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Observer Design for Hybrid Dynamical Systems with Approximately Known Jump Times

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Abstract

This paper proposes a general framework for the state estimation of plants modeled as hybrid dynamical systems with discrete events (or jumps) occurring at (approximately) known times. A candidate observer consists of a hybrid dynamical system with jumps that triggered when the plant jumps. With some information about the time elapsed between successive jumps, a Lyapunov-based analysis allows us to derive sufficient conditions for the design of the observer that renders the zero-estimation set uniformly asymptotically stable. The proposed sufficient conditions assure such properties when the observer has innovation terms in both the continuous regime (or flows) and at jumps, only at jumps, and also only during flows. These conditions apply to a large class of hybrid systems, including cases where the time between successive jumps is unbounded or tends to zero – namely, Zeno behavior –, as well as cases where detectability only holds during flows, at jumps, or neither. Building from these sufficient conditions, we study the robustness of this approach when the jumps of the observer are delayed with respect to those of the plant. Under some regularity and dwell-time conditions, we show that the estimation error remains bounded and satisfies a semi-global practical asymptotic stability-like property. The results are illustrated in several examples and applied to switched systems with state-triggered switches.

1. Introduction

1.1. Context

In many applications, estimating the state of a system is crucial, whether it be for control, supervision, or fault diagnosis purposes. Unfortunately, the problem of designing observers for hybrid systems of the form ([21])

$$\dot{x} = f(x) \quad x \in C, \quad x^+ = g(x) \quad x \in D \quad (1)$$

presenting both a continuous-time behavior in C and a discrete-time behavior in D , is unsolved, even when the flow/jump maps f and g are linear. The lack of general tools for such systems is mainly due to the fact that solutions from nearby initial conditions have different (unknown) domains of definition. Indeed, the times at which discrete events occur in the trajectories (jump times) typically depend on their unknown initial condition. Such a mismatch of time domains makes the formulation of observability/detectability and, in turn, observer design very challenging ([9]).

When the jumps of the observer cannot be triggered when the jumps of the plant occur, a standard error system approach for observer design does not apply since the jumps of the observer and of the plant are not necessarily synchronized. Very few observer results exist apart from particular settings as in [19], which requires the composition $g \circ g$ to be the identity map, and in [29], thanks to a change of coordinates transforming the jump map g into the identity map, in this way, removing the jumps. Another problem that is relevant in the context of switched systems is the problem of estimating the switching signal.

The observability properties of such a signal have been studied in [47, 30]. Observer designs based on the so-called *mode location observers*, capable of detecting and identifying properties of the switching signal appeared in [6, 31, 7, 22, 38, 50], to just list a few, which include the broad literature of fault tolerant control.

Impulsive systems consist of a class of continuous-time dynamical systems with state jumps that occur at pre-specified times, which are usually assumed to be separated by nonzero periods of flow – in particular, to avoid Zeno behavior. In that setting, the difficulties due to a possible mismatch of the domains of the solutions disappear since the jump times are assumed to be known. Observability/determinability thus reduce to comparing solutions with same output on the same time domain and have been extensively studied in [23, 49, 35, 51, 46]. As for observer design, results first appeared assuming each mode is observable [2], and then more generally in [36] (resp. in [46]), for impulsive systems (resp. switched impulsive systems) that are observable (resp. determinable) for any impulse time sequence containing more than a known finite number N of jumps.

Another important class of systems with hybrid-like dynamics for which observer results exist is when the system itself has continuous-time dynamics, but the measurements are available intermittently at specific time instances. For such a class of systems with sporadic events, observers have been designed under specific assumptions on the time elapsed between successive events or, in the case of periodic events, the sampling period. From

[44, 14], convergence of an impulsive observer with innovation terms triggered by the measurement events can be guaranteed when the sampling period is sufficiently small. Designs were then developed in [39, 15] for any constant sampling period provided that appropriate matrix inequalities are satisfied, and further extended in [40, 1, 18, 16, 42] to the case of sporadic measurements, i.e., when the time elapsed between sampling events varies in a known interval.

1.2. Content and Contributions

In this paper, we consider general hybrid systems as in [21]. Under the assumption that the jumps of the plant are instantaneously detected, a candidate observer is defined as a hybrid system that jumps at the same time as the plant does, and is fed with the measured output in either the flow map, the jump map, or both (Section 2). Assuming the plant has an average dwell-time or a reverse average dwell-time, or simply that the time between its successive jumps belongs to a known (possibly unbounded) closed set, we derive Lyapunov-based sufficient conditions so as to ensure uniform pre-asymptotic stability of the zero estimation error set (Section 3). Then, we provide additional design conditions in the particular cases where measurements are only available during flow (Section 4) or only at jumps (Section 5). Motivated by the fact that, in practice, it is difficult to instantaneously detect the jumps of the plant, we study the robustness of the observer when the jumps of the observer are slightly delayed relative to those of the plant (Section 6). Finally, we demonstrate how those results can be used for observer design of switched systems with state triggered jumps (Section 7).

Our main contributions compared to the above literature are as follows:

1. General hybrid systems are considered in a unified framework, only assuming knowledge about the time between successive jumps, which allows any type of solutions, from Zeno and eventually discrete, to eventually continuous trajectories;
2. Preliminary results were given in [10, 41], but restricted to hybrid systems with linear f and g , and when at least either the flow dynamics or the jump dynamics are detectable. Here, more general nonlinear sufficient conditions are obtained, even in the case where neither the continuous nor the discrete dynamics of the plant are detectable, but, the hybrid plant as a whole is;
3. When the plant has an average dwell-time and its continuous dynamics are strongly differentially observable, we prove that a hybrid observer can be obtained by copying the plant's discrete dynamics and designing a high-gain observer for its continuous dynamics, as long as the gain is taken sufficiently large compared to the average dwell-time and the Lipschitz constants of the flow and jump maps;
4. When the output measurements are only injected in the observer at jumps, we highlight that the innovation term in the observer, which only plays a role at jumps, should be designed based on an equivalent discrete-time system that models the hybrid plant sampled at jumps;

5. A robustness analysis with respect to delays in the triggering of the jumps of the observer jumps is provided: under some regularity and dwell-time conditions, we show that the estimation error remains bounded and semi-global practical stability holds outside the delay intervals between the plant's and the observer's jumps;
6. The generality of the framework enables us to recover and unify a significant part of the literature. In particular, the results apply well to switched systems with state-triggered switches: we show how a high-gain observer can be designed for switched systems with observable modes and average dwell-time, or how the output at the switching instants can be used when each mode is not observable on its own but the combination of them is.

1.3. Notation and preliminaries

\mathbb{R} (resp. \mathbb{N}) denotes the set of real numbers (resp. integers), $\mathbb{R}_{\geq 0} = [0, +\infty)$, $\mathbb{R}_{> 0} = (0, +\infty)$, and $\mathbb{N}_{> 0} = \mathbb{N} \setminus \{0\}$. For a square matrix P , $\text{eig}(P)$ denotes the set of its eigenvalues, and $\lambda_m(P)$ (resp. $\lambda_M(P)$) stands for its smallest (resp. largest) eigenvalue. The symbol \star in a matrix denotes the symmetric blocks. \mathbb{B} stands for a closed Euclidean ball of appropriate dimension, of radius 1 and centered at 0. A map $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is a \mathcal{K} -map if $\alpha(0) = 0$ and α is continuous and increasing, and a \mathcal{K}^∞ -map if it is also unbounded. For a set valued map $S : \mathbb{R}^{d_x} \rightrightarrows \mathbb{R}$ and a scalar c , writing $S(x) \leq c$ for some $x \in \mathbb{R}^{d_x}$ means that $s \leq c$ for all $s \in S(x)$. For a C^1 map $V : \mathbb{R}^{d_x} \rightarrow \mathbb{R}$, $L_S V(x)$ denotes the set of Lie derivatives along vector fields $s \in S(x)$, i.e. $\{\frac{dV}{dx}(x)s, s \in S(x)\}$. We consider hybrid dynamical systems of the form (1) ([21]) where f (resp. g) is the flow (resp. jump) map, and C (resp. D) is the flow (resp. jump) set. Solutions to such systems are defined on so-called *hybrid time-domains*. A subset E of $\mathbb{R}_{\geq 0} \times \mathbb{N}$ is a *compact hybrid time-domain* if $E = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$ for some finite sequence of times $0 = t_0 \leq t_1 \leq \dots \leq t_J$, and it is a hybrid time domain if for any $(T, J) \in E$, $E \cap [0, T] \times \{0, \dots, J\}$ is a compact hybrid time domain. For a solution $(t, j) \mapsto x(t, j)$ (see [21, Definition 2.6]), we denote $\text{dom } x$ its domain, $\text{dom}_t x$ (resp. $\text{dom}_j x$) its projection on the time (resp. jump) component, $T(x) := \sup \text{dom}_t x$, $J(x) := \sup \text{dom}_j x$, and for a positive integer j , $t_j(x)$ the only time defined by $(t_j, j) \in \text{dom } x$ and $(t_j, j-1) \in \text{dom } x$. When no ambiguity is possible, we will omit x and write T, J, t_j . Besides, $N(t, s)$ denotes the number of jumps occurring between times t and s . We say that x is *complete*, resp. *t-complete*, resp. *j-complete*, if $\text{dom } x$, resp. $\text{dom}_t x$, resp. $\text{dom}_j x$ is unbounded; x is *eventually continuous* (resp. *eventually discrete*) if $J < +\infty$ and $T > t_J$ (resp. $T < +\infty$ and $\text{dom } x \cap (T \times \mathbb{N})$ contains at least two points); x is *Zeno* if it is complete and $T < +\infty$.

2. Synchronized hybrid observer

2.1. Problem statement

We consider a hybrid plant of the form

$$\mathcal{H} \begin{cases} \dot{x} \in f(x), & y_c = h_c(x), & x \in C \\ x^+ \in g(x), & y_d = h_d(x), & x \in D \end{cases} \quad (2)$$

with state $x \in \mathbb{R}^{d_x}$, and output $y = (y_c, y_d) \in \mathbb{R}^{d_{y_c}} \times \mathbb{R}^{d_{y_d}}$, with y_c available during flow and y_d during jumps. For this class of hybrid systems, we are interested in estimating the state of (or part of the state of) \mathcal{H} when its solutions are initialized in a given subset $\mathcal{X}_0 \subseteq C \cup D$. We denote by $\mathcal{S}_{\mathcal{H}}(\mathcal{X}_0)$ the set of maximal solutions of \mathcal{H} with initial condition in \mathcal{X}_0 .

Definition 2.1. For a closed subset \mathcal{I} of $\mathbb{R}_{\geq 0}$ and a positive scalar τ^* , we will say that

- solutions have flow length within \mathcal{I} if for any $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}_0)$,

$$- 0 \leq t - t_j(x) \leq \sup \mathcal{I} \quad \forall (t, j) \in \text{dom } x$$

$$- t_{j+1}(x) - t_j(x) \in \mathcal{I} \text{ holds}$$

$$* \forall j \in \mathbb{N}_{>0} \text{ if } J(x) = +\infty$$

$$* \forall j \in \{1, \dots, J(x) - 1\} \text{ if } J(x) < +\infty.$$

For simplicity, we say that $\mathcal{C}_{\mathcal{X}_0}[\mathcal{I}]$ holds;

- solutions have an average dwell-time (ADT) τ^* if there exists $N_0 \in \mathbb{N}_{>0}$ such that for any $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}_0)$,

$$N(t, s) \leq \frac{(t - s)}{\tau^*} + N_0 \quad \forall t \geq s \geq 0.$$

For simplicity, we say that $\mathcal{C}_{\mathcal{X}_0}^{\text{av}}[\tau^*]$ holds;

- solutions have a reverse average dwell-time (rADT) τ^* if there exists $N_0 \in \mathbb{N}_{>0}$ such that for any $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}_0)$,

$$N(t, s) \geq \frac{(t - s)}{\tau^*} - N_0 \quad \forall t \geq s \geq 0;$$

For simplicity, we say that $\mathcal{C}_{\mathcal{X}_0}^{\text{rav}}[\tau^*]$ holds.

In the definition of $\mathcal{C}_{\mathcal{X}_0}[\mathcal{I}]$, the set \mathcal{I} describes the possible lengths of the flow intervals between successive jumps. The role of the first item is to bound the length of the intervals of flow which are not covered by the second item, namely possibly the first one, which is $[0, t_1]$, and the last one, which is $\text{dom}_t x \cap [t_J, +\infty)$ (when defined).

Properties $\mathcal{C}_{\mathcal{X}_0}^{\text{av}}[\tau^*]$ and $\mathcal{C}_{\mathcal{X}_0}^{\text{rav}}[\tau^*]$ correspond to the standard notions of *average dwell-time* and *reverse average dwell-time* respectively ([21, 26]). They enforce that the solutions jump, on average, at most (resp. at least) once per time interval of length τ^* . A particular case of $\mathcal{C}_{\mathcal{X}_0}^{\text{av}}[\tau^*]$ is when all the intervals of flow last at least τ^* , namely they have a *dwell-time*, which can also be modeled by $\mathcal{C}_{\mathcal{X}_0}[\mathcal{I}]$ with $\mathcal{I} = [\tau^*, +\infty)$.

We are now ready to state the observer problem of interest. Our goal is to design an observer assuming we know: 1) when the plant's jumps occur, 2) the outputs y_c during flows and/or y_d at jumps, 3) some information about the flow time between successive jumps of the type $\mathcal{C}_{\mathcal{X}_0}[\mathcal{I}]$, $\mathcal{C}_{\mathcal{X}_0}^{\text{av}}[\tau^*]$, or $\mathcal{C}_{\mathcal{X}_0}^{\text{rav}}[\tau^*]$. Note that $\mathcal{C}_{\mathcal{X}_0}[\mathbb{R}_{\geq 0}]$ always holds, but as we will see later, it is convenient to have as precise information about the duration of flow between successive jumps as possible.

Example 2.2. Consider a bouncing ball with gravity coefficient $\mathbf{g} > 0$ and restitution coefficient $\lambda > 0$, modeled as \mathcal{H} with

$$\begin{aligned} f(x) &= (x_2, -\mathbf{g}) \quad , \quad g(x) = (-x_1, -\lambda x_2) \\ C &= \mathbb{R}_{\geq 0} \times \mathbb{R} \quad , \quad D = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0, x_2 \leq 0\} \end{aligned} \quad (3)$$

If $\lambda < 1$, any maximal solution x is *Zeno*, i.e., such that $T(x) < +\infty$ and $J(x) = +\infty$. The time between two successive jumps $t_{j+1}(x) - t_j(x)$ tends to zero when j tends to $+\infty$, and its upper bound increases with $|x(0, 0)|$. So, if \mathcal{X}_0 is bounded, $\mathcal{C}_{\mathcal{X}_0}[\mathcal{I}]$ holds with \mathcal{I} of the form $\mathcal{I} = [0, \tau_M]$, with $\tau_M < +\infty$ depending on \mathcal{X}_0 .

If now $\lambda > 1$, any maximal solution initialized in $\mathbb{R}^2 \setminus \{(0, 0)\}$ is such that $T(x) = +\infty$, $J(x) = +\infty$. The time between two successive jumps $t_{j+1} - t_j(x)$ tends to $+\infty$ when j tends to $+\infty$, and its lower bound decreases with $|x(0, 0)|$. Therefore, if there exists $\delta > 0$ such that $\mathcal{X}_0 \subset \mathbb{R}^n \setminus \delta \mathbb{B}$, $\mathcal{C}_{\mathcal{X}_0}[\mathcal{I}]$ holds with \mathcal{I} of the form $\mathcal{I} = [\tau_m, +\infty)$, with $\tau_m > 0$ depending on \mathcal{X}_0 .

Example 2.3 (Switched systems). An important class of hybrid systems are the switched systems of the form (2) with

$$x = \begin{pmatrix} x_p \\ q \end{pmatrix}, \quad f(x) = \begin{pmatrix} f_q(x_p) \\ 0 \end{pmatrix}, \quad g(x) = \begin{pmatrix} g_q(x_p) \\ Q \end{pmatrix} \quad (4)$$

$$C = \bigcup_{q \in Q} C_q \times \{q\} \quad , \quad D = \bigcup_{q \in Q} D_q \times \{q\}$$

where $Q = \{1, \dots, q_{\max}\}$ and the discrete signal q indicates the mode in which the system evolves. When x_p is in D_q and a jump occurs, the mode either stays the same or is “switched” to a new value in Q . The plant then evolves according to the flow map f_q and jump map g_q , until q is switched to another value. Note that a way of forcing the mode to change at each jump is to take $g(x) = \begin{pmatrix} g_q(x_p) \\ Q \setminus \{q\} \end{pmatrix}$. By the way we have written (4), the switches are triggered by the state being in a certain region D_q : it is a state-dependent switching. The switches can also be triggered by an external signal called switching signal, in which case the switches are said time-dependent. This case could also be modeled by (2) by making some assumptions about the time between successive switches, which can take the form of $\mathcal{C}_{\mathcal{X}_0}^{\text{av}}[\tau^*]$, $\mathcal{C}_{\mathcal{X}_0}^{\text{rav}}[\tau^*]$, or $\mathcal{C}_{\mathcal{X}_0}[\mathcal{I}]$. See [32, 21] for more detail. In this paper, we assume the switching times are known or detected. Then, the output map is defined depending on the context and the available information : known or unknown mode q , measurements of x_p , etc. See Section 7.

Example 2.4 (Continuous-time system with sampled measurements). The proposed framework applies also to continuous-time systems

$$\dot{x}_p = f_p(x_p) \quad , \quad y = h_p(x_p)$$

whose output y is only available at discrete times t_j , which do not necessarily occur periodically. Assuming we know bounds on the time elapsed between two successive sampling events, or more generally that it belongs to a closed bounded set \mathcal{I} , namely $\mathcal{C}_{\mathcal{X}_0}[\mathcal{I}]$ holds, such a system can be modeled by \mathcal{H} with state $x = (x_p, \tau)$,

$$\begin{aligned} f(x) &= (f_p(x_p), 1) \quad , \quad g(x) = (x_p, 0) \quad (5) \\ C &= \mathbb{R}^{d_{x_p}} \times [0, \max \mathcal{I}] \quad , \quad D = \mathbb{R}^{d_{x_p}} \times \mathcal{I} \\ h_c(x) &= \emptyset \quad , \quad h_d(x) = (h_p(x_p), \tau) \end{aligned}$$

where τ models the (known) time elapsed since the previous jump. For instance, \mathcal{I} is a singleton in the case of periodic sampling [39, 15], and \mathcal{I} is a compact interval of $\mathbb{R}_{>0}$ in the case of aperiodic sampling with known bounds as considered for linear systems in [18, 42] and classes of nonlinear Lipschitz systems in [40, 1, 17, 34, 16]. Similarly, we could say that $\mathcal{C}_{\mathcal{X}_0}^{\text{av}}[\tau^*]$ holds if we know that measurements occur at most every τ^* units of time and adapt the model (5) accordingly ([10]). \triangle

2.2. Proposed hybrid observer

Since the plant's jump times are assumed to be known, it is natural to use an observer of the form

$$\hat{\mathcal{H}} \begin{cases} \dot{z} \in F(z, y_c) & \text{when } \mathcal{H} \text{ flows} \\ z^+ \in G(z, y_d) & \text{when } \mathcal{H} \text{ jumps} \end{cases} \quad (6)$$

that is synchronized with the plant, for some functions $F : \mathbb{R}^{d_z} \times \mathbb{R}^{d_{y_c}} \rightarrow \mathbb{R}^{d_z}$ and $G : \mathbb{R}^{d_z} \times \mathbb{R}^{d_{y_d}} \rightarrow \mathbb{R}^{d_z}$ to be chosen such that z asymptotically enables to reconstruct the plant state x , or part of it, as formalized next.

Since the plant and the observer jump simultaneously, the observer analysis and design can be carried out on the synchronized cascade system

$$\mathcal{H} - \hat{\mathcal{H}} \begin{cases} \left\{ \begin{array}{l} \dot{x} \in f(x) \\ \dot{z} \in F(z, h_c(x)) \end{array} \right\} (x, z) \in C \times \mathbb{R}^{d_z} \\ \left\{ \begin{array}{l} x^+ \in g(x) \\ z^+ \in G(z, h_d(x)) \end{array} \right\} (x, z) \in D \times \mathbb{R}^{d_z} \end{cases} \quad (7)$$

whose flow and jump map we denote

$$\mathcal{F}(x, z) = (f(x), F(z, h_c(x))) \quad (8a)$$

$$\mathcal{G}(x, z) = (g(x), G(z, h_d(x))) \quad (8b)$$

The observer problem can then be reformulated as a stabilization problem of a set $\mathcal{A} \subseteq \mathbb{R}^{d_x} \times \mathbb{R}^{d_z}$, which depends on the observation goal. For instance, if we want to estimate the full state x , we can first try to take $d_z = d_x$ and stabilize the zero estimation error set given by

$$\mathcal{A} = \{(x, z) \in (C \cup D \cup g(D)) \times \mathbb{R}^{d_z} : x = z\} \quad , \quad (9a)$$

which is nothing but the diagonal. In that case, z directly provides an asymptotic estimate of x . But sometimes, as for continuous-time systems, we need to change coordinates, or add some degrees of freedom, thus leading to $d_z \geq d_x$ and

$$\mathcal{A} = \{(x, z) \in (C \cup D \cup g(D)) \times \mathbb{R}^{d_z} : z = T(x)\} \quad , \quad (9b)$$

for some map $T : C \cup D \cup g(D) \rightarrow \mathbb{R}^{d_z}$. In that case, an estimate for x may be recovered from z by left-inversion if T is injective. We may also be interested in estimating only a part x_p of the state x , in the context of switched systems for instance, which can be translated into an appropriate choice of \mathcal{A} , i.e., more generally

$$\mathcal{A} = \{(x, z) \in (C \cup D \cup g(D)) \times \mathbb{R}^{d_z} : T(x, z) = 0\} \quad , \quad (9c)$$

for some map $T : C \cup D \cup g(D) \times \mathbb{R}^{d_z} \rightarrow \mathbb{R}^p$. The goal of this paper is finally to solve the following problem.

Problem 1. Given a set of initial conditions $\mathcal{X}_0 \subseteq \mathbb{R}^{d_x}$, a closed subset \mathcal{A} of $\mathbb{R}^{d_x} \times \mathbb{R}^{d_z}$ defining an observer problem, and assuming one of the conditions of Definition 2.1 holds, design maps $F : \mathbb{R}^{d_z} \times \mathbb{R}^{d_{y_c}} \rightarrow \mathbb{R}^{d_z}$ and $G : \mathbb{R}^{d_z} \times \mathbb{R}^{d_{y_d}} \rightarrow \mathbb{R}^{d_z}$ such that there exist a class- \mathcal{KL} function β and a subset \mathcal{Z}_0 of \mathbb{R}^{d_z} such that for every $\phi = (x, z) \in \mathcal{S}_{\mathcal{H} - \hat{\mathcal{H}}}(\mathcal{X}_0 \times \mathcal{Z}_0)$,

$$|\phi(t, j)|_{\mathcal{A}} \leq \beta(|\phi(0, 0)|_{\mathcal{A}}, t + j) \quad (10)$$

for all $(t, j) \in \text{dom } \phi$, namely \mathcal{A} is uniformly pre-asymptotically stable (UpAS) for $\mathcal{H} - \hat{\mathcal{H}}$ with basin of attraction including $\mathcal{X}_0 \times \mathcal{Z}_0$.

Note that for solutions $(x, z) \in \mathcal{S}_{\mathcal{H} - \hat{\mathcal{H}}}(\mathcal{X}_0 \times \mathcal{Z}_0)$, the set \mathcal{A} should also ensure that

$$x \text{ bounded and } |(x, z)|_{\mathcal{A}} \text{ bounded} \implies z \text{ bounded}$$

to guarantee from (10) that z cannot explode in finite time before x does. In other words, the observer solution is indeed defined as long as the plant's solution is. This is verified for \mathcal{A} defined in (9a) or (9b) if T is continuous.

Of course, the implementation of the observer $\hat{\mathcal{H}}$ then requires a perfect synchronization with the plant \mathcal{H} . Unfortunately, in practice, the detection of the plant's jumps often involve measurements and transmission of information which might entail some delays in the triggering of the observer's jumps. Therefore, the robustness of the UpAS property of \mathcal{A} given by Problem 1 will need to be analyzed. Such analysis is carried out in Section 6.

3. General sufficient conditions for $\mathcal{H} - \hat{\mathcal{H}}$

The following theorem gives a Lyapunov-based sufficient condition to solve Problem 1. It will be used throughout the paper in different cases.

Theorem 3.1. Assume there exist scalars $a_c, a_d \in \mathbb{R}$, \mathcal{K}^∞ -maps $\underline{\alpha}, \bar{\alpha}$, and a C^1 map $V : \mathbb{R}^{d_x} \times \mathbb{R}^{d_z} \rightarrow \mathbb{R}$ verifying

$$\begin{cases} \underline{\alpha}(|(x, z)|_{\mathcal{A}}) \leq V(x, z) \quad \forall (x, z) \in (C \cup D \cup g(D)) \times \mathbb{R}^{d_z} \\ V(x, z) \leq \bar{\alpha}(|(x, z)|_{\mathcal{A}}) \quad \forall (x, z) \in \mathcal{X}_0 \times \mathcal{Z}_0 \end{cases} \quad (11a)$$

$$L_{\mathcal{F}}V(x, z) \leq a_c V(x, z) \quad \forall (x, z) \in C \times \mathbb{R}^{d_z} \quad (11b)$$

$$V(\mathcal{G}(x, z)) \leq e^{a_d} V(x, z) \quad \forall (x, z) \in D \times \mathbb{R}^{d_z} \quad (11c)$$

with \mathcal{F} and \mathcal{G} defined in (8). Then, Problem 1 is solved if any of the following conditions (C) holds:

(C1) $a_c < 0$ and $\mathcal{C}_{\mathcal{X}_0}[\mathcal{I}]$ holds with $\min \mathcal{I} > \frac{a_d}{|a_c|}$.

(C1') $a_c < 0$ and $\mathcal{C}_{\mathcal{X}_0}^{\text{av}}[\tau^*]$ holds with $\tau^* > \frac{a_d}{|a_c|}$.

(C2) $a_d < 0$ and $\mathcal{C}_{\mathcal{X}_0}[\mathcal{I}]$ holds with $a_c \sup \mathcal{I} < |a_d|$.

(C2') $a_d < 0$ and $\mathcal{C}_{\mathcal{X}_0}^{\text{rav}}[\tau^*]$ holds with $a_c \tau^* < |a_d|$.

Proof. Let us assume either (C1) or (C2). Then $\mathcal{C}_{\mathcal{X}_0}[\mathcal{I}]$ holds and $a_c \tau + a_d < 0$ for all $\tau \in \mathcal{I}$. Therefore, there exists a positive scalar a such that $a_c \tau + a_d \leq -a(\tau + 1)$ for all $\tau \in \mathcal{I}$. We then easily prove that there exists M such that for any solution $\phi = (x, z)$ to $\mathcal{H} - \tilde{\mathcal{H}}$ initialized in $\mathcal{X}_0 \times \mathcal{Z}_0$, $a_c t + a_d j \leq M - a(t + j)$ $(t, j) \in \text{dom } \phi$, and Problem 1 is solved according to [21, Proposition 3.29] (See Remark 3.2 below). Similarly, in case of a (reverse) average dwell-time with either (C1') or (C2'), the proof is similar to [26, Corollary 1]. \square

Remark 3.2. In [21, Proposition 3.29], Condition (11a) is strengthened into

$$\underline{\alpha}(|(x, z)|_{\mathcal{A}}) \leq V(x, z) \leq \bar{\alpha}(|(x, z)|_{\mathcal{A}}) \\ \forall (x, z) \in (C \cup D \cup g(D)) \times \mathbb{R}^{d_z} \quad (12)$$

for easiness of presentation but actually the upper inequality is only needed on the initial conditions in the proof. We thus relax it to (11a), as it will be useful later.

From conditions (C), we recover the fact that in the case of a reverse average dwell-time or if $0 \in \mathcal{I}$, namely if there could be Zeno or eventually discrete solutions, then a_d is required to be negative, i.e., the innovation term in the discrete dynamics of the observer must make the error contractive at jumps; similarly in the case of average dwell-time or if $\sup \mathcal{I} = +\infty$, then a_c is required to be negative, i.e., the innovation term in the continuous dynamics must make the error contractive during flow. Finally, note that Theorem 3.1 allows the flow and jump maps of the plant and of the observer to be set valued, and hence, is suitable for the design of observers for plants modeled by differential inclusions or by difference inclusions, for which not many tools are available in the literature.

Example 3.3 (Linear flow/jump/output maps). Assume $f(x) = A_c x$, $g(x) = A_d x$, $h_c(x) = H_c x$, $h_d(x) = H_d x$, as considered in [10]. It is reasonable to consider \mathcal{A} defined as in (9a) and linear flow and jump maps in the observer, namely

$$F(z, y_c) = A_c z + L_c(y_c - H_c x) \\ G(z, y_d) = A_d z + L_d(y_d - H_d x)$$

with $L_c \in \mathbb{R}^{d_x \times d_{y_c}}$ and $L_d \in \mathbb{R}^{d_x \times d_{y_d}}$. Then the conditions in (11) hold for a quadratic Lyapunov function

$$V(x, z) = (x - z)^\top P(x - z)$$

if there exist scalars a_c and a_d , and a positive definite symmetric matrix $P \in \mathbb{R}^{d_x \times d_x}$ such that

$$(A_c - L_c H_c)^\top P + P(A_c - L_c H_c) \leq a_c P \quad (13a)$$

$$(A_d - L_d H_d)^\top P(A_d - L_d H_d) \leq e^{a_d} P \quad (13b)$$

The problem of finding common quadratic Lyapunov functions for continuous-time (resp. discrete-time) systems has been studied in the context of switched systems and quadratic stabilization (see e.g. [33]). But we are not aware of any result concerning the existence of a common quadratic Lyapunov function for both continuous and discrete dynamics as in (13). If a solution to (13) exists though, Problem 1 is solved if one of the conditions (C) holds. Note that if both (A_c, H_c) and (A_d, H_d) are detectable, (13) may be solvable with both $a_c \leq 0$ and $a_d \leq 0$, and (C) then holds directly if at least one of them is nonzero. By the Schur complement, this is equivalent to solving the LMIs

$$A_c^\top P + P A_c - (\tilde{L}_c H_c + H_c^\top \tilde{L}_c^\top) < 0 \\ \begin{pmatrix} P & (P A_d - \tilde{L}_d H_d)^\top \\ \star & P \end{pmatrix} > 0 \quad (14)$$

in $(P, \tilde{L}_c, \tilde{L}_d)$ and take $L_c = P^{-1} \tilde{L}_c$ and $L_d = P^{-1} \tilde{L}_d$. This has been done in [10, Example 3.3] for a bouncing ball modeled by (3) with a restitution coefficient $\lambda < 1$, and x_1 measured at all (hybrid) times.

Remark 3.4. In the favorable case where both the flow and jump dynamics of \mathcal{H} are detectable (such as the bouncing ball), it is not sufficient to choose independently a map F as a continuous-time observer of the flow and a map G as a discrete-time observer of the jumps. Indeed, their ‘‘contraction directions’’ could be incompatible: jumps could destroy what has been achieved during flow, or vice versa. For instance, with linear flow/jump/output maps, it is not enough to choose L_c, L_d such that $A_c - L_c H_c$ is Hurwitz and $A_d - L_d H_d$ is Schur. Actually, a necessary condition for convergence of the observer is that the error sampled at each jump converges to zero: this implies that the origin of the discrete system

$$\varepsilon_{k+1} = (A_d - L_d H_d) \exp\left((A_c - L_c H_c) \tau_k\right) \varepsilon_k$$

has to be asymptotically stable for $k \mapsto \tau_k \in \mathcal{I}$. If $\tau_k = \tau^* \in \mathcal{I}$ is constant, this is not verified for every choice of $A_d - L_d H_d$ Schur and $A_c - L_c H_c$ Hurwitz, as illustrated in Figure 1: $(A_d - L_d H_d) \exp\left((A_c - L_c H_c) \tau\right)$ is Schur only if $\tau^* \notin [0.1, 2]$. To avoid this phenomenon, (13a) and (13b) should be solved with the same P , and $a_c \leq 0$ and $a_d < 0$.

A drawback of Theorem 3.1 is that it requires at least a_c or a_d to be negative. Therefore, either the continuous or the discrete dynamics of \mathcal{H} has to admit an observer and thus be detectable. But take for instance the hybrid system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = 0 \\ \dot{x}_3 = 0 \end{cases}, \quad \begin{cases} x_1^+ = x_1 \\ x_2^+ = x_2 \\ x_3^+ = x_1 \end{cases} \quad (15)$$

with some arbitrary, but nonempty flow and jump sets. Suppose $h_c(x) = h_d(x) = x_1$. Neither the continuous nor the discrete dynamics is detectable, so Theorem 3.1 cannot apply. Nevertheless, this hybrid system as a whole is

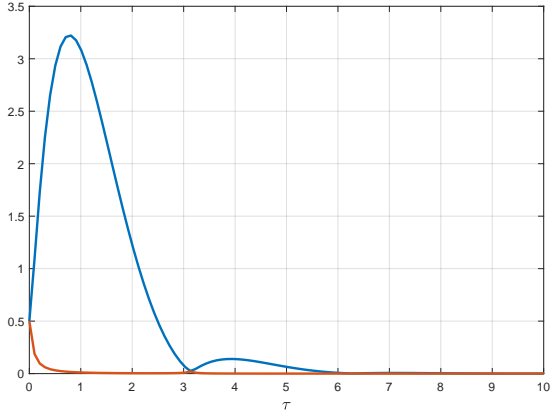


Figure 1: Absolute value of the eigenvalues of $(A_d - L_d H_d) \exp((A_c - L_c H_c)\tau)$ for $A_c = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $A_d = 5A_c$, $H_c = H_d = (1, 0)$, with L_c (resp. L_d) chosen such that $A_c - L_c H_c$ (resp. $A_d - L_d H_d$) is Hurwitz (resp. Schur) with eigenvalues $(-1+i, -1-i)$ (resp. $(0.5, -0.5)$).

determinable if there is at least one jump and one interval of flow. Indeed, pick a solution x to (15). If $y(t, j) = x_1(t, j) = 0$ for all (t, j) in the domain, the continuous dynamics give $x_2(t, j) = 0$ as soon as $[t, t + \tau) \times \{j\} \in \text{dom } x$ for some $\tau > 0$, and the discrete dynamics give $x_3(t, j) = 0$ for all j such that $(t, j - 1)$ and (t, j) are in the domain. We thus get determinability and we may hope to design an observer. We will see in this paper other methods that allow to build an observer for this system.

Another drawback is that Theorem 6.4 mixes constraints on the observer flow and jump maps which cannot be designed separately. This coupling appears through a_c and a_d in conditions (C). Even in the linear context of Example 3.3, the conditions are nonlinear, unless both a_c and a_d can be taken negative and (14) can be solved. In the following sections, we show how the design can be simplified when using innovation only in flow or only at jumps.

4. Flow-based hybrid observer

When the continuous dynamics of \mathcal{H} are detectable and *persistent* in the sense of an average dwell-time, it is tempting to use a continuous-time observer

$$\dot{z} = F(z, h_c(x)) \quad , \quad \hat{x} = \mathcal{T}(z) \quad (16)$$

as the observer's continuous dynamics F , and simply copy the discrete dynamics of \mathcal{H} in G . Indeed, intuitively, if the estimation error decreases more during flow than it increases at jumps, namely if the continuous-time observer (16) is sufficiently fast, the error is expected to converge to zero asymptotically. We thus need persistence of flow, namely conditions of the type $\mathcal{C}_{\mathcal{X}_0}[\mathcal{I}]$ with $\min \mathcal{I} > 0$, or more generally $\mathcal{C}_{\mathcal{X}_0}^{\text{av}}[\tau^*]$. In this section, we give conditions under which such a design works.

4.1. Sufficiently large average dwell-time

Assume the continuous-time observer (16) verifies (11a)-(11b) with $a_c < 0$. Then, if we can find G such that (11c) holds for some $a_d \in \mathbb{R}$, then Problem 1 is solved if the average dwell time is sufficiently large to satisfy (C1'). This result is very standard in the literature of switched systems as detailed in Section 7. Of course, if V does not increase at jumps, namely

$$V(\mathcal{G}(x, z)) \leq V(x, z) \quad \forall (x, z) \in D \times \mathbb{R}^{d_z} \quad ,$$

(11c) holds with $a_d = 0$. This is related to the notion of *non-expansiveness* of V for g in [26].

More generally, if (11a) is strengthened into (12) and if there exists a \mathcal{K}^∞ -map κ such that

$$|\mathcal{G}(x, z)|_{\mathcal{A}} \leq \kappa(|(x, z)|_{\mathcal{A}}) \quad \forall (x, z) \in D \times \mathbb{R}^{d_x} \quad (17a)$$

$$\bar{\alpha} \circ \kappa \circ \underline{\alpha}^{-1} \leq c \text{Id} \quad (17b)$$

for some positive scalar c , then (11c) automatically holds with $a_d = \ln(c)$. For instance, in the case where $d_z = d_x$ and \mathcal{A} is simply the diagonal set (9a), a map G satisfying (17a) is a simple copy of the plant's jump map g namely

$$G(z, y_d) = g(z) \quad , \quad (18)$$

if g is single-valued and κ -continuous on \mathbb{R}^{d_x} , namely

$$|g(x_a) - g(x_b)| \leq \kappa(|x_a - x_b|) \quad \forall (x_a, x_b) \in \mathbb{R}^{d_x} \times \mathbb{R}^{d_x} \quad .$$

Example 4.1 (Linear detectable flow and output maps). Assume $f(x) = A_c x$ and $h_c(x) = H_c x$, with the pair (A_c, H_c) detectable. Then, there exists a gain L such that $A_c - LH_c$ is Hurwitz and (11a)-(11b), and actually (12), hold with observer dynamics

$$F(z, y_c) = A_c z + L(y_c - H_c z) \quad , \quad (19)$$

A defined in (9a), V defined by

$$V(x, z) = (x - z)^\top P (x - z)$$

with the matrix $P \in \mathbb{R}^{d_x \times d_x}$ such that

$$(A_c - LH_c)^\top P + P(A_c - LH_c) \leq a_c P$$

for some $a_c < 0$ and with

$$\underline{\alpha}(\cdot) = \underline{\lambda}(P) (\cdot)^2 \quad , \quad \bar{\alpha}(\cdot) = \bar{\lambda}(P) (\cdot)^2 \quad .$$

Therefore, if g is Lipschitz with Lipschitz constant k_G , we take $G(z, y_d) = g(z)$ and Problem 1 is solved if the average dwell-time is larger than $\frac{1}{|a_c|} \ln \left(\frac{\bar{\lambda}(P)}{\underline{\lambda}(P)} k_G^2 \right)$. Note that if g is only locally Lipschitz and any $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}_0)$ remains in a compact set \mathcal{X} , g can be replaced by any Lipschitz function that agrees with g on \mathcal{X} and the above applies. Typically, we could take $G(z, y_d) = \text{sat}(g(z))$ where sat is an appropriate Lipschitz saturation map. Finally, if g is linear, namely $g(x) = A_d x$, then $G(z, y_d) = A_d z$, we can directly look for P and L solution to

$$(A_c - L_c H_c)^\top P + P(A_c - L_c H_c) \leq a_c P \quad (20a)$$

$$A_d^\top P A_d \leq e^{a_d} P \quad (20b)$$

$$a_c \tau^* + a_d < 0 \quad (20c)$$

with $a_c < 0$, as in [10, 41].

4.2. Observers with arbitrarily fast flow

The previous section requires the ADT to be sufficiently large. However, apart from switched systems where the switching signal may be a controlled input, ADT is typically not chosen for a general hybrid systems where the jumps are state-dependent. Therefore, the ADT (if it exists) is a property of the system and cannot be made “sufficiently fast”.

In that case, another way to satisfy (C1) or (C1') could be to choose a sufficiently fast continuous-time observer (16), i.e. satisfying (11a)-(11b) with $|a_c|$ sufficiently large. This *tunability* property requires the continuous dynamics to be *instantaneously observable* [4]. However, increasing a_c may require to change V , which in turns, modifies a_d . The following corollary shows that this compromise can be achieved under some conditions applying to the so-called “high-gain observers.”

Corollary 4.2. *Assume $\mathcal{C}_{\lambda_0}^{\text{av}}[\tau^*]$ holds for some $\tau^* > 0$, and there exist $\lambda > 0$, $\ell_0 > 0$, polynomials \underline{c} and \bar{c} , a continuous map $T : \mathbb{R}^{d_x} \rightarrow \mathbb{R}^{d_z}$, and for all $\ell > \ell_0$, maps $F_\ell : \mathbb{R}^{d_z} \times \mathbb{R}^{d_y} \rightarrow \mathbb{R}^{d_z}$ and $V_\ell : \mathbb{R}^{d_x} \times \mathbb{R}^{d_z} \rightarrow \mathbb{R}$ such that*

$$\underline{c}(\ell)|z - T(x)|^2 \leq V_\ell(x, z) \leq \bar{c}(\ell)|z - T(x)|^2 \\ \forall (x, z) \in (C \cup D \cup g(D)) \times \mathbb{R}^{d_z} \quad (21a)$$

$$L_{\mathcal{F}_\ell} V_\ell(x, z) \leq -\ell \lambda V_\ell(x, z) \quad \forall (x, z) \in C \times \mathbb{R}^{d_z} \quad (21b)$$

with $\mathcal{F}_\ell(x, z) = (f(x), F_\ell(z, h_c(x)))$. Then, there exists $\ell^* \geq \ell_0 d$ such that for all $\ell \geq \ell^*$, Problem 1 is solved for any compact sets $\mathcal{X}_0 \times \mathcal{Z}_0$, with \mathcal{A} defined in (9b), $F := F_\ell$, and any map $G : \mathbb{R}^{d_z} \times \mathbb{R}^{d_{y_d}} \rightarrow \mathbb{R}^{d_z}$, Lipschitz with respect to z (uniformly in $y_d \in h_d(D)$), verifying

$$G(T(x), h_d(x)) = T \circ g(x) \quad \forall x \in D. \quad (21c)$$

Note that the subscript ℓ highlights the dependency of V_ℓ and \mathcal{F}_ℓ with respect to the gain ℓ describing the decay rate in (21b), which can be chosen as large as necessary.

Proof. First, $|(x, z)|_{\mathcal{A}} \leq |z - T(x)|$ for all $(x, z) \in C \cup D \cup g(D)$ and on any compact sets $\mathcal{X}_0 \times \mathcal{Z}_0$, there exists a \mathcal{K}^∞ map α such that $|z - T(x)| \leq \alpha(|(x, z)|_{\mathcal{A}})$. Therefore, (21a) implies (11a) for all $\ell \geq \ell_0$. Then, (21b) implies (11b) with $a_c = -\ell \lambda$. Then, from the definition of \mathcal{G} in (8b) and from (21a), for all $(x, z) \in D \times \mathbb{R}^{d_z}$,

$$V_\ell(\mathcal{G}(x, z)) \leq \bar{c}(\ell) |G(z, h_d(x)) - T(g(x))|^2 \\ \leq \bar{c}(\ell) |G(z, h_d(x)) - G(T(x), h_d(x))|^2 \\ \leq \bar{c}(\ell) k_G^2 |z - T(x)|^2 \\ \leq \frac{\bar{c}(\ell)}{\underline{c}(\ell)} k_G^2 V_\ell(x, z)$$

where k_G is the Lipschitz constant of G with respect to z . Therefore, (11c) holds for all $\ell \geq \ell_0$ with $a_d = \ln \left(k_G^2 \frac{\bar{c}(\ell)}{\underline{c}(\ell)} \right)$. Exploiting exponential growth over polynomial growth, $-\ell \lambda \tau^* + \ln \left(k_G^2 \frac{\bar{c}(\ell)}{\underline{c}(\ell)} \right) < 0$ for ℓ sufficiently large and (C1') holds. \square

In other words, if we know a *high-gain* continuous-time observer for the continuous dynamics of the plant (2), verifying (21a)-(21b), then a possible hybrid observer is made of this continuous-time observer and a copy of the jump dynamics (written in the high-gain coordinates $z = T(x)$, i.e. verifying (21c)), with a gain ℓ sufficiently large compared to the average dwell-time and the Lipschitz constant of the jump dynamics.

Example 4.3 (Linear observable flow and output maps). *As in Example 4.1, assume $f(x) = A_c x$ and $h_c(x) = H_c x$, but this time with the pair (A_c, H_c) observable. The observer's eigenvalues can then be assigned arbitrarily fast. For that, we define $\mathcal{V} \in \mathbb{R}^{d_x \times d_x}$ a change of coordinates transforming (A_c, H_c) into a block-diagonal observable form, namely such that*

$$\mathcal{V} A_c \mathcal{V}^{-1} = \mathbf{A} + \mathbf{D} \mathbf{C} \quad , \quad H_c \mathcal{V}^{-1} = \mathbf{C}$$

with $\mathbf{A} := \text{blkdiag}(A_1, \dots, A_{d_{y_c}})$, $\mathbf{D} := \text{blkdiag}(D_1, \dots, D_{d_{y_c}})$, $\mathbf{C} := \text{blkdiag}(C_1, \dots, C_{d_{y_c}})$,

$$A_i = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & & \\ \vdots & \ddots & \ddots & \\ 0 & & 1 & 0 & 0 \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix} \in \mathbb{R}^{d_i \times d_i}$$

$$C_i = (0 \quad \dots \quad 0 \quad 1) \in \mathbb{R}^{1 \times d_i} \quad ,$$

$D_i \in \mathbb{R}^{d_i \times 1}$, and d_i integers such that $\sum_{i=1}^{d_{y_c}} d_i = d_{y_c}$. Consider vectors L_i such that $A_i - L_i C_i$ is Hurwitz, and for a positive scalar ℓ , define $L_i(\ell) := \text{diag}(\ell^{d_i-1}, \dots, \ell, 1)$. Then, let us take F defined by (19) with

$$L = \mathcal{V}^{-1} (\mathbf{D} + \ell \mathbf{L}(\ell) \mathbf{L}) \quad (22)$$

where $\mathbf{L} := \text{blkdiag}(L_1, \dots, L_{d_{y_c}})$, $\bar{L} := \text{blkdiag}(\bar{L}_1, \dots, \bar{L}_{d_{y_c}})$. We thus have $\text{eig}(A_c - L H_c) = \ell \text{eig}(\mathbf{A} - \mathbf{L} \mathbf{C})$. Consider a positive definite matrix $P \in \mathbb{R}^{d_x \times d_x}$ such that

$$(\mathbf{A} - \mathbf{L} \mathbf{C})^\top P + P(\mathbf{A} - \mathbf{L} \mathbf{C}) \leq -\lambda P$$

for some $\lambda > 0$. Then, (21a)-(21b) hold with $T = \text{Id}$,

$$V_\ell(x, z) = (x - z)^\top \mathcal{V}^\top L(\ell)^{-1} P L(\ell)^{-1} \mathcal{V} (x - z) \quad ,$$

$$\underline{\alpha}(\cdot) = \underline{\lambda} (\mathcal{V}^\top P \mathcal{V}) (\cdot)^2 \quad , \quad \bar{\alpha}(\cdot) = \bar{\lambda} (\mathcal{V}^\top P \mathcal{V}) \ell^{2(d-1)} (\cdot)^2$$

$d = \max d_i$. Therefore, whatever the average dwell-time is, Problem 1 is solved for ℓ sufficiently large by taking $G(z, y_d) = g(z)$ (resp. $G(z, y_d) = \text{sat}(g(z))$) if g is Lipschitz (resp. locally Lipschitz and the solutions x are bounded) as in Example 4.1.

Example 4.4 (Strongly differentially observable flow/output maps). *Assume that f and g are single-valued, with a single output ($d_{y_c} = 1$), and the flow dynamics of \mathcal{H} are strongly differentially observable of order d_z , namely the map $T : \mathbb{R}^{d_x} \rightarrow \mathbb{R}^{d_z}$ defined by*

$$T(x) = (h_c(x), L_f h_c(x), \dots, L_f^{d_z-1} h_c(x))$$

is an injective immersion on $C \cup D$. If in addition, there exists a Lipschitz map Φ verifying

$$\Phi(T(x)) = L_f^{d_z} h(x) \quad \forall x \in C \cup D,$$

then a high-gain observer as in [28] can be built for the flow dynamics, with

$$F_\ell(z, y) = Az + B\Phi(z) + \ell L(\ell)K(y - z_1),$$

$$A = \begin{pmatrix} 0 & 1 & & \dots & 0 \\ 0 & 0 & 1 & & \\ \vdots & \ddots & \ddots & \ddots & \\ \vdots & & & 0 & 1 \\ 0 & \dots & \dots & 0 & 0 \end{pmatrix} \in \mathbb{R}^{d_z}, \quad B = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$L(\ell) = \text{diag}(1, \ell, \ell^2, \dots, \ell^{d_z-1})$, and K such that $A - BK$ is Hurwitz. Note that if any $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}_0)$ evolves in a compact set $\mathcal{X} \subseteq C \cup D$, there exists a Lipschitz map $\mathcal{T} : \mathbb{R}^{d_z} \rightarrow \mathbb{R}^{d_x}$ such that

$$\mathcal{T}(T(x)) = x \quad \forall x \in \mathcal{X},$$

and Φ can simply be chosen as $\Phi = \text{sat} \circ L_f^{d_z} \circ \mathcal{T}$ where sat saturates outside of $L_f^{d_z}(\mathcal{X})$. Classical high gain computations [28] show that conditions (21a) and (21b) then hold for the Lyapunov function

$$V_\ell(x, z) = (T(x) - z)^\top L(\ell)^{-1} P L(\ell)^{-1} (T(x) - z),$$

with P a positive definite matrix such that

$$(A - BK)^\top P + P(A - BK) \leq -\lambda_0 P$$

for some $\lambda_0 > 0$, $\underline{c}(\ell) = \underline{\lambda}(P)$, $\bar{c}(\ell) = \bar{\lambda}(P)\ell^{2(d_z-1)}$, $\lambda > 0$ depending on the Lipschitz constant of Φ , and ℓ larger than a threshold also depending on that Lipschitz constant. Selecting G Lipschitz verifying (21c), finally provides an observer relative to \mathcal{A} defined in (9b), if the gain ℓ is sufficiently large according to Corollary 4.2. In particular, if any $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}_0)$ evolves in the compact set \mathcal{X} , we can choose $G(z, y_d) = \text{sat} \circ T \circ g \circ \mathcal{T}(z)$ where sat saturates outside of $T \circ g(D)$, and an estimate of x is obtained by $\hat{x} = \mathcal{T}(z)$. Note that the same tools can be used for multi-output triangular normal forms [25], and if z estimates only a part x_p of the state x , by replacing x by x_p everywhere.

For instance, consider a Lagrangian mechanical model with impacts of the form

$$\begin{cases} \dot{\theta} = \omega \\ \dot{\omega} = \alpha(\theta, \omega) \end{cases} \quad \begin{cases} \theta^+ = T(\theta) \\ \omega^+ = \Omega(\theta, \omega) \end{cases}$$

with $(\theta, \omega) \in \mathbb{R}^d \times \mathbb{R}^d$ the (bounded) positions and velocities, α , T and Ω locally Lipschitz functions, and θ measured. If the impacts can be detected and are known to have an average dwell-time, then an observer is simply given by

$$\begin{cases} \dot{\hat{\theta}} = \hat{\omega} - \ell(\hat{\theta} - \theta) \\ \dot{\hat{\omega}} = \text{sat}(\alpha(\hat{\theta}, \hat{\omega})) - \ell^2(\hat{\theta} - \theta) \end{cases} \quad \begin{cases} \hat{\theta}^+ = \text{sat}(T(\hat{\theta})) \\ \hat{\omega}^+ = \text{sat}(\Omega(\hat{\theta}, \hat{\omega})) \end{cases}$$

for ℓ sufficiently large, sat saturation functions as described above, and jumps triggered at the detected impacts.

5. Jump-based hybrid observer

We now consider the case where the output is rather used to create contraction of the Lyapunov function at jump times, namely we mostly exploit y_d . Without natural contraction in the continuous dynamics of \mathcal{H} , we thus need the jumps to be persistent and sufficiently frequent to inject sufficient information in the observer, i.e., conditions of the type $\mathcal{C}_{\mathcal{X}_0}^{\text{rav}}[\tau^*]$ or $\mathcal{C}_{\mathcal{X}_0}[\mathcal{I}]$ with \mathcal{I} bounded.

5.1. Sufficiently small reverse dwell time

Similarly to the previous section, we start by noting that when the discrete dynamics of \mathcal{H} admit a discrete-time observer verifying (11a) and (11c) with $a_d < 0$, we can try to find F such that (11b) holds for some $a_c \in \mathbb{R}$: Problem 1 will then be solved if a_d is sufficiently negative with respect to a_c and the amount of flow, or equivalently if the jumps are sufficiently frequent, i.e. either if $\max \mathcal{I}$ is sufficiently small to satisfy (C2), or the rADT is sufficiently small to satisfy (C2').

Example 5.1 (Linear detectable jump dynamics). Assume f is single-valued, $g(x) = A_d x$ and $h_d(x) = H_d x$, with the pair (A_d, H_d) detectable. There exists a gain L such that $A_d - LH_d$ is Schur and (11a),(11c) hold with jump dynamics

$$G(z, y_d) = A_d z + L(y_d - H_d z),$$

\mathcal{A} defined in (9a), V defined by

$$V(x, z) = (x - z)^\top P(x - z)$$

with the matrix $P \in \mathbb{R}^{d_x \times d_x}$ such that

$$(A_d - LH_d)^\top P(A_d - LH_d) \leq e^{a_d} P$$

for some $a_d < 0$. Then, choosing the flow dynamics F single-valued so that

$$|f(x) - F(z, h_c(x))| \leq c|x - z| \quad \forall (x, z) \in C \times \mathbb{R}^{d_z}$$

with some scalar c , ensures (11b) holds. For instance, if $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}_0)$ evolves in a compact set \mathcal{X} and f is locally Lipschitz, an intuitive choice is simply $F(z, y_c) = \text{sat} \circ f(z)$, where sat saturates outside of $f(\mathcal{X})$. In other words, F is simply a flow predictor. Then, Problem 1 is solved if the jumps are sufficiently frequent to satisfy (C2) or (C2').

This method has an interest only when the jumps are naturally sufficiently frequent (Zeno, eventually discrete solutions) or can be made so (switching systems). Otherwise, we need to take explicitly into account the potential increase of V during flow, to ensure both the conditions (11) and (C2') hold simultaneously.

Example 5.2. Let us come back to Example 5.1, but this time with a linear flow map, namely $f(x) = A_c x$. We can still take a copy of the continuous-time dynamics

$F(z, y_c) = A_c z$, but this time directly look for P and L solution to

$$A_c^\top P + P A_c \leq a_c P \quad (23a)$$

$$(A_d - L H_d)^\top P (A_d - L H_d) \leq e^{a_d} P \quad (23b)$$

$$a_c \tau^* + a_d < 0 \quad (23c)$$

with $a_d < 0$ as in [10, 41], where τ^* denotes the rADT or the maximal length of flow. In particular, [10, Example 4.2] shows that it can be done analytically for a bouncing ball exhibiting Zeno trajectories modeled by (3) with $\lambda < 1$, and with the output $h_d(x) = x_1$ only available at jumps. In this case, τ^* represents a known bound on the length of the flow intervals, which depends on the compact set of initial conditions (see Example 2.2). The presence of Zeno solutions is therefore not problematic to the observer design as long as it is properly taken into account through a rADT or with $0 \in \mathcal{I}$.

As noticed in [16] in the context of sampled systems ($A_d = I$), the design of Example 5.1 is extendable to particular classes of nonlinear continuous dynamics for which f is included in the convex hull of a finite number of linear maps. The LMI (23a) must then hold for each of those maps. Furthermore, [16] shows that (23) might be relaxed by allowing P and L_d to depend on the length τ of the flow intervals in a way that ensures contraction during both flows and jumps. But this requires the feasibility of some LMIs that are not necessarily related to observability.

In any case, the methods mentioned in this section require the detectability of the discrete dynamics of \mathcal{H} and a sufficient contraction of the error at jumps. When either the discrete dynamics are not detectable, or the coupling between flows and jumps makes the matrix inequalities not feasible, we propose in the next section to rather analyze an equivalent discrete-time system made of the plant sampled at the jump times, which naturally contains the information of both flows and jumps.

5.2. Equivalent discrete-time system

We now assume the jumps are persistent, i.e. $\mathcal{C}_{\mathcal{X}_0}[\mathcal{I}]$ holds with \mathcal{I} compact. We also suppose that absolutely continuous solutions of $\dot{x} \in f(x)$ are complete and denote Ψ_f the flow operator alongside f , i.e. $\Psi_f(x_0, \tau)$ denotes the set of points that can be reached at time τ by solutions to $\dot{x} \in f(x)$ initialized at x_0 at $\tau = 0$.

Now consider a solution $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}_0)$ and notice that $x_k := x(t_k, k)$ sampled after each jump and the output $y_k := h_d(x(t_k, k - 1))$ obtained before each jump verify

$$x_{k+1} \in g(\Psi_f(x_k, \tau_k)) \quad , \quad y_k \in h_d(\Psi_f(x_k, \tau_k)) \quad (24)$$

where $\tau_k = t_{k+1} - t_k$ denotes the length of the k th flow interval. It follows that with the discrete output y_d obtained at each jump, we are actually observing the equivalent discrete-time system (24). It is therefore the observability/determinability of (24) that counts, and we must look for F and G making \mathcal{A} UpAS for the reduced system

$$\begin{aligned} x_{k+1} &\in g(\Psi_f(x_k, \tau_k)) \\ z_{k+1} &\in G(\Psi_F(z_k, \tau_k), h_d(\Psi_f(x_k, \tau_k))) \end{aligned}$$

or equivalently

$$(x_{k+1}, z_{k+1}) \in \mathcal{G} \circ \Psi_{\mathcal{F}}((x_k, z_k), \tau_k) \quad , \quad (25)$$

with \mathcal{F} and \mathcal{G} defined in (8). Indeed, the following theorem shows that it is sufficient to prove UpAS of (25) with sequences $(\tau_k) \in \mathcal{I}^{\mathbb{N}}$ to solve Problem 1.

Theorem 5.3. *Assume that $\mathcal{C}_{\mathcal{X}_0}[\mathcal{I}]$ holds with \mathcal{I} compact and solutions to $(\dot{x}, \dot{z}) \in \mathcal{F}(x, z)$ are complete. Consider a set $\mathcal{X} \subseteq C \cup D$ such that any $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}_0)$ remains in \mathcal{X} at all times. Let $\tilde{\mathcal{X}}_0 \times \tilde{\mathcal{Z}}_0$ such that*

$$\begin{aligned} \mathcal{G} \circ \Psi_{\mathcal{F}}((x_0, z_0), \tau_0) &\subseteq \tilde{\mathcal{X}}_0 \times \tilde{\mathcal{Z}}_0 \\ \forall (x_0, z_0, \tau_0) &\in \mathcal{X}_0 \times \mathcal{Z}_0 \times [0, \max \mathcal{I}] \quad . \end{aligned}$$

Suppose there exists a class- \mathcal{KL} function β and $\mathcal{Z} \subseteq \mathbb{R}^{d_z}$ such that any solution (x, z) to (25) initialized in $\tilde{\mathcal{X}}_0 \times \tilde{\mathcal{Z}}_0$ and with input $k \mapsto \tau_k \in \mathcal{I}$ verifies $(x_k, z_k) \in \mathcal{X} \times \mathcal{Z}$ and

$$|(x_k, z_k)|_{\mathcal{A}} \leq \beta(|(x_0, z_0)|_{\mathcal{A}}, k) \quad \forall k \in \mathbb{N} \quad . \quad (26)$$

Then, Problem 1 is solved if there exists a class- \mathcal{K} function ρ such that for all $(x_0, z_0) \in \mathcal{X} \times \mathcal{Z}$ and for all $\tau \in [0, \max \mathcal{I}]$,

$$|\Psi_{\mathcal{F}}((x_0, z_0), \tau)|_{\mathcal{A}} \leq \rho(|(x_0, z_0)|_{\mathcal{A}}) \quad . \quad (27)$$

Proof. See Appendix A. □

Assumption (27) guarantees that the distance of (x, z) to \mathcal{A} during flow is continuous on the compact interval $[0, \max \mathcal{I}]$ with respect to the initial distance to \mathcal{A} . If \mathcal{A} is defined by (9a) and $f = F$ is locally Lipschitz, this regularity property is always satisfied when \mathcal{X} and \mathcal{Z} are compact.

The reduced system (25) may not be handier to use for design than (7) if Ψ_F and Ψ_f are not explicit, but it helps to understand the observability conditions that are at stake here. Besides, when f is linear, i.e. $f(x) = A_c x$, we can choose $F(z) = A_c z$, so that

$$\Psi_f(x_k, \tau_k) = \exp(A_c \tau_k) x_k \quad , \quad \Psi_F(z_k, \tau_k) = \exp(A_c \tau_k) z_k$$

and (27) immediately holds for \mathcal{A} diagonal defined in (9a).

When both g and f are linear we obtain the following constructive sufficient condition, that is weaker than (23).

Corollary 5.4. *Assume that $\mathcal{C}_{\mathcal{X}_0}[\mathcal{I}]$ holds with \mathcal{I} compact and f, g, h_d are defined by $f(x) = A_c x$, $g(x) = A_d x$ and $h_d(x) = H_d x$. Assume there exist a positive definite matrix $P \in \mathbb{R}^{d_x \times d_x}$ and a gain vector $L_d \in \mathbb{R}^{d_x \times d_{y_d}}$ such that*

$$\begin{aligned} (\exp(A_c \tau))^\top (A_d - L H_d)^\top P (A_d - L H_d) \exp(A_c \tau) &< P \\ \forall \tau \in \mathcal{I} \quad . \end{aligned} \quad (28)$$

Then, $F(z) = A_c z$ and $G(z, y_d) = A_d z + L(y_d - H_d z)$ solve Problem 1 with \mathcal{A} defined in (9a).

Proof. Follows from Theorem 5.3 by noticing that the distance to \mathcal{A} is the norm of the error $e = z - x$ with dynamics

$$e_{k+1} = (A_d - LH_d) \exp(A_c \tau) e_k \quad (29)$$

and using the Lyapunov function $V(x, z) = (x - z)^\top P(x - z)$. \square

The existence of the matrix P verifying (28) for a given τ is equivalent to $(A_d - L_d H_d) \exp(A_c \tau)$ being Schur for some gain L_d , which in turn is equivalent to the detectability of the discrete-time system

$$x_{k+1} = A_d \exp(A_c \tau) x_k \quad , \quad y_k = H_d \exp(A_c \tau) x_k \quad (30)$$

Thus, having (28) for any $\tau \in \mathcal{I}$ requires detectability of (30) for any $\tau \in \mathcal{I}$. It is not sufficient, however, because (28) must be verified with the same L_d and P for all $\tau \in \mathcal{I}$. So (28) requires in fact the detectability of the LTV or LPV discrete-time system

$$x_{k+1} = A_d \exp(A_c \tau_k) x_k \quad , \quad y_k = H_d \exp(A_c \tau_k) x_k \quad (31)$$

with input τ_k in the compact set \mathcal{I} , which is exactly (24). Actually, (28) is stronger because it requires a quadratic Lyapunov function with a matrix P , that is independent from the sequence $k \mapsto \tau_k$. This property is sometimes called ‘‘quadratic detectability’’ (see [48, 24, 11]).

Remark 5.5. *By the Schur complement, finding P and L_d satisfying (28) is equivalent to finding P and \tilde{L}_d satisfying the LMIs*

$$\begin{pmatrix} P & \exp(A_c \tau)^\top (P A_d - \tilde{L}_d H_d)^\top \\ \star & P \end{pmatrix} > 0 \quad \forall \tau \in \mathcal{I} \quad (32)$$

with $\tilde{L} = PL$. In the case where \mathcal{I} has infinitely many elements, an infinite number of LMIs must be solved which is not desirable. However, it is shown in [18] that it is always possible to compute numerically a polytopic decomposition of $\exp(A_c \tau)$, namely a finite number of matrices $\{M_1, M_2, \dots, M_\nu\}$ such that $\exp(A_c \tau)$ is in the convex hull of those matrices whenever $\tau \in \mathcal{I}$. Since (32) is convex in $\exp(A_c \tau)$, it is then sufficient to solve the finite number of LMIs

$$\begin{pmatrix} P & M_i^\top (P A_d - \tilde{L}_d H_d)^\top \\ \star & P \end{pmatrix} > 0 \quad \forall i \in \{1, 2, \dots, \nu\} \quad (33)$$

with common P and \tilde{L}_d . In particular, if A_c is nilpotent of order N , we have $\exp(A_c \tau) = \sum_{k=0}^{N-1} \frac{\tau^k}{k!} A_c^k$ so that for all τ in a compact subset \mathcal{I} of $\mathbb{R}_{>0}$, $\exp(A_c \tau)$ is in the convex hull of the $\nu = 2^{N-1}$ matrices $\left\{ I + \sum_{k=1}^{N-1} \frac{\tau_k^k}{k!} A_c^k \right\}$ with $\tau_k \in \{\min \mathcal{I}, \max \mathcal{I}\}$ for all k .

Example 5.6. *Consider the system (15) where $A_c = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $A_d = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, and $H_d = (1 \ 0 \ 0)$. Neither the continuous pair (A_c, H_c) nor the discrete pair (A_d, H_d) is detectable, so Theorem 3.1 cannot be used. However, the pair $A(\tau) := A_d \exp(A_c \tau) = \begin{pmatrix} 1 & \tau & 0 \\ 0 & 1 & 0 \\ 1 & \tau & 0 \end{pmatrix}$, $H(\tau) :=$*

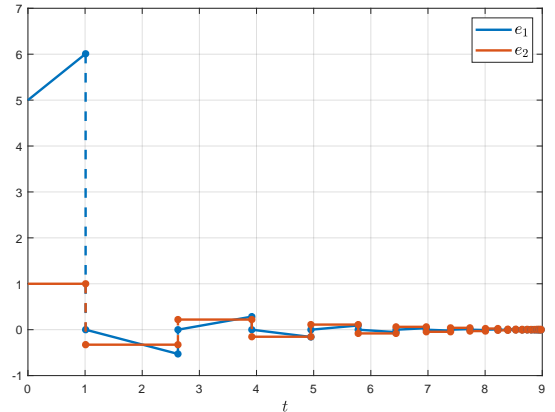


Figure 2: Error between a Zeno trajectory of system (3) with $\lambda = 0.8$ and a trajectory of observer (6) with F, G defined in Corollary 5.4 and $L = (-1, -0.1487)^\top$.

$H_d \exp(A_c \tau) = \begin{pmatrix} 1 & \tau & 0 \end{pmatrix}$ is detectable for any nonzero τ . Since A_c is nilpotent of order 2, according to Remark 5.5, for any \mathcal{I} compact subset of $\mathbb{R}_{>0}$, it is enough to solve the two LMIs given by

$$\begin{pmatrix} P & (I + \tau A_c)^\top (P A_d - \tilde{L} C)^\top \\ \star & P \end{pmatrix} > 0 \quad (34)$$

for $\tau = \tau_m := \min \mathcal{I} > 0$ and $\tau = \tau_M := \max \mathcal{I}$. If they are solvable (with a common P), then by Corollary 5.4, we obtain an observer. For instance, when choosing $\tau_m = 2$ and $\tau_M = 5$ and solving the LMIs via Yalmip for P and \tilde{L} , we get $L = P\tilde{L} = (1, 0.2259, 1)^\top$. \triangle

Example 5.7. *Consider again the bouncing ball (3) with restitution coefficient $\lambda < 1$, with measurements at jumps only, namely $H_c = 0$ and $H_d = (1, 0)$. As observed in Example 2.2, for any compact set of initial conditions \mathcal{X}_0 , there exists $\tau_M > 0$ such that $\mathcal{C}_{\mathcal{X}_0}(\mathcal{I})$ holds with $\mathcal{I} = [0, \tau_M]$. According to Corollary 5.4, it is enough to satisfy (28) instead of (23). Since A_c is nilpotent of order 2, we get again from Remark 5.5 that it is enough to solve (34) for $\tau = 0$ and $\tau = \tau_M$. With $\lambda = 0.8$ and $\tau_M = 5$, we obtain $L = (-1, -0.1487)$. The result of a simulation with initial condition $x_0 = (5, 0)$, $\hat{x}_0 = (10, 1)$ is shown on Figure 2. \triangle*

What makes the above work is the fact that the flow operator of the error $e = \hat{x} - x$, is explicitable through $\exp(A_c \tau)e$ and contained in the convex hull of a finite number of linear maps. In the context of sampled nonlinear systems, [5, 15] noticed that by copying the continuous dynamics in the observer, namely taking $F = f$, the error components evolve during flow according to

$$\dot{e}_i = f_i(\hat{x}) - f_i(x) = \frac{df_i}{dx}(v(t))e$$

for some v depending on x and \hat{x} , thanks to the mean value theorem. For certain classes of maps f [15], the error reachable set within a time $\tau \in \mathcal{I}$ may then be included

in the convex hull of a finite number of linear maps $\{e \mapsto M_i e\}_{i=1, \dots, \nu}$ if the Jacobian components of f are bounded. If g is linear, the discrete error system (29) is then replaced by

$$e_{k+1} = \sum_{i=1}^{\nu} \beta_{i,k} (A_d - LH_d) M_i e_k$$

with $\sum_{i=1}^{\nu} \beta_{i,k} = 1$, and following the same steps as in [5] with the Lyapunov function of Corollary 5.4, it is enough to ensure (28) with $\exp(A_c \tau)$ replaced by M_i , for each $i \in \{1, 2, \dots, \nu\}$, namely solve the LMIs (33).

The advantage of using a constant gain L is that it is sufficient to compute once the vertices M_i of the polytopic decomposition of the flow operator for $\tau \in \mathcal{I}$ and solve offline the finite number of LMIs (33). However, as mentioned above, those LMIs might not be solvable since they require a stronger property than detectability of (31). In that case, we may allow L to be time-varying, by adapting L to τ_k , as done in the particular case of sampled-data observers in [42]. Indeed, observe that the observer jump map G in (25) is applied after flowing τ_k units of time with F . Therefore, at the moment where G is used, τ_k represents the time elapsed since the previous jump and can be considered known and added to the output y_d . It follows that at each jump, at hybrid time (t_k, k) , the jump map can be adapted to the length of the previous intervals of flow, namely L is replaced by L_k in G defined in Corollary 5.4, in a way that makes

$$z_{k+1} = A(\tau_k) z_k + L_k (y_k - H(\tau_k) z_k)$$

with $A(\tau_k) = A_d \exp(A_c \tau_k)$ and $H(\tau_k) = H_d \exp(A_c \tau_k)$, an observer for (31), i.e. the error system

$$e_{k+1} = (A(\tau_k) - L_k H(\tau_k)) e_k \quad (35)$$

asymptotically stable. Since H is not constant, we cannot use the results obtained for LPV systems ([24] and references therein). However, an even simpler approach is to consider (31) as a LTV system and design L_k as the gain of a standard discrete-time Kalman filter. In the same spirit, if (31) is known to be observable after N jumps, [36] proposed to compute L_k based on the weighted observability Grammian over the past N jumps.

6. Robustness with respect to delays in jumps

We now study how the observer convergence is impacted if the observer jumps are delayed with respect to the plant's, thus leading to a mismatch between the observer jump times and those of the plant. For this, we start from the following assumption.

Assumption 6.1. $\mathcal{C}_{\mathcal{X}_0}[\mathcal{I}]$ holds with \mathcal{I} compact and $\min \mathcal{I} > 0$, and Problem 1 has been solved, namely the set \mathcal{A} is UpAS for $\mathcal{H} - \hat{\mathcal{H}}$ with basin of attraction including $\mathcal{X}_0 \times \mathcal{Z}_0$.

We choose to study the particular case where the value of the innovation term, implemented in the observer at

the delayed jump, is the one that would have been computed at the actual plant's jump time if there had been no delay. This covers the situations where the measurement and computation of the innovation $G(z, y_d)$ are instantaneous, but the implementation of the jump in the observer is delayed; or the measurement takes a known amount of time $\delta \geq 0$ to arrive to the observer, and the update of z is chosen as $G(z(t - \delta), y_d)$, thanks to a buffer in z or by backward integration of z . Inspired from [3], for any $\Delta \in [0, \min \mathcal{I}]$, this situation can be modeled as

$$\hat{\mathcal{H}}(\Delta) \left\{ \begin{array}{l} \dot{x} = f(x) \\ \dot{z} = F(z, h_c(x)) \\ \dot{\mu} = 0 \\ \dot{\tau}_\delta = -\min\{\tau_\delta + 1, 1\} \end{array} \right\} \mathbf{x} \in \hat{C}(\Delta) \quad (36)$$

$$\left\{ \begin{array}{l} x^+ = g(x) \\ z^+ = z \\ \mu^+ = G(z, h_d(x)) \\ \tau_\delta^+ \in [0, \Delta] \end{array} \right\} \mathbf{x} \in \hat{D}_{-1}(\Delta)$$

$$\left\{ \begin{array}{l} x^+ = x, \quad \mu^+ = 0 \\ z^+ = \mu, \quad \tau_\delta^+ = -1, \end{array} \right\} \mathbf{x} \in \hat{D}_0(\Delta)$$

with state $\mathbf{x} = (x, z, \mu, \tau_\delta)$, flow set

$$\hat{C}(\Delta) = \left(\hat{C} \times \{0\} \times \{-1\} \right) \cup \left(\hat{C} \times \mathbb{R}^{d_z} \times [0, \Delta] \right),$$

jump set $\hat{D}_{-1}(\Delta) \cup \hat{D}_0(\Delta)$ with

$$\hat{D}_{-1}(\Delta) = \hat{D} \times \{0\} \times \{-1\}$$

$$\hat{D}_0(\Delta) = (\hat{C} \cup \hat{D}) \times \mathbb{R}^{d_z} \times \{0\},$$

$$\hat{C} := C \times \mathbb{R}^{d_z}, \quad \hat{D} := D \times \mathbb{R}^{d_z}.$$

$\hat{\mathcal{H}}(\Delta)$ contains two new states μ and τ_δ evolving in \mathbb{R}^{d_z} and $[0, \Delta] \cup \{-1\}$ respectively. The state τ_δ is a timer modeling the delay between the plant's jump and the observer's jump. The role of μ is to store the update to be implemented in the observer at the end of the delay interval, when it actually jumps. More precisely, when $\tau_\delta = -1$ and x does not jump, $\hat{\mathcal{H}}(\Delta)$ flows, with μ and τ_δ remaining equal to 0 and -1 respectively. When x jumps, then the update that should have been instantaneously implemented in the observer is stored in the memory state μ , and τ_δ is set to a number in $[0, \Delta]$ thus starting a delay period: $\hat{\mathcal{H}}(\Delta)$ then flows and the time τ_δ decreases, until it reaches 0. At this point, a delay interval of length smaller than or equal to Δ has elapsed, and the observer state z is updated with the content of μ , while μ is reset to 0.

Note that the plant's state is not allowed to jump again before the delay expressed by τ_δ has expired. That is why this model only works in the case where $\Delta < \min \mathcal{I}$, i.e., the maximal delay is smaller than the smallest possible time between successive jumps of the plant.

In order to study the robustness of this property in presence of delay, we need to resort to compact attractors and some regularity properties of $\mathcal{H} - \hat{\mathcal{H}}$.

Assumption 6.2. *There exists a compact subset \mathcal{X} of $C \cup D$, such that any solution $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}_0)$ verifies $x(t, j) \in \mathcal{X}$ for all $(t, j) \in \text{dom } x$. Besides, $\mathcal{A}_{\mathcal{X}} := \mathcal{A} \cap (\mathcal{X} \times \mathbb{R}^{d_z})$ is compact.*

Assumption 6.3. *$\mathcal{H} - \hat{\mathcal{H}}$ defined in (7) satisfies the hybrid basic conditions defined in [21, Assumption 6.5], namely C and D are closed, $\mathcal{F}_{|_{\hat{C}}}$ and $\mathcal{G}_{|_{\hat{D}}}$ are outer semicontinuous and locally bounded, and $\mathcal{F}_{|_{\hat{C}}}$ takes convex values.*

It follows that the solutions of interest are also solution to (2) with flow set $C \cap \mathcal{X}$ and jump set $D \cap \mathcal{X}$, which are compact. The assumption that $\mathcal{A}_{\mathcal{X}}$ is compact, is typically satisfied whenever

$$(x, z) \in \mathcal{A} \iff z = T(x)$$

for some continuous map $T : \mathcal{O} \rightarrow \mathbb{R}^{d_z}$ with $\mathcal{X} \subseteq \mathcal{O}$, namely in all the examples considered above.

Let us define the set

$$\mathcal{A}' = (\mathcal{A}_{\mathcal{X}} \times \{0\} \times \{-1\}) \cup (\hat{G} \times \{0\})$$

$$\hat{G} := \{(g(x), z, G(z, h_d(x))) : x \in D, (x, z) \in \mathcal{A}_{\mathcal{X}}\}.$$

Theorem 6.4. *Suppose Assumptions 6.1, 6.2 and 6.3 hold. Then, the set \mathcal{A}' is UpAS for $\hat{\mathcal{H}}(0)$ with basin of attraction containing $\mathcal{X}_0 \times \mathcal{Z}_0 \times \{0\} \times \{-1\}$. Besides, there exist a \mathcal{KL} function β , $\epsilon^* > 0$ and $\Delta^* > 0$, $t^* \geq 0$, $j^* \in \mathbb{N}$, such that for any $\Delta \in [0, \Delta^*]$, any $\epsilon \in [0, \epsilon^*]$ and any solution $\phi = (x, z, \mu, \tau_\delta)$ to $\hat{\mathcal{H}}(\Delta)$ initialized in $\mathcal{X}_0 \times \mathcal{Z}_0 \times \{0\} \times \{-1\}$, we have*

$$|\phi(t, j)|_{\mathcal{A}'} \leq \beta(|\phi(0, 0)|_{\mathcal{A}'}, t + j) + \epsilon, \quad (37)$$

and $\text{dom } \phi = \mathcal{D}_{-1} \cup \mathcal{D}_0$ with $\mathcal{D}_k = \left(\bigcup_{j \in J_k} [t_j, t_{j+1}] \times \{j\} \right)$, $k \in \{0, -1\}$,

$$J_{-1} = \{j \in \mathbb{N} : \tau_\delta(t, j) = -1 \quad \forall t \in [t_j, t_{j+1}]\}$$

$$J_0 = \{j \in \mathbb{N} : \tau_\delta(t, j) \in [0, \Delta] \quad \forall t \in [t_j, t_{j+1}]\},$$

such that

- for all j in J_0 , $t_{j+1} - t_j \leq \Delta$,
- for all $(t, j) \geq (t^*, j^*)$ in \mathcal{D}_{-1} , $|(x, z)(t, j)|_{\mathcal{A}} \leq 2\epsilon$.

Proof. See Appendix B. \square

In other words, we achieve semiglobal practical stability of \mathcal{A} except possibly on the delay intervals (of maximal length Δ).

In fact, if \mathcal{A} is the diagonal set (9a), the mismatch during the delay intervals cannot be prevented if the jump map is not the identity. Indeed, after one jump of either x or z , one is close to x^- while the other is in $g(x^-)$, no matter how long the delay is. This well-known phenomenon, called *peaking*, was reported in the context of observation [19], but also more generally output-feedback and tracking

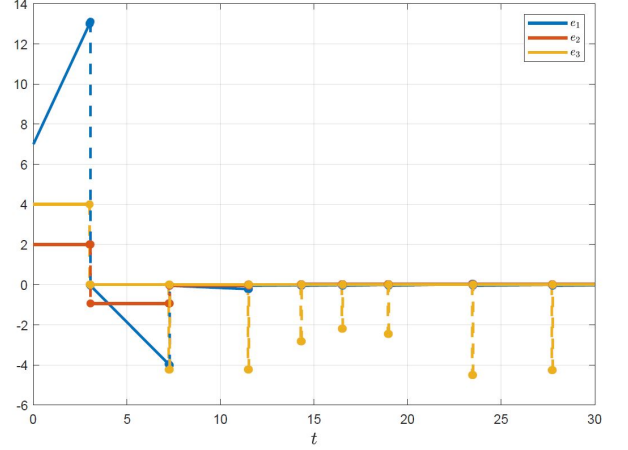


Figure 3: Error between a trajectory of system (15) with random interjump intervals in $\mathcal{I} = [2, 5]$ and observer (6) with $L_c = 0$ and $L = (1, 0.2259, 1)^\top$, and jumps triggered with a delay $\Delta = 0.05$.

[13]. This suggests that the Euclidian distance to evaluate the observer error is not appropriate and more general distances could be designed [12]. In particular, the expression of \mathcal{A}' shows that semi-global practical stability is actually ensured for the peaking free set

$$\tilde{\mathcal{A}} = \mathcal{A} \cup \{(x, z) \in \mathbb{R}^{d_x} \times \mathbb{R}^{d_z} : x \in g(x^-), (x^-, z) \in \mathcal{A}\}.$$

Note that in the limit case where $\min \mathcal{I} = 0$, namely Zeno or multiple jumps could happen, then an arbitrarily small delay in the observer jumps could lead to several jumps of delay, namely, one would need to consider

$$\tilde{\mathcal{A}} = \mathcal{A} \cup \{(x, z) \in \mathbb{R}^{d_x} \times \mathbb{R}^{d_z} : x \in g^{k^*}(x^-), (x^-, z) \in \mathcal{A}, g^k(x^-) \cap D \neq \emptyset \quad \forall k \in \{1, \dots, k^* - 1\}\}.$$

In the case of an average dwell-time, k^* would be limited by N_0 . Finally, in the extreme case where an infinite number of jumps happened during the delay interval, nothing could be done in presence of delay in jump detection: one would need to consider observers that automatically synchronize themselves with the plant.

Example 6.5. *We come back to Example 5.6 with a delay in the triggering of the observer's jump. The results are presented in Figures 3 -4 with delays of $\Delta = 0.05$ and $\Delta = 0.5$ respectively. Note that in this example, the assumption of boundedness of the plant's trajectory is not verified since x_1 and x_3 diverge. We can still see that the smaller the delay, the smaller the error outside the delay intervals. Since trajectories are here unbounded, it could also happen that the mismatch during the delay intervals grows larger and larger, although this is not the case here.*

7. Application to switched systems

We now show how the previously obtained results apply to the design of observers for switched systems with state

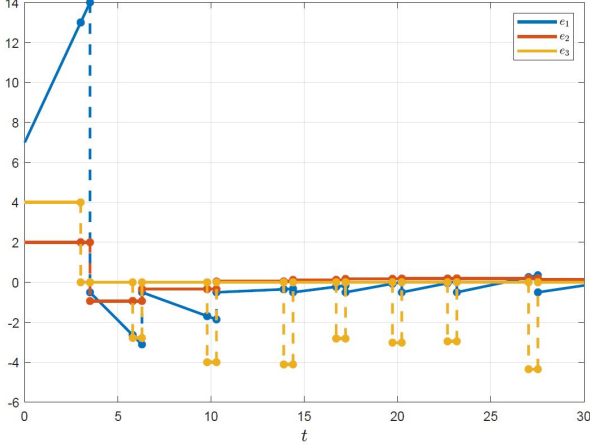


Figure 4: Error between a trajectory of system (15) with random interjump intervals in $\mathcal{I} = [2, 5]$ and observer (6) with $L_c = 0$ and $L = (1, 0.2259, 1)^\top$, and jumps triggered with a delay $\Delta = 0.5$.

$x = (x_p, q)$ as defined in (4) of Example 2.3. We focus on the case where the mode q is known at all times. Besides q , some additional measurements $h_q(x_p)$ of the continuous state x_p are available in each mode, either at all times, namely

$$y_c = y_d = (h_q(x_p), q) \quad (38a)$$

or only at the switching instants, namely

$$y_c = q \quad , \quad y_d = (h_q(x_p), q) \quad (38b)$$

7.1. Detectable individual flow dynamics

We start by assuming the output (38a) is available at all times and the continuous pair (f_q, h_q) of each mode is detectable for $q \in Q$. More precisely, assume that the individual continuous dynamics $\dot{x}_p = f_q(x_p)$ of each mode $q \in Q$ with output $h_q(x_p)$ admit a continuous-time observer

$$\dot{z} = F_q(z, h_q(x_p))$$

relative to some observation set $\mathcal{A}_q \subset \mathbb{R}^{d_{x_p}} \times \mathbb{R}^{d_z}$ and with a Lyapunov function $V_q : \mathbb{R}^{d_{x_p}} \times \mathbb{R}^{d_z} \rightarrow \mathbb{R}_{\geq 0}$ that verifies (11a)-(11b), namely

$$\begin{cases} \underline{\alpha}_q (|(x_p, z)|_{\mathcal{A}_q}) \leq V_q(x_p, z) \\ \forall (x_p, z) \in (C_q \cup D_q \cup g_q(D_q)) \times \mathbb{R}^{d_z}, \\ V_q(x_p, z) \leq \bar{\alpha}_q (|(x_p, z)|_{\mathcal{A}_q}) \quad \forall (x_p, z) \in \mathcal{X}_{p,0} \times \mathcal{Z}_0 \end{cases} \quad (39a)$$

$$L_{\mathcal{F}_q} V_q(x_p, z) \leq a_{c,q} V_q(x_p, z) \quad \forall (x_p, z) \in C_q \times \mathbb{R}^{d_z} \quad (39b)$$

for some scalar $a_{c,q} < 0$ and some class- \mathcal{K}^∞ maps $\underline{\alpha}_q, \bar{\alpha}_q$, and where $\mathcal{F}_q(x_p, z) = (f_q(x_p), F_q(z, h_q(x_p)))$.

Since we know an observer of each continuous mode, it is tempting to build an observer for the switched system by switching among these individual observers thanks to the knowledge of q . Because the decrease of the estimation error is brought by flowing in each mode, we need persistence of flow, namely conditions of the type $\mathcal{C}_{\mathcal{X}_0}[\mathcal{I}]$ with $\min \mathcal{I} > 0$, or more generally $\mathcal{C}_{\mathcal{X}_0}^{\text{av}}[\tau^*]$. However, it is well-known that switching among asymptotically stable

systems does not necessarily bring asymptotic stability, since whatever has been achieved in one mode could be destroyed by the following one, if the descent directions of the Lyapunov functions V_q are not compatible [32]. We show here how the results of Section 4 shed a new light on this problem.

For that, let us consider

$$\mathcal{A} := \{(x_p, q, z) \in \mathbb{R}^{d_{x_p}} \times Q \times \mathbb{R}^{d_z} : (x_p, z) \in \mathcal{A}_q\},$$

which verifies for $q \in Q$,

$$|(x_p, q, z)|_{\mathcal{A}} = 0 \iff |(x_p, z)|_{\mathcal{A}_q} = 0. \quad (40)$$

Therefore, designing an observer for the hybrid system (4) with state $x = (x_p, q)$ stabilizing \mathcal{A} indeed achieves the observation goal of each mode modeled by \mathcal{A}_q . From (38a), we can define

$$V((x_p, q), z) := V_q(x_p, z) \quad , \quad F(z, y_c) := F_q(z, h_q(x_p)). \quad (41)$$

Observing that $|(x_p, q, z)|_{\mathcal{A}} \leq |(x_p, z)|_{\mathcal{A}_q}$, we deduce from (39) and (40) that conditions (11a)-(11b) hold with $a_c := \max_{q \in Q} a_{c,q}$, $\mathcal{X}_0 = \mathcal{X}_{p,0} \times Q$, $\underline{\alpha} = \min_{q \in Q} \underline{\alpha}_q$, and some \mathcal{K}^∞ -map $\bar{\alpha}_q$ if $\mathcal{X}_{p,0} \times \mathcal{Z}_0$ is compact. It thus remains to satisfy (11c) and (C1) or (C1') to apply Theorem 3.1. Similarly to the literature, we consider the following two cases.

7.1.1. Common Lyapunov function

Assume $V_q = V_{q'} =: \bar{V}$ for all $(q, q') \in Q^2$. In that case, it is sufficient to check (11c) for \bar{V} independently, namely choose $G(z, y_d) = G_q(z, h_q(x_p))$ such that

$$\begin{aligned} \bar{V}(G_q(x_p, z)) &\leq e^{a_d} \bar{V}(x_p, z) \\ \forall (x_p, z) &\in D_q \times \mathbb{R}^{d_z}, \quad \forall q \in Q \end{aligned} \quad (42)$$

with $G_q(x_p, z) = (g_q(x_p), G_q(z, h_q(x_p)))$, and a scalar a_d . This is in particular satisfied with $a_d = 0$ if $G_q = \text{Id}$, thus directly satisfying (C1'). Otherwise, as noticed in Section 4.1, (42) holds if (12) holds for each q and there exist class- \mathcal{K}^∞ maps κ_q and a positive scalar c such that

$$\begin{aligned} |G_q(x_p, z)|_{\mathcal{A}_q} &\leq \kappa_q (|(x_p, z)|_{\mathcal{A}_q}) \quad \forall (x_p, z) \in D_q \times \mathbb{R}^{d_z} \\ \bar{\alpha}_q \circ \kappa_q \circ \underline{\alpha}_q^{-1} &\leq c \text{Id} \quad \forall q \in Q. \end{aligned}$$

(C1') then holds if the average dwell-time is sufficiently large, i.e. $\tau^* \geq \frac{\ln c}{|a_c|}$.

Example 7.1 (Linear switched systems with detectable modes). Assume $f_q(x_p) = A_q x_p$, $h_q(x_p) = H_q x_p$, and there exist gain vectors L_q and a positive definite matrix P such that

$$(A_q - L_q H_q)^\top P + P(A_q - L_q H_q) < 0 \quad \forall q \in Q.$$

Then, (39) holds with $F_q(z, y) = A_q z + L_q(y - H_q z)$, \mathcal{A}_q defined by

$$\mathcal{A}_q = \{(x_p, z) \in (C_q \cup D_q) \times \mathbb{R}^{d_z} : x_p = z\},$$

$$\bar{V}(x_p, z) = (x_p - z)^\top P(x_p - z)$$

$$\underline{\alpha}_q(\cdot) = \underline{\lambda}(P)(\cdot)^2 \quad , \quad \bar{\alpha}_q(\cdot) = \bar{\lambda}(P)(\cdot)^2 \quad .$$

If $g_q = \text{Id}$ for all q , we then take $G(z, y_d) = z$ and obtain an observer as in [2]. If the maps g_q are Lipschitz on D_q with an upper bound on the Lipschitz constants, we simply take a copy of the jump dynamics, namely $G(z, y_d) = g_q(z)$, and Problem 1 is solved if the average dwell-time is sufficiently large as used¹ in [8] (resp. [6]) with linear (resp. affine) jump maps g_q .

7.1.2. Multiple Lyapunov functions

In the more general case where the Lyapunov functions associated to each mode are not the same, some additional compatibility conditions appear between V_q and $V_{q'}$ to satisfy (11c), namely we need the stronger condition

$$V_{q'}(\mathcal{G}_q(x_p, z)) \leq e^{a_d} V_q(x_p, z) \quad \forall (x_p, z) \in D_q \times \mathbb{R}^{d_z} \quad \forall (q, q') \in Q^2 \quad . \quad (43)$$

In fact, it is well-known that even in the case where $g_q = \text{Id}$ and $\mathcal{G}_q = \text{Id}$, switching between asymptotically stable systems can lead to unstable dynamics if the descent directions of the Lyapunov functions are not ‘‘compatible’’. Examples of such phenomena appear in [32]. However, if condition (42) holds with a scalar $a_{d,0}$ and we add the assumption that there exists μ such that

$$V_{q'}(x_p, z) \leq \mu V_q(x_p, z) \quad \forall (x_p, z) \in (C_q \cup D_q) \times \mathbb{R}^{d_z} \quad \forall (q, q') \in Q^2 \quad (44)$$

then (43) holds with $a_d = \ln \mu + a_{d,0}$. Therefore, Problem 1 is solved if the switching is sufficiently slow *in average*, namely if $\tau^* > \frac{\ln \mu + a_{d,0}}{|a_c|}$. This is exactly the result of [27] in the context of linear switched systems and $\mathcal{G}_q = \text{Id}$, $a_{d,0} = 0$. Note that (44) always holds if (12) holds for each q and there exists μ such that

$$\bar{\alpha}_{q'} \circ \underline{\alpha}_q^{-1} \leq \mu \text{Id} \quad \forall (q, q') \in Q^2 \quad .$$

Example 7.2 (Linear switched systems with detectable modes). *Let us come back to Example 7.1. By detectability of each mode, there exist positive definite matrices P_q and gain vectors L_q such that*

$$(A_q - L_q H_q)^\top P_q + P_q (A_q - L_q H_q) < 0 \quad \forall q \in Q \quad .$$

Then, (39) holds with same F_q , same A_q and

$$V_q = (x_p - z)^\top P_q (x_p - z)$$

$$\underline{\alpha}_q(\cdot) = \underline{\lambda}(P_q)(\cdot)^2 \quad , \quad \bar{\alpha}_q(\cdot) = \bar{\lambda}(P_q)(\cdot)^2 \quad .$$

Besides, (44) holds with $\mu = \max\{\bar{\lambda}(P_{q'})/\underline{\lambda}(P_q)\}$ and Problem 1 is solved for sufficiently slow switching if the

maps g_q are uniformly Lipschitz as in Example 7.1. Actually a lot of effort has been made in the literature to find less conservative conditions on the switching signal. Recently, [20] exhibited generalized sufficient conditions relying on the framework of slowly-varying linear time-varying systems. The idea is that a switched linear system is stable if the overall variation of the matrix on a long interval is small: one way to achieve this, is to have slow-switching, but this switching can be faster if the variation $A_q - A_{q'}$ is small enough.

However, as mentioned in Section 4, restrictions on the (average) dwell-time makes sense only if the switching signal is chosen by the user, not if it is a property of the hybrid system.

A first way of avoiding constraints on the (average) dwell-time would be to use the information given by the output at the switches to ensure that V decreases through the switch. This is done in [37] for linear switched systems where $y_d = (H_q x_p, q)$ (see (38a)), noticing that when $g_q = \text{Id}$, the conditions on $(V_q, V_{q'})$ can be relaxed by ensuring $H_q z^+ = H_q x_p = y$ after the jump, through an oblique projection along the metric of P_q (instead of taking $z^+ = z$), namely

$$G(z, y_d) = z + P_q^{-1} H_q^\top (H_q P_q^{-1} H_q^\top)^{-1} (H_q x_p - H_q z) \quad .$$

Indeed, it can be shown via standard geometrical arguments that (42) then holds with $a_d = 0$ and (43) holds if the additional LMI

$$P_{q'} = P_q + d_{q,q'}^\top H_q + H_q^\top d_{q,q'}$$

is verified for some $d_{q,q'} \in \mathbb{R}^{d_{x_p} \times d_y}$.

More generally, when each mode is observable arbitrarily fast and admits a high-gain observer, we may avoid the restrictions on the average dwell-time by applying Corollary 4.2.

Example 7.3 (Linear switched systems with observable modes). *Assume now the pairs (A_q, H_q) are observable. Following Example 4.3, there exist matrices $\mathcal{V}_q \in \mathbb{R}^{d_{x_p} \times d_{x_p}}$ such that*

$$\mathcal{V}_q A_q \mathcal{V}_q^{-1} = A + D_q C \quad , \quad H_q \mathcal{V}_q^{-1} = C \quad .$$

Then, define

$$F_{q,\ell}(z, y) = A_q z + L_q(\ell)(y - H_q z)$$

with

$$L_q(\ell) = \mathcal{V}_q^{-1} (D_q + \ell L(\ell) \mathbf{L})$$

and

$$V_{\ell,q}(x_p, z) = (x_p - z)^\top \mathcal{V}_q^\top L(\ell)^{-1} P L(\ell)^{-1} \mathcal{V}_q (x_p - z) \quad .$$

Then, (21a)-(21b) hold with $T(x_p, q) = x_p$,

$$V_\ell((x_p, q), z) = V_{\ell,q}(x_p, z) \quad , \quad F_\ell(z, y) = F_{q,\ell}(z, h_q(x_p))$$

$$\underline{c}(\ell) = \min_{q \in Q} \underline{\lambda}(\mathcal{V}_q^\top P \mathcal{V}_q) \quad , \quad \bar{c}(\ell) = \ell^{2(d_x - 1)} \max_{q \in Q} \bar{\lambda}(\mathcal{V}_q^\top P \mathcal{V}_q) \quad .$$

¹In [8], the more general framework of unknown inputs is considered. Therefore, a first transformation is carried out to isolate the part of x_p that is not impacted by those unknown inputs.

Therefore, whatever the average dwell-time, Problem 1 is solved for ℓ sufficiently large by taking $G(z, y_d) = g_q(z)$ (resp. $G(z, y_d) = \text{sat}(g_q(z))$) if g is Lipschitz (resp. locally Lipschitz and the solutions x are uniformly bounded) as in Example 4.1.

This principle can also be applied to nonlinear switched systems.

Example 7.4 (Switched systems with strongly differentiable modes). Assume that f_q and g_q are single-valued with a single output ($d_y = 1$), and there exists $d_z \in \mathbb{N}$ such that each mode $q \in Q$ is strongly differentially observable of order d_z , namely the maps $T_q : C_q \cup D_q \rightarrow \mathbb{R}^{d_z}$ defined by

$$T_q(x_p) = (h_q(x_p), L_{f_q} h_q(x_p), \dots, L_{f_q}^{d_z-1} h_q(x_p))$$

are injective immersions on $C_q \cup D_q$. Then, following Example 4.4 or [28], if there exist Lipschitz maps Φ_q with uniformly bounded Lipschitz constants and verifying

$$\Phi_q(T_q(x_p)) = L_{f_q}^{d_z} h_q(x_p) \quad \forall x_p \in C_q \cup D_q,$$

a high-gain observer can be designed for each mode with

$$F_{q,\ell}(z, y) = Az + B\Phi_q(z) + \ell L(\ell)K(y - z_1),$$

and the Lyapunov function

$$V_{q,\ell}(x_p, z) = (T_q(x_p) - z)^\top L(\ell)^{-1} P L(\ell)^{-1} (T_q(x_p) - z),$$

where $A, B, K,$ and P are defined in Example 4.4. Then, (21a),(21b) hold with $T(x_p, q) = T_q(x_p)$,

$$V_\ell((x_p, q), z) = V_{q,\ell}(x_p, z) \quad , \quad F_\ell(z, y) = F_{q,\ell}(z, h_q(x_p)),$$

$\underline{c}(\ell) = \underline{\lambda}(P)$, $\bar{c}(\ell) = \bar{\lambda}(P)\ell^{2(d_z-1)}$, and $\lambda = \inf_{q \in Q} \lambda_q > 0$. It remains to choose G Lipschitz with respect to z such that (21c) holds to apply Corollary 4.2. But (21c) says we should have at the jumps

$$G(T_q(x_p), y_d) = T_{q^+}(g_q(x_p)) \quad (45)$$

where q^+ is the next mode. In other words, due to the fact that the change of coordinates T depends on q , we need to know, before the switch, both the current mode and the next. This is possible only if the switches are controlled or deterministic. Otherwise, we need to wait for the switch to happen before the observer can jump. This is exactly the issue studied in Section 6. Suppose the assumptions of Theorem 6.4 hold. If the observer jump (45) is implemented instantaneously after the switch (namely the new mode q^+ is detected instantaneously), $\mathcal{A}_\mathcal{X}$ is UpAS, except during the (instantaneous) delay, since this situation can be modeled by $\dot{\mathcal{H}}(0)$. In the more realistic case where there is a slight delay Δ between the switch and the detection of the new mode q^+ , then, $\mathcal{A}_\mathcal{X}$ is practically stable outside the delay intervals.

7.2. Unobservable individual flow dynamics

We now consider the case where each continuous pair (f_q, h_q) is not necessarily observable individually, but persistent switching between the modes brings determinability, namely the ability to reconstruct the current continuous state x_p by accumulating the past information provided by each mode. This case was handled in [45, 46, 43] by 1) using a partial state continuous-time observer during flow, which estimates the part of x_p that is observable from (f_q, h_q) ; 2) gathering and propagating forward the partial estimates obtained during the past N switches to produce a full estimate of x_p .

However, Section 5 sheds a new light on this problem. Indeed, let us assume that the switches are persistent, namely $\mathcal{C}_{\mathcal{X}_0}[\mathcal{I}]$ holds with \mathcal{I} compact, and that the output is available at the switches, namely y is given by (38b). Consider the switched system discretized at the switching times:

$$\begin{aligned} x_{p,k+1} &\in g_{q_k}(\Psi_{f_{q_k}}(x_k, \tau_k)) \\ y_k &= \left(h_{q_k}(\Psi_{f_{q_k}}(x_k, \tau_k)), \tau_k \right) \end{aligned} \quad (46)$$

where $\tau_k = t_{k+1} - t_k$ denotes the length of the k th switching interval. Note that τ_k was added to the output to encode the fact that it is known and can be used in the design of the observer jump map G . Inspired from Corollary 4.2, it is enough to look for F and G that make

$$z_{k+1} \in G(\Psi_F(z_k, \tau_k), y_k) \quad (47)$$

a UpAS observer for (46) relative to a set $\mathcal{A}_p \subset \mathbb{R}^{d_{x_p}} \times \mathbb{R}^{d_z}$, as long as there exists a class- \mathcal{K} function ρ such that for all $(x_p, z) \in \mathcal{X}_p \times \mathcal{Z}$, for all $q \in Q$, and for all $\tau \in [0, \max \mathcal{I}]$,

$$|(\Psi_{f_q}(x_p, \tau), \Psi_F(z, \tau))|_{\mathcal{A}_p} \leq \rho(|(x_p, z)|_{\mathcal{A}_p}), \quad (48)$$

where \mathcal{X}_p (resp. \mathcal{Z}) denotes a set where x_p (resp. solutions to (47)) evolves. Typically, when \mathcal{A}_p is the diagonal set, we simply take $F(z, y_c) = f_q(z)$ and (48) holds for bounded trajectories.

In particular, when $f_q(x_p) = A_q x_p$, $g_q(x_p) = D_q x_p$, $h_q(x_p) = H_q x_p$ are linear, we take

$$F(z, y_c) = A_q z \quad , \quad G(z, y_d) = D_q z - L_k(H_q z - H_q x_p)$$

where L_k is chosen at hybrid time (t_k, k) based on the known history of (q_k, τ_k) to make the LTV discrete-time error system

$$e_{k+1} = (D_{q_k} - L_k H_{q_k}) \exp(A_{q_k} \tau_k) e_k$$

asymptotically stable, for instance through a discrete Kalman filter.

8. Conclusion

Under the assumption that the plant's jumps can be detected, we have given Lyapunov-based sufficient conditions for asymptotic convergence of an observer for general hybrid systems. Constructive design methods have

been provided, in particular high-gain designs for differentially observable continuous dynamics and discrete-based designs when observability is ensured from the output at jump times. The obtained observers must be synchronized with the plant but we have shown its robustness with respect to delays in its jumps, namely semi-global practical stability of the estimation error outside the delay intervals. All those results provide a new insight for the design of observers for switched systems.

The next step is to develop observers able to synchronize automatically their jumps with the plant's, at least locally, to avoid relying on the often noisy and delayed jump detection. This problem represents a significant challenge since the entire analysis needs to be rethought to handle non-synchronous jumps and ensure contraction of the difference between jump times.

Appendix A. Proof of Theorem 5.3

Consider $\phi = (x, z) \in \mathcal{S}_{\mathcal{H}-\hat{\mathcal{H}}}(\mathcal{X}_0 \times \mathcal{Z}_0)$. By assumption, $x(t, j) \in \mathcal{X}$ for all $(t, j) \in \text{dom } \phi$. Denote $\tau_M = \max \mathcal{I}$. It is easy to see that $(\tilde{x}, \tilde{z}) : \text{dom}_j \phi \rightarrow \mathbb{R}^{d_x} \times \mathbb{R}^{d_z}$ defined by

$$\tilde{x}_k = x(t_k, k) \quad , \quad \tilde{z}_k = z(t_k, k)$$

verifies (25) with input τ defined by $\tau_k = t_{k+1} - t_k$, for all $k \in \text{dom}_j \phi \setminus \{J\}$. Besides, $(\tilde{x}_1, \tilde{z}_1) \in \tilde{\mathcal{X}}_0 \times \tilde{\mathcal{Z}}_0$ by definition. Also, it follows from $\mathcal{C}_{\mathcal{X}_0}[\mathcal{I}]$ that $\tau_0 \leq \tau_M$, and $\tau_k \in \mathcal{I}$ for all $k \in \mathbb{N}_{>0}$ if $J = +\infty$ and for all $k \in \{1, \dots, J-1\}$ otherwise. Therefore, $(x, z)(t_k, k) \in \mathcal{X} \times \mathcal{Z}$ for all k , and according to (26),(27), for all $k \in \text{dom}_j \phi \geq 1$,

$$\begin{aligned} |(x, z)(t_k, k)|_{\mathcal{A}} &\leq \beta(|(x, z)(t_1, 1)|_{\mathcal{A}}, k-1) \\ &\leq \beta(\rho(|(x, z)(0, 0)|_{\mathcal{A}}), k-1) . \end{aligned}$$

This latter inequality still holds for $k = 0$, by appropriately defining $\beta(s, t)$ for $t < 0$ so that $\beta(s, -1) \geq \rho^{-1}(s)$. Besides, we deduce that for all $j \in \text{dom}_j \phi$, $(x, z)(t_j, j) \in \mathcal{X} \times \mathcal{Z}$. By $\mathcal{C}_{\mathcal{X}_0}[\mathcal{I}]$, $t - t_j \in [0, \tau_M]$ for all $(t, j) \in \text{dom } \phi$, so that from (27),

$$\begin{aligned} |(x, z)(t, j)|_{\mathcal{A}} &\leq \rho(|(x, z)(t_j, j)|_{\mathcal{A}}) \\ &\leq \rho(\beta(\rho(|(x, z)(0, 0)|_{\mathcal{A}}), j-1)) \end{aligned}$$

But for all (t, j) in $\text{dom } \phi$, $t - t_j \leq \tau_M$ and $t_j - t_{j-1} \leq \tau_M$ for $j \geq 1$, so that $t_j \leq \tau_M j$ and $t \leq \tau_M(j+1)$. Thus,

$$|(x, z)(t, j)|_{\mathcal{A}} \leq \rho(\beta(\rho(|(x, z)(0, 0)|_{\mathcal{A}}), a(t+j) + b))$$

with $a = \frac{1}{\tau_M+1}$ and $b = \tau_M - 1$. Therefore, Problem 1 is solved.

Appendix B. Proof of Theorem 6.4

The proof relies on [3]. Take a solution $\phi_\delta = (x, z, \mu, \tau_\delta)$ to $\hat{\mathcal{H}}(\Delta)$ for some $\Delta \in [0, \min \mathcal{I}]$ with $(x, z)(0, 0) \in \mathcal{X}_0 \times \mathcal{Z}_0$. Observe that the component x is not impacted by the delay mechanism, therefore, from Assumption 6.2,

$x(t, j) \in \mathcal{X}$ for all $(t, j) \in \text{dom } x$. It follows that ϕ_δ is solution to a hybrid system $\hat{\mathcal{H}}_{\mathcal{X}}(\Delta)$ which has same dynamics as $\hat{\mathcal{H}}(\Delta)$ but with flow set $\hat{C}_{\mathcal{X}}(\Delta) := \hat{C}(\Delta) \cap (\mathcal{X} \times \mathbb{R}^{d_z+3})$ and jump set $\hat{D}_{\mathcal{X}}(\Delta) := \hat{D}(\Delta) \cap (\mathcal{X} \times \mathbb{R}^{d_z+3})$. In the framework of [3], $\hat{\mathcal{H}}_{\mathcal{X}}(\Delta)$ is then the delayed version of the nominal observer $\mathcal{H} - \hat{\mathcal{H}}$ with flow set $\hat{C}_{\mathcal{X}} = (C \cap \mathcal{X}) \times \mathbb{R}^{d_z}$, and jump set $\hat{D}_{\mathcal{X}} = (D \cap \mathcal{X}) \times \mathbb{R}^{d_z}$. By Assumption 6.1 (and by containment [21, Theorem 3.32]), the set \mathcal{A} is still UpAS for $\hat{\mathcal{H}}_{\mathcal{X}}$, and more precisely, the set $\mathcal{A}_{\mathcal{X}}$ (that is compact according to Assumption 6.2). With the hybrid basic conditions, we conclude from [3, Proposition 4.3, Remark 4.4] that the set \mathcal{A}' is UpAS for $\hat{\mathcal{H}}_{\mathcal{X}}(0)$ with basin of attraction containing $\mathcal{X}_0 \times \mathcal{Z}_0 \times \{0\} \times \{-1\}$. \hat{G} is compact by outer-semicontinuity and local boundedness of g and G . \mathcal{A}' is therefore compact. Besides, $\mathcal{X}_0 \times \mathcal{Z}_0 \times \{0\} \times \{-1\}$ is bounded and thus included in a compact subset of the basin of attraction, since the latter is open according to [21, Proposition 7.4]. Still from the hybrid basic conditions, \mathcal{A}' is actually semi-globally practically robustly \mathcal{KL} asymptotically stable for $\hat{\mathcal{H}}_{\mathcal{X}}(0)$ according to [21, Lemma 7.20]. This means that there exists a \mathcal{KL} function β such that for any $\epsilon > 0$, there exists $\rho > 0$ such that any solution ϕ to a ρ -perturbation of $\hat{\mathcal{H}}_{\mathcal{X}}(0)$ initialized in $\mathcal{X}_0 \times \mathcal{Z}_0 \times \{0\} \times \{-1\}$, verifies (37). Since $\hat{\mathcal{H}}_{\mathcal{X}}(\Delta)$ can be included in any outer-perturbation of $\hat{\mathcal{H}}_{\mathcal{X}}(0)$ by taking Δ sufficiently small, (37) holds along solutions of $\hat{\mathcal{H}}_{\mathcal{X}}(\Delta)$ for Δ sufficiently small. Now for ϵ sufficiently small and for sufficiently large $(t, j) \in \mathcal{D}_{-1}$, $|\phi(t, j)|_{\mathcal{A}'} = |\phi(t, j)|_{\mathcal{A}'_{-1}}$ and thus $|(x, z)(t, j)|_{\mathcal{A}} \leq |\phi(t, j)|_{\mathcal{A}'} \leq 2\epsilon$.

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