1, 0, 1)	0	(2, 2, 1)	$h_1h_3 - h_1h_2$
(1, 2, 2)	(1, 0, 1)	0	(2, 3, 0)	$2h_1h_3 - 2h_1h_2$
(1, 2, 2)	(1, 0, 1)	0	(2, 0, 3)	$-h_1h_3 + h_1h_2$
(1, 2, 2)	(0, 1, 2)	0	(2, 1, 2)	$-h_1h_3 + h_1h_2$
(1, 2, 2)	(0, 1, 2)	0	(2, 2, 1)	0
(1, 2, 2)	(0, 1, 2)	0	(2, 3, 0)	$h_1h_3 - h_1h_2$
(1, 2, 2)	(0, 1, 2)	0	(2, 0, 3)	$-2h_1h_3 + 2h_1h_2$
(1, 2, 2)	(1, 0, 2)	0	(2, 1, 2)	$-2h_2h_3 + h_1h_3$
(1, 2, 2)	(1, 0, 2)	0	(2, 2, 1)	$-2h_2h_3 + 2h_1h_3 - h_1h_2$
(1, 2, 2)	(1, 0, 2)	0	(2, 0, 3)	$-2h_2h_3 + 3h_1h_3 - 2h_1h_2$
(1, 2, 2)	(1, 0, 2)	0	(2, 0, 3)	$-2h_2h_3 + h_1h_2$

Only few cases are actually admissible in the above table and we will examine them in the following.

When the class of E is h_1h_2 , then E is a line and we have an exact sequence of the form

$$0 \longrightarrow \mathcal{O}_F(-h_1 - h_2) \longrightarrow \mathcal{O}_F(-h_1) \oplus \mathcal{O}_F(-h_2) \longrightarrow \mathcal{I}_{E|F} \longrightarrow 0.$$

It follows that $\mathcal{I}_{E|F}(h) \cong \mathcal{I}_{E|F}(c_1 - D)$ is globally generated. Thus we conclude again that \mathcal{E} should be globally generated, contradicting Lemma 5.1. A similar argument holds when the class of E is h_1h_3 .

When the class of E is $h_1h_3 + h_1h_2$, then we cannot repeat the above argument. In this case we have that $h^2(F, \mathcal{O}_F(D - h)) = h^3(F, \mathcal{O}_F(D - 2h)) = 0$. Moreover \mathcal{E} is aCM, hence we obtain

$$h^1(F, \mathcal{I}_{E|F}(c_1 - D - h)) = h^2(F, \mathcal{I}_{E|F}(c_1 - D - 2h)) = 0.$$

The cohomology of Sequence (3) twisted by $\mathcal{O}_F(c_1 - D - 3h)$ implies that

$$h^3(F, \mathcal{I}_{E|F}(c_1 - D - 3h)) \leq h^3(F, \mathcal{O}_F(c_1 - D - 3h)).$$

Since $c_1 - D - 3h = -2h_1 - h_2 - 2h_3$ we conclude that the last dimension is zero. It follows that $\mathcal{I}_{E|F}(c_1 - D)$ is regular in the sense of Castelnuovo–Mumford, hence it is globally generated. As in the previous cases we also obtain that \mathcal{E} is globally generated, contradicting again Lemma 5.1. A similar argument also proves that the class of E cannot be $2h_1h_2$, $2h_1h_3$.

We summarize the above discussion in the following statement.

Proposition 5.2. *If \mathcal{E} is an indecomposable initialized aCM bundle of rank 2 on F , then the zero-locus $(s)_0$ of a general section $s \in H^0(F, \mathcal{E})$ has pure dimension 2.*

By combining Propositions 3.5, 4.2 and 5.2 we have the proof of Theorem A.

The next sections are devoted to the classification of all indecomposable initialized aCM bundles of rank 2 on F , or, in other words, to the proof of Theorem B stated in the introduction.

6. THE STANDARD BOUND

In this section we deal with the extremal cases $c_1 = 0$ and $c_1 = 2h$. In the first case Proposition 3.5 implies that the zero-locus $E := (s)_0$ of a general section $s \in H^0(F, \mathcal{E})$ has codimension 2 inside F . Formula (14) and Lemma 4.1 (notice that we know that $D = 0$) give $\deg(E) = hc_2 = 1$, thus E is a line. In particular, if its class in $A^2(F)$ is $\beta_1 h_2 h_3 + \beta_2 h_1 h_3 + \beta_3 h_1 h_2$, then necessarily all the β_i 's are zero but one which is 1, i.e. the class of E in $A^2(F)$ is either $h_2 h_3$ or $h_1 h_3$ or $h_1 h_2$.

Conversely we show that each line $E \subseteq F$ arises as the zero locus of a section of an indecomposable initialized aCM bundle of rank 2 on F with $c_1 = 0$. To this purpose, assume that the class of E is $h_2 h_3$. We know that $\omega_E \cong \mathcal{O}_E(-2h)$, thus adjunction formula on F implies $\det(\mathcal{N}_{E|F}) \cong \mathcal{O}_E$. Theorem 2.3 with $\mathcal{L} := \mathcal{O}_F$ guarantees the existence of a vector bundle \mathcal{E} of rank 2 fitting into a sequence of the form

$$0 \longrightarrow \mathcal{O}_F \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_{E|F} \longrightarrow 0.$$

Hence $h^1(F, \mathcal{E}(th)) \leq h^1(F, \mathcal{I}_{E|F}(th))$, $t \in \mathbb{Z}$. The vanishing $h^1(\mathbb{P}^7, \mathcal{I}_{E|\mathbb{P}^7}(th)) = h^2(\mathbb{P}^7, \mathcal{I}_{F|\mathbb{P}^7}(th)) = 0$ (recall that both E and F are aG), imply $h^1(F, \mathcal{I}_{E|F}(th)) = 0$, i.e. $h^1(F, \mathcal{E}(th)) = 0$. Since $c_1 = 0$, it follows that $\mathcal{E}^\vee \cong \mathcal{E}$, thus Serre's duality also yields $h^2(F, \mathcal{E}(th)) = 0$. We conclude that \mathcal{E} is an aCM bundle. It is trivial to check that \mathcal{E} is initialized. Finally, if \mathcal{E} were decomposable, then $\mathcal{E} \cong \mathcal{M} \oplus \mathcal{M}^{-1}$ because $c_1 = 0$. Thus $h_2 h_3 = c_2 = -c_1(\mathcal{M})^2$. But if $c_1(\mathcal{M}) = \mu_1 h_1 + \mu_2 h_2 + \mu_3 h_3$, then $c_1(\mathcal{M})^2 = 2(\mu_2 \mu_3 h_2 h_3 + \mu_1 \mu_3 h_1 h_3 + \mu_1 \mu_2 h_1 h_2)$ which cannot coincide with $h_2 h_3$.

Therefore we have proven the following theorem.

Theorem 6.1. *There exist indecomposable initialized aCM bundles \mathcal{E} of rank 2 with $c_1 = 0$.*

Moreover the zero-locus of a general section of \mathcal{E} is a line and each line on F can be obtained in this way.

Now we turn our attention to the case $c_1 = 2h$. Proposition 4.2 guarantees that \mathcal{E} is Ulrich, hence globally generated. Thus also in this case the zero-locus $E := (s)_0$ of a general section $s \in H^0(F, \mathcal{E})$ is a smooth curve inside F .

Using Sequence (2) we obtain that $h^1(F, \mathcal{I}_{E|F}) = h^1(F, \mathcal{I}_{E|F}(h)) = 0$, hence both $h^0(E, \mathcal{O}_E) = 1$ and E is linearly normal (use Sequence (3)). We conclude that we can assume E to be a smooth curve which is not contained in any hyperplane (from now on we will briefly refer to such a property by saying that E is non-degenerate). Moreover, combining Lemma 4.1 and Formula (14), we obtain $\deg(E) = hc_2 = 8$ and that E is elliptic.

Conversely, let E be a smooth, non-degenerate elliptic curve of degree 8 on F . We know by adjunction that $\mathcal{O}_E \cong \omega_E \cong \det(\mathcal{N}_{E|F}) \otimes \mathcal{O}_F(-2h)$, hence $\det(\mathcal{N}_{E|F}) \cong \mathcal{O}_F(2h) \otimes \mathcal{O}_E$. The invertible sheaf $\mathcal{O}_F(2h)$ satisfies the vanishing of the Theorem 2.3, thus E is the zero locus of a section s of a vector bundle \mathcal{E} of rank 2 on F with $c_1 = 2h$ and the class of E in $A^2(F)$ is c_2 .

We conclude that in order to show the existence of indecomposable, initialized, aCM bundles \mathcal{E} of rank 2 with $c_1 = 2h$, it suffices to prove the existence of smooth, non-degenerate elliptic curve of degree 8 on F . The remaining part of this section is devoted to the proof of the existence of such curves.

Due to Proposition 1.1 and Corollary 2.2 of [17], we know that E is aCM, then $h^1(\mathbb{P}^7, \mathcal{I}_{E|\mathbb{P}^7}(th)) = 0$. Since F is aG, hence $h^2(\mathbb{P}^7, \mathcal{I}_{F|\mathbb{P}^7}(th)) = 0$, taking the cohomology of sequence

$$(18) \quad 0 \longrightarrow \mathcal{I}_{F|\mathbb{P}^7} \longrightarrow \mathcal{I}_{E|\mathbb{P}^7} \longrightarrow \mathcal{I}_{E|F} \longrightarrow 0,$$

it also follows that $h^1(F, \mathcal{I}_{E|F}(th)) = 0$. The cohomology of the Sequence (2) yields $h^1(F, \mathcal{E}(th)) = 0$. Such a vanishing also implies $h^2(F, \mathcal{E}(th)) = 0$ by Serre's duality. We conclude that \mathcal{E} is aCM. Thanks to Proposition 4.2 we also know that \mathcal{E} is Ulrich. Thus non-degenerate elliptic curves on a del Pezzo threefold F correspond to Ulrich bundles on F with $c_1 = 2h$.

First we deal with the possible values of c_2 .

Lemma 6.2. *If \mathcal{E} is an indecomposable initialized aCM bundle of rank 2 with $c_1 = 2h$, then c_2 is either $2h_2h_3 + 2h_1h_3 + 4h_1h_2$ or $2h_2h_3 + 3h_1h_3 + 3h_1h_2$ up to permutation of the h_i 's.*

Proof. The linear system $|h_i|$ on F has dimension 1. Let $D_i \in |h_i|$ be general. The cohomology of the exact sequence

$$0 \longrightarrow \mathcal{O}_F(h - h_i) \longrightarrow \mathcal{O}_F(h) \longrightarrow \mathcal{O}_{D_i}(h) \longrightarrow 0$$

yields $h^0(D_i, \mathcal{O}_{D_i}(h)) = 4$, thus D_i spans a space of dimension 3 in \mathbb{P}^7 .

Let E be the zero locus of a general section of \mathcal{E} . Since E is non-degenerate, we know that the restriction to E of $|h_i|$ has dimension at least 1. Since E is elliptic, it follows that its degree, which is β_i , is greater than 2, thanks to Riemann–Roch theorem on the curve E . Now the statement follows from the fact that $\deg(E) = hc_2 = \beta_1 + \beta_2 + \beta_3 = 8$. \square

We now proceed to construct explicitly such curves and bundles. To this purpose we will make use of degenerate elliptic curves. Since the proofs of the results below depend only on the fact that F is a del Pezzo threefold, we will state and prove such results in general. Thus, in what follows, F will be a del Pezzo threefold of degree $3 \leq d \leq 7$, embedded in \mathbb{P}^{d+1} . Let H be a general hyperplane in \mathbb{P}^{d+1} , and define $S := F \cap H$. The surface S is smooth and connected of degree d . Moreover, adjunction on F implies that S is a del Pezzo surface. In particular we know that S can be represented as the blow up of \mathbb{P}^2 in $9 - d$ points in general position.

Thus the Picard group of S is freely generated by the class ℓ of the pull-back of a general line in \mathbb{P}^2 and by the classes of the $9 - d$ exceptional divisors e_1, \dots, e_{9-d} of the blow up.

Clearly we have a natural map $A^1(S) \rightarrow A^2(F)$ and we want to inspect which elliptic curves on F come from curves on S . We will make such an analysis later on in the section. We now make some comments on the deformation theory of degenerate elliptic curves on a del Pezzo threefolds.

Proposition 6.3. *Let F be a del Pezzo threefold of degree $3 \leq d \leq 7$. If $H \subseteq \mathbb{P}^{d+1}$ is a general hyperplane and $C \subseteq S := F \cap H$ is a smooth, connected elliptic curve of degree δ , then*

$$h^0(S, \mathcal{O}_S(C)) = \delta + 1, \quad h^0(C, \mathcal{N}_{C|F}) = 2\delta, \quad h^1(S, \mathcal{O}_S(C)) = h^1(C, \mathcal{N}_{C|F}) = 0.$$

Proof. We have the isomorphisms $\mathcal{N}_{C|S} \cong \mathcal{O}_C(C)$ and $\mathcal{N}_{S|F} \cong \mathcal{O}_S(h)$. Thus we have the exact sequences

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_C(C) \longrightarrow \mathcal{N}_{C|F} \longrightarrow \mathcal{O}_C(h) \longrightarrow 0, \\ 0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_S(C) \longrightarrow \mathcal{O}_C(C) \longrightarrow 0, \end{aligned}$$

hence $\chi(\mathcal{N}_{C|F}) = \chi(\mathcal{O}_C(C)) + \chi(\mathcal{O}_C(h)) = \chi(\mathcal{O}_S(C)) + \chi(\mathcal{O}_C(h)) - 1$. Since $p_a(C) = 1$, it follows that $h^1(C, \mathcal{O}_C(h)) = 0$, hence Riemann–Roch on C implies $h^0(C, \mathcal{O}_C(h)) = \delta$.

Let $a\ell - \sum_{i=1}^d b_i e_i$ be the class of C in $A^1(S)$. The conditions $\deg(C) = \delta$ and $p_a(C) = 1$ yield $C^2 = a^2 - \sum_{i=1}^d b_i^2 = \delta > 0$. Thus Riemann–Roch theorem on S gives us $\chi(\mathcal{O}_S(C)) = \delta + 1$, hence

$$(19) \quad \chi(\mathcal{N}_{E|F}) = 2\delta.$$

The surface S is del Pezzo, then $h^i(S, \mathcal{O}_S) = 0$, $i = 1, 2$. It follows from the cohomology of the second sequence above and by Serre's duality on S that

$$h^1(C, \mathcal{O}_C(C)) = h^1(S, \mathcal{O}_S(C)) = h^1(S, \mathcal{O}_S(-C - h)).$$

The last dimension is 0. Indeed it suffices to take the cohomology of the second sequence above twisted by $\mathcal{O}_S(-h)$. At this point the cohomology of the first sequence above yields $h^1(E, \mathcal{N}_{E|F}) = 0$. Combining this last vanishing with equality (19) we obtain the last equality of the statement. \square

Now we come back to analyse the case $F \cong \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, identifying the elliptic curves on F contained in a smooth hyperplane section. We use the same notation introduced at the end of the previous section: thus H is a general hyperplane in \mathbb{P}^7 , and $S := F \cap H$.

Lemma 6.4. *C is a smooth, connected elliptic curve of degree 8 on S if and only if, up to automorphisms of S , C is a smooth connected element in the class either $3\ell - e_1$, or $4\ell - 2e_1 - 2e_2$.*

Proof. Let $a\ell - \sum_{i=1}^3 b_i e_i$ be the class of C in $A^1(S)$. Since C is effective, it follows that $a, b_i \geq 0$. Moreover, one easily checks that the conditions $\deg(C) = 8$ and $p_a(C) = 1$ yield

$$(20) \quad 3a - \sum_{i=1}^3 b_i = 8, \quad a^2 - \sum_{i=1}^3 b_i^2 = 8.$$

Schwarz's inequality implies $(\sum_{i=1}^3 b_i)^2 \leq 3 \sum_{i=1}^3 b_i^2$. Combining such an inequality with equalities (20), we finally obtain $6a^2 - 48a + 88 \leq 0$, i.e. $a = 3, 4, 5$.

Now, again by equalities (20), we obtain the cases $3\ell - e_1$, $4\ell - 2e_1 - 2e_2$, $5\ell - 3e_1 - 2e_2 - 2e_3$. The automorphism of S induced by the quadratic transformation of \mathbb{P}^2 centered at the blown up points maps (ℓ, e_1, e_2, e_3) to $(2\ell - e_1 - e_2 - e_3, \ell - e_2 - e_3, \ell - e_1 - e_3, \ell - e_1 - e_2)$, thus it transforms $5\ell - 3e_1 - 2e_2 - 2e_3$ in $3\ell - e_1$. It follows that we can restrict our attention to $3\ell - e_1$ and $4\ell - 2e_1 - 2e_2$ only. \square

We recall that if E is a non-degenerate, smooth elliptic curve on F , its class in $A^2(F)$ is either $2h_2h_3 + 3h_1h_3 + 3h_1h_2$ or $2h_2h_3 + 2h_1h_3 + 4h_1h_2$ (see Lemma 6.2). We will show in the following the existence of non-degenerate, smooth, connected elliptic curves on F of both types. To this purpose we will first show the existence of degenerate elliptic curves in both classes and we will postpone the proof of the existence of non-degenerate curves at the end of the present section.

Lemma 6.5. *Let $H \subseteq \mathbb{P}^7$ be a general hyperplane and let C be a smooth, connected elliptic curve of degree 8 on $S := F \cap H$. The class of C in $A^2(F)$ is either $2h_2h_3 + 3h_1h_3 + 3h_1h_2$ or $2h_2h_3 + 2h_1h_3 + 4h_1h_2$.*

Proof. First we deal with the classes of ℓ and e_i in $A^2(F)$. Recall that the class of S inside $A^1(F)$ is the class of the hyperplane section, i.e. $h = h_1 + h_2 + h_3$: thus, e.g. $h_1S = h_1h = h_1h_2 + h_1h_3$. Moreover the class of the hyperplane section of S is $3\ell - e_1 - e_2 - e_3$ in $A^1(S)$.

We have $h_i h^2 = 2h_1h_2h_3$, thus $h_i S = h_i h$ is the class of a conic on S . Arguing as in the proof of Lemma 6.4 one easily checks that the classes of conics on S are $\ell - e_i$. Thus we can assume that $h_i S = \ell - e_i$ in $A^1(S)$.

Now, we turn our attention to ℓ . Let $\gamma_1 h_2 h_3 + \gamma_2 h_1 h_3 + \gamma_3 h_1 h_2$ its class in $A^2(F)$. We have $\gamma_i = \ell h_i \geq 0$ and

$$\gamma_1 + \gamma_2 + \gamma_3 = \ell h = \deg(\ell) = \ell(3\ell - e_1 - e_2 - e_3) = 3$$

If $\gamma_1 = 0$, then $e_1 h_3 = (\ell - h_1 S) h_3 = -h_1 h_2 h_3$. Since e_1 and h_3 are both effective, we obtain an absurd. Thus $\gamma_1 \geq 1$ and, similarly, $\gamma_2, \gamma_3 \geq 1$. We conclude that the classes of ℓ, e_1, e_2, e_3 in $A^2(F)$ are $h_2 h_3 + h_1 h_3 + h_1 h_2, h_2 h_3, h_1 h_3, h_1 h_2$ respectively.

Thanks to Lemma 6.4, the assertion about the class of C in $A^2(F)$ now follows from direct substitution. \square

As explained at the beginning of our analysis, the curves C described in Lemma 6.5, being degenerate, do not correspond to the bundles we are interested in: indeed such bundles correspond to non-degenerate curves. Thus we have finally to show the existence of non-degenerate curves in the same classes. We will check this with a well-known deformation argument (see [23]).

Proposition 6.6. *If $E \subseteq F$ is a general element in the class either $2h_2h_3 + 3h_1h_3 + 3h_1h_2$ or $2h_2h_3 + 2h_1h_3 + 4h_1h_2$. Then E is a non-degenerate smooth, connected elliptic curve.*

Proof. Let \mathcal{H} be the union of the components of $\text{Hilb}^{8t}(F)$ containing non-degenerate smooth connected elliptic curves of degree 8. Take the incidence variety

$$\mathcal{X} := \{ (C, H) \in \mathcal{H} \times (\mathbb{P}^7)^\vee \mid C \subseteq H \}$$

Let $(C, H) \in \mathcal{X}$ be general. In particular $H \subseteq \mathbb{P}^7$ is general, then $S := F \cap H$ is a smooth del Pezzo surface of degree 6 as above and $C \subseteq S$ is a smooth connected elliptic curve whose class in $A^2(F)$ is $\gamma_1 h_2 h_3 + \gamma_2 h_1 h_3 + \gamma_3 h_1 h_2$. Anyhow $\deg(C) = 8$, hence Proposition 6.3 yields $h^0(C, \mathcal{N}_{C|S}) = 16$ and $h^1(C, \mathcal{N}_{C|S}) = 0$. It follows that \mathcal{H} is smooth at the point corresponding to C (briefly we say

that C is unobstructed) and it has dimension 16. Assume that all deformation of C inside F were degenerate.

On the one hand, the image of the projection $\mathcal{X} \rightarrow \mathcal{H}$ would contain an open neighbourhood of the point corresponding to C . Since the points in the fibre over C are parametrized by the hyperplanes containing C , it follows that such a projection would be generically injective, hence \mathcal{X} would contain an irreducible component $\widehat{\mathcal{X}}$ of dimension 16 containing (C, H) .

On the other hand the fibre of the projection $\widehat{\mathcal{X}} \rightarrow (\mathbb{P}^7)^\vee$ over H is isomorphic to the projectivization of $H^0(S, \mathcal{O}_S(C))$ which is \mathbb{P}^8 (see Proposition 6.3). We conclude that $\dim(\widehat{\mathcal{X}}) \leq 15$, a contradiction. We conclude that each general deformation of C inside F is non-degenerate in \mathbb{P}^7 .

Let $\mathcal{C} \subseteq F \times B \rightarrow B$ a flat family of curves in \mathcal{H} with special fibre $\mathcal{C}_{b_0} \cong C$ over b_0 . Since C is unobstructed, we can assume that B is integral. Since $\mathcal{C} \rightarrow B$ is flat and C is integral, it follows that \mathcal{C} is integral too. Take a general element in the class h_i , say Q_i , and consider the family $\mathcal{Q}_i := \mathcal{C} \cap (Q_i \times B) \rightarrow B$. Since Q_i is general and h_i is globally generated, we can assume that \mathcal{Q}_i is a family of 0-cycles of F . Up to a proper choice of homogeneous coordinates $t_0^{(i)}, t_1^{(i)}$ on the i^{th} copy of \mathbb{P}^1 inside the product F , we can assume $\mathcal{O}_{\mathcal{Q}_i} \cong \mathcal{O}_{\mathcal{C}}/(t_1^{(i)})$. Thanks to the choice of Q_i , the element $t_1^{(i)}$ is regular element in $\mathcal{O}_{\mathcal{C}}$. Thus the Corollary of Theorem 22.5 of [34], implies that \mathcal{Q}_i is flat over B . By semicontinuity the degree of the fibre of \mathcal{Q}_i over b , which is $\mathcal{C}_b h_i$, is upper semicontinuous, thus there exists an open subset $B_i \subseteq B$ containing b_0 such that $\mathcal{C}_b h_i \leq \gamma_i$. Since

$$\gamma_1 + \gamma_2 + \gamma_3 = 8 = \deg(\mathcal{C}_b) = \mathcal{C}_b h = \sum_{i=1}^3 \mathcal{C}_b h_i \leq \gamma_1 + \gamma_2 + \gamma_3,$$

at each $b \in B' := B_1 \cap B_2 \cap B_3 \subseteq B$, we finally conclude that $\mathcal{C}_b h_i = \gamma_i$, i.e. \mathcal{C}_b is in the same class of C inside $A^2(F)$. Hence each general element $E \subseteq F$ in the classes $2h_2h_3 + 3h_1h_3 + 3h_1h_2$ and $2h_2h_3 + 2h_1h_3 + 4h_1h_2$ is non-degenerate. \square

We finally state our existence result.

Theorem 6.7. *There exist indecomposable initialized aCM bundles \mathcal{E} of rank 2 with $c_1 = 2h$ and c_2 either $2h_2h_3 + 3h_1h_3 + 3h_1h_2$ or $2h_2h_3 + 2h_1h_3 + 4h_1h_2$ up to permutations of the h_i 's. Moreover the zero-locus of a general section of \mathcal{E} is an elliptic normal curve.*

Conversely, up to permutations of the h_i 's, each elliptic normal curve on F can be obtained as the zero locus of a general section of an initialized aCM bundles \mathcal{E} of rank 2 with $c_1 = 2h$ and c_2 either $2h_2h_3 + 3h_1h_3 + 3h_1h_2$ or $2h_2h_3 + 2h_1h_3 + 4h_1h_2$. The bundle \mathcal{E} is indecomposable if and only if it is not the complete intersection of two divisors $Q_1 \in |2h_2 + h_3|$ and $Q_2 \in |2h_1 + h_3|$.

Proof. Almost all the statement follows from the previous proposition and from the correspondence between non-degenerate smooth elliptic curves and bundles with $c_1 = 2h$. The only unproved assertion is that there are indecomposable bundles of each type.

Assume that $\mathcal{E} \cong \mathcal{L}_1 \oplus \mathcal{L}_2$ for some $\mathcal{L}_i \in \text{Pic}(F)$. Thus we know that \mathcal{L}_i is aCM. Moreover either both the \mathcal{L}_i are initialized or one of them is initialized and the other has no sections.

Thanks to Lemma 2.4 and to the equalities

$$\begin{aligned} h^0(F, \mathcal{L}_1) + h^0(F, \mathcal{L}_2) &= h^0(F, \mathcal{E}) = 12, \\ c_1(\mathcal{L}_1) + c_1(\mathcal{L}_2) &= c_1 = 2h \end{aligned}$$

it is easy to check that $\mathcal{E} \cong \mathcal{O}_F(2h_2 + h_3) \oplus \mathcal{O}_F(2h_1 + h_3)$ necessarily. In particular in the case $(2, 3, 3)$, i.e. $c_2 = 2h_2h_3 + 3h_1h_3 + 3h_1h_2$, the vector bundle \mathcal{E} is indecomposable.

Consider the other case $(2, 2, 4)$, i.e. $c_2 = 2h_2h_3 + 2h_1h_3 + 4h_1h_2$. We have

$$\dim_k(\text{Ext}_F^1(\mathcal{O}_F(2h_2 + h_3), \mathcal{O}_F(2h_1 + h_3))) = h^1(F, \mathcal{O}_F(2h_1 - 2h_2)) = 3,$$

thus there are non-trivial extensions of the form

$$(21) \quad 0 \longrightarrow \mathcal{O}_F(2h_1 + h_3) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_F(2h_2 + h_3) \longrightarrow 0.$$

Notice that \mathcal{E} is aCM, because both $\mathcal{O}_F(2h_2 + h_3)$ and $\mathcal{O}_F(2h_1 + h_3)$ are aCM.

If \mathcal{E} is decomposable, then we checked above that $\mathcal{E} \cong \mathcal{O}_F(2h_2 + h_3) \oplus \mathcal{O}_F(2h_1 + h_3)$. Since

$$\dim(\mathrm{Hom}_F(\mathcal{O}_F(2h_1 + h_3), \mathcal{O}_F(2h_2 + h_3))) = h^0(F, \mathcal{E}(-h_1 + 2h_2 - h_3)) = 0,$$

it follows that the epimorphism $\mathcal{E} \rightarrow \mathcal{O}_F(2h_2 + h_3)$ has a section, thus Sequence (21) splits. In particular \mathcal{E} splits if and only if the extension (21) splits too. Thus each non-zero section of $\mathrm{Ext}_F^1(\mathcal{O}_F(2h_2 + h_3), \mathcal{O}_F(2h_1 + h_3))$ induces an indecomposable bundle. \square

7. THE SPORADIC EXTREMAL CASE

In this section we deal with the case $c_1 = h_1 + 2h_2 + 3h_3$. Again Proposition 4.2 guarantees that \mathcal{E} is Ulrich, hence globally generated. It follows that the zero-locus $E := (s)_0$ of a general section $s \in H^0(F, \mathcal{E})$ is a smooth curve inside F .

Notice that $\mathcal{O}_F(-c_1)$ is aCM. This fact and the cohomology of Sequence (2) yield $h^1(F, \mathcal{I}_{E|F}) = h^1(F, \mathcal{I}_{E|F}(h)) = 0$. It follows that both $h^0(E, \mathcal{O}_E) = 1$ and E is linearly normal (use Sequence (3)). We conclude that we can assume E to be a non-degenerate, smooth, irreducible curve. Moreover, again by combining Lemma 4.1 with $D = 0$ and Formula (14), one easily obtains $\deg(E) = hc_2 = 7$. It follows that E is a rational normal curve in \mathbb{P}^7 .

Conversely, let E be a non-degenerate, smooth, rational curve of degree 7 in the class $\beta_1 h_2 h_3 + \beta_2 h_1 h_3 + \beta_3 h_1 h_2$ on $A^2(F)$. We know by adjunction that $\mathcal{O}_{\mathbb{P}^1}(-2) \cong \omega_E \cong \det(\mathcal{N}_{E|F}) \otimes \mathcal{O}_F(-2h)$, thus $\det(\mathcal{N}_{E|F}) \cong \mathcal{O}_{\mathbb{P}^1}(12)$. The restriction to E of $|h_i|$, $i = 1, 2, 3$, is a linear system of divisors on \mathbb{P}^1 of degree β_i , hence its dimension is β_i . Since E is non-degenerate and $|h_i|$ is a pencil on F , it follows that $\beta_i \geq 1$. The same argument applied to $|h_i + h_j|$, $i, j = 1, 2, 3$ and $i \neq j$, similarly shows that $\beta_i + \beta_j \geq 3$. We conclude that the possible values of $(\beta_1, \beta_2, \beta_3)$ are $(3, 2, 2)$, $(4, 1, 2)$, $(3, 3, 1)$ up to permutations of the h_i 's.

Let us now consider the first case, i.e. the class of E is $3h_2 h_3 + 2h_1 h_3 + 2h_1 h_2$. Such an E cannot be the zero locus of a section of any aCM bundle \mathcal{E} with $c_2 = 3h_2 h_3 + 2h_1 h_3 + 2h_1 h_2$. On the one hand, Lemma 4.1 would imply that $c_1 c_2 = 12$. It follows that if $c_1 = \alpha_1 h_1 + \alpha_2 h_2 + \alpha_3 h_3$, then $(\alpha_1, \alpha_2, \alpha_3) = (1, 2, 3)$, up to permutations. On the other hand, the first of the equalities of System (15) shows that there is no permutation of the h_i 's such that $c_1 c_2 = 12$.

Remark 7.1. *Smooth rational curves in the class $3h_2 h_3 + 2h_1 h_3 + 2h_1 h_2$ actually exist. Indeed choose three general maps from \mathbb{P}^1 to \mathbb{P}^1 of degrees 3, 2, 2. Their product gives a morphism $\mathbb{P}^1 \rightarrow F$ whose image E is a rational curve.*

Using any computer algebra software (e.g. CoCoA: see [18]) one can compute the homogeneous ideal of E in \mathbb{P}^7 checking that it is minimally generated by quadratic forms, hence E is non-degenerate. In the same way one can also check that the Hilbert polynomial of E is $7t + 1$. It follows that the arithmetic genus of E is 0. We conclude that E is a smooth rational curve of degree 7.

The sheaf $\mathcal{L} := \mathcal{O}_F(2h_1 + 2h_2 + h_3)$ satisfies the hypothesis of Theorem 2.3. Thus E is the zero locus of a section of rank 2 bundle \mathcal{E} on F with $c_1 = 2h_1 + 2h_2 + h_3$ and $c_2 = 3h_2 h_3 + 2h_1 h_3 + 2h_1 h_2$. The cohomology of Sequence (2) twisted by $\mathcal{O}_F(-h)$ shows that \mathcal{E} is initialized. An analysis of all the possible cases also shows that \mathcal{E} is indecomposable. Nevertheless, \mathcal{E} cannot be aCM, thanks to the discussion above.

Let us examine the cases $(4, 1, 2)$ and $(3, 3, 1)$. The sheaf $\mathcal{L} := \mathcal{O}_F(h_1 + 2h_2 + 3h_3)$ satisfies the hypothesis of Theorem 2.3. Thus E is the zero locus of a section s of a vector bundle \mathcal{E} of rank 2 on F with $c_1 = h_1 + 2h_2 + 3h_3$ and c_2 either $4h_2 h_3 + h_1 h_3 + 2h_1 h_2$, or $3h_2 h_3 + 3h_1 h_3 + h_1 h_2$ respectively. Moreover, E is aCM (see Proposition 1.1 and Corollary 2.2 of [17]), and the same holds for F .

Lemma 7.2. *Let E be a non-degenerate, smooth, rational curve of degree 7 whose class in $A^2(F)$ is either $4h_2 h_3 + h_1 h_3 + 2h_1 h_2$, or $3h_2 h_3 + 3h_1 h_3 + h_1 h_2$. Then the bundle \mathcal{E} defined above is Ulrich.*

Proof. Since E is aCM, it follows that $h^1(\mathbb{P}^7, \mathcal{I}_{E|\mathbb{P}^7}(th)) = h^2(\mathbb{P}^7, \mathcal{I}_{F|\mathbb{P}^7}(th)) = 0$. Again the cohomology of Sequence (18) implies $h^1(F, \mathcal{I}_{E|F}(th)) = 0$. The cohomology of Sequence (2) twisted by $\mathcal{O}_F(-c_1)$ implies for $t \in \mathbb{Z}$

$$h^2(F, \mathcal{E}(th)) = h^1(F, \mathcal{E}^\vee((-2-t)h)) = h^1(F, \mathcal{I}_{E|F}((-2-t)h)) = 0.$$

We check that $h^1(F, \mathcal{E}(th)) = h^1(F, \mathcal{I}_{E|F}(c_1 + th)) = 0$, $t \in \mathbb{Z}$. Notice that $\deg(\mathcal{O}_E(c_1 + th)) = (c_1 + th)c_2 = 12 + 7t$. Hence $h^1(F, \mathcal{I}_{E|F}(c_1 + th)) \leq h^0(E, \mathcal{O}_E(c_1 + th)) = 0$ when $t \leq -2$, because $E \cong \mathbb{P}^1$.

Let $t = -1$. We have $(h_2 + 2h_3)E = 5$. Hence $h^0(E, \mathcal{O}_E(h_2 + 2h_3)) = 6$. The cohomologies of Sequences (3) and (2) suitably twisted yield

$$h^1(F, \mathcal{E}(-h)) = h^0(F, \mathcal{E}(-h)) = h^0(F, \mathcal{I}_{E|F}(h_2 + 2h_3)).$$

Assume the existence of a surface $M \subseteq F$ in $|\mathcal{O}_F(h_2 + 2h_3)|$ through E . We have $\deg(M) = 6$. If M were not integral, then E , being integral, would be contained in an integral surface $M' \subseteq F$ of degree at most 5. In particular M' would be degenerate in \mathbb{P}^7 , thus the same would hold for E , a contradiction. It follows that M is integral and it is easy to check that it is non-degenerate, thus a surface of minimal degree in \mathbb{P}^7 . Hence it is either a smooth rational normal scroll, or it is the cone onto a smooth rational normal curve of degree 6 contained in a hyperplane H with vertex a point $V \notin H$ (see [25], p. 522).

In second case the tangent space at M in V is the whole \mathbb{P}^7 . Since such a space is contained in the tangent space at F in V , which is a smooth variety, we get a contradiction. Hence $M \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))$ for some integer $e \geq 0$. Let ξ be the tautological divisor on M and f the class of a fibre. The embedding $M \subseteq \mathbb{P}^7$ is given by some linear system $|\xi + af|$ with $a > 0$.

Since a rational normal curve has no trisecant curves because it is cut out by quadrics (see [25], p. 530), it follows that E is either in $|(\xi + af) + bf|$, or in $|2(\xi + af) - cf|$ for suitable integers b and c . The conditions $\deg(M) = 6$ and $\deg(E) = 7$ imply $b = 1$ and $c = -5$.

The class of $|\xi + af|$ in $A^2(F)$ is $hM = 3h_2h_3 + 2h_1h_3 + h_1h_2$. The linear system $|\mathcal{O}_M(h_3)|$ is a pencil of lines on M . An easy computation in $\text{Pic}(M)$ shows that the only pencil of lines on M is $|f|$, thus the class of f inside $A^2(F)$ is h_2h_3 . By combining the above results we finally obtain that the class of E in $A^2(F)$ must be either $4h_2h_3 + 2h_1h_3 + h_1h_2$, or $h_2h_3 + 4h_1h_3 + 2h_1h_2$, contradicting the hypothesis on E . We conclude that $h^1(F, \mathcal{E}(-h)) = h^0(F, \mathcal{E}(-h)) = 0$. In particular \mathcal{E} is initialized.

Let us examine the case $t \geq 0$. In this case we have a commutative diagram

$$\begin{array}{ccc} H^0(F, \mathcal{O}_F((t+1)h)) \otimes H^0(F, \mathcal{O}_F(M)) & \xrightarrow{\varphi_t \otimes \vartheta_{-1}} & H^0(E, \mathcal{O}_E((t+1)h)) \otimes H^0(E, \mathcal{O}_E(M)) \\ \downarrow & & \downarrow \vartheta_t \\ H^0(F, \mathcal{O}_F(c_1 + th)) & \longrightarrow & H^0(E, \mathcal{O}_E(c_1 + th)) \end{array}$$

where φ_t is the natural restriction map: it follows that the top horizontal arrow is surjective. Since $E \cong \mathbb{P}^1$ the right vertical arrow is an epimorphism. Thus ϑ_t is surjective too, hence $h^1(F, \mathcal{I}_{E|F}(c_1 + th)) = 0$, $t \geq 0$. The vanishing of $h^i(F, \mathcal{E}(th)) = 0$, $i = 1, 2$ and $t \in \mathbb{Z}$, proved above implies that \mathcal{E} is aCM, hence \mathcal{E} is an Ulrich bundle thanks to Proposition 4.2. \square

We now construct examples of Ulrich bundles with $c_1 = h_1 + 2h_2 + 3h_3$. We have that

$$\dim_k(\text{Ext}_F^1(\mathcal{O}_F(2h_2 + h_3), \mathcal{O}_F(h_1 + 2h_3))) = h^1(F, \mathcal{O}_F(h_1 - 2h_2 + h_3)) = 4,$$

thus there are non-trivial extensions of the form

$$0 \longrightarrow \mathcal{O}_F(h_1 + 2h_3) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_F(2h_2 + h_3) \longrightarrow 0.$$

The bundle \mathcal{E} has Chern classes are $c_1 = h_1 + 2h_2 + 3h_3$ and $c_2 = 4h_2h_3 + h_1h_3 + 2h_1h_2$.

Since both $\mathcal{O}_F(2h_2 + h_3)$ and $\mathcal{O}_F(h_1 + 2h_3)$ are aCM, looking at the cohomology of above sequence twisted by $\mathcal{O}_F(th)$, it follows that the same is true for \mathcal{E} . Moreover, the same cohomology

sequence and vanishing also yield that \mathcal{E} is initialized. Arguing as in the previous section we are also able to show that \mathcal{E} is indecomposable, with the exception of the trivial extension.

The above discussion partially proves the following result.

Theorem 7.3. *There exist indecomposable initialized aCM bundles \mathcal{E} of rank 2 with $c_1 = h_1 + 2h_2 + 3h_3$ and c_2 either $4h_2h_3 + h_1h_3 + 2h_1h_2$ or $3h_2h_3 + 3h_1h_3 + h_1h_2$ up to permutation of the h_i 's. Moreover the zero-locus of a general section of \mathcal{E} is a rational normal curve.*

Conversely, up to permutations of the h_i 's, each rational normal curve on F whose class in $A^2(F)$ is not $3h_2h_3 + 2h_1h_3 + 2h_1h_2$ can be obtained as the zero locus of a general section of an initialized aCM bundles \mathcal{E} of rank 2 with $c_1 = h_1 + 2h_2 + 3h_3$ and c_2 either $4h_2h_3 + h_1h_3 + 2h_1h_2$ or $3h_2h_3 + 3h_1h_3 + h_1h_2$. The bundle \mathcal{E} is indecomposable if and only if it is not the complete intersection of two divisors $Q_1 \in |2h_2 + h_3|$ and $Q_2 \in |h_1 + 2h_3|$.

Proof. The above discussion proves the existence of a 3-dimensional family of initialized aCM vector bundles with $c_2 = 4h_2h_3 + h_1h_3 + 2h_1h_2$. The only decomposable bundle in such a family is $\mathcal{O}_F(h_1 + 2h_3) \oplus \mathcal{O}_F(2h_2 + h_3)$, whose sections are trivially complete intersection curves inside F .

Now we deal with the case $c_2 = 3h_2h_3 + 3h_1h_3 + h_1h_2$. Again the above discussion shows that if such a vector bundle exists, then it is indecomposable. In order to show its existence we again work by deforming suitable curves on a hyperplane section of F as we did in the previous section. Let $S := H \cap F$ be a general hyperplane section of F . S is a del Pezzo surface of degree 6 in \mathbb{P}^6 . We know by the proof of Lemma 6.5 that the classes of ℓ, e_1, e_2, e_3 in $A^2(F)$ are $h_2h_3 + h_1h_3 + h_1h_2, h_2h_3, h_1h_3, h_1h_2$ respectively. Using the same notations of Lemma 6.4 one easily checks that S contains rational curves of degree 7 and their classes are $3\ell - 2e_3, 4\ell - e_1 - e_2 - 3e_3$ which are both in the class of $3h_2h_3 + 3h_1h_3 + h_1h_2$.

Take a smooth rational curve $C \subseteq S$ of degree 7. We have $\mathcal{N}_{C|S} \cong \mathcal{O}_C(C)$ and $\mathcal{N}_{S|F} \cong \mathcal{O}_S(hS)$, thus the exact sequences

$$\begin{aligned} 0 &\longrightarrow \mathcal{O}_C(C) \longrightarrow \mathcal{N}_{C|F} \longrightarrow \mathcal{O}_C(h) \longrightarrow 0, \\ 0 &\longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_S(C) \longrightarrow \mathcal{O}_C(C) \longrightarrow 0, \end{aligned}$$

Since $C \cong \mathbb{P}^1$, it follows that $\mathcal{O}_C(C) \cong \mathcal{O}_{\mathbb{P}^1}(5)$, $\mathcal{O}_C(h) \cong \mathcal{O}_{\mathbb{P}^1}(7)$, thus $h^0(S, \mathcal{O}_S(C)) = 7$, $h^0(C, \mathcal{N}_{E|F}) = 14$, $h^1(C, \mathcal{N}_{C|F}) = 0$.

Arguing as in the proof of Proposition 6.6 one checks the existence of a non-degenerate smooth rational curve E whose class is $3h_2h_3 + 3h_1h_3 + h_1h_2$. \square

8. THE INTERMEDIATE CASES

In this section we will show which other indecomposable, initialized, aCM bundles \mathcal{E} of rank 2 are actually admissible besides the ones occurring in the extremal cases. We know that for the general section $s \in H^0(F, \mathcal{E})$ its zero-locus $E := (s)_0$ has pure dimension 1.

System (15), Inequality (16) with $D = 0$ and Formulas (13), (14) yield the following list (up to permutations).

Case	$(\alpha_1, \alpha_2, \alpha_3)$	$(\beta_1, \beta_2, \beta_3)$	$\deg(E)$	$p_a(E)$
L	$(0, 0, 2)$	$(0, 0, 0)$	0	-1
M	$(0, 0, 1)$	$(1, 0, 0)$	1	0
N	$(0, 1, 1)$	$(1, 0, 0)$	1	0
P	$(0, 1, 2)$	$(1, 0, 0)$	1	0
Q	$(1, 1, 1)$	$(1, 1, 0)$	2	0
R	$(1, 1, 1)$	$(2, 0, 0)$	2	0
S	$(0, 2, 2)$	$(2, 0, 0)$	2	-1
T	$(1, 1, 2)$	$(1, 1, 1)$	3	0
U	$(1, 1, 2)$	$(0, 2, 1)$	3	0
V	$(1, 2, 2)$	$(2, 2, 1)$	5	0
W	$(1, 2, 2)$	$(2, 3, 0)$	5	0

It is immediate to exclude the case L because $E \neq \emptyset$ due to Lemma 3.3.

Let \mathcal{E} be in the list above. Then $\mathcal{E}^\vee(th)$ is an indecomposable aCM bundle too. We take t in such a way that it is also initialized. Thus $0 \leq 2t - \alpha_1 \leq 2$ (see Proposition 4.2), hence $t = 1$. We have

$$\begin{aligned} c_1(\mathcal{E}^\vee(h)) &= (2 - \alpha_1)h_1 + (2 - \alpha_2)h_2 + (2 - \alpha_3)h_3, \\ c_2(\mathcal{E}^\vee(h)) &= (\beta_1 + 2 - (\alpha_2 + \alpha_3))h_2h_3 + \\ &\quad + (\beta_2 + 2 - (\alpha_1 + \alpha_3))h_1h_3 + (\beta_3 + 2 - (\alpha_1 + \alpha_2))h_1h_2 \end{aligned}$$

In case S, then $c_2(\mathcal{E}^\vee(h)) = 0$, thus we can again exclude it. In case U (resp. W), then $c_2(\mathcal{E}^\vee(h)) = -h_2h_3 + h_1h_3 + h_1h_2$ (resp. $2h_1h_3 - h_1h_2$). Due to the positivity of the β_i 's (see Formula (16)) also these two cases cannot occur. The same argument shows that cases T and V occur if and only if cases N and M occur respectively.

We now examine the remaining cases one by one, checking which of them actually occurs. We have to analyze cases M, N, P, Q, R.

8.1. The case M (and V). In case M we know by the table that E must be a line.

Let $E \subseteq F$ be a line in the class $h_2h_3 \in A^2(F)$. Since $\omega_E \cong \mathcal{O}_E(-2h)$, it follows that $\det(\mathcal{N}_{E|F}) \cong \mathcal{O}_E$ which thus coincides with $\mathcal{O}_F(h_3) \otimes \mathcal{O}_E$. Taking into account the vanishing $h^2(F, \mathcal{O}_F(-h_3)) = 0$, we thus know the existence of an exact sequence

$$0 \longrightarrow \mathcal{O}_F \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_{E|F}(h_3) \longrightarrow 0.$$

where \mathcal{E} is a vector bundle of rank 2 on F (see Theorem 2.3). It is immediate to check that \mathcal{E} is initialized. We now check that \mathcal{E} is aCM. Trivially we have $h^1(F, \mathcal{E}(th)) = h^1(F, \mathcal{I}_{E|F}(th + h_3))$.

The cohomology of Sequence (3) twisted by $\mathcal{O}_F(th + h_3)$ gives rise to the exact sequence

$$H^0(F, \mathcal{O}_F(th + h_3)) \longrightarrow H^0(E, \mathcal{O}_E(th + h_3)) \longrightarrow H^1(F, \mathcal{I}_{E|F}(th + h_3)) \longrightarrow 0.$$

The multiplication by h_3 gives rise to the commutative diagram

$$\begin{array}{ccc} H^0(F, \mathcal{O}_F(th)) & \xrightarrow{\varphi_t} & H^0(E, \mathcal{O}_E(th)) \\ \downarrow & & \downarrow \psi_t \\ H^0(F, \mathcal{O}_F(th + h_3)) & \longrightarrow & H^0(E, \mathcal{O}_E(th + h_3)) \end{array}$$

Since both E and F are aG, it is easy to check that φ_t is surjective (see the proof of the analogous fact in Section 6). Moreover E is in the class of h_2h_3 , thus ψ_t is an isomorphism. It follows that the map below is surjective, hence $h^1(F, \mathcal{I}_{E|F}(th + h_3)) = 0$.

Now we look at the vanishing of H^2 . We know that we can also write the exact sequence

$$0 \longrightarrow \mathcal{O}_F(-h_3) \longrightarrow \mathcal{E}^\vee \longrightarrow \mathcal{I}_{E|F} \longrightarrow 0.$$

We thus obtain

$$h^2(F, \mathcal{E}(th)) = h^1(F, \mathcal{E}^\vee(-(2+t)h)) = h^1(F, \mathcal{I}_{E|F}(-(2+t)h)).$$

But the last number is the dimension of the cokernel of the map $\varphi_{-(2+t)}$ which is zero. We conclude that \mathcal{E} is aCM.

Assume that \mathcal{E} is decomposable. Thus it is the sum of two aCM invertible sheaves, one of them being initialized, the other one either initialized or without sections. Such sheaves are listed in Lemma 2.4. Looking at c_1 the unique possibilities are then $\mathcal{O}_F \oplus \mathcal{O}_F(h_3)$, $\mathcal{O}_F(-h_2) \oplus \mathcal{O}_F(h_2 + h_3)$, $\mathcal{O}_F(-h_2 - h_3) \oplus \mathcal{O}_F(h_2 + 2h_3)$. But, in these cases, c_2 is 0, $-h_2h_3$, $-3h_2h_3$ respectively. We conclude that \mathcal{E} is indecomposable.

In particular both the cases M and V are admissible.

8.2. The case N (and T). In case N we know by the table that $E \subseteq F$ is a line in the class $h_2h_3 \in A^2(F)$. On the one hand we know that $\det(\mathcal{N}_{E|F}) \cong \mathcal{O}_F(h_2 + h_3) \otimes \mathcal{O}_E$. Since $h^2(F, \mathcal{O}_F(-h_2 - h_3)) = 0$, it follows the existence of an exact sequence

$$0 \longrightarrow \mathcal{O}_F(-h_2 - h_3) \longrightarrow \mathcal{E}^\vee \longrightarrow \mathcal{I}_{E|F} \longrightarrow 0.$$

where \mathcal{E} is a vector bundle of rank 2 on F . Let $i = 2, 3$: there are $D_i \in |h_i|$ containing E , because $c_2h_i = 0$. Thus $E = D_2 \cap D_3$ necessarily. The associated Koszul complex is the exact sequence

$$0 \longrightarrow \mathcal{O}_F(-h_2 - h_3) \longrightarrow \mathcal{O}_F(-h_2) \oplus \mathcal{O}_F(-h_3) \longrightarrow \mathcal{I}_{E|F} \longrightarrow 0.$$

We have $h^1(F, \mathcal{O}_F(-h_2 - h_3)) = 0$ (see Lemma 2.4), hence the two sequences above must be isomorphic due to Theorem 2.3. It follows that $\mathcal{E} \cong \mathcal{O}_F(h_2) \oplus \mathcal{O}_F(h_3)$.

In particular both the cases N and T are not admissible in the sense that the sheaf \mathcal{E} exists, but it is decomposable.

8.3. The case P. Arguing as in the case N we check that $\mathcal{E} \cong \mathcal{O}_F(h_2 + h_3) \oplus \mathcal{O}_F(h_3)$. Hence this case is not admissible in the sense that the sheaf \mathcal{E} exists, but it is decomposable.

8.4. The cases Q and R. In these cases E is a curve of degree 2 of genus 0. E is either irreducible or the union of two concurrent lines, because $p_a(E) = 0$, or a double line.

In case Q only the first two cases are possible and we can easily construct the exact sequence

$$0 \longrightarrow \mathcal{O}_F(-h) \longrightarrow \mathcal{E}^\vee \longrightarrow \mathcal{I}_{E|F} \longrightarrow 0$$

where \mathcal{E} is a vector bundle of rank 2 on F .

We have $h_3c_2 = 0$: since $|h_3|$ is base-point-free, it follows the existence of $Q \in |h_3|$ containing E . Q is a smooth quadric surface, E is cut out on Q by a hyperplane. Since $\mathcal{O}_F(h) \otimes \mathcal{O}_Q \cong \mathcal{O}_F(h_1 + h_3) \otimes \mathcal{O}_Q$, it follows the existence of $D \in |h_1 + h_2|$ such that $D \cap Q = E$. In particular we have a corresponding Koszul complex

$$0 \longrightarrow \mathcal{O}_F(-h) \longrightarrow \mathcal{O}_F(-h_1 - h_2) \oplus \mathcal{O}_F(-h_3) \longrightarrow \mathcal{I}_{E|F} \longrightarrow 0.$$

As in the previous cases N and P, we deduce that $\mathcal{E} \cong \mathcal{O}_F(h_1 + h_2) \oplus \mathcal{O}_F(h_3)$.

Let us examine case R. We claim that E is not reduced. E.g., assume E is the union of two distinct concurrent lines, say L, M , and let $p := L \cap M$ be their intersection point. Thus there are $Q_i \in |h_i|$, $i = 1, 2$ such that $p \in Q_1 \cap Q_2$. Since $LQ_i + MQ_i = EQ_i = 0$, it would follow that $E = L \cup M \subseteq Q_1 \cap Q_2$, hence its class inside $A^2(F)$ would be h_2h_3 . But we know that its class is $c_2 = 2h_2h_3$, a contradiction. If E is irreducible we can argue similarly.

Thus E is a double structure on a line E_{red} whose class in $A^2(F)$ is h_2h_3 necessarily. The general theory of double structures (see e.g. [7]) gives us an exact sequence of the form

$$0 \longrightarrow \mathcal{C}_{E_{red}|E} \longrightarrow \mathcal{O}_E \longrightarrow \mathcal{O}_{E_{red}} \longrightarrow 0.$$

The conormal sheaf $\mathcal{C}_{E_{red}|E}$ is an invertible sheaf on $E_{red} \cong \mathbb{P}^1$, thus $\mathcal{C}_{E_{red}|E} \cong \mathcal{O}_{\mathbb{P}^1}(-a)$. Moreover $a = 1$, because $p_a(E) = 0$ (see Section 2 of [7]).

Recall that the Hilbert scheme Γ of lines in F is the union of three components Γ_i , each of them isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ (Proposition 3.5.6 of [28]).

On the one hand, all the lines on F in the class $h_2h_3 \in A^2(F)$ are parameterized by the same copy of $\mathbb{P}^1 \times \mathbb{P}^1$ and we denote it by Γ_1 . The variety F does not contain planes, hence the universal family $S_1 \rightarrow \Gamma_1$ maps surjectively on F (see Proposition 3.3.5 of [28]). Lemma 3.3.4 of [28] implies that the line L corresponding to the general point in Γ_1 satisfies $\mathcal{C}_{L|F} \cong \mathcal{N}_{L|F}^\vee \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}$. Clearly the induced action of the automorphism group of F on Γ_1 is transitive, hence there is an automorphism of F fixing the class h_2h_3 and sending E_{red} to L . We conclude that $\mathcal{C}_{E_{red}|F} \cong \mathcal{C}_{L|F} \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}$.

On the other hand, the inclusion $E_{red} \subseteq E \subseteq F$ implies the existence of an epimorphism $\mathcal{C}_{E_{red}|F} \rightarrow \mathcal{C}_{E_{red}|E}$. We obtain an epimorphism $\mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \rightarrow \mathcal{O}_{\mathbb{P}^1}(-1)$, clearly an absurd.

In particular both the cases Q and R are not admissible. In the former case the bundle \mathcal{E} exists, but it is decomposable, while in the latter \mathcal{E} does not exist at all.

We summarize the results of this section in the following result.

Theorem 8.1. *Let \mathcal{E} be an indecomposable initialized aCM bundle of rank 2 on F and let $c_1 = \alpha_1 h_1 + \alpha_2 h_2 + \alpha_3 h_3$ with $\alpha_1 \leq \alpha_2 \leq \alpha_3$. Assume that c_1 is neither 0, nor $2h$, nor $h_1 + 2h_2 + 3h_3$. Then (c_1, c_2) is either $(h_3, h_2 h_3)$, or $(h_1 + 2h_2 + 2h_3, 2h_2 h_3 + 2h_1 h_3 + h_1 h_2)$.*

Conversely, for each such a pair, there exists an indecomposable, initialized, aCM bundle \mathcal{E} of rank 2 on F with Chern classes c_1 and c_2 .

Moreover the zero-locus of a general section of \mathcal{E} is respectively a line or a, possibly reducible, quintic with arithmetic genus 0. Each line on F can be obtained in this way.

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