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# Observer Design for Hybrid Dynamical Systems with Approximately Known Jump Times

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## Abstract

This paper proposes a general framework for the state estimation of plants modeled as hybrid dynamical systems with discrete events (or jumps) occurring at (approximately) known times. A candidate observer consists of a hybrid dynamical system with jumps triggered when the plant jumps. With some information about the time elapsed between successive jumps, a Lyapunov-based analysis allows us to derive sufficient conditions for the design of the observer that renders the zero-estimation set uniformly asymptotically stable. In particular, we show that a high-gain flow-based observer, with innovation during flow only, can be designed when the flow dynamics are strongly differentially observable. On the other hand, a jump-based observer, with innovation at jumps only, should be designed based on an equivalent discrete-time system corresponding to the hybrid system discretized at jump times, and presenting the observability of the combination of both flows and jumps. In the linear context, this reasoning leads us to a constructive hybrid Kalman filter. These designs apply to a large class of hybrid systems, including cases where the time between successive jumps is unbounded or tends to zero – namely, Zeno behavior –, as well as cases where detectability only holds during flows, at jumps, or neither. Building from these sufficient conditions, we study the robustness of this approach when the jumps of the observer are delayed with respect to those of the plant. Under some regularity and dwell-time conditions, we show that the estimation error remains bounded and satisfies a semi-global practical asymptotic stability-like property. The results are illustrated in several examples and applications, including mechanical systems with impacts, spiking neurons, and switched systems.

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## 1. Introduction

### 1.1. Context

In many applications, estimating the state of a system is crucial, whether it be for control, supervision, or fault diagnosis purposes. Unfortunately, the problem of designing observers for hybrid systems of the form ([1])

$$\dot{x} = f(x) \quad x \in C, \quad x^+ = g(x) \quad x \in D \quad (1)$$

presenting both a continuous-time behavior in  $C$  and a discrete-time behavior in  $D$ , is still largely unsolved, even when the flow/jump maps  $f$  and  $g$  are linear. A major difficulty lies in the fact that the plant's jump times, that is, the times at which discrete events occur in the plant's solution, generally depend on its initial condition, which is unknown in the context of observer design. From there, one may distinguish two scenarios: when the plant's jump times are detected by sensors (or known a priori), and the considerably more complex scenario when the plant's jump times are truly unknown.

In the second scenario, the jumps of the observer cannot be triggered when the jumps of the plant occur. It follows that the domain of definition of the plant's and observer's solutions are different and a standard error system approach for observer design does not apply. This mismatch of time domains makes the formulation of observability/detectability and, in turn, observer design very challenging ([2]). Very few observer results for plants of

the form (1) exist apart from particular settings as in [3], which requires the composition  $g \circ g$  to be the identity map, and in [4], thanks to a change of coordinates transforming the jump map  $g$  into the identity map, in this way, removing the jumps. Note that in the particular context of switched systems, this mismatch issue translates into the problem of estimating the switching signal. The observability properties of such a signal have been studied in [5, 6]. Observer designs based on the so-called *mode location observers*, capable of detecting and identifying properties of the switching signal appeared in [7, 8, 9, 10, 11, 12], to list a few, which include the broad literature of fault tolerant control.

On the other hand, in the first scenario where the plant's jump times are known/detected, the observer jumps can be triggered at the same time as the plant's (up to small detection delays). The difficulties due to a possible mismatch of time domains thus disappear, and observability analysis also reduces to comparing solutions with same output on the same time domain.

A first class of systems falling into this first scenario, is the so-called *impulsive systems*. It consists of continuous-time dynamical systems (possibly switching among different flow dynamics) with state jumps (or switch) that occur at pre-specified times, which are usually assumed to be separated by nonzero periods of flow – in particular, to avoid Zeno behavior. The impulsive systems literature is

rich and includes a variety of models of impulsive systems. In particular, models of impulsive systems in which the state includes a logic variable that selects the right-hand side of the differential equation governing the dynamics in between impulses are referred to as switched impulsive systems, or also as switched systems with known jump times. In that setting, observability/determinability have been extensively studied in [13, 14, 15, 16, 17]. As for observer design, results first appeared assuming observability of each flow dynamics [18], and then more generally in [19] (resp. in [17]), for impulsive systems (resp. switched impulsive systems) that are observable (resp. determinable) for any impulse time sequence containing more than a known finite number  $N$  of jumps. In other words, the information available during a single flow interval is not sufficient to reconstruct the full state, but it becomes sufficient after  $N$  jumps. In [19], the observer consists of an impulsive system synchronized with the plant, with innovation terms at jumps only. Those innovations are linear in the error, with a time-varying gain that is related to a weighted observability Grammian over the past  $N$  jumps. In [17], the authors develop an observation procedure based on the continuous-time estimation of the observable states of each of the past  $N$  modes: after some time, putting together the information given by each mode enables to reconstruct the whole state.

Another important class of hybrid systems falling into the second scenario is when the system itself has continuous-time dynamics, but the measurements are sampled and available intermittently at specific time instances. For such a class of systems with sporadic events, observers have been designed under specific assumptions on the time elapsed between successive events or, in the case of periodic events, the sampling period. From [20, 21], convergence of an impulsive observer with innovation terms triggered by the measurement events can be guaranteed when the sampling period is sufficiently small. Designs were then developed in [22, 23] for any constant sampling period provided that appropriate matrix inequalities are satisfied, and further extended in [24, 25, 26, 27, 28] to the case of sporadic measurements, i.e., when the time elapsed between sampling events varies in a known interval.

In this paper, we propose to address the problem of state estimation for general hybrid systems (1), in the context of the first scenario, namely when the jump times of the plant's solution are (approximately) known, and in an attempt to unify most of the previously cited approaches. Preliminary results in this direction were given in [29, 30], in the particular case where  $f$  and  $g$  are linear, and when at least either the flow dynamics or the jump dynamics are detectable. We extend those results here to nonlinear dynamics, also when neither the continuous nor the discrete dynamics of the plant are detectable, but, the hybrid plant as a whole is.

## 1.2. Content and Contributions

First, under the assumption that the jumps of the plant are instantaneously detected, a candidate observer is defined as a hybrid system that jumps at the same time as

the plant does, and is fed with the measured output in either the flow map, the jump map, or both (Section 2). Assuming the plant has an average dwell-time or a reverse average dwell-time, or simply that the time between its successive jumps belongs to a known (possibly unbounded) closed set, we derive Lyapunov-based sufficient conditions so as to ensure uniform pre-asymptotic stability of the zero estimation error set (Section 3). Then, we provide additional design conditions for special cases of the general observer problem and proposed hybrid observer. In Section 4, we consider the case when measurements are only used during flow, for which we propose a hybrid observer, which we call *flow-based hybrid observer*. Similarly, but for the situation when output measurements are used only at jumps, Section 5 introduces a *jump-based hybrid observer* and associated design conditions. Motivated by the fact that, in practice, the jumps of the plant cannot always be *instantaneously* detected, we study the robustness of the observer when the jumps of the observer are slightly delayed relative to those of the plant (Section 6). Finally, we demonstrate how those results can be used for observer design of switched systems with state triggered jumps (Section 7.3).

Our main contributions compared to the above literature are as follows:

1. General hybrid systems (1) are considered in a unified framework, only assuming knowledge about the time between successive jumps, which allows any type of solutions, from Zeno and eventually discrete, to eventually continuous trajectories;
2. When the plant has an average dwell-time and its continuous dynamics are strongly differentially observable, we prove that a hybrid observer can be obtained by copying the plant's discrete dynamics and designing a high-gain observer for its continuous dynamics, as long as the gain is taken sufficiently large compared to the average dwell-time and the Lipschitz constants of the flow and jump maps;
3. When the output measurements are only injected in the observer at jumps, we highlight that the innovation term in the observer, which only plays a role at jumps, should be designed based on an equivalent discrete-time system that models the hybrid plant sampled at jumps. In the linear context, this reasoning leads us to a constructive hybrid Kalman filter;
4. A robustness analysis with respect to delays in the triggering of the jumps of the observer jumps is provided: under some regularity and dwell-time conditions, we show that the estimation error remains bounded and semi-global practical stability holds outside the delay intervals between the plant's and the observer's jumps;
5. The generality of the framework enables us to recover and unify a significant part of the literature. In particular, the results apply well to switched systems with state-triggered switches: we show how a high-gain observer can be designed for switched systems with observable modes and average dwell-time, or how the output at the switching instants can be used when each

mode is not observable on its own but the combination of them is.

These results are motivated by several applications, which are presented in detail in Section 7, featuring hybrid dynamics for which the estimation of the state and of parameters is crucial.

### 1.3. Notation and Preliminaries

$\mathbb{R}$  (resp.  $\mathbb{N}$ ) denotes the set of real numbers (resp. integers),  $\mathbb{R}_{\geq 0} = [0, +\infty)$ ,  $\mathbb{R}_{> 0} = (0, +\infty)$ , and  $\mathbb{N}_{> 0} = \mathbb{N} \setminus \{0\}$ . For a square matrix  $P$ ,  $\text{eig}(P)$  denotes the set of its eigenvalues, and  $\lambda_m(P)$  (resp.  $\lambda_M(P)$ ) stands for its smallest (resp. largest) eigenvalue. The symbol  $\star$  in a matrix denotes the symmetric blocks.  $\mathbb{B}$  stands for a closed Euclidian ball of appropriate dimension, of radius 1 and centered at 0. A map  $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  is a  $\mathcal{K}$ -map if  $\alpha(0) = 0$  and  $\alpha$  is continuous and increasing, and a  $\mathcal{K}^\infty$ -map if it is also unbounded. A map  $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  is a  $\mathcal{KL}$ -map if for all  $t \in \mathbb{R}_{\geq 0}$ ,  $r \mapsto \beta(r, t)$  is increasing and for all  $r \in \mathbb{R}_{\geq 0}$ ,  $\lim_{t \rightarrow \infty} \beta(r, t) = 0$ . For a set valued map  $S : \mathbb{R}^{d_x} \rightrightarrows \mathbb{R}$  and a scalar  $c$ , writing  $S(x) \leq c$  for some  $x \in \mathbb{R}^{d_x}$  means that  $s \leq c$  for all  $s \in S(x)$ . For a  $C^1$  map  $V : \mathbb{R}^{d_x} \rightarrow \mathbb{R}$ ,  $L_S V(x)$  denotes the set of Lie derivatives along vector fields  $s \in S(x)$ , i.e.  $\{\frac{dV}{dx}(x)s, s \in S(x)\}$ . We consider hybrid dynamical systems of the form (1) ([1]) where  $f$  (resp.  $g$ ) is the flow (resp. jump) map, and  $C$  (resp.  $D$ ) is the flow (resp. jump) set. Solutions to such systems are defined on so-called *hybrid time-domains*. A subset  $E$  of  $\mathbb{R}_{\geq 0} \times \mathbb{N}$  is a *compact hybrid time-domain* if  $E = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$  for some finite sequence of times  $0 = t_0 \leq t_1 \leq \dots \leq t_J$ , and it is a hybrid time domain if for any  $(T, J) \in E$ ,  $E \cap [0, T] \times \{0, \dots, J\}$  is a compact hybrid time domain. For a solution  $(t, j) \mapsto x(t, j)$  (see [1, Definition 2.6]), we denote  $\text{dom } x$  its domain,  $\text{dom}_t x$  (resp.  $\text{dom}_j x$ ) its projection on the time (resp. jump) component,  $T(x) := \sup \text{dom}_t x$ ,  $J(x) := \sup \text{dom}_j x$ , and for a positive integer  $j$ ,  $t_j(x)$  the only time defined by  $(t_j, j) \in \text{dom } x$  and  $(t_j, j-1) \in \text{dom } x$ . When no ambiguity is possible, we will omit  $x$  and write  $T$ ,  $J$ ,  $t_j$ . Besides,  $N(t, s)$  denotes the number of jumps occurring between times  $t$  and  $s$ . We say that  $x$  is *complete*, resp. *t-complete*, resp. *j-complete*, if  $\text{dom } x$ , resp.  $\text{dom}_t x$ , resp.  $\text{dom}_j x$  is unbounded ;  $x$  is *eventually continuous* (resp. *eventually discrete*) if  $J < +\infty$  and  $T > t_J$  (resp.  $T < +\infty$  and  $\text{dom } x \cap (T \times \mathbb{N})$  contains at least two points) ;  $x$  is *Zeno* if it is complete and  $T < +\infty$ .

## 2. Synchronized Hybrid Observer

### 2.1. Mathematical Modeling

We consider a hybrid plant of the form

$$\mathcal{H} \begin{cases} \dot{x} \in f(x), & y_c = h_c(x), & x \in C \\ x^+ \in g(x), & y_d = h_d(x), & x \in D \end{cases} \quad (2)$$

with state  $x \in \mathbb{R}^{d_x}$ , and output  $y = (y_c, y_d) \in \mathbb{R}^{d_{y_c}} \times \mathbb{R}^{d_{y_d}}$ , with  $y_c$  available during flow and  $y_d$  during jumps. For this class of hybrid systems, we are interested in estimating

the state of (or part of the state of)  $\mathcal{H}$  when its solutions are initialized in a given subset  $\mathcal{X}_0 \subseteq C \cup D$ . We denote by  $\mathcal{S}_{\mathcal{H}}(\mathcal{X}_0)$  the set of maximal solutions of  $\mathcal{H}$  with initial condition in  $\mathcal{X}_0$ .

**Definition 2.1.** For a closed subset  $\mathcal{I}$  of  $\mathbb{R}_{\geq 0}$  and a positive scalar  $\tau^\star$ , we will say that

- *solutions have flow length within  $\mathcal{I}$  if for any  $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}_0)$ ,*
  - $0 \leq t - t_j(x) \leq \sup \mathcal{I} \quad \forall (t, j) \in \text{dom } x$
  - $t_{j+1}(x) - t_j(x) \in \mathcal{I}$  holds
  - \*  $\forall j \in \mathbb{N}_{> 0}$  if  $J(x) = +\infty$
  - \*  $\forall j \in \{1, \dots, J(x) - 1\}$  if  $J(x) < +\infty$ .

For simplicity, we say that  $\mathcal{C}_{\mathcal{X}_0}[\mathcal{I}]$  holds;

- *solutions have an average dwell-time (ADT)  $\tau^\star$  if there exists  $N_0 \in \mathbb{N}_{> 0}$  such that for any  $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}_0)$ ,*

$$N(t, s) \leq \frac{(t - s)}{\tau^\star} + N_0 \quad \forall t \geq s \geq 0.$$

For simplicity, we say that  $\mathcal{C}_{\mathcal{X}_0}^{\text{av}}[\tau^\star]$  holds;

- *solutions have a reverse average dwell-time (rADT)  $\tau^\star$  if there exists  $N_0 \in \mathbb{N}_{> 0}$  such that for any  $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}_0)$ ,*

$$N(t, s) \geq \frac{(t - s)}{\tau^\star} - N_0 \quad \forall t \geq s \geq 0;$$

For simplicity, we say that  $\mathcal{C}_{\mathcal{X}_0}^{\text{rav}}[\tau^\star]$  holds.

In the definition of  $\mathcal{C}_{\mathcal{X}_0}[\mathcal{I}]$ , the set  $\mathcal{I}$  describes the possible lengths of the flow intervals between successive jumps. The role of the first item is to bound the length of the intervals of flow which are not covered by the second item, namely possibly the first one, which is  $[0, t_1]$ , and the last one, which is  $\text{dom}_t x \cap [t_J, +\infty)$  (when defined).

Properties  $\mathcal{C}_{\mathcal{X}_0}^{\text{av}}[\tau^\star]$  and  $\mathcal{C}_{\mathcal{X}_0}^{\text{rav}}[\tau^\star]$  correspond to the standard notions of *average dwell-time* and *reverse average dwell-time* respectively ([1, 31]). They enforce that the solutions jump, on average, at most (resp. at least) once per time interval of length  $\tau^\star$ . A particular case of  $\mathcal{C}_{\mathcal{X}_0}^{\text{av}}[\tau^\star]$  is when all the intervals of flow last at least  $\tau^\star$ , namely they have a *dwell-time*, which can also be modeled by  $\mathcal{C}_{\mathcal{X}_0}[\mathcal{I}]$  with  $\mathcal{I} = [\tau^\star, +\infty)$ .

We are now ready to state the observer problem of interest. Our goal is to design an observer assuming we know: 1) when the plant's jumps occur, 2) the outputs  $y_c$  during flows and/or  $y_d$  at jumps, 3) some information about the flow time between successive jumps of the type  $\mathcal{C}_{\mathcal{X}_0}[\mathcal{I}]$ ,  $\mathcal{C}_{\mathcal{X}_0}^{\text{av}}[\tau^\star]$ , or  $\mathcal{C}_{\mathcal{X}_0}^{\text{rav}}[\tau^\star]$ . Note that  $\mathcal{C}_{\mathcal{X}_0}[\mathbb{R}_{\geq 0}]$  always holds, but as we will see later, it is convenient to have as precise information about the duration of flow between successive jumps as possible.

**Example 2.2** (Mechanical systems with impacts). A *Lagrangian mechanical system with impacts is typically modeled as  $\mathcal{H}$  with state  $x = (x_1, x_2) \in \mathbb{R}^p \times \mathbb{R}^p$  capturing its (generalized) position and velocity, flow of the form*

$f(x) = (x_2, \alpha(x))$ ,  $g(x)$  translating the velocity discontinuity at the impact,  $D$  characterizing the impact condition and  $C = \text{cl}(\mathbb{R}^p \setminus D)$ . A complete model is given in Section 7.1. If there is loss of energy at impacts, we typically know that for a bounded set of initial conditions  $\mathcal{X}_0$ , the time between impacts is bounded, so that  $\mathcal{C}_{\mathcal{X}_0}[\mathcal{I}]$  holds with  $\mathcal{I}$  of the form  $\mathcal{I} = [0, \tau_M]$ , with  $\tau_M > 0$ . This case does not exclude Zeno behavior close to the impact set  $D$ . On the other hand, we may know that solutions have a dwell-time, for instance if at least  $\tau_m > 0$  amount of time is needed to flow from  $g(D)$  to  $D$ . Then,  $\mathcal{C}_{\mathcal{X}_0}[\mathcal{I}]$  holds with  $\mathcal{I}$  of the form  $\mathcal{I} = [\tau_m, +\infty)$  or  $\mathcal{I} = [\tau_m, \tau_M]$ .

In the particular case of a bouncing ball with gravity coefficient  $\mathbf{g} > 0$  and restitution coefficient  $\lambda > 0$ , modeled as  $\mathcal{H}$  with

$$\begin{aligned} f(x) &= (x_2, -\mathbf{g}) \quad , \quad g(x) = (-x_1, -\lambda x_2) \\ C &= \mathbb{R}_{\geq 0} \times \mathbb{R} \quad , \quad D = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0, x_2 \leq 0\} \end{aligned} \quad (3)$$

If  $\lambda < 1$ , any maximal solution  $x$  is Zeno, i.e., such that  $T(x) < +\infty$  and  $J(x) = +\infty$ . The time between two successive jumps  $t_{j+1}(x) - t_j(x)$  tends to zero when  $j$  tends to  $+\infty$ , and its upper bound increases with  $|x(0, 0)|$ . So, if  $\mathcal{X}_0$  is bounded,  $\mathcal{C}_{\mathcal{X}_0}[\mathcal{I}]$  holds with  $\mathcal{I}$  of the form  $\mathcal{I} = [0, \tau_M]$ , with  $\tau_M < +\infty$  depending on  $\mathcal{X}_0$ .

If now  $\lambda > 1$ , any maximal solution initialized in  $\mathbb{R}^2 \setminus \{(0, 0)\}$  is such that  $T(x) = +\infty$ ,  $J(x) = +\infty$ . The time between two successive jumps  $t_{j+1} - t_j(x)$  tends to  $+\infty$  when  $j$  tends to  $+\infty$ , and its lower bound decreases with  $|x(0, 0)|$ . Therefore, if there exists  $\delta > 0$  such that  $\mathcal{X}_0 \subset \mathbb{R}^n \setminus \delta\mathbb{B}$ ,  $\mathcal{C}_{\mathcal{X}_0}[\mathcal{I}]$  holds with  $\mathcal{I}$  of the form  $\mathcal{I} = [\tau_m, +\infty)$ , with  $\tau_m > 0$  depending on  $\mathcal{X}_0$ .

**Example 2.3** (Switched systems). The important class of switched systems also falls in the framework with

$$x = \begin{pmatrix} x_p \\ q \end{pmatrix}, \quad f(x) = \begin{pmatrix} f_q(x_p) \\ 0 \end{pmatrix}, \quad g(x) = \begin{pmatrix} g_q(x_p) \\ Q \end{pmatrix} \quad (4)$$

$$C = \bigcup_{q \in Q} C_q \times \{q\} \quad , \quad D = \bigcup_{q \in Q} D_q \times \{q\}$$

where  $Q = \{1, \dots, q_{max}\}$  and the discrete signal  $q$  indicates the mode in which the system evolves. When  $x_p$  is in  $D_q$  and a jump occurs, the mode either stays the same or is “switched” to a new value in  $Q$ . The plant then evolves according to the flow map  $f_q$  and jump map  $g_q$ , until  $q$  is switched to another value. Note that a way of forcing the mode to change at each jump is to take  $g(x) = \begin{pmatrix} g_q(x_p) \\ Q \setminus \{q\} \end{pmatrix}$ . By the way we have written (4), the switches are triggered by the state being in a certain region  $D_q$ : it is a state-dependent switching. The switches can also be triggered by an external signal called switching signal, in which case the switches are said time-dependent. This case could also be modeled by (2) by making some assumptions about the time between successive switches, which can take the form of  $\mathcal{C}_{\mathcal{X}_0}^{\text{rav}}[\tau^*]$ ,  $\mathcal{C}_{\mathcal{X}_0}^{\text{rav}}[\tau^*]$ , or  $\mathcal{C}_{\mathcal{X}_0}[\mathcal{I}]$ .

See [32, 1] for more detail. In this paper, we assume the switching times are known or detected. The output map is defined depending on the available information: known or unknown mode  $q$ , measurements of  $x_p$ , etc. See Section 7.3.

**Example 2.4** (Continuous-time system with sampled measurements). The proposed framework applies also to continuous-time systems

$$\dot{x}_p = f_p(x_p) \quad , \quad y = h_p(x_p)$$

whose output  $y$  is only available at discrete times  $t_j$ , which do not necessarily occur periodically. Assuming we know bounds on the time elapsed between two successive sampling events, or more generally that it belongs to a closed bounded set  $\mathcal{I}$ , namely  $\mathcal{C}_{\mathcal{X}_0}[\mathcal{I}]$  holds, such a system can be modeled by  $\mathcal{H}$  with state  $x = (x_p, \tau)$ ,

$$f(x) = (f_p(x_p), 1) \quad , \quad g(x) = (x_p, 0) \quad (5)$$

$$C = \mathbb{R}^{d_{x_p}} \times [0, \max \mathcal{I}] \quad , \quad D = \mathbb{R}^{d_{x_p}} \times \mathcal{I}$$

$$h_c(x) = \emptyset \quad , \quad h_d(x) = (h_p(x_p), \tau)$$

where  $\tau$  models the (known) time elapsed since the previous jump. For instance,  $\mathcal{I}$  is a singleton in the case of periodic sampling [22, 23], and  $\mathcal{I}$  is a compact interval of  $\mathbb{R}_{>0}$  in the case of aperiodic sampling with known bounds as considered for linear systems in [26, 28] and classes of nonlinear Lipschitz systems in [24, 25, 33, 34, 27]. Similarly, we could say that  $\mathcal{C}_{\mathcal{X}_0}^{\text{rav}}[\tau^*]$  holds if we know that measurements occur at most every  $\tau^*$  units of time and adapt the model (5) accordingly ([29]).

## 2.2. Problem Statement and Proposed Hybrid Observer

Since the plant’s jump times are assumed to be known, it is natural to use an observer for (2) of the form

$$\hat{\mathcal{H}} \begin{cases} \dot{z} \in F(z, y_c) & \text{when } \mathcal{H} \text{ flows} \\ z^+ \in G(z, y_d) & \text{when } \mathcal{H} \text{ jumps} \end{cases} \quad (6)$$

that is synchronized with the plant, for some functions  $F : \mathbb{R}^{d_z} \times \mathbb{R}^{d_{y_c}} \rightarrow \mathbb{R}^{d_z}$  and  $G : \mathbb{R}^{d_z} \times \mathbb{R}^{d_{y_d}} \rightarrow \mathbb{R}^{d_z}$  to be chosen such that  $z$  asymptotically enables to reconstruct the plant state  $x$ , or part of it, as formalized next.

Since the plant and the observer jump simultaneously, the observer analysis and design can be carried out on the cascade system

$$\mathcal{H} - \hat{\mathcal{H}} \begin{cases} \dot{x} \in f(x) \\ \dot{z} \in F(z, h_c(x)) \end{cases} (x, z) \in C \times \mathbb{R}^{d_z} \\ \left\{ \begin{array}{l} x^+ \in g(x) \\ z^+ \in G(z, h_d(x)) \end{array} \right\} (x, z) \in D \times \mathbb{R}^{d_z} \quad (7)$$

whose flow and jump map we denote

$$\mathcal{F}(x, z) = (f(x), F(z, h_c(x))) \quad (8a)$$

$$\mathcal{G}(x, z) = (g(x), G(z, h_d(x))) \quad (8b)$$

The observer problem can then be reformulated as a stabilization problem of a set  $\mathcal{A} \subseteq \mathbb{R}^{d_x} \times \mathbb{R}^{d_z}$ , which depends on the observation goal. For instance, if we want to estimate the full state  $x$ , we can first try to take  $d_z = d_x$  and stabilize the *zero estimation error set* given by

$$\mathcal{A} = \{(x, z) \in (C \cup D \cup g(D)) \times \mathbb{R}^{d_z} : x = z\}, \quad (9a)$$

which is nothing but the diagonal. In that case,  $z$  directly provides an asymptotic estimate of  $x$ . But sometimes, as for continuous-time systems, we need to change coordinates, or add some degrees of freedom, thus leading to  $d_z \geq d_x$  and

$$\mathcal{A} = \{(x, z) \in (C \cup D \cup g(D)) \times \mathbb{R}^{d_z} : z = T(x)\}, \quad (9b)$$

for some map  $T : C \cup D \cup g(D) \rightarrow \mathbb{R}^{d_z}$ . In that case, an estimate for  $x$  may be recovered from  $z$  by left-inversion if  $T$  is injective. We may also be interested in estimating only a part  $x_p$  of the state  $x$ , in the context of switched systems for instance, which can be translated into an appropriate choice of  $\mathcal{A}$ , i.e., more generally

$$\mathcal{A} = \{(x, z) \in (C \cup D \cup g(D)) \times \mathbb{R}^{d_z} : T(x, z) = 0\}, \quad (9c)$$

for some map  $T : C \cup D \cup g(D) \times \mathbb{R}^{d_z} \rightarrow \mathbb{R}^p$ . The goal of this paper is finally to solve the following problem.

**Problem (O).** *Given a set of initial conditions  $\mathcal{X}_0 \subseteq \mathbb{R}^{d_x}$ , a closed subset  $\mathcal{A}$  of  $\mathbb{R}^{d_x} \times \mathbb{R}^{d_z}$  defining an observer problem, and assuming one of the conditions of Definition 2.1 holds, design maps  $F : \mathbb{R}^{d_z} \times \mathbb{R}^{d_{y_c}} \rightarrow \mathbb{R}^{d_z}$  and  $G : \mathbb{R}^{d_z} \times \mathbb{R}^{d_{y_d}} \rightarrow \mathbb{R}^{d_z}$  such that there exist a class- $\mathcal{KL}$  function  $\beta$  and a subset  $\mathcal{Z}_0$  of  $\mathbb{R}^{d_z}$  such that for every  $\phi = (x, z) \in \mathcal{S}_{\mathcal{H}-\hat{\mathcal{H}}}(\mathcal{X}_0 \times \mathcal{Z}_0)$ ,*

$$|\phi(t, j)|_{\mathcal{A}} \leq \beta(|\phi(0, 0)|_{\mathcal{A}}, t + j) \quad (10)$$

for all  $(t, j) \in \text{dom } \phi$ , namely  $\mathcal{A}$  is uniformly pre-asymptotically stable (UpAS) for  $\mathcal{H} - \hat{\mathcal{H}}$  with basin of attraction including  $\mathcal{X}_0 \times \mathcal{Z}_0$ .

Note that for solutions  $(x, z) \in \mathcal{S}_{\mathcal{H}-\hat{\mathcal{H}}}(\mathcal{X}_0 \times \mathcal{Z}_0)$ , the set  $\mathcal{A}$  should also ensure that

$$x \text{ bounded and } |(x, z)|_{\mathcal{A}} \text{ bounded} \implies z \text{ bounded}$$

to guarantee from (10) that  $z$  cannot explode in finite time before  $x$  does. In other words, the observer solution is indeed defined as long as the plant's solution is. This is verified for  $\mathcal{A}$  defined in (9a) or (9b) if  $T$  is continuous.

**Remark 2.5.** *The implementation of the observer  $\hat{\mathcal{H}}$  requires a perfect jump synchronization with the plant  $\mathcal{H}$ . Unfortunately, the practical detection of the plant's jumps often involve measurements and transmission of information which might entail some delays in the triggering of the observer's jumps. The robustness of the UpAS property of  $\mathcal{A}$  given by Problem (O) is thus analyzed in Section 6.*

### 3. A General Sufficient Condition for the Design of Hybrid Observers

The following theorem gives a Lyapunov-based sufficient condition to solve Problem (O). It will be used throughout the paper in different cases.

**Theorem 3.1.** *Assume there exist scalars  $a_c, a_d \in \mathbb{R}$ ,  $\mathcal{K}^\infty$ -maps  $\underline{\alpha}, \bar{\alpha}$ , and a  $C^1$  map  $V : \mathbb{R}^{d_x} \times \mathbb{R}^{d_z} \rightarrow \mathbb{R}$  verifying*

$$\begin{cases} \underline{\alpha}(|(x, z)|_{\mathcal{A}}) \leq V(x, z) & \forall (x, z) \in (C \cup D \cup g(D)) \times \mathbb{R}^{d_z} \\ V(x, z) \leq \bar{\alpha}(|(x, z)|_{\mathcal{A}}) & \forall (x, z) \in \mathcal{X}_0 \times \mathcal{Z}_0 \end{cases} \quad (11a)$$

$$L_{\mathcal{F}}V(x, z) \leq a_c V(x, z) \quad \forall (x, z) \in C \times \mathbb{R}^{d_z} \quad (11b)$$

$$V(\mathcal{G}(x, z)) \leq e^{a_d} V(x, z) \quad \forall (x, z) \in D \times \mathbb{R}^{d_z} \quad (11c)$$

with  $\mathcal{F}$  and  $\mathcal{G}$  defined in (8). Then, Problem (O) is solved if any of the following conditions (C) holds:

$$(C1) \quad a_c < 0 \text{ and } \mathcal{C}_{\mathcal{X}_0}[\mathcal{I}] \text{ holds with } \min \mathcal{I} > \frac{a_d}{|a_c|}.$$

$$(C1') \quad a_c < 0 \text{ and } \mathcal{C}_{\mathcal{X}_0}^{\text{av}}[\tau^*] \text{ holds with } \tau^* > \frac{a_d}{|a_c|}.$$

$$(C2) \quad a_d < 0 \text{ and } \mathcal{C}_{\mathcal{X}_0}[\mathcal{I}] \text{ holds with } a_c \sup \mathcal{I} < |a_d|.$$

$$(C2') \quad a_d < 0 \text{ and } \mathcal{C}_{\mathcal{X}_0}^{\text{rav}}[\tau^*] \text{ holds with } a_c \tau^* < |a_d|.$$

*Proof.* Let us assume either (C1) or (C2). Then  $\mathcal{C}_{\mathcal{X}_0}[\mathcal{I}]$  holds and the jumps occur according to a timer following the dynamics

$$\begin{cases} \dot{\tau} = 1 & \tau \in [0, \sup \mathcal{I}] \\ \tau^+ = 0 & \tau \in \mathcal{I} \end{cases}$$

The map  $V'(x, z, \tau) = e^{a\tau} V(x, z)$  then verifies conditions of the type (11) with both  $a'_c = a_c + a < 0$  and  $a'_d < 0$  if:  $0 < a < |a_c|$  is chosen such that  $a'_d = a_d - a \min \mathcal{I} < 0$  under (C1); or  $a < 0$  is chosen such that  $a'_c = a_c + a < 0$  and  $a'_d = a_d - a \sup \mathcal{I} < 0$  under (C2). It follows from [1, Theorem 3.18] that Problem (O) is solved (See Remark 3.2 below).

Similarly, in case of a (reverse) average dwell-time with either (C1') or (C2'), [35, Propositions IV.1 and IV.4] show how to modify  $V$  to obtain a Lyapunov function  $V'$  which decreases both during flow and jumps. Under (C1'), denoting  $\delta^* = \frac{1}{\tau^*}$  the jumps can be modeled by a timer

$$\begin{cases} \dot{\tau} \in [0, \delta^*] & \tau \in [0, N_0] \\ \tau^+ = \tau - 1 & \tau \in [1, N_0] \end{cases}$$

so that  $V'(x, z, \tau) = e^{a\tau} V(x, z)$  with  $a \in (a_d, |a_c| \tau^*)$  is a strict Lyapunov function. On the other hand, under (C2'), the jumps can be modeled by a timer

$$\begin{cases} \dot{\tau} = 1 & \tau \in [0, N_0 \delta^*] \\ \tau^+ = \max\{0, \tau - \delta^*\} & \tau \in [0, N_0 \delta^*] \end{cases}$$

so that  $V'(x, z, \tau) = e^{-a\tau} V(x, z)$  with  $a \in (a_c, |a_d| \tau^*)$  is a strict Lyapunov function.  $\square$

**Remark 3.2.** In [1, Theorem 3.18], Condition (11a) is strengthened into

$$\underline{\alpha} (|(x, z)|_{\mathcal{A}}) \leq V(x, z) \leq \bar{\alpha} (|(x, z)|_{\mathcal{A}}) \quad \forall (x, z) \in (C \cup D \cup g(D)) \times \mathbb{R}^{d_z} \quad (12)$$

for easiness of presentation but actually the upper inequality is only needed on the initial conditions in the proof. We thus relax it to (11a), because it will be useful later.

From conditions (C), we recover the fact that in the case of a reverse average dwell-time or if  $0 \in \mathcal{I}$ , namely if there could be Zeno or eventually discrete solutions, then  $a_d$  is required to be negative, i.e., the innovation term in the discrete dynamics of the observer must make the error contractive at jumps; similarly in the case of average dwell-time or if  $\sup \mathcal{I} = +\infty$ , then  $a_c$  is required to be negative, i.e., the innovation term in the continuous dynamics must make the error contractive during flow. Finally, note that Theorem 3.1 allows the flow and jump maps of the plant and of the observer to be set valued, and hence, is suitable for the design of observers for plants modeled by differential inclusions or by difference inclusions, for which not many tools are available in the literature.

**Example 3.3** (Linear flow/jump/output maps). The case where  $f(x) = A_c x$ ,  $g(x) = A_d x$ ,  $h_c(x) = H_c x$ ,  $h_d(x) = H_d x$  has been studied in [29]. It is reasonable to consider  $\mathcal{A}$  defined as in (9a) and linear flow and jump maps in the observer, namely

$$F(z, y_c) = A_c z + L_c (y_c - H_c x) \quad (13a)$$

$$G(z, y_d) = A_d z + L_d (y_d - H_d x) \quad (13b)$$

with  $L_c \in \mathbb{R}^{d_x \times d_{y_c}}$  and  $L_d \in \mathbb{R}^{d_x \times d_{y_d}}$ . Then, the conditions in (11) hold for a quadratic Lyapunov function  $V(x, z) = (x - z)^\top P (x - z)$  if there exist scalars  $a_c$  and  $a_d$ , and a positive definite symmetric matrix  $P \in \mathbb{R}^{d_x \times d_x}$  such that

$$(A_c - L_c H_c)^\top P + P (A_c - L_c H_c) \leq a_c P \quad (14a)$$

$$(A_d - L_d H_d)^\top P (A_d - L_d H_d) \leq e^{a_d} P \quad (14b)$$

The problem of finding common quadratic Lyapunov functions for continuous-time (resp. discrete-time) systems has been studied in the context of switched systems and quadratic stabilization (see e.g. [36]). But we are not aware of any result concerning the existence of a common quadratic Lyapunov function for both continuous and discrete dynamics as in (14). If a solution to (14) exists, Problem (O) is solved if one of the conditions (C) holds. Note that if both  $(A_c, H_c)$  and  $(A_d, H_d)$  are detectable, (14) may be solvable with both  $a_c \leq 0$  and  $a_d \leq 0$ , and (C) then holds directly if at least one of them is nonzero. By the Schur complement, this is equivalent to solving the LMIs

$$\begin{aligned} A_c^\top P + P A_c - (\tilde{L}_c H_c + H_c^\top \tilde{L}_c^\top) &< 0 \\ \begin{pmatrix} P & (P A_d - \tilde{L}_d H_d)^\top \\ \star & P \end{pmatrix} &> 0 \end{aligned} \quad (15)$$

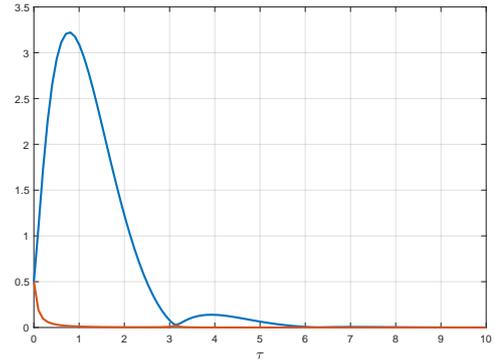


Figure 1: Absolute value of the eigenvalues of  $(A_d - L_d H_d) \exp((A_c - L_c H_c) \tau)$  for  $A_c = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ ,  $A_d = 5A_c$ ,  $H_c = H_d = (1, 0)$ , with  $L_c$  (resp.  $L_d$ ) chosen such that  $A_c - L_c H_c$  (resp.  $A_d - L_d H_d$ ) is Hurwitz (resp. Schur).

in  $(P, \tilde{L}_c, \tilde{L}_d)$  and take  $L_c = P^{-1} \tilde{L}_c$  and  $L_d = P^{-1} \tilde{L}_d$ . This has been done in [29, Example 3.3] for a bouncing ball with a restitution coefficient  $\lambda < 1$ , and position  $\tau_k$  measured at all (hybrid) times.

**Remark 3.4.** In the favorable case where both the flow and jump dynamics of  $\mathcal{H}$  are detectable, it is not sufficient to choose independently a map  $F$  as a continuous-time observer of the flow and a map  $G$  as a discrete-time observer of the jumps. Indeed, their “contraction directions” could be incompatible: jumps could destroy what has been achieved during flow, or vice versa. For instance, with linear flow/jump/output maps, it is not enough to choose  $L_c, L_d$  such that  $A_c - L_c H_c$  is Hurwitz and  $A_d - L_d H_d$  is Schur. Actually, a necessary condition for convergence of the observer is that the error sampled at each jump converges to zero: this implies that the origin of the discrete system

$$\varepsilon_{k+1} = (A_d - L_d H_d) \exp((A_c - L_c H_c) \tau_k) \varepsilon_k$$

has to be asymptotically stable for  $k \mapsto \tau_k \in \mathcal{I}$ . If  $\tau_k = \tau^* \in \mathcal{I}$  is constant, this is not verified for every choice of  $A_d - L_d H_d$  Schur and  $A_c - L_c H_c$  Hurwitz, as illustrated in Figure 1:  $(A_d - L_d H_d) \exp((A_c - L_c H_c) \tau)$  is Schur only if  $\tau^* \notin [0.1, 2]$ . To avoid this phenomenon, (11b) and (11c) should hold with the same Lyapunov function  $V$ .

A drawback of Theorem 3.1 is that it mixes constraints on the observer flow and jump maps which cannot be designed separately. This coupling appears through  $a_c$  and  $a_d$  in conditions (C) and it is not clear how it can be solved in the general nonlinear context. Even in the linear case as in Example 3.3, the conditions are nonlinear, unless both  $a_c$  and  $a_d$  can be taken negative and (15) can be solved. In Sections 4 and 5, we show how this loop can be broken by using innovation only in flow or only at jumps, through high-gain in the former and by considering an equivalent discrete-time system for the latter.

Another drawback is that Theorem 6.4 requires at least  $a_c$  or  $a_d$  to be negative. Therefore, either the continuous or the discrete dynamics of  $\mathcal{H}$  has to admit an observer and thus be detectable. But it could happen that neither the continuous nor the discrete dynamics are observable, and yet the system as a whole is. An application featuring an hybrid system with such a property is given in Section 7.2. Another example is

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = 0 \\ \dot{x}_3 = 0 \end{cases}, \quad \begin{cases} x_1^+ = x_1 \\ x_2^+ = x_2 \\ x_3^+ = x_1 \end{cases} \quad (16)$$

with some arbitrary, but nonempty flow and jump sets. Suppose  $h_c(x) = h_d(x) = x_1$ . Neither the continuous nor the discrete dynamics is detectable, so Theorem 3.1 cannot apply. Nevertheless, this hybrid system as a whole is determinable if there is at least one interval of flow (providing  $x_2$ ) and one jump (providing  $x_3$ ). Actually, Section 5 will show that we should rather study an equivalent discrete-time system, containing both the continuous and discrete dynamics and providing insight for observer design.

#### 4. Flow-based Hybrid Observer

When the continuous dynamics of  $\mathcal{H}$  are detectable and *persistent* in the sense of an average dwell-time, it is tempting to use a continuous-time observer

$$\dot{z} = F(z, h_c(x)) \quad , \quad \hat{x} = \mathcal{T}(z) \quad (17)$$

as the observer's continuous dynamics  $F$ , and simply copy the discrete dynamics of  $\mathcal{H}$  in  $G$ . Indeed, intuitively, if the estimation error decreases more during flow than it increases at jumps, namely if the continuous-time observer (17) is sufficiently fast, the error is expected to converge to zero asymptotically. We thus need persistence of flow, namely conditions of the type  $\mathcal{C}_{\mathcal{X}_0}[\mathcal{I}]$  with  $\min \mathcal{I} > 0$ , or more generally  $\mathcal{C}_{\mathcal{X}_0}^{\text{av}}[\tau^*]$ . In this section, we give conditions under which such a design works.

##### 4.1. Sufficiently Large Average Dwell-time

The first thing to notice is that if the continuous-time observer (17) verifies (11a)-(11b) with  $a_c < 0$  and if  $G$  is chosen such that (11c) holds for some  $a_d \in \mathbb{R}$ , then Problem (O) is solved if the average dwell time (ADT) is sufficiently large to satisfy (C1'). This result is very standard in the literature of switched systems as detailed in Section 7.3. Of course, if  $V$  does not increase at jumps, namely

$$V(\mathcal{G}(x, z)) \leq V(x, z) \quad \forall (x, z) \in D \times \mathbb{R}^{d_z} \quad ,$$

(11c) holds with  $a_d = 0$ . This is related to the notion of *non-expansiveness* of  $V$  for  $g$  in [31]. Actually, if (11a) is strengthened into (12) and if there exists a  $\mathcal{K}^\infty$ -map  $\kappa$  such that

$$|\mathcal{G}(x, z)|_{\mathcal{A}} \leq \kappa(|(x, z)|_{\mathcal{A}}) \quad \forall (x, z) \in D \times \mathbb{R}^{d_z} \quad (18a)$$

$$\bar{\alpha} \circ \kappa \circ \underline{\alpha}^{-1} \leq c \text{Id} \quad (18b)$$

for some positive scalar  $c$ , then (11c) automatically holds with  $a_d = \ln(c)$ . For instance, in the case where  $d_z = d_x$  and  $\mathcal{A}$  is simply the diagonal set (9a), a map  $G$  satisfying (18a) is a simple copy of the plant's jump map  $g$  namely

$$G(z, y_d) = g(z) \quad , \quad (19)$$

if  $g$  is single-valued and  $\kappa$ -continuous, namely

$$|g(x) - g(\hat{x})| \leq \kappa(|x - \hat{x}|) \quad \forall (x, \hat{x}) \in D \times \mathbb{R}^{d_x} \quad . \quad (20)$$

In particular, if  $g$  is Lipschitz with Lipschitz constant  $k_G$  and  $V$  is quadratic with  $\underline{\alpha} = \underline{\lambda}(\cdot)^2$  and  $\bar{\alpha} = \bar{\lambda}(\cdot)^2$ , then Problem (O) is solved if the ADT is larger than  $\frac{1}{|a_c|} \ln \left( \frac{\bar{\lambda}(P)}{\underline{\lambda}(P)} k_G^2 \right)$ . Note that if  $g$  is only locally Lipschitz and any  $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}_0)$  remains in a compact set  $\mathcal{X}$ , it is enough to guarantee (20) on  $(D \cap \mathcal{X}) \times \mathbb{R}^{d_x}$  by taking

$$G(z, y_d) = \text{sat}(g(z)) \quad (21)$$

where  $\text{sat}$  is an appropriate saturation map active outside of  $g(\mathcal{X})$ .

However, apart from switched systems where the switching signal may be a controlled input, the ADT is typically not chosen for a general hybrid systems where the jumps are state-dependent. Therefore, the ADT (if it exists) is a property of the system and cannot be made "sufficiently fast". When the flow/jump/output maps are linear, this problem is overcome in [29, 30] by using (13) with  $L_d = 0$  and  $L_c$  such that there exists  $P > 0$  solution to

$$(A_c - L_c H_c)^\top P + P(A_c - L_c H_c) \leq a_c P \quad (22a)$$

$$A_d^\top P A_d \leq e^{a_d} P \quad (22b)$$

$$a_c \tau^* + a_d < 0 \quad (22c)$$

with  $a_c < 0$ . However existence of a solution to (22) is a priori not guaranteed, and more importantly, this method is not viable for general nonlinear systems.

##### 4.2. Arbitrary Average Dwell-time

Another way to satisfy (C1) or (C1') is to choose a sufficiently fast continuous-time observer (17), i.e. satisfying (11a)-(11b) with  $|a_c|$  sufficiently large. This *tunability* property requires the continuous dynamics to be *instantaneously observable* [37]. However, increasing  $a_c$  may require to change  $V$ , which in turns, modifies  $a_d$ . The following novel corollary shows that this compromise can be achieved under some conditions applying to the so-called "high-gain observers."

**Corollary 4.1.** *Assume  $\mathcal{C}_{\mathcal{X}_0}^{\text{av}}[\tau^*]$  holds for some  $\tau^* > 0$ , and there exist  $\lambda > 0$ ,  $\ell_0 > 0$ , polynomials  $\underline{c}$  and  $\bar{c}$ , a continuous map  $T : \mathbb{R}^{d_x} \rightarrow \mathbb{R}^{d_z}$ , and for all  $\ell > \ell_0$ , maps  $F_\ell : \mathbb{R}^{d_z} \times \mathbb{R}^{d_y} \rightarrow \mathbb{R}^{d_z}$  and  $V_\ell : \mathbb{R}^{d_x} \times \mathbb{R}^{d_z} \rightarrow \mathbb{R}$  such that*

$$\underline{c}(\ell)|z - T(x)|^2 \leq V_\ell(x, z) \leq \bar{c}(\ell)|z - T(x)|^2 \quad \forall (x, z) \in (C \cup D \cup g(D)) \times \mathbb{R}^{d_z} \quad (23a)$$

$$L_{\mathcal{F}_\ell} V_\ell(x, z) \leq -\ell \lambda V_\ell(x, z) \quad \forall (x, z) \in C \times \mathbb{R}^{d_z} \quad (23b)$$

with  $\mathcal{F}_\ell(x, z) = (f(x), F_\ell(z, h_c(x)))$ . Then, there exists  $\ell^* \geq \ell_0 d$  such that for all  $\ell \geq \ell^*$ , Problem (O) is solved for any compact sets  $\mathcal{X}_0 \times \mathcal{Z}_0$ , with  $\mathcal{A}$  defined in (9b),  $F := F_\ell$ , and any map  $G : \mathbb{R}^{d_z} \times \mathbb{R}^{d_{y_d}} \rightarrow \mathbb{R}^{d_z}$ , Lipschitz with respect to  $z$  (uniformly in  $y_d \in h_d(D)$ ), verifying

$$G(T(x), h_d(x)) = T \circ g(x) \quad \forall x \in D. \quad (23c)$$

Note that the subscript  $\ell$  highlights the dependency of  $V_\ell$  and  $\mathcal{F}_\ell$  with respect to the gain  $\ell$  describing the decay rate in (23b), which can be chosen as large as necessary.

*Proof.* First,  $|(x, z)|_{\mathcal{A}} \leq |z - T(x)|$  for all  $(x, z) \in C \cup D \cup g(D)$  and on any compact sets  $\mathcal{X}_0 \times \mathcal{Z}_0$ , there exists a  $\mathcal{K}^\infty$  map  $\alpha$  such that  $|z - T(x)| \leq \alpha(|(x, z)|_{\mathcal{A}})$ . Therefore, (23a) implies (11a) for all  $\ell \geq \ell_0$ . Then, (23b) implies (11b) with  $a_c = -\ell\lambda$ . Then, from the definition of  $\mathcal{G}$  in (8b) and from (23a), for all  $(x, z) \in D \times \mathbb{R}^{d_z}$ ,

$$\begin{aligned} V_\ell(\mathcal{G}(x, z)) &\leq \bar{c}(\ell) |G(z, h_d(x)) - T(g(x))|^2 \\ &\leq \bar{c}(\ell) |G(z, h_d(x)) - G(T(x), h_d(x))|^2 \\ &\leq \bar{c}(\ell) k_G^2 |z - T(x)|^2 \\ &\leq \frac{\bar{c}(\ell)}{\underline{c}(\ell)} k_G^2 V_\ell(x, z) \end{aligned}$$

where  $k_G$  is the Lipschitz constant of  $G$  with respect to  $z$ . Therefore, (11c) holds for all  $\ell \geq \ell_0$  with  $a_d = \ln \left( k_G^2 \frac{\bar{c}(\ell)}{\underline{c}(\ell)} \right)$ . Exploiting exponential growth over polynomial growth,  $-\ell\lambda\tau^* + \ln \left( k_G^2 \frac{\bar{c}(\ell)}{\underline{c}(\ell)} \right) < 0$  for  $\ell$  sufficiently large and (C1') holds.  $\square$

In other words, if we know a *high-gain* continuous-time observer for the continuous dynamics of the plant (2), verifying (23a)-(23b), then a possible hybrid observer is made of this continuous-time observer and a copy of the jump dynamics (written in the high-gain coordinates  $z = T(x)$ , i.e. verifying (23c)), with a gain  $\ell$  sufficiently large compared to the average dwell-time and the Lipschitz constant of the jump dynamics. Compared to [29, 30], this result guarantees the existence of a solution to (22) in the linear context and provides a constructive way to compute it as detailed in Example (4.2). More importantly, the result applies to general nonlinear dynamics whose flow dynamics are strongly differentially observable, such as Lagrangian models with impacts, as detailed in Example 4.3.

**Example 4.2** (Linear observable flow/output maps). Assume  $f(x) = A_c x$  and  $h_c(x) = H_c x$  with the pair  $(A_c, H_c)$  observable. The observer's eigenvalues can then be assigned arbitrarily fast. For that, we define  $\mathcal{V} \in \mathbb{R}^{d_x \times d_x}$  a change of coordinates transforming  $(A_c, H_c)$  into a block-diagonal observable form, namely such that

$$\mathcal{V} A_c \mathcal{V}^{-1} = \mathbf{A} + \mathbf{D} \mathbf{H} \quad , \quad H_c \mathcal{V}^{-1} = \mathbf{H}$$

with  $\mathbf{A} := \text{blkdiag}(A_1, \dots, A_{d_{y_c}})$ ,  $\mathbf{D} :=$

$\text{blkdiag}(D_1, \dots, D_{d_{y_c}})$ ,  $\mathbf{H} := \text{blkdiag}(H_1, \dots, H_{d_{y_c}})$ ,

$$A_i = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & & \\ \vdots & \ddots & \ddots & \\ 0 & & 1 & 0 & 0 \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix} \in \mathbb{R}^{d_i \times d_i}$$

$$H_i = (0 \quad \dots \quad 0 \quad 1) \in \mathbb{R}^{1 \times d_i} ,$$

$D_i \in \mathbb{R}^{d_i \times 1}$ , and  $d_i$  integers such that  $\sum_{i=1}^{d_{y_c}} d_i = d_{y_c}$ . Consider vectors  $L_i$  such that  $A_i - L_i H_i$  is Hurwitz, and for a positive scalar  $\ell$ , define  $\mathcal{L}_i(\ell) := \text{diag}(\ell^{d_i-1}, \dots, \ell, 1)$ . Then, let us take  $F$  defined by (13a) with

$$L_c = \mathcal{V}^{-1} (\mathbf{D} + \ell \mathcal{L}(\ell) \mathbf{L}) \quad (24)$$

where  $\mathbf{L} := \text{blkdiag}(L_1, \dots, L_{d_{y_c}})$ ,  $\mathcal{L} := \text{blkdiag}(\mathcal{L}_1, \dots, \mathcal{L}_{d_{y_c}})$ . We thus have

$$\text{eig}(A_c - L_c H_c) = \ell \text{eig}(\mathbf{A} - \mathbf{L} \mathbf{H}) .$$

Consider a positive definite matrix  $P \in \mathbb{R}^{d_x \times d_x}$  such that

$$(\mathbf{A} - \mathbf{L} \mathbf{H})^\top P + P(\mathbf{A} - \mathbf{L} \mathbf{H}) \leq -\lambda P$$

for some  $\lambda > 0$ . Then, (23a)-(23b) hold with  $T = \text{Id}$ ,

$$V_\ell(x, z) = (x - z)^\top \mathcal{V}^\top \mathcal{L}(\ell)^{-1} P \mathcal{L}(\ell)^{-1} \mathcal{V} (x - z) ,$$

$$\underline{\alpha}(\cdot) = \lambda (\mathcal{V}^\top P \mathcal{V}) (\cdot)^2 \quad , \quad \bar{\alpha}(\cdot) = \bar{\lambda} (\mathcal{V}^\top P \mathcal{V}) \ell^{2(d-1)} (\cdot)^2$$

$d = \max d_i$ . Therefore, whatever the average dwell-time is, Problem (O) is solved for  $\ell$  sufficiently large by taking  $G$  as in (19) (resp. (21)) if  $g$  is Lipschitz (resp. locally Lipschitz and the solutions  $x$  are bounded).

**Example 4.3** (Strongly differentially observable flow/output maps). Assume that  $f$  and  $g$  are single-valued, with a single output ( $d_{y_c} = 1$ ), and the flow dynamics of  $\mathcal{H}$  are strongly differentially observable of order  $d_z$ , namely the map  $T : \mathbb{R}^{d_x} \rightarrow \mathbb{R}^{d_z}$  defined by

$$T(x) = (h_c(x), L_f h_c(x), \dots, L_f^{d_z-1} h_c(x)) \quad (25)$$

is an injective immersion on  $C \cup D$ . If in addition, there exists a Lipschitz map  $\Phi$  verifying

$$\Phi(T(x)) = L_f^{d_z} h(x) \quad \forall x \in C \cup D ,$$

then a high-gain observer as in [38] can be built for the flow dynamics, with

$$F_\ell(z, y) = A z + B \Phi(z) + \ell \mathcal{L}(\ell) K (y - z_1) ,$$

$$A = \begin{pmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & 1 & \\ \vdots & \ddots & \ddots & \ddots \\ \vdots & & & 0 & 1 \\ 0 & \dots & \dots & 0 & 0 \end{pmatrix} \in \mathbb{R}^{d_z} , \quad B = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$\mathcal{L}(\ell) = \text{diag}(1, \ell, \ell^2, \dots, \ell^{d_z-1})$ , and  $K$  such that  $A - KH$  is Hurwitz with  $H = (1, 0, \dots, 0)$ . Note that if any  $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}_0)$  evolves in a compact set  $\mathcal{X} \subseteq C \cup D$ , there exists a Lipschitz map  $\mathcal{T} : \mathbb{R}^{d_z} \rightarrow \mathbb{R}^{d_x}$  such that

$$\mathcal{T}(\mathcal{T}(x)) = x \quad \forall x \in \mathcal{X},$$

and  $\Phi$  can simply be chosen as  $\Phi = \text{sat} \circ L_f^{d_z} \circ \mathcal{T}$  where  $\text{sat}$  saturates outside of  $L_f^{d_z}(\mathcal{X})$ . Classical high gain computations [38] show that conditions (23a) and (23b) then hold for the Lyapunov function

$$V_\ell(x, z) = (T(x) - z)^\top \mathcal{L}(\ell)^{-1} P \mathcal{L}(\ell)^{-1} (T(x) - z),$$

with  $P$  a positive definite matrix such that

$$(A - KH)^\top P + P(A - KH) \leq -\lambda_0 P$$

for some  $\lambda_0 > 0$ ,  $\underline{c}(\ell) = \underline{\lambda}(P)$ ,  $\bar{c}(\ell) = \bar{\lambda}(P)\ell^{2(d_z-1)}$ ,  $\lambda > 0$  depending on the Lipschitz constant of  $\Phi$ , and  $\ell$  larger than a threshold also depending on that Lipschitz constant. Selecting  $G$  Lipschitz verifying (23c), finally provides an observer relative to  $\mathcal{A}$  defined in (9b), if the gain  $\ell$  is sufficiently large according to Corollary 4.1. In particular, if any  $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}_0)$  evolves in the compact set  $\mathcal{X}$ , we can choose  $G(z, y_d) = \text{sat} \circ T \circ g \circ \mathcal{T}(z)$  where  $\text{sat}$  saturates outside of  $T \circ g(D)$ , and an estimate of  $x$  is obtained by  $\hat{x} = \mathcal{T}(z)$ . This design is illustrated in Sections 7.1 and 7.2. Note that the same tools can be used for multi-output triangular normal forms [39], and if  $z$  estimates only a part  $x_p$  of the state  $x$ , by replacing  $x$  by  $x_p$  everywhere.

## 5. Jump-based Hybrid Observer

We now consider the case where the output is rather used to create contraction of the Lyapunov function at jump times, namely we mostly exploit  $y_d$ . Without natural contraction in the continuous dynamics of  $\mathcal{H}$ , we thus need the jumps to be persistent and sufficiently frequent to inject sufficient information in the observer, i.e., conditions of the type  $\mathcal{C}_{\mathcal{X}_0}^{\text{rav}}[\tau^*]$  or  $\mathcal{C}_{\mathcal{X}_0}[\mathcal{I}]$  with  $\mathcal{I}$  bounded.

### 5.1. Sufficiently Small Reverse Dwell-time

Similarly to the previous section, we can start by noting that when the discrete dynamics of  $\mathcal{H}$  admit a discrete-time observer verifying (11a) and (11c) with  $a_d < 0$ , we may choose  $F$  such that (11b) holds for some  $a_c \in \mathbb{R}$  and Problem (O) will then be solved if  $a_d$  is sufficiently negative with respect to  $a_c$  and the amount of flow, or equivalently if the jumps are sufficiently frequent, i.e. either if  $\max \mathcal{I}$  is sufficiently small to satisfy (C2), or the rADT is sufficiently small to satisfy (C2'). When  $f$  is single-valued and  $\mathcal{A}$  defined as in (9a), one may choose  $F$  single-valued so that

$$|f(x) - F(z, h_c(x))| \leq c|x - z| \quad \forall (x, z) \in C \times \mathbb{R}^{d_z}$$

with some scalar  $c$ . For instance, if  $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}_0)$  evolves in a compact set  $\mathcal{X}$  and  $f$  is locally Lipschitz, one may simply take  $F(z, y_c) = \text{sat}(f(z))$ , where  $\text{sat}$  saturates outside of  $f(\mathcal{X})$ . In other words,  $F$  is simply a flow predictor.

However, again, this method has an interest only when the jumps are naturally sufficiently frequent (Zeno, eventually discrete solutions) or can be made so (switching systems). Otherwise, we need to take explicitly into account the potential increase of  $V$  during flow, to ensure the conditions (11) and (C2') hold simultaneously. When the flow/jump/output maps are linear, one may choose  $F$  and  $G$  as in (13) with  $L_c = 0$  and  $L_d$  such that there exists  $P$  positive definite solution to

$$A_c^\top P + P A_c \leq a_c P \quad (26a)$$

$$(A_d - L_d H_d)^\top P (A_d - L_d H_d) \leq e^{a_d} P \quad (26b)$$

$$a_c \tau^* + a_d < 0 \quad (26c)$$

with  $a_d < 0$  as in [29, 30], where  $\tau^*$  denotes the rADT or the maximal length of flow. In particular, [29, Example 4.2] shows that it can be done analytically for a bouncing ball exhibiting Zeno trajectories, and with the position only measured at jumps. In this case,  $\tau^*$  represents a known bound on the length of the flow intervals, which depends on the compact set of initial conditions. The presence of Zeno solutions is therefore not problematic to the observer design as long as it is properly taken into account through a rADT or with  $0 \in \mathcal{I}$ .

As noticed in [27] in the context of sampled systems ( $A_d = I$ ), this design is extendable to particular classes of nonlinear continuous dynamics for which  $f$  is included in the convex hull of a finite number of linear maps. The LMI (26a) must then hold for each of those maps. Furthermore, [27] shows that (26) might be relaxed by allowing  $P$  and  $L_d$  to depend on the length  $\tau$  of the flow intervals in a way that ensures contraction during both flows and jumps. But this requires the feasibility of some LMIs that are not necessarily related to observability.

In any case, the methods mentioned in this section require the detectability of the discrete dynamics of  $\mathcal{H}$  and a sufficient contraction of the error at jumps. When either the discrete dynamics are not detectable, or the coupling between flows and jumps makes the matrix inequalities not feasible, we show in the next section that we should rather analyze an equivalent discrete-time system made of the plant sampled at the jump times, which naturally contains the information of both flows and jumps.

### 5.2. Arbitrary Reverse Dwell-Time

We now assume the jumps are persistent, i.e.  $\mathcal{C}_{\mathcal{X}_0}[\mathcal{I}]$  holds with  $\mathcal{I}$  compact, but without any constraint on the upper-bound of  $\mathcal{I}$ . We also suppose that absolutely continuous solutions of  $\dot{x} \in f(x)$  are complete and we denote  $\Psi_f$  the flow operator alongside  $f$ , i.e.  $\Psi_f(x_0, \tau)$  denotes the set of points that can be reached at time  $\tau$  by solutions to  $\dot{x} \in f(x)$  initialized at  $x_0$  at  $\tau = 0$ .

Now consider a solution  $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}_0)$  and notice that  $x_k := x(t_k, k)$  sampled after each jump and the output  $y_k := h_d(x(t_k, k-1))$  obtained before each jump verify

$$x_{k+1} \in g(\Psi_f(x_k, \tau_k)) \quad , \quad y_k \in h_d(\Psi_f(x_k, \tau_k)) \quad (27)$$

where  $\tau_k = t_{k+1} - t_k$  denotes the length of the  $k$ th flow interval. It follows that with the discrete output  $y_d$  obtained

at each jump, we are actually observing the equivalent discrete-time system (27). It is therefore the observability/determinability of (27) that counts, and we must look for  $F$  and  $G$  making  $\mathcal{A}$  UpAS for the reduced system

$$\begin{aligned} x_{k+1} &\in g(\Psi_f(x_k, \tau_k)) \\ z_{k+1} &\in G(\Psi_F(z_k, \tau_k), h_d(\Psi_f(x_k, \tau_k))) \end{aligned}$$

or equivalently

$$(x_{k+1}, z_{k+1}) \in \mathcal{G} \circ \Psi_{\mathcal{F}}((x_k, z_k), \tau_k) , \quad (28)$$

with  $\mathcal{F}$  and  $\mathcal{G}$  defined in (8). Indeed, the following theorem shows that it is sufficient to prove UpAS of (28) with sequences  $(\tau_k) \in \mathcal{I}^{\mathbb{N}}$  to solve Problem (O).

**Theorem 5.1.** *Assume that  $\mathcal{C}_{\mathcal{X}_0}[\mathcal{I}]$  holds with  $\mathcal{I}$  compact and solutions to  $(\dot{x}, \dot{z}) \in \mathcal{F}(x, z)$  are complete. Consider a set  $\mathcal{X} \subseteq C \cup D$  such that any  $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}_0)$  remains in  $\mathcal{X}$  at all times. Let  $\tilde{\mathcal{X}}_0 \times \tilde{\mathcal{Z}}_0$  such that*

$$\begin{aligned} \mathcal{G} \circ \Psi_{\mathcal{F}}((x_0, z_0), \tau_0) &\subseteq \tilde{\mathcal{X}}_0 \times \tilde{\mathcal{Z}}_0 \\ \forall (x_0, z_0, \tau_0) &\in \mathcal{X}_0 \times \mathcal{Z}_0 \times [0, \max \mathcal{I}] . \end{aligned}$$

Suppose there exists a class- $\mathcal{KL}$  function  $\beta$  and  $\mathcal{Z} \subseteq \mathbb{R}^{d_z}$  such that any solution  $(x, z)$  to (28) initialized in  $\tilde{\mathcal{X}}_0 \times \tilde{\mathcal{Z}}_0$  and with input  $k \mapsto \tau_k \in \mathcal{I}$  verifies  $(x_k, z_k) \in \mathcal{X} \times \mathcal{Z}$  and

$$|(x_k, z_k)|_{\mathcal{A}} \leq \beta(|(x_0, z_0)|_{\mathcal{A}}, k) \quad \forall k \in \mathbb{N} . \quad (29)$$

Then, Problem (O) is solved if there exists a class- $\mathcal{K}$  function  $\rho$  such that for all  $(x_0, z_0) \in \mathcal{X} \times \mathcal{Z}$  and for all  $\tau \in [0, \max \mathcal{I}]$ ,

$$|\Psi_{\mathcal{F}}((x_0, z_0), \tau)|_{\mathcal{A}} \leq \rho(|(x_0, z_0)|_{\mathcal{A}}) . \quad (30)$$

*Proof.* See Appendix A.  $\square$

The condition (30) guarantees that the distance of  $(x, z)$  to  $\mathcal{A}$  during flow is continuous on the compact interval  $[0, \max \mathcal{I}]$  with respect to the initial distance to  $\mathcal{A}$ . If  $\mathcal{A}$  is defined by (9a) and  $f = F$  is locally Lipschitz, this regularity property is always satisfied when  $\mathcal{X}$  and  $\mathcal{Z}$  are compact.

It is important to note that  $\Psi_f$  and  $\Psi_F$  need not be computed for the implementation of the observer (6): they are only used in the analysis in order to design the maps  $F$  and  $G$  to be used in (6). Although the reduced system (28) may not be handier to use for design than (7), it helps to understand the observability conditions that are at stake here. Besides, when  $f$  is linear, i.e.  $f(x) = A_c x$ , we can choose  $F(z) = A_c z$ , so that

$$\Psi_f(x_k, \tau_k) = \exp(A_c \tau_k) x_k \quad , \quad \Psi_F(z_k, \tau_k) = \exp(A_c \tau_k) z_k$$

and (30) immediately holds for  $\mathcal{A}$  diagonal defined in (9a).

When both  $g$  and  $f$  are linear we obtain the following constructive sufficient condition, that is weaker than (26).

**Corollary 5.2.** *Assume that  $\mathcal{C}_{\mathcal{X}_0}[\mathcal{I}]$  holds with  $\mathcal{I}$  compact and  $f, g, h_d$  are defined by  $f(x) = A_c x$ ,  $g(x) = A_d x$  and  $h_d(x) = H_d x$ . Assume there exist a positive definite matrix  $P \in \mathbb{R}^{d_x \times d_x}$  and a gain vector  $L_d \in \mathbb{R}^{d_x \times d_{y_d}}$  such that*

$$(\exp(A_c \tau))^\top (A_d - L_d H_d)^\top P (A_d - L_d H_d) \exp(A_c \tau) < P \quad \forall \tau \in \mathcal{I} . \quad (31)$$

Then,  $F$  and  $G$  defined in (13) with  $L_c = 0$ , solve Problem (O) with  $\mathcal{A}$  defined in (9a).

*Proof.* Follows from Theorem 5.1 by noticing that the distance to  $\mathcal{A}$  is the norm of the error  $e = z - x$  with dynamics

$$e_{k+1} = (A_d - L_d H_d) \exp(A_c \tau) e_k \quad (32)$$

and using the Lyapunov function  $V(x, z) = (x - z)^\top P (x - z)$ .  $\square$

**Remark 5.3.** *We say that (31) is weaker than (26) because (26) implies (31). Indeed, denoting  $f(\tau) = (\exp(A_c \tau))^\top (A_d - L_d H_d)^\top P (A_d - L_d H_d) \exp(A_c \tau)$ , (26b) implies that  $f(\tau) \leq e^{a_d} (\exp(A_c \tau))^\top P \exp(A_c \tau)$ . Define the function  $f_\varepsilon : \mathbb{R} \rightarrow \mathbb{R}$  by*

$$f_\varepsilon(\tau) = \varepsilon^\top (\exp(A_c \tau))^\top P \exp(A_c \tau) \varepsilon$$

With (26a), we have

$$\begin{aligned} \frac{df_\varepsilon}{d\tau}(\tau) &= e^\top (\exp(A_c \tau))^\top [A_c^\top P + P A_c] \exp(A_c \tau) e \\ &\leq a_c f_\varepsilon(\tau) . \end{aligned}$$

It follows that for all  $\tau \geq 0$ ,  $f_\varepsilon(\tau) \leq e^{a_c \tau} f_\varepsilon(0)$  and since this is valid for all  $e$  in  $\mathbb{R}^n$ , we get

$$(\exp(A_c \tau))^\top P \exp(A_c \tau) \leq e^{a_c \tau} P$$

and  $f(\tau) \leq e^{a_c \tau + a_d} P < P$  with (26c).

The existence of the matrix  $P$  verifying (31) for a given  $\tau$  is equivalent to  $(A_d - L_d H_d) \exp(A_c \tau)$  being Schur for some gain  $L_d$ , which in turn is equivalent to the detectability of the discrete-time system

$$x_{k+1} = A_d \exp(A_c \tau) x_k \quad , \quad y_k = H_d \exp(A_c \tau) x_k . \quad (33)$$

Thus, having (31) for any  $\tau \in \mathcal{I}$  requires detectability of (33) for any  $\tau \in \mathcal{I}$ . It is not sufficient, however, because (31) must be verified with the same  $L_d$  and  $P$  for all  $\tau \in \mathcal{I}$ . So (31) requires in fact the detectability of the LTV or LPV discrete-time system

$$x_{k+1} = A_d \exp(A_c \tau_k) x_k \quad , \quad y_k = H_d \exp(A_c \tau_k) x_k \quad (34)$$

with input  $\tau_k$  in the compact set  $\mathcal{I}$ , which is exactly (27). Actually, (31) is stronger because it requires a quadratic Lyapunov function with a matrix  $P$ , that is independent from the sequence  $k \mapsto \tau_k$ . This property is sometimes called ‘‘quadratic detectability’’ (see [40, 41, 42]).

**Remark 5.4.** By the Schur complement, finding  $P$  and  $L_d$  satisfying (31) is equivalent to finding  $P$  and  $\tilde{L}_d$  satisfying the LMIs

$$\begin{pmatrix} P & \exp(A_c\tau)^\top (PA_d - \tilde{L}_d H_d)^\top \\ \star & P \end{pmatrix} > 0 \quad \forall \tau \in \mathcal{I} \quad (35)$$

with  $\tilde{L}_d = PL_d$ . In the case where  $\mathcal{I}$  has infinitely many elements, an infinite number of LMIs must be solved which is not desirable. However, it is shown in [26] that it is always possible to compute numerically a polytopic decomposition of  $\exp(A_c\tau)$ , namely a finite number of matrices  $\{M_1, M_2, \dots, M_\nu\}$  such that  $\exp(A_c\tau)$  is in the convex hull of those matrices whenever  $\tau \in \mathcal{I}$ . Since (35) is convex in  $\exp(A_c\tau)$ , it is then sufficient to solve the finite number of LMIs

$$\begin{pmatrix} P & M_i^\top (PA_d - \tilde{L}_d H_d)^\top \\ \star & P \end{pmatrix} > 0 \quad \forall i \in \{1, 2, \dots, \nu\} \quad (36)$$

with common  $P$  and  $\tilde{L}_d$ . An example is given in Section 7.1. In particular, if  $A_c$  is nilpotent of order  $N$ , we have  $\exp(A_c\tau) = \sum_{k=0}^{N-1} \frac{\tau^k}{k!} A_c^k$  so that for all  $\tau$  in a compact subset  $\mathcal{I}$  of  $\mathbb{R}_{\geq 0}$ ,  $\exp(A_c\tau)$  is in the convex hull of the  $\nu = 2^{N-1}$  matrices  $\left\{ I + \sum_{k=1}^{N-1} \frac{\tau^k}{k!} A_c^k \right\}$  with  $\tau_k \in \{\min \mathcal{I}, \max \mathcal{I}\}$  for all  $k$ . See Section 7.2.

**Example 5.5.** Consider the system (16) where  $A_c = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ ,  $A_d = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ , and  $H_d = (1 \ 0 \ 0)$ . Neither the continuous pair  $(A_c, H_c)$  nor the discrete pair  $(A_d, H_d)$  is detectable, so Theorem 3.1 cannot be used. However, the pair  $A(\tau) := A_d \exp(A_c\tau) = \begin{pmatrix} 1 & \tau & 0 \\ 0 & 1 & 0 \\ 1 & \tau & 0 \end{pmatrix}$ ,  $H(\tau) := H_d \exp(A_c\tau) = (1 \ \tau \ 0)$  is detectable for any nonzero  $\tau$ . Since  $A_c$  is nilpotent of order 2, according to Remark 5.4, for any  $\mathcal{I}$  compact subset of  $\mathbb{R}_{> 0}$ , it is enough to solve the two LMIs given by

$$\begin{pmatrix} P & (I + \tau A_c)^\top (PA_d - \tilde{L}C)^\top \\ \star & P \end{pmatrix} > 0 \quad (37)$$

for  $\tau = \tau_m := \min \mathcal{I} > 0$  and  $\tau = \tau_M := \max \mathcal{I}$ . If they are solvable (with a common  $P$ ), then by Corollary 5.2, we obtain an observer. For instance, when choosing  $\tau_m = 2$  and  $\tau_M = 5$  and solving the LMIs via Yalmip for  $P$  and  $\tilde{L}_d$ , we get  $L_d = P\tilde{L}_d = (1, 0.2259, 1)^\top$ .  $\triangle$

What makes the above work is the fact that the flow operator of the error  $e = \hat{x} - x$ , is explicitable through  $\exp(A_c\tau)e$  and is contained in the convex hull of a finite number of linear maps. In the context of sampled nonlinear systems, [43, 23] noticed that by copying the continuous dynamics in the observer, namely taking  $F = f$ , the error components evolve during flow according to

$$\dot{e}_i = f_i(\hat{x}) - f_i(x) = \frac{df_i}{dx}(v(t))e$$

for some  $v$  depending on  $x$  and  $\hat{x}$ , thanks to the mean value theorem. For certain classes of maps  $f$  [23], the error

reachable set within a time  $\tau \in \mathcal{I}$  may then be included in the convex hull of a finite number of linear maps  $\{e \mapsto M_i e\}_{i=1, \dots, \nu}$  if the Jacobian components of  $f$  are bounded. If  $g$  is linear, the discrete error system (32) is then replaced by

$$e_{k+1} = \sum_{i=1}^{\nu} \beta_{i,k} (A_d - L_d H_d) M_i e_k$$

with  $\sum_{i=1}^{\nu} \beta_{i,k} = 1$ , and following the same steps as in [43] with the Lyapunov function of Corollary 5.2, it is enough to ensure (31) with  $\exp(A_c\tau)$  replaced by  $M_i$ , for each  $i \in \{1, 2, \dots, \nu\}$ , namely solve the LMIs (36).

The advantage of using a constant gain  $L_d$  is that it is sufficient to compute only once the vertices  $M_i$  of the polytopic decomposition of the flow operator for  $\tau \in \mathcal{I}$  and solve offline the finite number of LMIs (36). However, as mentioned above, those LMIs might not be solvable since they require a stronger property than detectability of (34). In that case, we may allow  $L_d$  to be time-varying, by adapting  $L_d$  to  $\tau_k$ , as done in the particular case of sampled-data observers in [28]. Indeed, observe that the observer jump map  $G$  in (28) is applied after flowing  $\tau_k$  units of time with  $F$ . Therefore, at the moment where  $G$  is used,  $\tau_k$  represents the time elapsed since the previous jump and is known to the observer. This can be modeled in this framework with a timer  $\tau$  added to the observer state, where  $\tau$  flows according to  $\dot{\tau} = 1$  and jumps according to  $\tau^+ = 0$ . It follows that at each jump, the gain  $L_d$  in the jump map  $G$  defined in Corollary 5.2 can be adapted to the length of the previous interval of flow, in a way that makes

$$\hat{x}_{k+1} = A(\tau_k) \hat{x}_k + L_{d,k} (y_k - H(\tau_k) \hat{x}_k) \quad (38)$$

an observer for (34), where

$$A(\tau_k) = A_d \exp(A_c \tau_k) \quad , \quad H(\tau_k) = H_d \exp(A_c \tau_k) \quad .$$

Since  $H$  is not constant, we cannot use the results obtained for LPV systems ([41]). However, an even simpler approach is to consider (34) as a LTV system and design  $L_{d,k}$  as the gain of a discrete Kalman filter. More precisely, we use an observer with state  $z = (\hat{x}, \tau, K, P)$ , flow dynamics  $F(z) = (A_c \hat{x}, 1, 0, 0)$  and jump dynamics

$$G(z, y_d) = \begin{pmatrix} A_d \hat{x} + A(\tau)K(y_d - H_d \hat{x}) \\ 0 \\ \chi(P, \tau) \\ (I - \chi(P, \tau)H(\tau))p(P, \tau) \end{pmatrix} \quad (39)$$

where the maps  $p$  and  $\chi$  are defined by

$$p(P, \tau) = A(\tau)PA(\tau)^\top + Q$$

$$\chi(P, \tau) = p(P, \tau)H(\tau)^\top \left( H(\tau)p(P, \tau)H(\tau)^\top + R \right)^{-1}$$

It is important to note that the innovation of  $\hat{x}_{k+1}$  in (38) must be with  $y_k$ , instead of  $y_{k+1}$  in a standard Kalman filter. That is why we use in (39) a Kalman filter with prediction after innovation, where the gain writes  $L_d = A(\tau)K$  with  $K$  the Kalman gain computed at the previous jump. In the same spirit, if (34) is known to be observable after  $N$  jumps, [19] proposed to compute  $L_{d,k}$  based on the weighted observability Grammian over the past  $N$  jumps.

## 6. Robustness with Respect to Delays in Jumps

We now study how the observer convergence is impacted if the observer jumps are delayed with respect to the plant's, thus leading to a mismatch between the observer jump times and those of the plant. For this, we start from the following assumption.

**Assumption 6.1.**  $\mathcal{C}_{\mathcal{X}_0}[\mathcal{I}]$  holds with  $\mathcal{I}$  compact and  $\min \mathcal{I} > 0$  (dwell-time), and Problem (O) has been solved, namely the set  $\mathcal{A}$  is UpAS for  $\mathcal{H} - \hat{\mathcal{H}}$  with basin of attraction including  $\mathcal{X}_0 \times \mathcal{Z}_0$ .

We choose to study the particular case where the value of the innovation term, implemented in the observer at the delayed jump, is the one that would have been computed at the actual plant's jump time if there had been no delay. This covers the situations where the measurement and computation of the innovation  $G(z, y_d)$  are instantaneous, but the implementation of the jump in the observer is delayed; or the measurement takes a known amount of time  $\delta \geq 0$  to arrive to the observer, and the update of  $z$  is chosen as  $G(z(t - \delta), y_d)$ , thanks to a buffer in  $z$  or by backward integration of  $z$ . Inspired from [44], for any delay  $\Delta \in [0, \min \mathcal{I}]$ , this situation can be modeled as

$$\hat{\mathcal{H}}(\Delta) \left\{ \begin{array}{l} \dot{x} = f(x) \\ \dot{z} = F(z, h_c(x)) \\ \dot{\mu} = 0 \\ \dot{\tau}_\delta = -\min\{\tau_\delta + 1, 1\} \end{array} \right\} \mathbf{x} \in \hat{C}(\Delta)$$

$$\hat{\mathcal{H}}(\Delta) \left\{ \begin{array}{l} x^+ = g(x) \\ z^+ = z \\ \mu^+ = G(z, h_d(x)) \\ \tau_\delta^+ \in [0, \Delta] \end{array} \right\} \mathbf{x} \in \hat{D}_{-1}(\Delta) \quad (40)$$

$$\hat{\mathcal{H}}(\Delta) \left\{ \begin{array}{l} x^+ = x, \quad \mu^+ = 0 \\ z^+ = \mu, \quad \tau_\delta^+ = -1, \end{array} \right\} \mathbf{x} \in \hat{D}_0(\Delta)$$

with state  $\mathbf{x} = (x, z, \mu, \tau_\delta)$ , flow set

$$\hat{C}(\Delta) = \left( \hat{C} \times \{0\} \times \{-1\} \right) \cup \left( \hat{C} \times \mathbb{R}^{d_z} \times [0, \Delta] \right),$$

jump set  $\hat{D}_{-1}(\Delta) \cup \hat{D}_0(\Delta)$  with

$$\hat{D}_{-1}(\Delta) = \hat{D} \times \{0\} \times \{-1\}$$

$$\hat{D}_0(\Delta) = (\hat{C} \cup \hat{D}) \times \mathbb{R}^{d_z} \times \{0\},$$

$$\hat{C} := C \times \mathbb{R}^{d_z}, \quad \hat{D} := D \times \mathbb{R}^{d_z}.$$

$\hat{\mathcal{H}}(\Delta)$  contains two new states  $\mu$  and  $\tau_\delta$  evolving in  $\mathbb{R}^{d_z}$  and  $[0, \Delta] \cup \{-1\}$  respectively. The state  $\tau_\delta$  is a timer modeling the delay between the plant's jump and the observer's jump. The role of  $\mu$  is to store the update to be implemented in the observer at the end of the delay interval, when it actually jumps. More precisely, when  $\tau_\delta = -1$  and  $x$  does not jump,  $\hat{\mathcal{H}}(\Delta)$  flows, with  $(x, z)$  flowing according to  $\mathcal{F}$  as in  $\mathcal{H} - \hat{\mathcal{H}}$ , while  $\mu$  and  $\tau_\delta$  remain equal to 0 and  $-1$  respectively. When the plant's state  $x$  jumps,

then the update  $G(z, h_d(x))$  that should have been instantaneously implemented in the observer state  $z$  is stored in the memory state  $\mu$ , and  $\tau_\delta$  is set to a number in  $[0, \Delta]$  thus starting a delay period:  $\hat{\mathcal{H}}(\Delta)$  then flows and the time  $\tau_\delta$  decreases, until it reaches 0. At this point, a delay interval of length smaller than or equal to  $\Delta$  has elapsed, and the observer state  $z$  is updated with the content of  $\mu$ , while  $\mu$  is reset to 0 and  $\tau_\delta$  switched back to  $-1$ .

Note that the plant's state  $x$  is not allowed to jump again before the delay expressed by  $\tau_\delta$  has expired. That is why this model only works in the case where  $\Delta < \min \mathcal{I}$ , i.e., the maximal delay is smaller than the smallest possible time between successive jumps of the plant.

In order to study the robustness of this property in presence of delay, we need to resort to compact attractors and some regularity properties of  $\mathcal{H} - \hat{\mathcal{H}}$ .

**Assumption 6.2.** There exists a compact subset  $\mathcal{X}$  of  $C \cup D$ , such that any solution  $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}_0)$  verifies  $x(t, j) \in \mathcal{X}$  for all  $(t, j) \in \text{dom } x$ . Besides,  $\mathcal{A}_{\mathcal{X}} := \mathcal{A} \cap (\mathcal{X} \times \mathbb{R}^{d_z})$  is compact.

**Assumption 6.3.**  $\mathcal{H} - \hat{\mathcal{H}}$  defined in (7) satisfies the hybrid basic conditions defined in [1, Assumption 6.5], namely  $C$  and  $D$  are closed,  $\mathcal{F}|_{\hat{C}}$  and  $\mathcal{G}|_{\hat{D}}$  are outer semicontinuous and locally bounded, and  $\mathcal{F}|_{\hat{C}}$  takes convex values.

It follows that the plant solutions of interest are also solution to (2) with flow set  $C \cap \mathcal{X}$  and jump set  $D \cap \mathcal{X}$ , which are compact. The assumption that  $\mathcal{A}_{\mathcal{X}}$  is compact, is typically satisfied whenever

$$(x, z) \in \mathcal{A} \iff z = T(x)$$

for some continuous map  $T : \mathcal{O} \rightarrow \mathbb{R}^{d_z}$  with  $\mathcal{X} \subseteq \mathcal{O}$ , namely in all the examples considered above.

Let us define the set

$$\mathcal{A}' = (\mathcal{A}_{\mathcal{X}} \times \{0\} \times \{-1\}) \cup (\hat{G} \times \{0\})$$

$$\hat{G} := \{(g(x), z, G(z, h_d(x))) : x \in D, (x, z) \in \mathcal{A}_{\mathcal{X}}\}.$$

**Theorem 6.4.** Suppose Assumptions 6.1, 6.2 and 6.3 hold. Then, the set  $\mathcal{A}'$  is UpAS for  $\hat{\mathcal{H}}(0)$  with basin of attraction containing  $\mathcal{X}_0 \times \mathcal{Z}_0 \times \{0\} \times \{-1\}$ . Besides, there exist a  $\mathcal{KL}$ -map  $\beta$ , scalars  $t^* \geq 0$  and  $j^* \in \mathbb{N}$ , and for any  $\epsilon > 0$ , there exists  $\Delta^* > 0$ , such that any solution  $\phi = (x, z, \mu, \tau_\delta)$  to  $\hat{\mathcal{H}}(\Delta)$  with  $\Delta < \Delta^*$  and initialized in  $\mathcal{X}_0 \times \mathcal{Z}_0 \times \{0\} \times \{-1\}$  verifies

$$|\phi(t, j)|_{\mathcal{A}'} \leq \beta(|\phi(0, 0)|_{\mathcal{A}'}, t + j) + \epsilon, \quad (41)$$

and  $\text{dom } \phi = \mathcal{D}_{-1} \cup \mathcal{D}_0$  with  $\mathcal{D}_k = \left( \bigcup_{j \in J_k} [t_j, t_{j+1}] \times \{j\} \right)$ ,  $k \in \{0, -1\}$ ,

$$J_{-1} = \{j \in \mathbb{N} : \tau_\delta(t, j) = -1 \quad \forall t \in [t_j, t_{j+1}]\}$$

$$J_0 = \{j \in \mathbb{N} : \tau_\delta(t, j) \in [0, \Delta] \quad \forall t \in [t_j, t_{j+1}]\},$$

such that

- for all  $(t, j) \geq (t^*, j^*)$  in  $\mathcal{D}_{-1}$ ,  $|(x, z)(t, j)|_{\mathcal{A}} \leq 2\epsilon$ ,
- for all  $j$  in  $J_0$ ,  $t_{j+1} - t_j \leq \Delta$ .

*Proof.* See Appendix B.  $\square$

In other words, we achieve semi-global practical stability of  $\mathcal{A}$  except possibly on the delay intervals. More precisely, for any  $\epsilon > 0$ , there exists a maximal delay  $\Delta^*$  between the plant's and observer's jumps, such that the distance of  $(x, z)$  to  $\mathcal{A}$  is asymptotically smaller than  $2\epsilon$ , except possibly during the delay intervals in-between the plant's and observer's jump of length smaller than  $\Delta^*$ . This is illustrated in Section 7.2.

In fact, if  $\mathcal{A}$  is the diagonal set (9a), the mismatch during the delay intervals cannot be prevented if the jump map is not the identity. Indeed, after one jump of either  $x$  or  $z$ , one is close to  $x^-$  while the other is in  $g(x^-)$ , no matter how long the delay is. This well-known phenomenon, called *peaking*, was reported in the context of observation [3], but also more generally output-feedback and tracking [45]. This suggests that the Euclidian distance to evaluate the observer error is not appropriate and more general distances could be designed [46]. In particular, the expression of  $\mathcal{A}$  shows that semi-global practical stability is actually ensured for the peaking free set

$$\tilde{\mathcal{A}} = \mathcal{A} \cup \{(x, z) \in \mathbb{R}^{d_x} \times \mathbb{R}^{d_z} : x \in g(x^-), (x^-, z) \in \mathcal{A}\}.$$

Note that in the limit case where  $\min \mathcal{I} = 0$ , namely Zeno or multiple jumps could happen, then an arbitrarily small delay in the observer jumps could lead to several jumps of delay, namely, one would need to consider

$$\tilde{\mathcal{A}} = \mathcal{A} \cup \{(x, z) \in \mathbb{R}^{d_x} \times \mathbb{R}^{d_z} : x \in g^{k^*}(x^-), (x^-, z) \in \mathcal{A}, g^k(x^-) \cap D \neq \emptyset \quad \forall k \in \{1, \dots, k^* - 1\}\}.$$

In the case of an average dwell-time,  $k^*$  would be limited by  $N_0$ .

## 7. Applications

The results in the previous sections are exercised in applications. Section 7.1 introduces a model that covers a class of mechanical systems with impacts, including juggling systems and walking robots; see [47] and [48], and the references therein. Section 7.2 presents a second application that pertains to a parameterized model capturing the dynamics exhibited by a wide range of cortical neurons. This model, introduced in [49], has been widely used by the neuroscience community due to its capabilities of reproducing a variety of spiking and bursting behaviors by properly choosing its parameters. Finally, in Section 7.3, the results are applied to the design of observers for the general class of switched systems defined in Example 2.3.

### 7.1. Mechanical system with impacts

Consider a system evolving according to

$$\begin{cases} \dot{\theta} = \omega \\ \dot{\omega} = \alpha(\theta, \omega) \end{cases} \quad \begin{cases} \theta^+ = g_\theta(\theta, \omega) \\ \omega^+ = g_\omega(\theta, \omega) \end{cases}$$

with  $x = (\theta, \omega) \in \mathbb{R}^d \times \mathbb{R}^d$  the (bounded) positions and velocities,  $\alpha$ ,  $g_\theta$  and  $g_\omega$  locally Lipschitz functions, the position  $y = \theta$  measured and jumps occurring at the impacts of  $\theta$  on a surface  $\mathcal{W}$ , typically modeled by a jump set of the form

$$D = \{(\theta, \omega) \in \mathbb{R}^d \times \mathbb{R}^d, \theta \in \mathcal{W}, \langle \omega, \nabla_{\mathcal{W}} \rangle \leq 0\}$$

where the second condition ensures the velocity is pointing inwards  $\mathcal{W}$ . The flow dynamics are clearly strongly differentially observable of order  $d_z = 2$ , since  $(y, \dot{y}) = x$  defines an injective immersion (with  $T$  simply the identity map). Therefore, if the impacts are detected (for instance through force sensors) and are known to have an average dwell-time, then an observer is simply given by

$$\begin{cases} \dot{\hat{\theta}} = \hat{\omega} - \ell(\hat{\theta} - y) \\ \dot{\hat{\omega}} = \text{sat}(\alpha(\hat{\theta}, \hat{\omega})) - \ell^2(\hat{\theta} - y) \end{cases} \quad \begin{cases} \hat{\theta}^+ = \text{sat}(g_\theta(\hat{\theta}, \hat{\omega})) \\ \hat{\omega}^+ = \text{sat}(g_\omega(\hat{\theta}, \hat{\omega})) \end{cases}$$

for  $\ell$  sufficiently large,  $\text{sat}$  saturation functions saturating outside the bounds within which  $x$  is known to evolve, and jumps triggered at the detected impacts.

On the other hand, if the mechanical system possibly exhibits Zeno behavior (i.e. with  $T(x) < +\infty$  and  $J(x) = +\infty$ ), for instance due to gravity, a jump-based observer should be used instead. For instance, consider a vertical bouncing ball with

$$\begin{aligned} f(\theta, \omega) &= (\omega, -\rho\omega - \mathbf{g}) \quad , \quad g(\theta, \omega) = (-\theta, -\lambda\omega) \quad (42) \\ C &= \mathbb{R}_{\geq 0} \times \mathbb{R} \quad , \quad D = \{(\theta, \omega) \in \mathbb{R}^2 : \theta = 0, \omega \leq 0\} \end{aligned}$$

with  $\mathbf{g}$  the gravity constant,  $\rho$  the friction coefficient, and  $\lambda < 1$  the impact restitution coefficient. Assume the measurement  $y_d = \theta$  is only available at jumps, namely only impact sensors are used. We know that any maximal solution  $x$  is Zeno. More precisely, the time between two successive jumps  $t_{j+1}(x) - t_j(x)$  tends to zero when  $j$  tends to  $+\infty$ , and its upper bound increases with  $|x(0, 0)|$ . Hence, for any bounded set of initial conditions  $\mathcal{X}_0$ ,  $\mathcal{C}_{\mathcal{X}_0}[\mathcal{I}]$  holds with  $\mathcal{I}$  of the form  $\mathcal{I} = [0, \tau_M]$ , with  $\tau_M < +\infty$  depending on  $\mathcal{X}_0$ . Since the system is linear, we implement a linear observer (13), with  $L_c = 0$  and  $L_d$  chosen such that (31) holds, where  $A_c = \begin{pmatrix} 0 & 1 \\ 0 & -\rho \end{pmatrix}$ ,  $A_d = \begin{pmatrix} -1 & 0 \\ 0 & -\lambda \end{pmatrix}$  and  $H_d = (1, 0)$ . As in Remark 5.4, we compute a polytopic decomposition of  $\exp(A_c \tau)$  based on the residues of  $A_c$ . Because one eigenvalue of  $A_c$  equals zero and  $\tau_m = 0$ , we obtain that  $\exp(A_c \tau)$  is in the convex hull of only two matrices  $M_1 = I$  and  $M_2 = \begin{pmatrix} 1 & 3.9347 \\ 0 & 0.6065 \end{pmatrix}$  for  $\tau_m = 0$ ,  $\tau_M = 5$ ,  $\lambda = 0.8$ , and  $\rho = 0.1$ . Solving (36) with Yalmip then gives  $L_d = (-1, -0.1085)$ . The result of a simulation with initial condition  $x_0 = (5, 0)$ ,  $\hat{x}_0 = (10, 2)$  is shown on Figure 2. One could also use the hybrid Kalman filter (39).

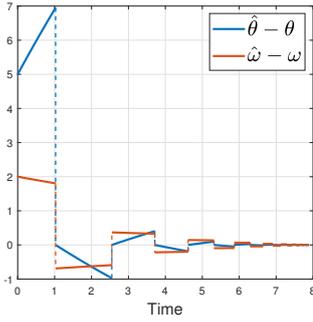


Figure 2: Jump-based estimation of a Zeno bouncing ball (42).

## 7.2. Spiking Neurons

The parameterized model of a spiking neuron in [49] results in a hybrid system  $\mathcal{H}$  as in (2) with state  $(x_1, x_2) \in \mathbb{R}^2$  and data given by

$$\begin{aligned} f(x) &= (0.04x_1^2 + 5x_1 + 140 - x_2 + I_{\text{ext}}, a(bx_1 - x_2)) \\ g(x) &= (c, x_2 + d) \quad , \quad h_c(x) = h_d(x) = x_1 \\ C &= \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq v_m\} \\ D &= \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = v_m\} \end{aligned} \quad (43)$$

where  $x_1$  is the membrane potential,  $x_2$  is the recovery variable, and  $I_{\text{ext}}$  represents the (constant) synaptic current or injected DC current. The value of the input  $I_{\text{ext}}$  and the model parameters  $a$ ,  $b$ ,  $c$ , and  $d$ , as well as the threshold voltage  $v_m$  characterize the neuron type and its firing pattern [50]. The solutions are known to have an average dwell-time (actually a dwell-time), and the jump times can be detected from the discontinuities of the output  $y = x_1$ . According to [49], this hybrid system model combines the biological plausibility of Hodgkin-Huxley-type dynamics and the computational efficiency of integrate-and-fire neurons, making it versatile for the study of the dynamics of a wide class of neurons. We consider two scenarios for estimation of state variables and parameters.

Let us first assume that  $c$  and  $d$  are known and we want to estimate the state  $x_2$  as well as the parameters  $a$  and  $b$ . Adding the constant parameters  $a$  and  $b$  to the state, the nonlinear map  $T$  made of the successive derivatives of  $h$  along  $f$  defined by (25) for  $d_z = 4$  is an injective immersion with respect to  $x = (x_1, x_2, a, b)$  if the matrix  $\begin{pmatrix} x_1 & x_2 \\ \dot{x}_1 & \dot{x}_2 \end{pmatrix}$  is invertible along the plant trajectories. We can therefore use the high-gain design of Example 4.3. The result of a simulation with  $I_{\text{ext}} = 10$ ,  $a = 0.02$ ,  $b = 0.2$ ,  $c = -55$ ,  $d = 4$ ,  $x(0, 0) = (-55, -6, a, b)$ ,  $\hat{x}_0 = (-50, 0, 0.1, 0.1)$ ,  $z(0, 0) = T(\hat{x}_0)$ ,  $\ell = 4$ ,  $K = (3.0777, 4.2361, 3.0777, 1)$  and appropriate saturations is shown on Figure 3.

Let us now consider the case where  $a$ ,  $b$  are known but  $d$  is not. Neither the continuous dynamics nor the discrete dynamics are observable for  $(x_1, x_2, d)$  with output  $x_1$ , so a flow-based observer cannot be used. However,  $x_2$  is observable from the flow and  $d$  impacts  $x_2$  at jumps, so the system as a whole could be observable thanks to the jumps. Actually, we observe that the model is linear in the unknowns  $x_2$  and  $d$ , namely the dynamics of  $(x_1, x_2, d)$  are

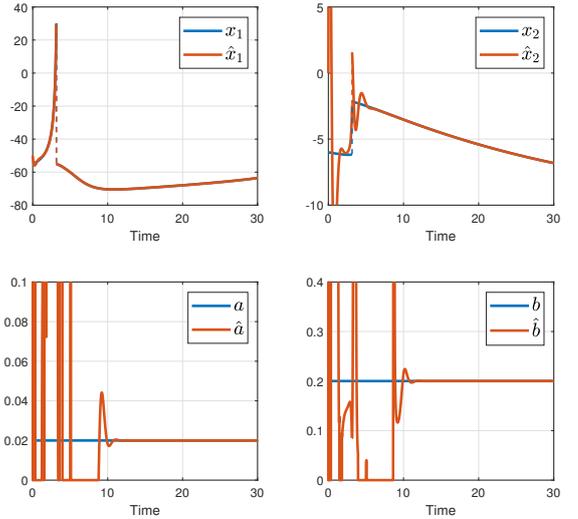


Figure 3: Flow-based estimation of  $(x_1, x_2, a, b)$  in the neuron model (43) through high-gain observer.

characterized by the matrices

$$A_c = \begin{pmatrix} 0 & -1 & 0 \\ 0 & -a & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad , \quad A_d = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad , \quad H_d = (1, 0, 0)$$

modulo output injection. One can check that the equivalent discrete-time model given by (33), namely the pair  $A(\tau) := A_d \exp(A_c \tau) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 - a\tau & 0 \\ 0 & 0 & 1 \end{pmatrix}$ ,  $H(\tau) := H_d \exp(A_c \tau) = (1 - \tau, 0)$  is observable for any nonzero  $\tau$ . Let's say that from the measurements, the flow intervals in-between firing times are known to be within a compact set  $\mathcal{I} = [\tau_m, \tau_M]$  with  $\tau_m > 0$ . Since  $A_c$  is nilpotent of order 2, from Remark 5.4, it is enough to solve the two LMIs given by

$$\begin{pmatrix} P & (I + \tau A_c)^\top (P A_d - \tilde{L}_d H_d)^\top \\ \star & P \end{pmatrix} > 0 \quad (44)$$

for  $\tau = \tau_m$  and  $\tau = \tau_M$ . If they are solvable (with common  $P$  and  $\tilde{L}_d$ ), then with Corollary 5.2, we obtain an observer with state  $z = (\hat{x}_1, \hat{x}_2, \hat{d})$  by taking

$$\begin{aligned} F(\hat{x}_1, \hat{x}_2, \hat{d}, y_c) &= (f(y_c, \hat{x}_2), 0) \\ G(\hat{x}_1, \hat{x}_2, \hat{d}, y_d) &= (c, \hat{x}_2 + \hat{d}, \hat{d}) + L_d(y_d - \hat{x}_1) . \end{aligned}$$

For instance, for  $\tau_m = 30$  and  $\tau_M = 50$ , solving the LMIs via Yalmip for  $P$  and  $\tilde{L}_d$ , we get  $L_d = (0, 0.0028, -0.0063)$ . The results of a simulation are provided on the top and bottom left of Figure 4. Of course, because information from the output is injected at the jumps only, the estimation error takes a longer time to converge than the flow-based observer where instantaneous observability is guaranteed during flow. Note that theoretically, we could have used the estimate  $\hat{x}_1$  instead of the measurement  $y_c$  in the observer flow  $F$  for the linear terms in  $x_1$ , and take it into account in  $A_c$ . However, this does not work well

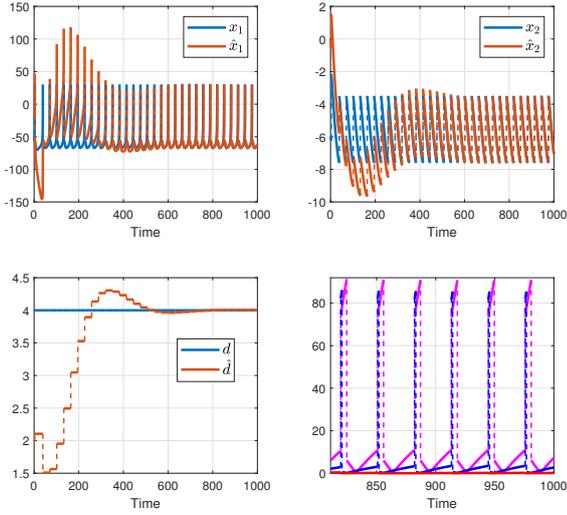


Figure 4: Top and bottom left : jump-based estimation of  $(x_1, x_2, d)$  in the neuron model (43) without delay. Bottom right : norm of the residual estimation error without delay (red), with delay  $\Delta = 1$  (blue), with delay  $\Delta = 5$  (pink).

with the term  $5x_1$  because it produces an error growing in  $e^{5\tau}$  during flow, which reaches  $10^{86}$  for  $\tau = 40$ . In other words, a jump-based observer will only work numerically if the eigenvalues of  $A_c$  are reasonable compared to the length of the flow intervals. Finally, we plot on the bottom right of Figure 4, the norm of the estimation error  $\|(\hat{x}_1, \hat{x}_2, \hat{d}) - (x_1, x_2, d)\|$  in steady state, when the jumps of the observer are triggered with some delay after the plant's state. We observe the error peaking to around 80 during the delay intervals whatever the delay's value ; and then the smaller the delay the smaller the error outside of the delay intervals, as predicted by Theorem 6.4.

### 7.3. Application to Switched Systems

We now show how the results apply to the design of observers for switched systems with state  $x = (x_p, q)$  as defined in (4) of Example 2.3. We focus on the case where the mode  $q$  is known at all times. Besides  $q$ , some additional measurements  $h_q(x_p)$  of the continuous state  $x_p$  are available in each mode, either at all times, namely

$$y_c = y_d = (h_q(x_p), q) \quad (45a)$$

or only at the switching times, namely

$$y_c = q \quad , \quad y_d = (h_q(x_p), q) . \quad (45b)$$

#### 7.3.1. Flow-based observer

We start by assuming the output (45a) is available at all times and the continuous pair  $(f_q, h_q)$  of each mode is detectable for  $q \in Q$ . More precisely, assume that the individual continuous dynamics  $\dot{x}_p = f_q(x_p)$  of each mode  $q \in Q$  with output  $h_q(x_p)$  admit a continuous-time observer

$$\dot{z} = F_q(z, h_q(x_p))$$

relative to some observation set  $\mathcal{A}_q \subset \mathbb{R}^{d_{x_p}} \times \mathbb{R}^{d_z}$  and with a Lyapunov function  $V_q : \mathbb{R}^{d_{x_p}} \times \mathbb{R}^{d_z} \rightarrow \mathbb{R}_{\geq 0}$  that verifies (11a)-(11b), namely

$$\begin{cases} \underline{\alpha}_q (|(x_p, z)|_{\mathcal{A}_q}) \leq V_q(x_p, z) \\ \forall (x_p, z) \in (C_q \cup D_q \cup g_q(D_q)) \times \mathbb{R}^{d_z}, \\ V_q(x_p, z) \leq \bar{\alpha}_q (|(x_p, z)|_{\mathcal{A}_q}) \quad \forall (x_p, z) \in \mathcal{X}_{p,0} \times \mathcal{Z}_0 \end{cases} \quad (46a)$$

$$L_{\mathcal{F}_q} V_q(x_p, z) \leq a_{c,q} V_q(x_p, z) \quad \forall (x_p, z) \in C_q \times \mathbb{R}^{d_z} \quad (46b)$$

for some scalar  $a_{c,q} < 0$  and some class- $\mathcal{K}^\infty$  maps  $\underline{\alpha}_q, \bar{\alpha}_q$ , and where  $\mathcal{F}_q(x_p, z) = (f_q(x_p), F_q(z, h_q(x_p)))$ .

Since we know an observer of each continuous mode, it is tempting to build an observer for the switched system by switching among these individual observers thanks to the knowledge of  $q$ . Because the decrease of the estimation error is brought by flowing in each mode, we need persistence of flow, namely conditions of the type  $\mathcal{C}_{\mathcal{X}_0}[\mathcal{I}]$  with  $\min \mathcal{I} > 0$ , or more generally  $\mathcal{C}_{\mathcal{X}_0}^{\text{av}}[\tau^*]$ . However, it is well-known that switching among asymptotically stable systems does not necessarily bring asymptotic stability, since whatever has been achieved in one mode could be destroyed by the following one, if the descent directions of the Lyapunov functions  $V_q$  are not compatible [32]. We show here how the results of Section 4 shed a new light on this problem.

For that, let us consider

$$\mathcal{A} := \{(x_p, q, z) \in \mathbb{R}^{d_{x_p}} \times Q \times \mathbb{R}^{d_z} : (x_p, z) \in \mathcal{A}_q\} ,$$

which verifies for  $q \in Q$ ,

$$|(x_p, q, z)|_{\mathcal{A}} = 0 \iff |(x_p, z)|_{\mathcal{A}_q} = 0 . \quad (47)$$

Therefore, designing an observer for the hybrid system (4) with state  $x = (x_p, q)$  stabilizing  $\mathcal{A}$  indeed achieves the observation goal of each mode modeled by  $\mathcal{A}_q$ . From (45a), we can define

$$V((x_p, q), z) := V_q(x_p, z) \quad , \quad F(z, y_c) := F_q(z, h_q(x_p)) .$$

Observing that  $|(x_p, q, z)|_{\mathcal{A}} \leq |(x_p, z)|_{\mathcal{A}_q}$ , we deduce from (46) and (47) that conditions (11a)-(11b) hold with  $a_c := \max_{q \in Q} a_{c,q}$ ,  $\mathcal{X}_0 = \mathcal{X}_{p,0} \times Q$ ,  $\underline{\alpha} = \min_{q \in Q} \underline{\alpha}_q$ , and some  $\mathcal{K}^\infty$ -map  $\bar{\alpha}_q$  if  $\mathcal{X}_{p,0} \times \mathcal{Z}_0$  is compact. It thus remains to satisfy (11c) and (C1) or (C1') to apply Theorem 3.1.

#### Common Lyapunov function

The first case considered in the literature is when  $V_q = V_{q'} =: \bar{V}$  for all  $(q, q') \in Q^2$ . In that case, it is sufficient to check (11c) for  $\bar{V}$  independently, namely choose  $G(z, y_d) = G_q(z, h_q(x_p))$  such that

$$\begin{aligned} \bar{V}(\mathcal{G}_q(x_p, z)) &\leq e^{a_d} \bar{V}(x_p, z) \\ \forall (x_p, z) &\in D_q \times \mathbb{R}^{d_z} , \quad \forall q \in Q \end{aligned} \quad (48)$$

with  $\mathcal{G}_q(x_p, z) = (g_q(x_p), G_q(z, h_q(x_p)))$ , and a scalar  $a_d$ . This is in particular satisfied with  $a_d = 0$  if  $\mathcal{G}_q = \text{Id}$ , thus directly satisfying (C1'). Otherwise, as noticed in Section

4, (48) holds if (12) holds for each  $q$  and there exist class- $\mathcal{K}^\infty$  maps  $\kappa_q$  and a positive scalar  $c$  such that

$$\begin{aligned} |\mathcal{G}_q(x_p, z)|_{\mathcal{A}_q} &\leq \kappa_q(|(x_p, z)|_{\mathcal{A}_q}) \quad \forall (x_p, z) \in D_q \times \mathbb{R}^{d_x} \\ \bar{\alpha}_q \circ \kappa_q \circ \underline{\alpha}_q^{-1} &\leq c \text{Id} \quad \forall q \in Q. \end{aligned}$$

(C1') then holds if the average dwell-time is sufficiently large, i.e.  $\tau^* \geq \frac{\ln c}{|a_c|}$ . In particular, this is what is done for linear modes in [18] when  $g_q = \text{Id}$  for all  $q$ , and in [51] (resp. [7]) with linear (resp. affine) jump maps  $g_q$ .

**Example 7.1** (Linear switched systems with detectable modes). Assume  $f_q(x_p) = A_q x_p$ ,  $h_q(x_p) = H_q x_p$ , and there exist gain vectors  $L_q$  and a positive definite matrix  $P$  such that

$$(A_q - L_q H_q)^\top P + P(A_q - L_q H_q) < 0 \quad \forall q \in Q.$$

Then, (46) holds with  $F_q(z, y) = A_q z + L_q(y - H_q z)$ ,  $\mathcal{A}_q$  defined by

$$\begin{aligned} \mathcal{A}_q &= \{(x_p, z) \in (C_q \cup D_q) \times \mathbb{R}^{d_z} : x_p = z\}, \\ \bar{V}(x_p, z) &= (x_p - z)^\top P(x_p - z) \\ \underline{\alpha}_q(\cdot) &= \underline{\lambda}(P)(\cdot)^2, \quad \bar{\alpha}_q(\cdot) = \bar{\lambda}(P)(\cdot)^2. \end{aligned}$$

If  $g_q = \text{Id}$  for all  $q$ , we then take  $G(z, y_d) = z$  and obtain an observer as in [18]. If the maps  $g_q$  are Lipschitz on  $D_q$  with an upper bound on the Lipschitz constants, we simply take a copy of the jump dynamics, namely  $G(z, y_d) = g_q(z)$ , and Problem (O) is solved if the average dwell-time is sufficiently large as used<sup>1</sup> in [51] (resp. [7]) with linear (resp. affine) jump maps  $g_q$ .

Multiple Lyapunov functions In the more general case where the Lyapunov functions associated to each mode are not the same, some additional compatibility conditions appear between  $V_q$  and  $V_{q'}$  to satisfy (11c), namely we need the stronger condition

$$\begin{aligned} V_{q'}(\mathcal{G}_q(x_p, z)) &\leq e^{a_d} V_q(x_p, z) \\ \forall (x_p, z) \in D_q \times \mathbb{R}^{d_z} \quad \forall (q, q') \in Q^2. \end{aligned} \quad (49)$$

However, if condition (48) holds with a scalar  $a_{d,0}$  and we add the assumption that there exists  $\mu$  such that

$$\begin{aligned} V_{q'}(x_p, z) &\leq \mu V_q(x_p, z) \\ \forall (x_p, z) \in (C_q \cup D_q) \times \mathbb{R}^{d_z} \quad \forall (q, q') \in Q^2 \end{aligned} \quad (50)$$

then (49) holds with  $a_d = \ln \mu + a_{d,0}$ . Therefore, Problem (O) is solved if the switching is sufficiently slow *in average*, namely if  $\tau^* > \frac{\ln \mu + a_{d,0}}{|a_c|}$ . This is exactly the result of [52] in the context of linear switched systems and  $\mathcal{G}_q = \text{Id}$ ,  $a_{d,0} = 0$ . Note that (50) always holds if (12) holds for each  $q$  and there exists  $\mu$  such that

$$\bar{\alpha}_{q'} \circ \underline{\alpha}_q^{-1} \leq \mu \text{Id} \quad \forall (q, q') \in Q^2,$$

as for linear systems.

<sup>1</sup>In [51], the more general framework of unknown inputs is considered. Therefore, a first transformation is carried out to isolate the part of  $x_p$  that is not impacted by those unknown inputs.

**Example 7.2** (Linear switched systems with detectable modes). Let us come back to Example 7.1. By detectability of each mode, there exist positive definite matrices  $P_q$  and gain vectors  $L_q$  such that

$$(A_q - L_q H_q)^\top P_q + P_q(A_q - L_q H_q) < 0 \quad \forall q \in Q.$$

Then, (46) holds with same  $F_q$ , same  $\mathcal{A}_q$  and

$$V_q = (x_p - z)^\top P_q(x_p - z)$$

$$\underline{\alpha}_q(\cdot) = \underline{\lambda}(P_q)(\cdot)^2, \quad \bar{\alpha}_q(\cdot) = \bar{\lambda}(P_q)(\cdot)^2.$$

Besides, (50) holds with  $\mu = \max\{\bar{\lambda}(P_{q'})/\underline{\lambda}(P_q)\}$  and Problem (O) is solved for sufficiently slow switching if the maps  $g_q$  are uniformly Lipschitz. Actually a lot of effort has been made in the literature to find less conservative conditions on the switching signal. Recently, [53] exhibited generalized sufficient conditions relying on the framework of slowly-varying linear time-varying systems. The idea is that a switched linear system is stable if the overall variation of the matrix on a long interval is small: one way to achieve this, is to have slow-switching, but this switching can be faster if the variation  $A_q - A_{q'}$  is small enough.

However, as mentioned in Section 4, restrictions on the (average) dwell-time makes sense only if the switching signal is chosen by the user, not if it is a property of the hybrid system.

A first way of avoiding constraints on the (average) dwell-time would be to use the information given by the output at the switches to ensure that  $V$  decreases through the switch. This is done in [54] for linear switched systems where  $y_d = (H_q x_p, q)$  (see (45a)), noticing that when  $g_q = \text{Id}$ , the conditions on  $(V_q, V_{q'})$  can be relaxed by ensuring  $H_q z^+ = H_q x_p = y$  after the jump, through an oblique projection along the metric of  $P_q$  (instead of taking  $z^+ = z$ ), namely

$$G(z, y_d) = z + P_q^{-1} H_q^\top (H_q P_q^{-1} H_q^\top)^{-1} (H_q x_p - H_q z).$$

Indeed, it can be shown via standard geometrical arguments that (48) then holds with  $a_d = 0$  and (49) holds if the additional LMI

$$P_{q'} = P_q + d_{q,q'}^\top H_q + H_q^\top d_{q,q'}$$

is verified for some  $d_{q,q'} \in \mathbb{R}^{d_{x_p} \times d_y}$ .

More generally, the contribution of this paper is to realize that when each mode is observable arbitrarily fast and admits a high-gain observer, we can avoid the restrictions on the average dwell-time by applying Corollary 4.1. We show in the next two examples how this applies to any switched systems with strongly differentially observable continuous modes.

**Example 7.3** (Switched systems with linear observable modes). Assume  $f_q(x_p) = A_q x_p$ ,  $h_q(x_p) = H_q x_p$ , with the pairs  $(A_q, H_q)$  observable. Following Example 4.2, there exist matrices  $\mathcal{V}_q \in \mathbb{R}^{d_{x_p} \times d_{x_p}}$  such that

$$\mathcal{V}_q A_q \mathcal{V}_q^{-1} = \mathbf{A} + \mathbf{D}_q \mathbf{C}, \quad H_q \mathcal{V}_q^{-1} = \mathbf{C}.$$

Then, define

$$F_{q,\ell}(z, y) = A_q z + L_q(\ell)(y - H_q z)$$

with

$$L_q(\ell) = \mathcal{V}_q^{-1}(\mathbf{D}_q + \ell \mathcal{L}(\ell) \mathbf{L})$$

and

$$V_{\ell,q}(x_p, z) = (x_p - z)^\top \mathcal{V}_q^\top \mathcal{L}(\ell)^{-1} P \mathcal{L}(\ell)^{-1} \mathcal{V}_q (x_p - z) .$$

Then, (23a)-(23b) hold with  $T(x_p, q) = x_p$ ,

$$V_\ell((x_p, q), z) = V_{q,\ell}(x_p, z) , \quad F_\ell(z, y) = F_{q,\ell}(z, h_q(x_p))$$

$$\underline{c}(\ell) = \min_{q \in Q} \underline{\lambda}(\mathcal{V}_q^\top P \mathcal{V}_q) , \quad \bar{c}(\ell) = \ell^{2(d_x-1)} \max_{q \in Q} \bar{\lambda}(\mathcal{V}_q^\top P \mathcal{V}_q) .$$

Therefore, whatever the average dwell-time, Problem (O) is solved for  $\ell$  sufficiently large by taking  $G(z, y_d) = g_q(z)$  (resp.  $G(z, y_d) = \text{sat}(g_q(z))$ ) if  $g$  is Lipschitz (resp. locally Lipschitz and the solutions  $x$  are uniformly bounded), according to (19) (resp. (21)).

**Example 7.4** (Switched systems with strongly differentiable modes). Assume that  $f_q$  and  $g_q$  are single-valued with a single output ( $d_y = 1$ ), and there exists  $d_z \in \mathbb{N}$  such that each mode  $q \in Q$  is strongly differentially observable of order  $d_z$ , namely the maps  $T_q : C_q \cup D_q \rightarrow \mathbb{R}^{d_z}$  defined by

$$T_q(x_p) = (h_q(x_p), L_{f_q} h_q(x_p), \dots, L_{f_q}^{d_z-1} h_q(x_p))$$

are injective immersions on  $C_q \cup D_q$ . Then, following Example 4.3 or [38], if there exist Lipschitz maps  $\Phi_q$  with uniformly bounded Lipschitz constants verifying

$$\Phi_q(T_q(x_p)) = L_{f_q}^{d_z} h_q(x_p) \quad \forall x_p \in C_q \cup D_q ,$$

a high-gain observer can be designed for each mode with

$$F_{q,\ell}(z, y) = A z + B \Phi_q(z) + \ell \mathcal{L}(\ell) K (y - z_1) ,$$

and the Lyapunov function

$$V_{q,\ell}(x_p, z) = (T_q(x_p) - z)^\top \mathcal{L}(\ell)^{-1} P \mathcal{L}(\ell)^{-1} (T_q(x_p) - z) ,$$

where  $A, B, K$ , and  $P$  are defined in Example 4.3. Then, (23a),(23b) hold with  $T(x_p, q) = T_q(x_p)$ ,

$$V_\ell((x_p, q), z) = V_{q,\ell}(x_p, z) \quad , \quad F_\ell(z, y) = F_{q,\ell}(z, h_q(x_p)) ,$$

$\underline{c}(\ell) = \underline{\lambda}(P)$ ,  $\bar{c}(\ell) = \bar{\lambda}(P) \ell^{2(d_z-1)}$ , and  $\lambda = \inf_{q \in Q} \lambda_q > 0$ . It remains to choose  $G$  Lipschitz with respect to  $z$  such that (23c) holds to apply Corollary 4.1. But (23c) says we should have at the jumps

$$G(T_q(x_p), y_d) = T_{q^+}(g_q(x_p)) \quad (51)$$

where  $q^+$  is the next mode. In other words, due to the fact that the change of coordinates  $T$  depends on  $q$ , we need to know, before the switch, both the current mode and the next. This is possible only if the switches are controlled or deterministic. Otherwise, we need to wait for the switch

to happen before the observer can jump. This is exactly the issue studied in Section 6. Suppose the assumptions of Theorem 6.4 hold. If the observer jump (51) is implemented instantaneously after the switch (namely the new mode  $q^+$  is detected instantaneously),  $\mathcal{A}_X$  is UpAS, except during the (instantaneous) delay, since this situation can be modeled by  $\hat{H}(0)$ . In the more realistic case where there is a slight delay  $\Delta$  between the switch and the detection of the new mode  $q^+$ , then,  $\mathcal{A}_X$  is practically stable outside the delay intervals.

### 7.3.2. Jump-based observer

We now consider the case where each continuous pair  $(f_q, h_q)$  is not necessarily observable individually, but persistent switching between the modes brings determinability, namely the ability to reconstruct the current continuous state  $x_p$  by accumulating the past information provided by each mode. This case was handled in [55, 17, 56] by 1) using a partial state continuous-time observer during flow, which estimates the part of  $x_p$  that is observable from  $(f_q, h_q)$ ; 2) gathering and propagating forward the partial estimates obtained during the past  $N$  switches to produce a full estimate of  $x_p$ .

However, Section 5 sheds a new light on this problem. Indeed, let us assume that the switches are persistent, namely  $\mathcal{C}_{\mathcal{X}_0}[\mathcal{I}]$  holds with  $\mathcal{I}$  compact, and that the output is available at the switches, namely  $y$  is given by (45b). Consider the switched system discretized at the switching times:

$$x_{p,k+1} \in g_{q_k}(\Psi_{f_{q_k}}(x_k, \tau_k))$$

$$y_k = \left( h_{q_k} \left( \Psi_{f_{q_k}}(x_k, \tau_k) \right), \tau_k \right) \quad (52)$$

where  $\tau_k = t_{k+1} - t_k$  denotes the length of the  $k$ th switching interval. Note that  $\tau_k$  was added to the output to encode the fact that it is known and can be used in the design of the observer jump map  $G$ . Inspired from Corollary 4.1, it is enough to look for  $F$  and  $G$  that make

$$z_{k+1} \in G(\Psi_F(z_k, \tau_k), y_k) \quad (53)$$

a UpAS observer for (52) relative to a set  $\mathcal{A}_p \subset \mathbb{R}^{d_x} \times \mathbb{R}^{d_z}$ , as long as there exists a class- $\mathcal{K}$  function  $\rho$  such that for all  $(x_p, z) \in \mathcal{X}_p \times \mathcal{Z}$ , for all  $q \in Q$ , and for all  $\tau \in [0, \max \mathcal{I}]$ ,

$$|(\Psi_{f_q}(x_p, \tau), \Psi_F(z, \tau))|_{\mathcal{A}_p} \leq \rho(|(x_p, z)|_{\mathcal{A}_p}) , \quad (54)$$

where  $\mathcal{X}_p$  (resp.  $\mathcal{Z}$ ) denotes a set where  $x_p$  (resp. solutions to (53)) evolves. Typically, when  $\mathcal{A}_p$  is the diagonal set, we simply take  $F(z, y_c) = f_q(z)$  and (54) holds for bounded trajectories.

In particular, when  $f_q(x_p) = A_q x_p$ ,  $g_q(x_p) = D_q x_p$ ,  $h_q(x_p) = H_q x_p$  are linear, we take  $z = (\hat{x}_p, \tau)$  and

$$F(z, y_c) = (A_q \hat{x}_p, 1)$$

$$G(z, y_d) = (D_q \hat{x}_p - L(\tau)(H_q z - H_q x_p), 0)$$

where  $L(\tau)$  is chosen based on the known history of  $(q, \tau)$  to make the LTV discrete-time error system

$$e_{k+1} = (D_{q_k} - L(\tau_k) H_{q_k}) \exp(A_{q_k} \tau_k) e_k$$

asymptotically stable, for instance through a discrete Kalman filter as in (39).

## 8. Conclusion

Under the assumption that the plant's jumps can be detected, we have given Lyapunov-based sufficient conditions for asymptotic convergence of an observer for general hybrid systems. Design methods have been provided, in particular high-gain designs for nonlinear differentially observable continuous dynamics, and discrete-based designs when observability is ensured from the output at jump times. Jumps in the observers must be triggered at the same time as the plant's but we have shown their robustness with respect to detection delays, namely semi-global practical stability of the estimation error outside the delay intervals. Those results provide a new insight for the design of observers for switched systems.

However, unlike the flow-based designs which are inherently made for nonlinear dynamics, the nonlinear jump-based designs are limited by the computation of the flow reachable set, as well as the limits already existing for the design of nonlinear discrete-time observers. Future work consists in combining the flow-based and jump-based designs via high-gain ISS interconnections in order to enlarge the class of systems for which those designs are constructive. Besides, further work is needed to evaluate how those observers can be used in the context of output-feedback, as was done on a particular example of biped robot in [57], with precisely the high-gain observer of Example 4.3.

More importantly, observers able to synchronize automatically their jumps with the plant's still need to be developed, at least locally, to avoid relying on the often noisy/delayed jump detection. Indeed, the robust practical stability result of Theorem 6.4 would then enable to combine such local auto-synchronizing observers with the global observers of this paper. This problem represents a significant challenge since the entire analysis needs to be rethought to handle non-simultaneous jumps and ensure contraction of the difference between jump times.

## Appendix A. Proof of Theorem 5.1

Consider  $\phi = (x, z) \in \mathcal{S}_{\mathcal{H}-\hat{\mathcal{H}}}(\mathcal{X}_0 \times \mathcal{Z}_0)$ . By assumption,  $x(t, j) \in \mathcal{X}$  for all  $(t, j) \in \text{dom } \phi$ . Denote  $\tau_M = \max \mathcal{I}$ . It is easy to see that  $(\tilde{x}, \tilde{z}) : \text{dom}_j \phi \rightarrow \mathbb{R}^{d_x} \times \mathbb{R}^{d_z}$  defined by

$$\tilde{x}_k = x(t_k, k) \quad , \quad \tilde{z}_k = z(t_k, k)$$

verifies (28) with input  $\tau$  defined by  $\tau_k = t_{k+1} - t_k$ , for all  $k \in \text{dom}_j \phi \setminus \{J\}$ . Besides,  $(\tilde{x}_1, \tilde{z}_1) \in \tilde{\mathcal{X}}_0 \times \tilde{\mathcal{Z}}_0$  by definition. Also, it follows from  $\mathcal{C}_{\mathcal{X}_0}[\mathcal{I}]$  that  $\tau_0 \leq \tau_M$ , and  $\tau_k \in \mathcal{I}$  for all  $k \in \mathbb{N}_{>0}$  if  $J = +\infty$  and for all  $k \in \{1, \dots, J-1\}$  otherwise. Therefore,  $(x, z)(t_k, k) \in \mathcal{X} \times \mathcal{Z}$  for all  $k$ , and according to (29),(30), for all  $k \in \text{dom}_j \phi \geq 1$ ,

$$\begin{aligned} |(x, z)(t_k, k)|_{\mathcal{A}} &\leq \beta(|(x, z)(t_1, 1)|_{\mathcal{A}}, k-1) \\ &\leq \beta(\rho(|(x, z)(0, 0)|_{\mathcal{A}}), k-1) . \end{aligned}$$

This latter inequality still holds for  $k = 0$ , by appropriately defining  $\beta(s, t)$  for  $t < 0$  so that  $\beta(s, -1) \geq \rho^{-1}(s)$ . Besides, we deduce that for all  $j \in \text{dom}_j \phi$ ,  $(x, z)(t_j, j) \in \mathcal{X} \times \mathcal{Z}$ . By  $\mathcal{C}_{\mathcal{X}_0}[\mathcal{I}]$ ,  $t - t_j \in [0, \tau_M]$  for all  $(t, j) \in \text{dom } \phi$ , so that from (30),

$$\begin{aligned} |(x, z)(t, j)|_{\mathcal{A}} &\leq \rho(|(x, z)(t_j, j)|_{\mathcal{A}}) \\ &\leq \rho(\beta(\rho(|(x, z)(0, 0)|_{\mathcal{A}}), j-1)) \end{aligned}$$

But for all  $(t, j)$  in  $\text{dom } \phi$ ,  $t - t_j \leq \tau_M$  and  $t_j - t_{j-1} \leq \tau_M$  for  $j \geq 1$ , so that  $t_j \leq \tau_M j$  and  $t \leq \tau_M(j+1)$ . Thus,

$$|(x, z)(t, j)|_{\mathcal{A}} \leq \rho(\beta(\rho(|(x, z)(0, 0)|_{\mathcal{A}}), a(t+j) + b))$$

with  $a = \frac{1}{\tau_M+1}$  and  $b = \tau_M - 1$ . Therefore, Problem (O) is solved.

## Appendix B. Proof of Theorem 6.4

The proof relies on [44]. Take a solution  $\phi_\delta = (x, z, \mu, \tau_\delta)$  to  $\hat{\mathcal{H}}(\Delta)$  for some  $\Delta \in [0, \min \mathcal{I})$  with  $(x, z)(0, 0) \in \mathcal{X}_0 \times \mathcal{Z}_0$ . Observe that the component  $x$  is not impacted by the delay mechanism, therefore, from Assumption 6.2,  $x(t, j) \in \mathcal{X}$  for all  $(t, j) \in \text{dom } x$ . It follows that  $\phi_\delta$  is solution to a hybrid system  $\hat{\mathcal{H}}_{\mathcal{X}}(\Delta)$  which has same dynamics as  $\hat{\mathcal{H}}(\Delta)$  but with flow set  $\hat{C}_{\mathcal{X}}(\Delta) := \hat{C}(\Delta) \cap (\mathcal{X} \times \mathbb{R}^{d_z+3})$  and jump set  $\hat{D}_{\mathcal{X}}(\Delta) := \hat{D}(\Delta) \cap (\mathcal{X} \times \mathbb{R}^{d_z+3})$ . In the framework of [44],  $\hat{\mathcal{H}}_{\mathcal{X}}(\Delta)$  is then the delayed version of the nominal observer  $\mathcal{H} - \hat{\mathcal{H}}$  with flow set  $\hat{C}_{\mathcal{X}} = (C \cap \mathcal{X}) \times \mathbb{R}^{d_z}$ , and jump set  $\hat{D}_{\mathcal{X}} = (D \cap \mathcal{X}) \times \mathbb{R}^{d_z}$ . By Assumption 6.1 (and by containment [1, Theorem 3.32]), the set  $\mathcal{A}$  is still UpAS for  $\hat{\mathcal{H}}_{\mathcal{X}}$ , and more precisely, the set  $\mathcal{A}_{\mathcal{X}}$  (that is compact according to Assumption 6.2). With the hybrid basic conditions, we conclude from [44, Proposition 4.3, Remark 4.4] that the set  $\mathcal{A}'$  is UpAS for  $\hat{\mathcal{H}}_{\mathcal{X}}(0)$  with basin of attraction containing  $\mathcal{X}_0 \times \mathcal{Z}_0 \times \{0\} \times \{-1\}$ .  $\hat{G}$  is compact by outer-semicontinuity and local boundedness of  $g$  and  $G$ .  $\mathcal{A}'$  is therefore compact. Besides,  $\mathcal{X}_0 \times \mathcal{Z}_0 \times \{0\} \times \{-1\}$  is bounded and thus included in a compact subset of the basin of attraction, since the latter is open according to [1, Proposition 7.4]. Still from the hybrid basic conditions,  $\mathcal{A}'$  is actually semi-globally practically robustly  $\mathcal{KL}$  asymptotically stable for  $\hat{\mathcal{H}}_{\mathcal{X}}(0)$  according to [1, Lemma 7.20]. This means that there exists a  $\mathcal{KL}$  function  $\beta$  such that for any  $\epsilon > 0$ , there exists  $\rho > 0$  such that any solution  $\phi$  to a  $\rho$ -perturbation of  $\hat{\mathcal{H}}_{\mathcal{X}}(0)$  initialized in  $\mathcal{X}_0 \times \mathcal{Z}_0 \times \{0\} \times \{-1\}$ , verifies (41). Since  $\hat{\mathcal{H}}_{\mathcal{X}}(\Delta)$  can be included in any outer-perturbation of  $\hat{\mathcal{H}}_{\mathcal{X}}(0)$  by taking  $\Delta$  sufficiently small, (41) holds along solutions of  $\hat{\mathcal{H}}_{\mathcal{X}}(\Delta)$  for  $\Delta$  sufficiently small. Now for  $\epsilon$  sufficiently small and for sufficiently large  $(t, j) \in \mathcal{D}_{-1}$ ,  $|\phi(t, j)|_{\mathcal{A}'} = |\phi(t, j)|_{\mathcal{A}'_{-1}}$  and thus  $|(x, z)(t, j)|_{\mathcal{A}} \leq |\phi(t, j)|_{\mathcal{A}'} \leq 2\epsilon$ .

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